Muon $g-2$:
Reclaiming the theoretical calculation of the leading QCD contribution

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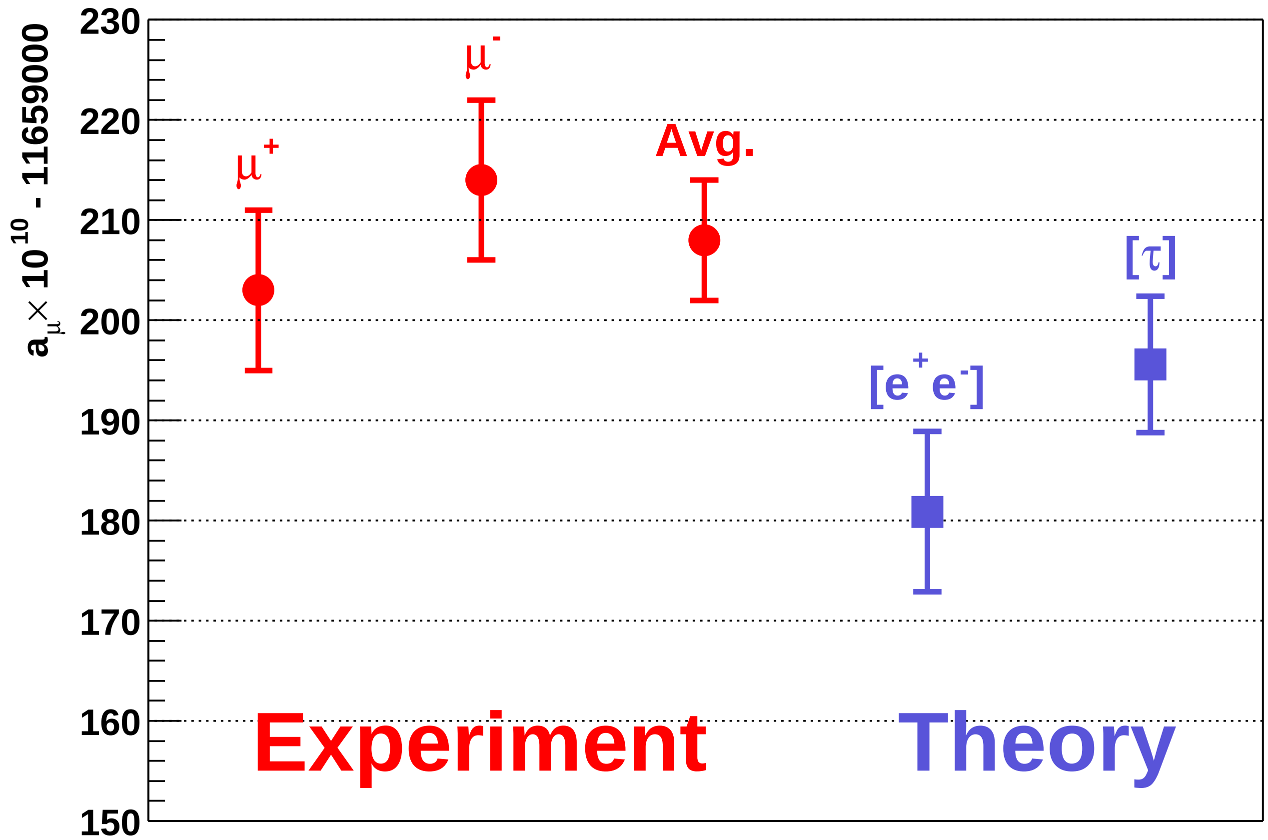
Stupid Question: Why?

- Currently:
  - Experiment: Very precise — 0.5*ppm* (BNL)
  - "Theory":
    - Low compared with experiment
    - Relates $g-2$ to
      1. $e^+e^- \rightarrow$ hadrons cross section and
      2. τ decay cross section
    - Discrepancy with experiment: $0.7\sigma \rightarrow 2.7\sigma$
- Lattice: method to extract hadronic contributions without experimental input

Theory vs. Experiment



$$a_\mu^{\rm exp} = \left(\frac{g-2}{2}\right)^{\rm exp} = 11\,659\,208(6) \times 10^{-10}$$

Outline

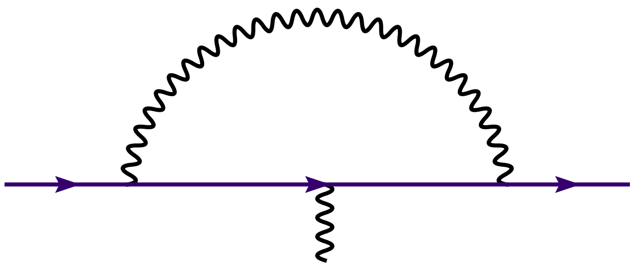
- Muon $g-2$ and current theoretical predictions
- Calculating $g-2$ on the lattice, with Lattice Gauge Theory and Chiral Perturbation Theory
- $O(\alpha^2)$ Contribution: Vacuum Polarization
- Lattice results for vacuum polarization
- Fits and preliminary results for $g-2$

Muon $g-2$

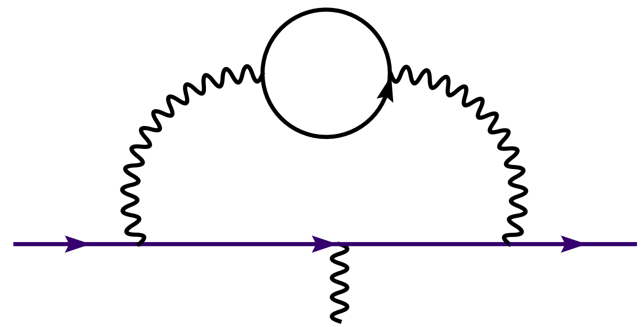
Full muon-photon vertex:

$$\Gamma^\mu = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu} q_\mu}{2m_\mu} F_2(q^2) \qquad a_\mu = \frac{g-2}{2} = F_2(0)$$

