What's Missing from Braginskii and Drift Kinetics

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I.) Can't correctly model tokamak drift wave phenomena or obtain edge and core fluid equations from Braginskii: recall $p\vec{V}_{\perp} \sim \vec{q}_{\perp}$

Realized by Mikhailovskii and Tyspin first, but they made errors

II.) Hazeltine drift kinetics missing the perpendicular portions of gyroviscosity as well as the perpendicular viscosity

Can't use to directly evaluate Neoclassical radial electric field

III.) Gyrokinetics

IV.) Summary & Comments: need hybrid fluid-kinetic codes

LOGO

I. What's Missing From Braginskii?

MHD vs. Drift Ordering (Short Mean Free Path)

MHD Ordering: seldom appropriate for tokamaks

Mean ion flow velocity = $\vec{V} \sim v_i$ = ion thermal speed

so $p\vec{V} >> \vec{q}$ = ion heat flux (with p = ion pressure)

Braginskii: Zh. Exp. Teor. Fiz. **33**, 459 (1957) [Sov. Phys. JETP **6**, 358 (1958)]& *Reviews of Plasma Physics*, V.1, p.205 (1965)
Robinson and Bernstein: Ann. Phys. **18**, 110 (1962)

But typically $pV \sim q$:

$$\vec{V} \approx (c/B)\vec{b} \times \nabla\Phi + (c/enB)\vec{b} \times \nabla p + \vec{V}_{||}$$
$$\vec{q} \approx (5cp/2eB)\vec{b} \times \nabla T - (125p/32M\nu)\vec{b} \cdot \nabla T$$

Note: Braginskii assumes all components of $\vec{V} \sim v_i$

MHD vs. Drift Ordering (Short Mean Free Path)

Drift Ordering: tokamak ordering

Mikhailovskii: Sov. Phys. JETP **25**, 623 (1967)

Mikhailovskii and Tsypin: Plasma Phys. **13**, 785 (1971) and Beitr. Plasmaphys. **24**, 335 (1984)

$p\vec{V} \sim \vec{q}$ = ion heat flux (with p = ion pressure)

M&T recovered <u>all</u> of Braginskii plus new terms in viscosity,
$\vec{\vec{\pi}} = M\int d^3 v f(\vec{v}\vec{v} - \vec{\vec{I}}v^2/3) = \vec{\vec{\pi}}_{||} + \vec{\vec{\pi}}_g + \vec{\vec{\pi}}_{\perp}$, associated with heat flux

For example, in gyroviscosity $p\nabla\vec{V} \rightarrow p\nabla\vec{V} + (2/5)\nabla\vec{q}$

Also, M&T did not consider electrons

BUT Need to Correct Mikhailovskii and Tsypin

M&T use a truncated polynomial expansion for the ion distribution function = f and a moment approach.

1) They neglect terms in the ion viscosity due to the full non-linear form of the collision operator: their parallel, $\vec{\vec{\pi}}_{\|}$, and perpendicular, $\vec{\vec{\pi}}_{\perp}$, ion viscosities are missing $q_{\|}^2$ and $q_{\perp}^2$ terms (their gyro-viscosity $\vec{\vec{\pi}}_g$ is ok to lowest order)

2) They truncate the polynomial expansion of the gyrophase dependent part of f too soon: their perpendicular ion viscosity is incorrect ($\vec{\vec{\pi}}_g$ ok to lowest order) - need full gyrophase dependence

3) Their gyroviscosity is missing the collisional heat flow: the $\vec{q}$ in $p\nabla\vec{V} \rightarrow p\nabla\vec{V} + (5/2)\nabla\vec{q}$ should include the collisional heat flux as well as the diamagnetic and parallel

Details: Catto & Simakov, PoP **11**, 90 (2004) & **12**, 114503 (2006)