$O(\alpha)$:



$O(\alpha^2)$:

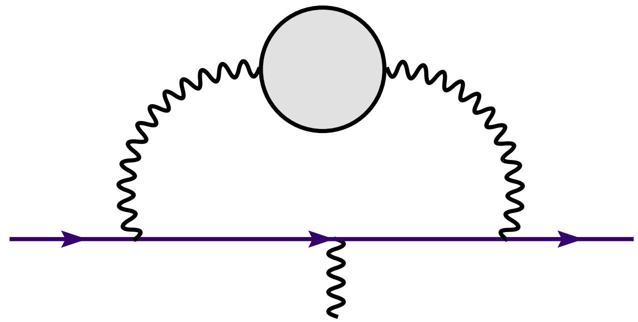


$\vdots$

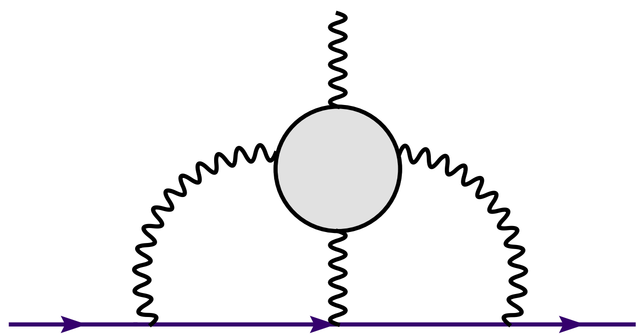
$\vdots$

Hadronic Contributions

$O(\alpha^2)$, Hadronic contribution to the photon vacuum polarization:



$O(\alpha^3)$, Light-by-light scattering:



- Hadronic contributions are 7×10^{-5} times smaller than leading corrections

Leading Hadronic Contribution

The $O(\alpha^2)$ hadronic contribution, a_μ^{HLO}, cannot be calculated in perturbation theory

Using the Optical Theorem, one can evaluate it using
the cross section for $e^+e^- \rightarrow$hadrons:

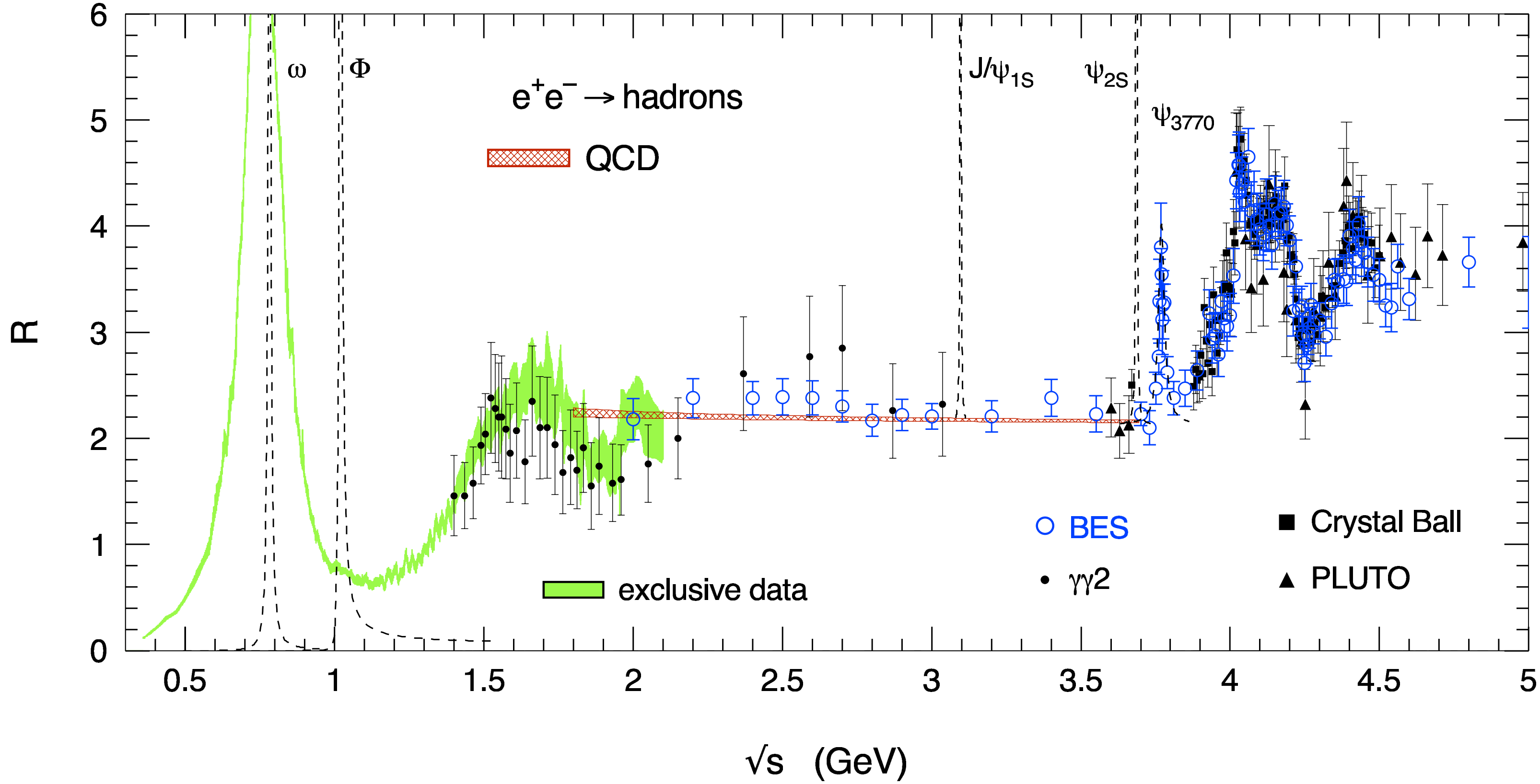
$$a_\mu^{HLO} = \frac{\alpha^2}{3\pi^2}\int_{4m_\pi^2}^{\infty}\frac{ds}{s}K(s)R(s)$$

The kernel, $K(s)$ is known (dominated by small s), and $R(s)$ can be measured experimentally.