C&S Drift Ordering Expansion Parameters

Solve for f to obtain the viscosity by expanding in:

$$\delta = \rho/L_\perp << 1 \quad \text{and} \quad \Delta = \lambda/L_{||} << 1$$

where ρ = ion gyroradius, λ = mean free path, $L_\perp$ = perpendicular scale length, & $L_{||}$ = parallel scale length

We allow $L_\perp << L_{||}$ as well as $L_\perp \lesssim L_{||}$ to permit

$$\delta \sim \Delta$$

The $\delta \sim \Delta$ ordering allows $\vec{q}_\perp \sim \vec{q}_{||}$ and retains all turbulent and neoclassical effects in tokamaks - more general than Pfirsch-Schlüter ordering $\delta << \Delta$

If $\nabla T = 0$, then rigid rotating ion Maxwellian is the solution - all complications are due to ∇T

Example: Gyroviscosity

The species gyroviscosity is of the form

$$\vec{\vec{\pi}}_g = \frac{p}{4\Omega}\left[\vec{b}\times(\vec{\vec{\alpha}}+\vec{\vec{\alpha}}^T)\cdot(\vec{\vec{I}}+3\vec{b}\vec{b})-(\vec{\vec{I}}+3\vec{b}\vec{b})\cdot(\vec{\vec{\alpha}}+\vec{\vec{\alpha}}^T)\times\vec{b}\right]$$

with

Braginskii: $\vec{\vec{\alpha}} = \nabla\vec{V}$

Mikhailovskii & Tsypin: $\vec{\vec{\alpha}} = \nabla\vec{V} + (2/5p)\nabla(\vec{q}_{||} + \vec{q}_{dia})$

Catto & Simakov: $\vec{\vec{\alpha}} = \nabla\vec{V} + (2/5p)\nabla(\vec{q}_{||} + \vec{q}_{dia} + \vec{q}_{coll})$

Expressions are also available for

ion $\vec{\vec{\pi}}_{||}$: a bit simpler than $\vec{\vec{\pi}}_g$

electron $\vec{\vec{\pi}}_{||}$: slightly messier than the ion expression

ion $\vec{\vec{\pi}}_{\perp}$: really messy, but neoclassics only requires $\nabla\vec{V}$ terms

(Braginskii good enough for $\vec{\vec{\pi}}_{\perp}$ in a tokamak as $\vec{\vec{\pi}}_g$ is q^2 larger)

Aside: need all of $\vec{\vec{\pi}}_{\perp}$ unless have drift departures from surfaces

Parallel Ion Viscosity:

$$\overleftrightarrow{\pi}_{\|} = \frac{0.32}{\nu}(\overleftrightarrow{I} - 3\vec{b}\vec{b}) : [(p\nabla\vec{V} + \tfrac{2}{5}\nabla\vec{q}) + 0.25(\nabla\vec{q} - \vec{q}\nabla \ln p + \tfrac{4}{15}\nabla\vec{q}_{\|})](3\vec{b}\vec{b} - \overleftrightarrow{I})$$

$$+ \frac{M}{3pT}[0.41 q_{\|}^2 - 0.06 q^2](3\vec{b}\vec{b} - \overleftrightarrow{I})$$

Perpendicular Ion Viscosity: $\overleftrightarrow{\pi}_{\perp} = \overleftrightarrow{\pi}_{\perp 1} + \overleftrightarrow{\pi}_{\perp 2}$

$$\overleftrightarrow{\pi}_{\perp 1} = -\frac{3\nu}{10\Omega^2}[\overleftrightarrow{W} + 3\vec{n}\vec{n}\cdot\overleftrightarrow{W} + 3\overleftrightarrow{W}\cdot\vec{n}\vec{n} + (1/2)(\overleftrightarrow{I} - 15\vec{n}\vec{n})(\vec{n}\cdot\overleftrightarrow{W}\cdot\vec{n})$$

$$-(1/2)(\overleftrightarrow{I} - \vec{n}\vec{n})(\overleftrightarrow{W}:\overleftrightarrow{I})]$$

$$\overleftrightarrow{\pi}_{\perp 2} = -\frac{9M\nu}{200pT\Omega}[(\vec{q} + \frac{31}{15}\vec{q}_{\|})(\vec{n}\times\vec{q}) + (\vec{n}\times\vec{q})(\vec{q} + \frac{31}{15}\vec{q}_{\|})]$$

$$\overleftrightarrow{W} \equiv p\nabla\vec{V} + \frac{2}{5}\nabla\vec{q} - \frac{3}{10p}(p\nabla\vec{q} - \vec{q}\nabla p) - \frac{1}{100p}(3p\nabla\vec{q}_{\|} + 5\vec{q}_{\|}\nabla p)$$

$$-\frac{1}{400T}(90\vec{q} - 13\vec{q}_{\|})\nabla T + \text{Transpose}$$

Heat Flow Implies Momentum Transport

Viscosity associated with heat flow leads to flows $\Rightarrow V(r,t) \neq 0$

$$\frac{\partial V}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\chi\frac{\partial}{\partial r}\left(V + \frac{2q}{5p}\right)\right] \quad \text{and} \quad V(r, t = 0) = 0$$

- Assume fewer hot ions carry heat in the positive direction on A than cold negative flowing ions: no net particle flow but a positive directed heat flow

- Collisional colds radially random walk faster than the few hots: resulting in a negative particle flow on B - a radial toroidal angular momentum flux has occurred

- Transient push off a wall imparts positive plasma spin

Collisional Moment Equations and Closure

The conservation of number, momentum and energy equations plus the Maxwell equations are to be solved

The most straightforward procedure is the one used in MHD:
1) total momentum $\Rightarrow \vec{V}$
2) electron momentum $\Rightarrow \vec{E}$
3) Faraday's law $\Rightarrow \vec{B}$ ($\nabla \cdot \vec{B} = 0$ automatic)
4) Ampere's law $\Rightarrow \vec{J}$ ($\nabla \cdot \vec{J} = 0$ automatic, no need for vorticity)

Vorticity equations in reduced two fluid descriptions derived by making approximations unable to recover neoclassical radial $\vec{E}$

Collisional moment equations retain turbulence & neoclassical:
a) $\vec{q}_{||}$ & $\vec{\pi}_{||}$ evaluated to lowest order in $\Delta \sim \delta$ & δ^2, respectively
b) diamagnetic & collisional parts of $\vec{q}_{\perp}$ of order δ & $\nu\delta/\Omega$
c) $\vec{\pi}_g$ is of order δ^2 and $\vec{\pi}_{\perp}$ is of order $\nu\delta^2/\Omega$

Self-Consistent Closure: Multiple Length Scales ($\delta \sim \Delta$)

1) perpendicular momentum: flows diamagnetic $\Rightarrow \vec{V}_\perp \sim \delta v_i$

2) continuity: $\partial n/\partial t \sim \nabla_\perp \cdot (n\vec{V}_\perp) \Rightarrow \omega \sim \delta^2 \Omega$

3) parallel momentum: $Mn\partial V_{\|}/\partial t \sim \nabla_{\|} p \Rightarrow V_{\|}/v_i \sim v_i/\omega L_{\|} \sim L_\perp/\delta L_{\|}$

4) $\vec{V}_\perp \sim V_{\|} \Rightarrow L_\perp/L_{\|} \sim \delta^2$

note: $Mn\partial V_{\|}/\partial t \sim \vec{b}\cdot(\nabla\cdot\vec{\vec{\pi}}_g) \sim p\delta^2/L_\perp$, but $\vec{b}\cdot(\nabla\cdot\vec{\vec{\pi}}_{\|}) \sim p\delta^2/L_{\|}$

5) check: energy balance: $n\partial T/\partial t \sim \nabla_\perp \cdot \vec{q}_{dia} \Rightarrow \omega p \sim p\delta v_i/L_\perp$