Not a theoretical problem since 1961!

$R(s)$

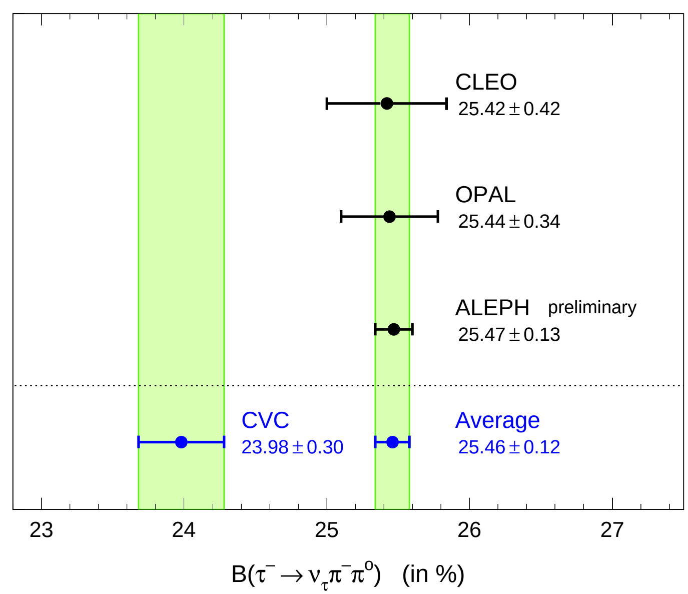
The precision of the Standard Model prediction is limited by the experimental measurement of $R(s)$.



(Davier et al, hep-ph/0208177)

Using τ decay

- Introduced by Alemany et al (hep-ph/9607319)
- In isospin limit, relate τ spectral data to isovector part of $\sigma(e^+e^-)$ using Conserved Vector Current (CVC) relations
- Result for $g-2$ is higher than “standard method”
- Contraversial: Studies have conflicting results on validity of CVC relations
- Either way, still is an experimental calculation, and we want a theoretical one

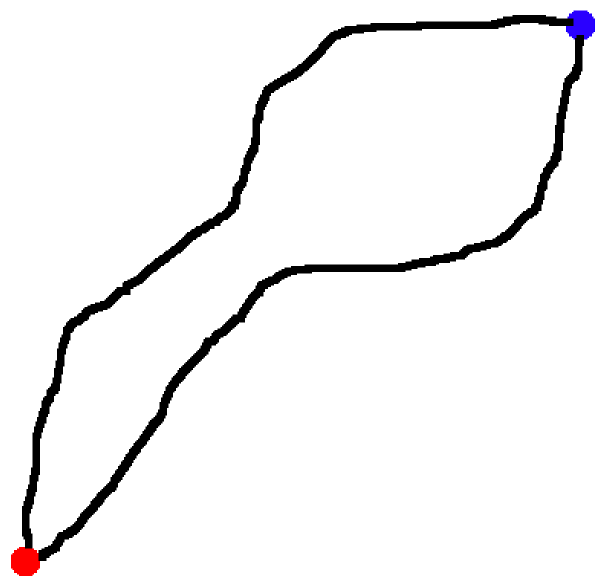


(Davier et al, hep-ph/0208177)

Field Theory

Path Integral: $Z[J] = \int_{\phi(x_a)}^{\phi(x_b)} \mathcal{D}\phi \exp\left\{ i \int d^4x \, [\mathcal{L}[\phi(x)] + J(x)\phi(x)] \right\}$

($\mathcal{D} =$Sum over all paths)

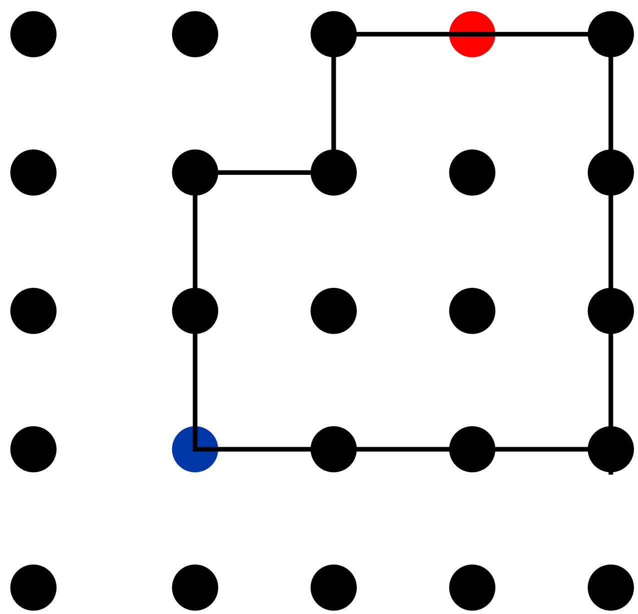


- There are an infinite number of paths!
- Use PT if coupling constant is small (high-E QCD, QED)

Field theory on a lattice 1

To calculate Z (and physical quantities) on the lattice:

- Continue to Euclidean space: $t \rightarrow -it_E$
- Discretize space and time (with a lattice spacing a) and put system in a finite volume V



- Now a finite dimensional path integral

Field theory on a lattice 2

- This is still non-trivial: Finite, but large dimensional integral
- Use Monte Carlo techniques to evaluate Z and whatever matrix element you want (within reason)
- In the end, take $a \to 0$ and $V \to \infty$ (the "continuum limit") and continue back to Minkowski space
- A few comments:
  - We can vary external (valence) and internal (sea) quark masses separately
  - Often $m_{\rm sea} \to \infty$ (Quenched approx) due to limited computational power
  - Finite volume$\Rightarrow$discrete momenta.
  - $p_{\rm min} = 2\pi/T$, where T is the size of the largest direction
  - Quarks on the lattice are a problem...