Next order corrections: $L_\perp/L_{\|} \sim \delta^2 \sim \nu/\Omega$

1) continuity: $\nabla_{\|}\cdot(n\vec{V})/\nabla_\perp\cdot(n\vec{V}) \sim \delta^2$

2) perpendicular momentum: $\vec{\vec{\pi}}_{\|} + \vec{\vec{\pi}}_g \sim \delta^2 p$

3) parallel momentum: $\vec{b}\cdot(\nabla\cdot\vec{\vec{\pi}}_{\|})/\nabla_{\|} p \sim \delta^2 \sim \vec{b}\cdot(\nabla\cdot\vec{\vec{\pi}}_\perp)/\nabla_{\|} p \sim \nu/\Omega$

4) energy balance: viscous heating is a δ^2 correction &

$$\nabla_{\|}\cdot \vec{q}_{\|}/\nabla_\perp\cdot\vec{q}_{dia} \sim L_\perp/L_{\|} \sim \delta^2 \sim \nabla_\perp\cdot\vec{q}_{coll}/\nabla_\perp\cdot\vec{q}_{dia} \sim \nu/\Omega$$

Also: Pfirsch-Schlüter closure for $\delta << \Delta$ & $L_\perp \sim L_{\|}$

and $V_{\|} \sim v_i >> \vec{V}_\perp \sim \vec{q}/p$ and $L_\perp/L_{\|} \sim \delta \sim \nu/\Omega$

Pfirsch-Schlüter Radial Electric Field: ($\delta << \Delta$ & $L_\perp \sim L_\parallel$)

Knowing viscosity can evaluate radial electric field from toroidal angular momentum conservation: $\langle R^2 \nabla\zeta \cdot \vec{\vec{\pi}} \cdot \nabla\psi \rangle = 0$ for $\vec{B} = I\nabla\zeta + \nabla\zeta \times \nabla\psi$ with ζ the toroidal angle variable [details Catto & Simakov, PoP **12**, 012501 (2005)]

Why doesn't $\vec{E}$ agree with Hazeltine drift kinetic PS result? [Hazeltine PF **17**, 961 (1974)] - instead, $\vec{E}$ agrees with large aspect ratio result of Claassen & Gerhauser, Czech. J. Phys. **49**, 69 (1999)

1) Must use $\vec{B} \cdot (\nabla p + en\nabla\Phi) = 0$ not $\nabla p = 0 = \nabla\Phi$

2) Hazeltine drift kinetic equation [Hazeltine Plasma Phys. **15**, 77 (1973)] missing parallel parts of gyroviscosity: direct evaluation gives $p\nabla\vec{V}_\parallel + (5/2)\nabla\vec{q}_\parallel$ instead of $p\nabla\vec{V} + (5/2)\nabla\vec{q}$ - need more than just $\langle R^2 \nabla\zeta \cdot \vec{\vec{\pi}} \cdot \nabla\psi \rangle \to \left\langle MR^2 \nabla\zeta \cdot \int d^3v \tilde{f}^H \vec{v}\vec{v} \cdot \nabla\psi \right\rangle$

II. What's Missing From Drift Kinetics?

Hazeltine derives the drift kinetic equation by writing $f = \bar{f} + \tilde{f}$,
with $\bar{f} = (2\pi)^{-1}\oint d\varphi f = \langle f \rangle_\varphi$ and $\tilde{f} = f - \bar{f}$ and using $\varepsilon = v^2/2$ & μ

He keeps $\bar{f}$ to all orders, but in $\tilde{f}$ he keeps only order δ terms and assumes $B^{-1}\partial f/\partial\mu \sim \partial f/\partial\varepsilon$, where $\delta = \rho / L_\perp$

In tokamaks, f is isotropic (Maxwellian) to lowest order so that
$\vec{J} \times \vec{B} = c\nabla p$ and $B^{-1}\partial f/\partial\mu \sim \delta\partial f/\partial\varepsilon$

Therefore, Hazeltine's $\tilde{f}$ contains some, but not all, order δ^2 terms

To get all δ^2 order terms in $\tilde{f}$ need to solve

$\tilde{f} = \Omega^{-1}\int^{\varphi} d\varphi(\dot{f} + \Omega\partial f/\partial\varphi - \langle \dot{f} \rangle_\varphi)$ by inserting $\tilde{f}$ to order δ on right

where $\dot{f} = \partial f/\partial t + \vec{v}\cdot\nabla f + \vec{a}\cdot\nabla_v f = -\Omega\partial f/\partial\varphi + \ldots$

Check: Old + New Terms Give Full Gyroviscosity

Denote Hazeltine $\tilde{f}$ by $\tilde{f}^H$ and rest of $\tilde{f}$ by $\tilde{f}^N$, then $\tilde{f}=\tilde{f}^H+\tilde{f}^N$ to order δ^2, giving an arbitrary collisionality gyroviscosity that reduces to the correct collisional expression for $\vec{\pi}_g = M\int d^3v\tilde{f}\vec{v}\vec{v}$

In the collisional limit, find for a direct evaluation

$$\vec{\pi}_g = \frac{p}{4\Omega}\left[\vec{b}\times(\vec{\alpha}+\vec{\alpha}^T)\cdot(\vec{I}+3\vec{b}\vec{b})-(\vec{I}+3\vec{b}\vec{b})\cdot(\vec{\alpha}+\vec{\alpha}^T)\times\vec{b}\right]+\text{inertial}$$

with $\vec{\alpha}=\vec{\alpha}^H+\vec{\alpha}^N$ [see Simakov & Catto, PoP **12**, 012105 (2005)]

Hazeltine terms only: $\vec{\alpha}^H=\nabla\vec{V}_{\parallel}+(2/5p)\nabla\vec{q}_{\parallel}$

New S&C terms: $\vec{\alpha}^N=\nabla\vec{V}_{\perp}+(2/5p)\nabla\vec{q}_{\perp}$

Inertial: due to the difference between $\vec{v}$ and $\vec{w}=\vec{v}-\vec{V}$ moments and $\exp(-Mv^2/2T)$ and $\exp(-Mw^2/2T)$

A moment evaluation of $\vec{\pi}_g$ only requires lowest order $\tilde{f}^H$

Gyrophase Dependent Part of Distribution Function

In the Hazeltine $\tilde{f}$, $\tilde{f}^H$, the $\partial\bar{f}/\partial\mu$ terms are order δ^2

$$\tilde{f}^H = \vec{v}\cdot[\Omega^{-1}\vec{b}\times\nabla|_\mu \bar{f} - \vec{v}_E\partial\bar{f}/\partial\varepsilon - (\vec{v}_E + \vec{v}_M)B^{-1}\partial\bar{f}/\partial\mu]$$
$$-[(\vec{v}_\perp\vec{v}\times\vec{b} + \vec{v}\times\vec{b}\vec{v}_\perp):\nabla\vec{b}](v_\parallel/4\Omega B)\partial\bar{f}/\partial\mu]$$

The following order δ^2 terms in $\nabla|_\mu \bar{f}$ and $\partial\bar{f}/\partial\varepsilon$ are missing

$$\tilde{f}^N = \Omega^{-1}[(\vec{v}_\parallel + \tfrac{1}{4}\vec{v}_\perp)\vec{v}\times\vec{b} + \vec{v}\times\vec{b}(\vec{v}_\parallel + \tfrac{1}{4}\vec{v}_\perp)]:\vec{h}$$

where $\vec{h} = \nabla\vec{g}_\perp + (e\vec{E}/M)\partial\vec{g}_\perp/\partial\varepsilon$ and $\vec{g}_\perp = \Omega^{-1}\vec{b}\times\nabla\bar{f}_0 - \vec{v}_E\partial\bar{f}_0/\partial\varepsilon$