Simulating Quarks

Quarks are anti-commuting fields $\longrightarrow$ Must integrate over them first in the path integral:

$$Z = \int_{A_\mu,\psi,\bar{\psi}} e^{-S_{QCD}} = \int_{A_\mu} \det K[A] e^{-S_{\text{gluons}}}$$

- $K[A]$ is the Dirac operator for a given set of gauge fields
- $\det K$ is slow to simulate (very non-local), quenched approximation sets this to 1
- For example, pion propagator:

$$\langle \pi^+ \pi^- \rangle = \frac{1}{Z} \int_{A_\mu,\psi,\bar{\psi}} (\bar{u}\gamma_5 d)(\bar{d}\gamma_5 u) e^{-S_{QCD}}$$

Wick contract the quarks to give us quark propagators, which we can evaluate on a given gauge background

Lattice Quarks

Discrete version of the theory has the 15 “doubling symmetries”

$$\psi_x \to e^{i\pi x\cdot p}\Gamma_p \psi_x \qquad \bar{\psi}_x \to e^{i\pi x\cdot p}\bar{\psi}_x \Gamma_p^\dagger$$

$$\begin{aligned} ap &\in \{(1,0,0,0),(0,1,0,0),\ldots,\\ &\quad (1,1,0,0),\ldots,(1,1,1,1)\} \\ \Gamma_p &= \prod_\mu (i\gamma_5\gamma_\mu)^{ap_\mu} \end{aligned}$$

$\Rightarrow$16 species (“tastes”) when $a \to 0$

If ψ_x^0 satisfies the lattice Dirac equation, we have 15 other solutions, ψ_x^p, which are degenerate in mass in the continuum limit

Lattice Quarks

- Many solutions to the doubling problem:
  - Wilson quarks: Slow, breaks chiral symmetry at finite a, difficult to renormalize, but gets rid of all doublers
  - Staggered quarks: Fast, has a remnant chiral symmetry at finite a, still has four species as $a \to 0$
  - Domain-Wall quarks: Slow, has controlled and small chiral symmetry breaking at finite a, no doubling remnants
  - Overlap quarks: VERY slow, but perfect chiral symmetry
- For now we'll choose staggered:
  - Dynamical simulations with Full QCD with very light quark masses
  - Lightest quark masses$\Rightarrow$easier to take chiral limit
  - Largest volumes
  - These lattices already exist (MILC Collaboration)

Staggered Quarks

- On the lattice, the usual continuum $SO(4)$ rotation symmetry is broken to allow only hypercubic rotations
- A unitary transformation on ψ can diagonalize the γ matrices
- This decouples the four spinor components of the fermion $\Rightarrow$ we can keep only one component per species
- We have 16 one-component fields, *staggered* on separate sites of a hypercube $\Rightarrow$ 4 four-component Dirac *tastes*, degenerate as $a \rightarrow 0$

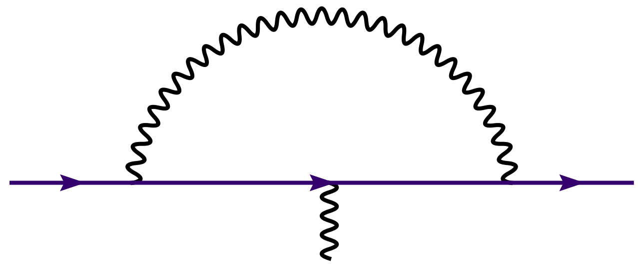
Aside: $4 \to 1$ tastes

- Evaluate the staggered quark path integral$\Rightarrow \det K$
- $\det K$ describes four degenerate tastes in the continuum limit
- $\Rightarrow (\det K)^{1/4}$ describes 1 taste

- Can we do this *before* taking the continuum limit?
- At finite a, we have violations of the taste symmetry (ie the four quark species are not degenerate in mass for $a \neq 0$)

- Won't worry about this now:
  - There is evidence that this isn't a problem
  - ***Lots*** of people trying to figure out if it is/isn't a problem
- "Fourth-root" can be taken into account in chiral perturbation theory with staggered quarks.

Vertex Correction



- Apply Feynman rules, take external $q^2 \to 0$, go to Euclidean space, and performing angular rotations, we get

$$a_\mu^{(1)} = \frac{\alpha}{\pi} \int_0^\infty dK^2 f(K^2)$$

- $f(K^2)$ is a known function of K^2 and m_μ^2

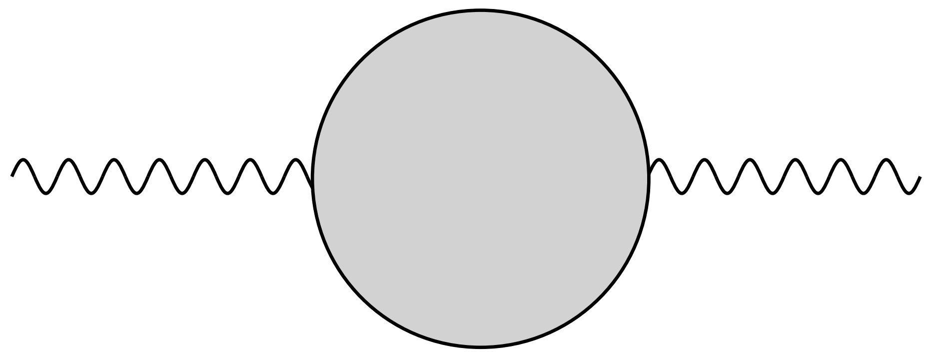
- Integral is finite and gives precisely

$$a_\mu^{(1)} = \frac{\alpha}{2\pi}$$

- Lot of work for something we already know...