$\tilde{f} = \tilde{f}^H + \tilde{f}^N$ is needed to directly evaluate the perpendicular viscosity [see Simakov & Catto PoP **12**, 012105 (2005)]

Full Drift Kinetic Equation (to order δ^2)

The full drift kinetic equation with $\tilde{f}$ to order δ^2 is given by

$$\langle \dot{f} \rangle_\varphi = \langle \dot{\bar{f}} \rangle_\varphi + \langle \dot{\tilde{f}}^H \rangle_\varphi + \langle \dot{\tilde{f}}^N \rangle_\varphi$$

with the Hazeltine drift kinetic equation given by

$$\langle \dot{\bar{f}} \rangle_\varphi + \langle \dot{\tilde{f}}^H \rangle_\varphi = \frac{\partial \bar{f}}{\partial t} + [(v_\parallel + v_p)\vec{b} + \vec{v}_d] \cdot \nabla\big|_\mu \bar{f} + \left\{ \frac{e\vec{E}}{M} \cdot [(v_\parallel + v_p)\vec{b} + \vec{v}_d] + \mu \frac{\partial B}{\partial t} \right\} \frac{\partial \bar{f}}{\partial \varepsilon} + \dot{\mu}_{gc} \frac{\partial \bar{f}}{\partial \mu}$$

and the (somewhat awkward) new term by

$$\langle \dot{\tilde{f}}^N \rangle_\varphi = \left[v_\parallel \left(\vec{v}_d + v_p \vec{b} + \frac{5\mu B}{4\Omega} \vec{\kappa} \times \vec{b} \right) + \left(\vec{v}_d + v_p \vec{b} + \frac{5\mu B}{4\Omega} \vec{\kappa} \times \vec{b} \right) v_\parallel \right] : \overleftrightarrow{h}$$

$$+ \frac{v_\parallel \mu B}{4\Omega} [\vec{b} \times \nabla \vec{b} + (\vec{b} \times \nabla \vec{b})^T - \nabla \vec{b} \times \vec{b} - (\nabla \vec{b} \times \vec{b})^T] : \overleftrightarrow{h}$$

$$+ \mu B (\vec{v}_\parallel \vec{v}_E + \vec{v}_E \vec{v}_\parallel) : \partial \overleftrightarrow{h} / \partial \varepsilon - - \frac{v_\parallel \mu B}{\Omega} \vec{b} \cdot \nabla \times (\overleftrightarrow{h} \cdot \vec{b} + \vec{b} \cdot \overleftrightarrow{h})$$

with $\overleftrightarrow{h} = \nabla \vec{g}_\perp + (e\vec{E}/M) \partial \vec{g}_\perp / \partial \varepsilon$ and $\vec{g}_\perp = \Omega^{-1} \vec{b} \times \nabla \bar{f}_0 - \vec{v}_E \partial \bar{f}_0 / \partial \varepsilon$

Lowest Order Drift Kinetic Equation (to order δ)

The lowest order drift kinetic equation is

$$\frac{\partial \bar{f}}{\partial t} + [(v_{\parallel} + v_p)\vec{b} + \vec{v}_d] \cdot \nabla|_{\mu} \bar{f} + \left\{ \frac{e\vec{E}}{M} \cdot [(v_{\parallel} + v_p)\vec{b} + \vec{v}_d] + \mu \frac{\partial B}{\partial t} \right\} \frac{\partial \bar{f}}{\partial \varepsilon} = \langle C \rangle_{\varphi}$$

which can be written in a conservative form

Hybrid lowest order drift kinetic - fluid closure plus $\tilde{f} = \tilde{f}^H + \tilde{f}^N$:

1) Evaluate $\vec{q}_{\parallel}$ directly from $\bar{f}$ and all (diamagnetic + collisional) $\vec{q}_{\perp}$ from the $Mv^2\vec{v}/2$ moment of the full kinetic equation for f