Leading Hadronic Contribution

We want to insert the quark loop
into the vacuum polarization:



- We can apply this procedure to the $O(\alpha^2)$ hadronic contribution to a_μ to get (Blum, 2003)

$$a_\mu^{(2)\mathrm{had,LO}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2)$$

$$\hat{\Pi}(K^2) = 4\pi^2 \sum_i Q_i^2 [\Pi_i(K^2) - \Pi_i(0)]$$

Leading Hadronic Contribution

So now we just need to evaluate $\Pi(q^2)$ on the lattice,
and plug it into our expression for a_μ

First some comments about $f(K^2)$:

- $f(K^2) \sim 1/(2m_\mu\sqrt{K^2})$ for small K^2
- diverges as $K^2 \to 0 \Longrightarrow$ dominated by low momentum region
- Need large lattices to reach these low momenta

Lattice Calculation of $\Pi^{\mu\nu}$

- Calculate the vacuum polarization using the conserved current

$$\Pi^{\mu\nu}(q) = \int d^4x e^{iq\cdot(x-y)} \langle J^\mu(x) J^\nu(y) \rangle = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$$

- Continuum J^μ satisfies $\partial_\mu J^\mu = 0$:

$$J^\mu = \bar{\psi}\gamma^\mu\psi$$

- On the lattice this is a point-split current:

$$J_\mu(x) = \frac{1}{2}\left[\bar{\psi}(x+a\hat{\mu})U_\mu^\dagger(x)(1+\gamma^\mu)\psi(x) - \bar{\psi}(x)U_\mu(x)(1-\gamma^\mu)\psi(x+a\hat{\mu})\right]$$

- Satisfies

$$\sum_\mu \frac{J_\mu(x) - J_\mu(x-a\mu)}{a} = 0$$

Lattice Calculation of $\Pi^{\mu\nu}$

- Discrete version satisfies a discrete Ward Identity, so

$$\Pi^{\mu\nu}(q) = (\hat{q}^2 \delta^{\mu\nu} - \hat{q}^\mu \hat{q}^\nu)\Pi(\hat{q}^2)$$

  with

$$\hat{q}^\mu = \frac{2}{a} \sin\left(\frac{aq^\mu}{2}\right)$$

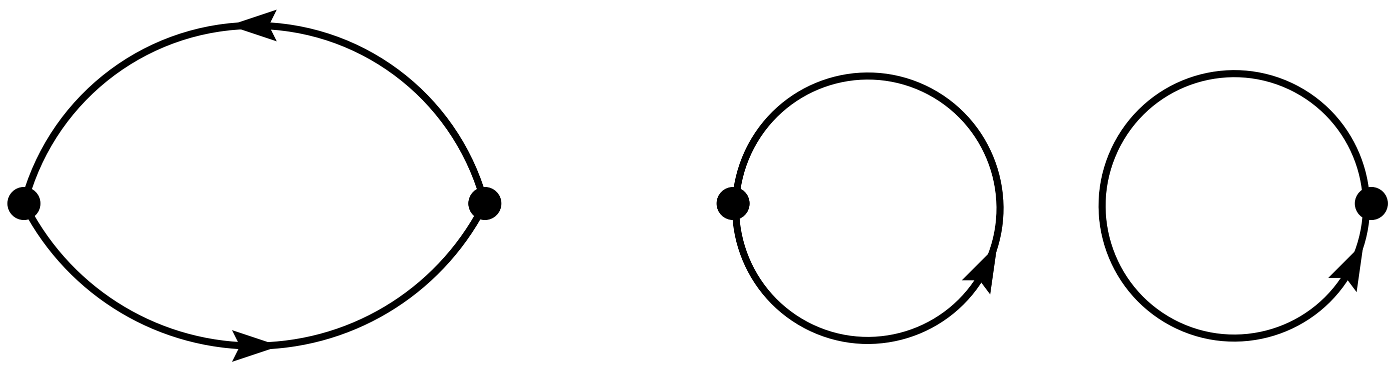
  and

$$q^\mu = \frac{2\pi n^\mu}{aL_\mu}$$

- WI provides strong check on the calculation!

Lattice Calculation of $\Pi^{\mu\nu}$

- To perform lattice calculation: Wick contract the quark fields in $\langle J^\mu(x) J^\nu(y) \rangle$, giving two types of contractions:



- Fourier transform to get $\Pi^{\mu\nu}$
- We neglect second contraction (suppressed, also *very noisy*)
- For more details on the lattice calculation, see
  - T. Blum, PRL 91 052001, 2003—Quenched Domain-Wall Quarks
  - T. Blum, Confinement 2003 (hep-lat/0310064)—Includes staggered calculations

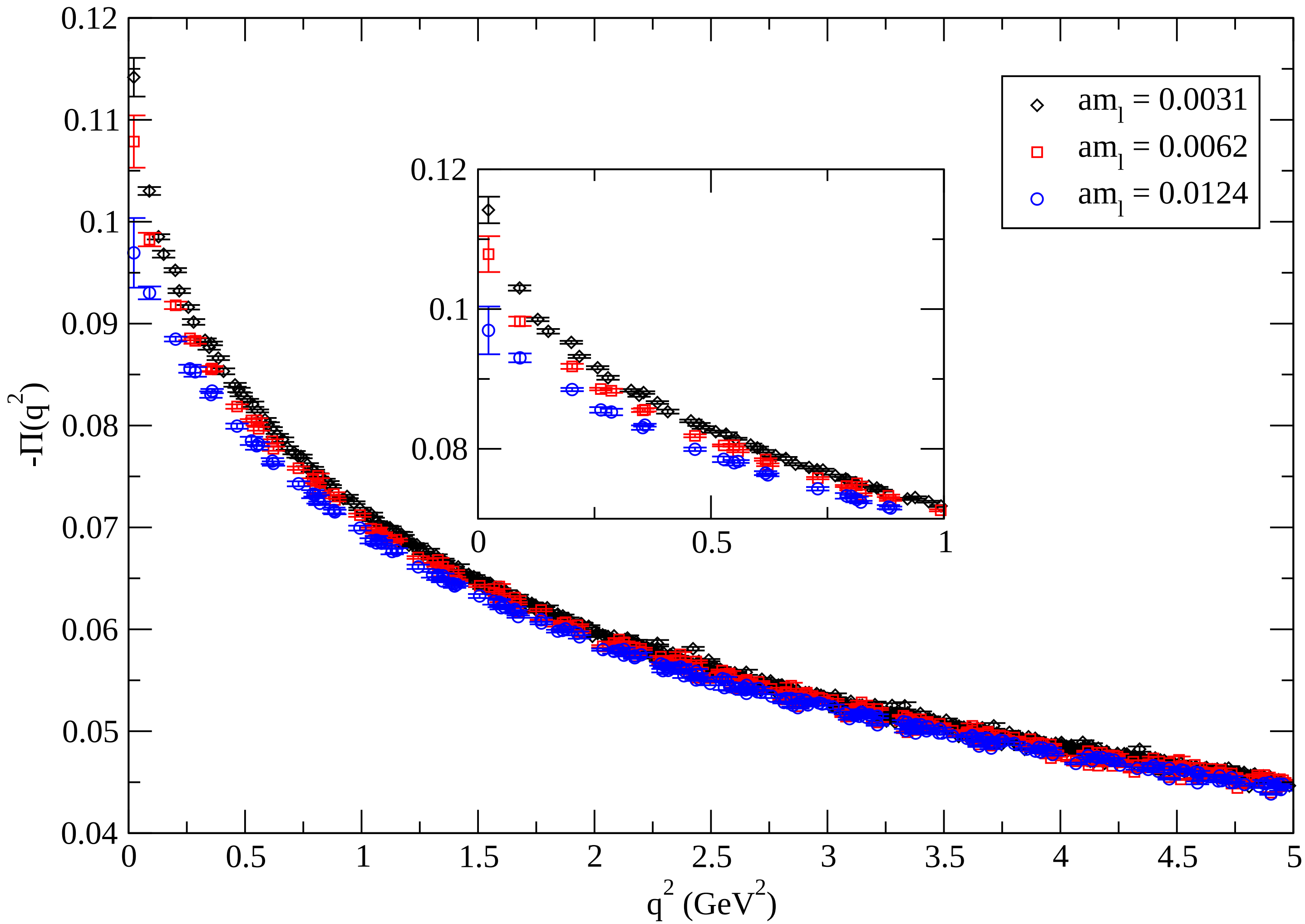
Simulation parameters

- On the lattice, “Full QCD”$= 2+1$ flavors (c, b, t integrated out):
  - 1 “heavy” flavor, the strange quark at physical m_s
  - 2 light flavors: $m_u = m_d \equiv m_l \gtrsim m_s/10$ (can’t yet simulate at “real” m_u or m_d)
- These are “Improved staggered” configurations (so we have smaller lattice spacing errors)

MILC $2+1$-flavor Configurations

a (fm)	Volume	am_l	am_s	$am_{\rm val}$
0.086(2)	$28^3 \times 96$	0.0124	0.031	0.031
0.086(2)	$28^3 \times 96$	0.0124	0.031	0.0124
0.086(2)	$28^3 \times 96$	0.0062	0.031	0.031
0.086(2)	$28^3 \times 96$	0.0062	0.031	0.0062
0.086(2)	$40^3 \times 96$	0.0031	0.031	0.0031
0.086(2)	$40^3 \times 96$	0.0031	0.031	0.031

Simulation Results ($2+1$ Staggered)



Fitting $\Pi(q^2)$

- High-q^2 easy: Use continuum PT
- Low-q^2 is tough:
  - Simple polynomials? These undershoot the data for lowest q^2
  - Physics-based models, like Chiral Perturbation Theory (χPT)?
    - χPT is an expansion in mass/energy of pions
    - Since it's good for low-energy processes, could work here, for the low-q^2 region

χPT—Chiral Symmetry

As $m_q \to 0$ $(q = u, d, s)$, QCD has an $SU(3)_L \times SU(3)_R$ chiral symmetry.

$$q_L \to L q_L \,, \quad q_R \to R q_R$$

$SU(3)_L \times SU(3)_R \to SU(3)_V$ by a nonvanishing quark condensate

$$\langle \bar{q}_R q_L \rangle \neq 0$$

$\Rightarrow$ 8 massless bosons: $\pi^{\pm}, \pi^0, K^{\pm}, K^0, \overline{K}^0, \eta$

Put the pions in the field Σ ($\Sigma \to L \Sigma R^{\dagger}$ under the chiral symmetry)
To leading order in the pion momentum

$$\mathcal{L}_{\text{kin}} \propto \mathrm{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma^{\dagger}]$$

Mass in χPT

We know the pions are not massless, and neither are the light quarks.

Mass term in QCD looks like

$$\mathcal{L}_{\mathrm{QCD},m} = \bar{q}_L M q_R + \bar{q}_R M q_L$$

where M is the 3×3 light quark mass matrix.