2) Evaluate $\vec{\vec{\pi}}_{\parallel}$ directly from $\bar{f}$ and $\vec{\vec{\pi}}_g$ & $\vec{\vec{\pi}}_{\perp}$ from the $M\vec{v}\vec{v}$ moment of the full kinetic equation for f ($\tilde{f}^H + \tilde{f}^N$ is needed to evaluate $\vec{\vec{\pi}}_{\perp}$)

Retains turbulence, zonal flows, and neoclassical (& classical)

a) $\vec{q}_{\parallel}$ and $\vec{\vec{\pi}}_{\parallel}$ are only evaluated to lowest order in $(p_{\parallel} - p_{\perp})/p$

b) diamagnetic and collisional parts of $\vec{q}_{\perp}$ are order δ and $\nu\delta/\Omega$

c) $\vec{\vec{\pi}}_g$ is of order $pV_{\parallel}/\Omega L_{\perp}$ and $\vec{\vec{\pi}}_{\perp}$ is of order $p\nu V_{\parallel}/\Omega^2 L_{\perp}$

If $(p_{\parallel} - p_{\perp})/p \sim \delta^2$ closure proceeds as for collisional limit

Stronger Anisotropy, Low Collisional Ordering: $(p_{\parallel} - p_{\perp})/p \sim \delta$

1) perpendicular momentum: flows diamagnetic $\Rightarrow \vec{V}_{\perp} \sim \delta v_i$

2) continuity: $\partial n/\partial t \sim \nabla_{\perp} \cdot (n\vec{V}_{\perp}) \Rightarrow \omega \sim \delta^2 \Omega$

3) parallel momentum: $Mn\partial V_{\parallel}/\partial t \sim \nabla_{\parallel} p \Rightarrow V_{\parallel}/v_i \sim v_i/\omega L_{\parallel} \sim L_{\perp}/\delta L_{\parallel}$

4) $\delta \sim V_{\perp}/v_i << V_{\parallel}/v_i \sim L_{\perp}/\delta L_{\parallel} << 1 \Rightarrow \delta^2 << L_{\perp}/L_{\parallel} << \delta$

note: $Mn\partial V_{\parallel}/\partial t \sim \vec{b} \cdot (\nabla \cdot \vec{\vec{\pi}}_g) \sim \nabla_{\parallel} p >> \vec{b} \cdot (\nabla \cdot \vec{\vec{\pi}}_{\parallel}) \sim p\delta/L_{\parallel}$

5) check: energy balance: $n\partial T/\partial t \sim \nabla_{\perp} \cdot \vec{q}_{dia} \Rightarrow \omega p \sim p\delta v_i/L_{\perp}$

Next order corrections: $\lambda \sim L_{\parallel}$ & $\delta \sim (L_{\perp}/L_{\parallel})^{2/3} \sim (\nu/\Omega)^{2/5}$

1) continuity: assume $\nabla_{\parallel} \cdot (n\vec{V}_{\parallel})/\nabla_{\perp} \cdot (n\vec{V}_{\perp}) \sim (L_{\perp}/\delta L_{\parallel})^2 \sim \delta \Rightarrow$

$(L_{\perp}/L_{\parallel})^2 \sim \delta^3 \Rightarrow V_{\parallel}/v_i \sim \delta^{1/2}$

2) viscosities: $\vec{\vec{\pi}}_{\parallel} \sim \delta p$, $\vec{\vec{\pi}}_g \sim \delta^{3/2} p$ & $\vec{\vec{\pi}}_{\perp} \sim \nu \delta^{3/2} p/\Omega$

3) parallel momentum: $\vec{b} \cdot (\nabla \cdot \vec{\vec{\pi}}_{\parallel})/\nabla_{\parallel} p \sim \delta$ and assume