Mass term in χPT should transform like the QCD mass term, so we have

$$\mathcal{L}_{\mathrm{mass}} \propto \mathrm{Tr}[M\Sigma + \Sigma^\dagger M]$$

Staggered χPT for 3 light flavors

Lee & Sharpe, PRD 60, 114503; CA & Bernard, PRD 68 034014 & 074011

- Light mesons: $\Sigma = \exp(i\Phi/f)$, with

$$\Phi = \begin{pmatrix} U & \pi^+ & K^+ \\ \pi^- & D & K^0 \\ K^- & \bar{K}^0 & S \end{pmatrix}$$

- Components above are 4×4 matrices
- Under chiral $SU(12)_L \times SU(12)_R$: $\Sigma \to L\Sigma R^\dagger$
- $\mathcal{L}$ is an expansion in
  - $m_\pi^2 \sim m_q$; m_q is a light quark mass
  - a^2, the lattice spacing

Staggered χPT

$$\mathcal{L} = \frac{f^2}{8}\,\mathrm{Tr}[\partial_\mu \Sigma \partial^\mu \Sigma] + \frac{\mu f^2}{4}\,\mathrm{Tr}[\mathcal{M}(\Sigma + \Sigma^\dagger)] - a^2 \mathcal{V}_\Sigma$$

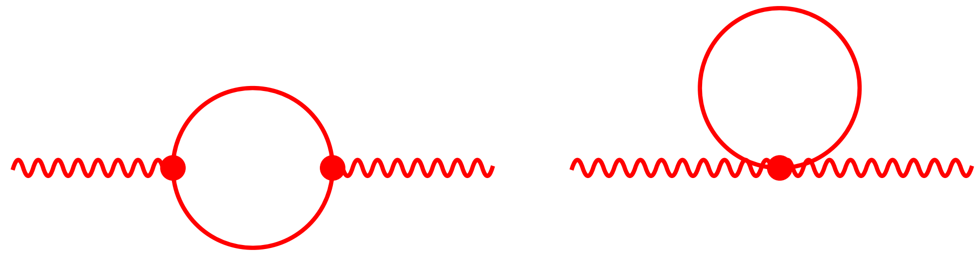
- $\mathcal{M}$: Light quark mass matrix
- $\mathcal{V}_\Sigma$: Taste-breaking potential arising from four-quark operators.
- f: tree-level pion decay constant
- For each pion: 16 tastes [in degenerate $SO(4)$ representations: P, A, T, V, S] with masses:

$$m_t^2 = \mu(m_a + m_b) + a^2 \Delta_t, \quad (t = P, A, T, V, S)$$

- Taste violations at finite lattice spacing $\Rightarrow$ $\Delta_t \neq 0$
- Remnant chiral symmetry$\Rightarrow$ $\Delta_P = 0$
- To include photons:

$$\partial_\mu \Sigma \to \partial_\mu \Sigma + ieA_\mu [Q, \Sigma]$$

One-loop pion contribution



- One-loop pion/kaon contribution:

$$\Pi_M(q^2) = \frac{\alpha}{4\pi}\left\{\frac{1}{3}(1+x_M)^{3/2}\ln\left(\frac{\sqrt{1+x_M}+1}{\sqrt{1+x_M}-1}\right) - \frac{2x_M}{3} - \frac{8}{9} + \frac{1}{3}\ln\left(\frac{m_M^2}{\Lambda^2}\right)\right\}$$

$$\Pi(q^2) = \frac{1}{16}\sum_t\left[\Pi_{\pi_t}(q^2) + \Pi_{K_t}(q^2)\right] + \text{c. t.}$$

$$x = 4m^2/q^2$$

- Sum over t is a sum over the 16 tastes
- Nice: No free parameters (besides counterterm—this is just a constant)
- Bad: Two orders of magnitude too small!

SχPT with vectors

- Without sea quarks (quenched), $\Pi(q^2)$ is dominated by effects of the ρ (QCDSF), perhaps they play a role here...
- Use resonance formalism of Ecker, Gasser, and Pich [NPB 321 311 (1989)]
- Incorporate vectors into field $V_{\mu\nu}$ so that under chiral $SU(12)_L \times SU(12)_R$:

$$V_{\mu\nu} \to U V_{\mu\nu} U^\dagger$$

where $U \in SU(12)$ is defined as

$$\sigma \to L\sigma U^\dagger = U\sigma R^\dagger$$

with $\sigma^2 = \Sigma$

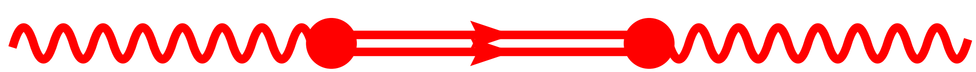
SχPTwith vectors

- So we have the interaction Lagrangian

$$\mathcal{L}_{\rm vec} = \frac{f_V}{2\sqrt{2}}\,{\rm Tr}\left[V_{\mu\nu}(\sigma F^{\mu\nu}\sigma^\dagger + \sigma^\dagger F^{\mu\nu}\sigma)\right] + \ldots$$

$$F^{\mu\nu} = eQ(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

- $V_{\mu\nu}$ is a 12×12 matrix with the 8 lightest vector mesons (each with 16 tastes)
- Empirically taste violations among vectors are small–Will ignore them here
- Leading contribution to the photon vacuum polarization is at tree level:



SχPT with ρ

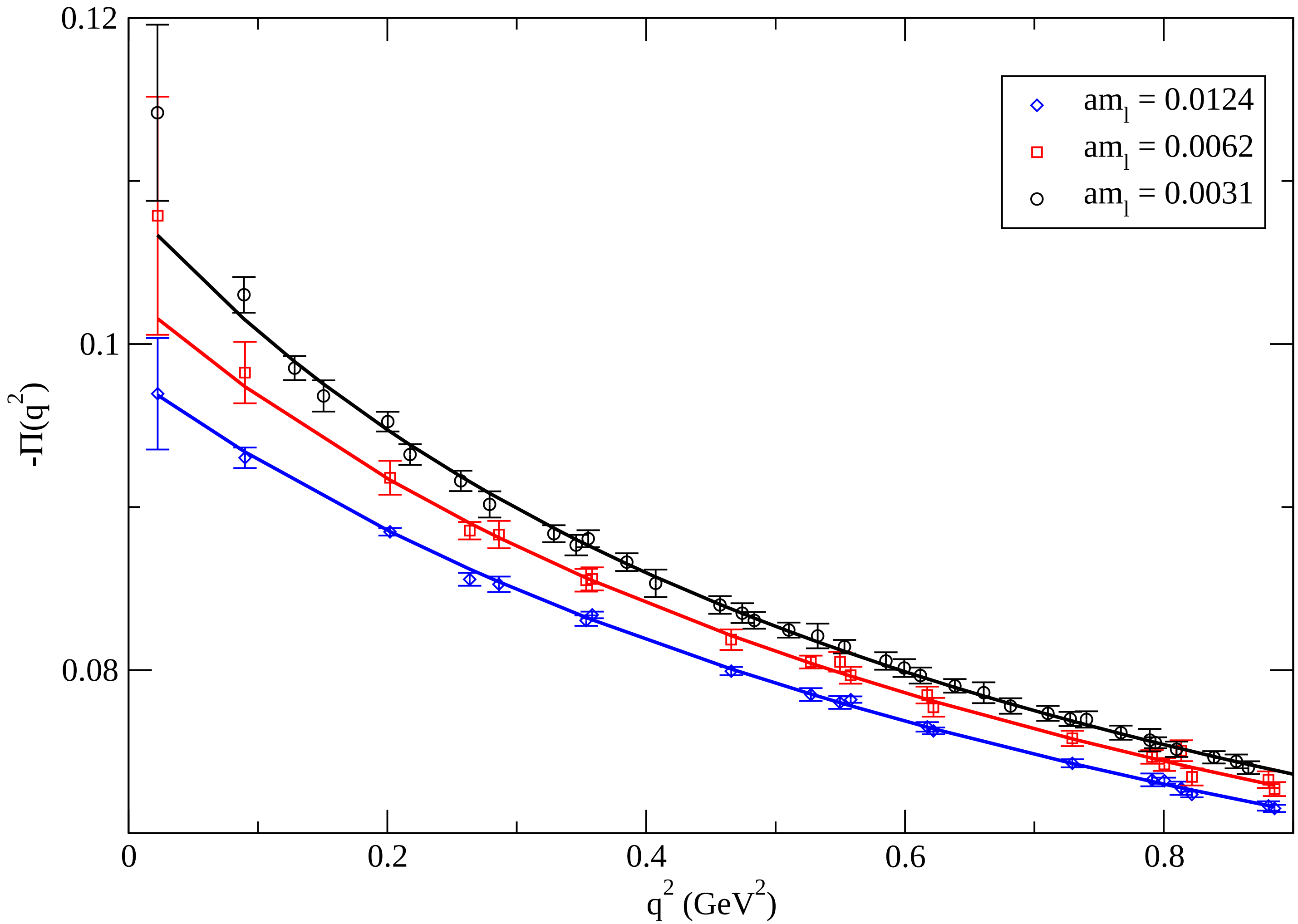
- Tree-level result:

$$\Pi_V(q^2) = -\frac{\alpha}{4\pi}\frac{(4\pi)^2 f_V^2}{3}\frac{4}{3}\frac{1}{q^2+m_\rho^2}$$

- Although the masses are heavy, the numerator has enhancement of $(4\pi)^2 f_V^2$.
- There are no free parameters: The masses and f_V can be measured directly in the simulations (f_V not measured yet)
- One-loop calculation: only tadpole corrections to $\rho-$photon vertex

$$\Pi_V^{1-\mathrm{loop}}(q^2) = \frac{\alpha}{4\pi}\left(\frac{8f_V^2}{f^2}\right)\frac{1}{q^2+m_\rho^2}\sum_t\left[m_{\pi_t}^2\ln m_{\pi_t}^2 + m_{K_t}^2\ln m_{K_t}^2\right]$$

Fit to χPT$+\rho$ result



Preliminary Results

$$
\begin{aligned}
a_\mu^{\mathrm{had,VP}}(\infty) &= 367(12) \times 10^{-10} \\
a_\mu^{\mathrm{had,VP}}(0.0124) &= 431(7) \times 10^{-10} \\
a_\mu^{\mathrm{had,VP}}(0.0062) &= 509(14) \times 10^{-10} \\
a_\mu^{\mathrm{had,VP}}(0.0031) &= 636(8) \times 10^{-10} \\
a_\mu^{\mathrm{had,VP,pert}}(\mathrm{phys}) &\lesssim 10 \times 10^{-10} \\
a_\mu^{\mathrm{had,disp}}(\mathrm{phys}) &= 693.4(5.3)(3.5) \times 10^{-10}
\end{aligned}
$$

- Statistical errors only
- Possibly large uncertainties:
  - Low-q^2: Still undershoots at small mass, although not as much as a simple polynomial fit
- Last line is from e^+e^- data and dispersion relation [A. Hocker, ICHEP 2004]

Preliminary Results

How to extrapolate?

- We have f_V, m_V, and pion/kaon masses all as functions of the light quark mass $\Rightarrow$ Could extrapolate these to physical point $am_l \approx 0.001$...
- Extrapolation to physical point: Must go through the 2π threshold (and m_V is not a linear function of m_l for light quark masses)
- The three values for $a_\mu^{\mathrm{had,VP}}$ show significant curvature as a function of m_l: Quadratic fit?

Quadratic fit of $a_\mu^{\mathrm{had,VP}}$ vs. m_l gives:

$$a_\mu^{\mathrm{had,VP}}(\mathrm{phys}) \approx 726 \times 10^{-10}$$

(Errors are not shown on purpose!)

Summary

- Haven't included "disconnected diagrams" in lattice calculation (noisy)
- Functional form from χPT+ρ fits well to lattice data with few unknown parameters, but not ideal
- Need to understand why fit undershoots data: Bad fitting form or are we missing something?
- Issues/Future needs:
  - Study possible finite volume problems
  - Starting calculations on coarse MILC lattices $\Rightarrow$Continuum limit!
  - Twisted BCs to get more low-q^2 points?

Thanks to RIKEN & US DOE for calculations
Thanks to MILC for configurations