$\vec{b} \cdot (\nabla \cdot \vec{\vec{\pi}}_{\perp})/\nabla_{\parallel} p \sim (\nu/\Omega)(V_{\parallel} L_{\parallel}/\Omega L_{\perp}^2) \sim \nu/\Omega \sim \delta L_{\perp}/L_{\parallel}$

4) energy balance: assume $q_{\parallel} \sim pV_{\parallel}$ so viscous heating $\pi_{\parallel} V_{\parallel}/q_{\parallel} \sim \delta$

$\nabla_{\parallel} \cdot \vec{q}_{\parallel}/\nabla_{\perp} \cdot \vec{q}_{dia} \sim (L_{\perp}/\delta L_{\parallel})^2 \sim \delta$ and $\nabla_{\perp} \cdot \vec{q}_{coll}/\nabla_{\perp} \cdot \vec{q}_{dia} \sim \nu/\Omega$

Reminder: Lowest Order Maxwellian

Requires either

a) Collisional plasma - scrape off layer

$C\{f_0\} = 0$ or f_0 Maxwellian

b) Magnetic surfaces (or closed field lines) - inside separatrix

lowest order $v_{||}\vec{b}\cdot\nabla f_0 = C\{f_0\}$ for $\lambda \sim L_{||}$, then annihilate

streaming to find $\oint d\theta C\{f_0\}/v_{||}\vec{b}\cdot\nabla\theta = 0 \Rightarrow f_0$ Maxwellian

Pressure anisotropy is weak $(p_{||} - p_{\perp})/p << 1$

Lowest order pressure balance is just $\vec{J}\times\vec{B} = c\nabla p$

III. Gyrokinetics

Versions implemented in core codes retain all $k_\perp\rho \sim 1$ effects, where $k_\perp$ is the perpendicular wavenumber of the turbulence, but normally retain only order δ corrections in f from $\vec{R} = \vec{r} + \Omega^{-1}\vec{v}\times\vec{b}$: like lowest order drift kinetics with $k_\perp\rho \sim 1$

Cannot evolve the neoclassical radial electric field, which like zonal flows, results in an axisymmetric global flow

Inverse energy cascade in tokamaks: turbulence gives rise to zonal flows, that will alter the neoclassical flow and evolve on the transport time scale

Sheared and coupled zonal and neoclassical global flows can alter the saturation level on transport time scales

Need a moment approach similar to drift kinetics but with $k_\perp\rho \sim 1$

Rigidly Rotating Maxwellian: $\nabla T = 0$

$$f_M = n(M/2\pi T)^{3/2} \exp[-(M/2T)(\vec{v} - \omega R^2 \nabla\zeta)^2]$$

Use $E = v^2/2 + (e\Phi/M)$ & $\psi_* = \psi - (Mc/e)R^2\nabla\zeta\cdot\vec{v}$, then

$$f_M = n_0 (M/2\pi T)^{3/2} \exp[-(ME/T) - (e\omega\psi_*/cT)]$$

with

$$n_0 \equiv n \exp[(e\Phi/T) + (e\omega\psi/cT) - (M\omega^2 R^2/2T)] = \text{constant}$$

so that momentum conservation is satisfied

$$Mn\omega^2 R^2 \nabla\zeta\cdot\nabla(R^2\nabla\zeta) + en[\nabla\Phi - (\omega/c)R^2\nabla\zeta \times \vec{B}] + T\nabla n = 0$$

Rigidly rotating ion Maxwellian satisfies ion-ion collision operator and Vlasov operator if $\nabla T = 0$ and $\partial\Phi/\partial t = 0$ (recall C_{ie} small)

IV. Summary

Braginskii usually not appropriate in tokamaks since perpendicular flows are weak

Hazeltine drift kinetics has to be used with care as does gyrokinetics - normally need to use moments equations

Need to develop hybrid kinetic-fluid descriptions:
*For fluid codes to handle long mean free path effects
*For gyrokinetic codes to handle transport time scales

Question: Is there any fluid or kinetic code correctly retaining neoclassics and turbulence? - need both!