Renormalization-group analysis of the validity of staggered-fermion QCD with the fourth-root recipe

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hep-lat/0607007

with input from

C. Bernard, PRD 73 (2006) 114503 [hep-lat/0603011]

C. Bernard, M. Golterman, YS, PRD 73 (2006) 114511 [hep-lat/0604017]

C. Bernard, M. Golterman, YS, S. Sharpe, hep-lat/0603027

Doubling problem $\longrightarrow$ 4th-root staggered fermions

Naive-fermion propagator $= \dfrac{a}{i\sum_\mu \gamma_\mu \sin(ap_\mu)}$

$\Longrightarrow$ 16 poles!

Elimination ("taste" $=$ quark)

naive	$\longrightarrow$	staggered	$\longrightarrow$	4th-root staggered
16 tastes	$\longrightarrow$	4 tastes	$\longrightarrow$	1 taste
	exact symmetry		brute force!	
4 components	$\longrightarrow$	1 component	$\longrightarrow$	?

4th-root staggered: $Z = \int \mathcal{DU} \exp(-S_g(U)) \prod_{i=1}^{N_f} \det{}^{1/4}(D_{stag}(m_i))$

- cheap (one-component per color per site) $+$ non-anomalous chiral symmetry

$\Longrightarrow$ light quark masses

But: taste symmetry broken $\Longrightarrow$ **non-locality!!**

Really non-local and non-unitary? ($a \neq 0$)

Consider Goldstone-Pion sector (one-flavor theory)

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- continuum: no pions

- staggered: 1 Goldstone pions + 14 approx Goldstone pions = 15 in C.L.

right number of (approx) pions for 4-quark theory, not for 1-quark theory?!

- The physical states: taste-singlets (only η' !!).

Must achieve cancellation of contributions of all unphysical states.

Not possible at $a \neq 0$: (tasty) pion masses have $O(a^2)$ corrections.

This talk: Argue that it should work in the continuum limit, because <u>exact taste symmetry</u> is recovered.

Non-locality?? So What?!

Q: Do 4th-root staggered fermions provide a valid regularization of QCD?

Who cares? Just a numerical trick, leave it to the engineers.

Unacceptable answer!

Continuum and chiral extrapolations require analytic control!

A (1): Bother! Because of claimed high-precision QCD results: V

$f_K/f_\pi = 1.210(0.3\%)(1.0\%) \longrightarrow$ best $|V_{us}|$ (MILC '04, Marciano '04)

Heavy-Heavy, Heavy-Light (HPQCD, UKQCD, MILC, Fermilab)

"mixed" ....

A (2): Bother! Advanced-level, exciting field-theory problem! V

Renormalization-Group blocking

- Problem: UV taste violations are always $O(1)$.
- Solution: RG blocking, hold coarse-lattice spacing $a_c \ll \Lambda_{QCD}^{-1}$ fixed:

original fields	$U_\mu,\ \chi,\ \overline{\chi},$	$a_f = a_c\, 2^{-n-1}$
$k^{\rm th}$ step fields	$V_\mu^{(k)},\ \psi_{i\alpha}^{(k)},\ \overline{\psi}_{i\alpha}^{(k)},$	$a_k = a_c\, 2^{k-n}$
coarse-lattice fields	$V_\mu^{(n)},\ \psi_{i\alpha}^{(n)},\ \overline{\psi}_{i\alpha}^{(n)},$	$a_n = a_c$

- Starting point is one-component theory: need all of its symmetries!
- Perform $n+1$ blocking steps, $k=0$ step is special:
transition from one-component basis to taste basis (next slide)
- RG projects onto small-momentum states $\Longrightarrow$ acts naturally in taste basis.
- Two mechanisms to avoid doublers (next slide)

From one-component basis to taste basis

- Free taste-basis Dirac operator (block 2^4 site variables into 4×4 field):

$$D_{taste} = a_0^{-1} \sum_\mu \big([\gamma_\mu \otimes \mathbf{1}] i \sin(p_\mu a_0) + [\gamma_5 \otimes \xi_5 \xi_\mu](1 - \cos(p_\mu a_0)) \big) + m$$

where $\mathbf{1} =$ identity $4 \otimes 4$ matrix in taste space, and $a_0 = 2a_f$.

Get a feeling: the $U(1)_\epsilon$ chiral symmetry is $\delta\psi = i[\gamma_5 \otimes \xi_5]\psi$

Doublers removed by Wilson-like term (irrelevant; breaks taste symmetry).

- QCD: $D_{stag} \Longrightarrow D_{taste}$ via gaussian "RG" step ($Q^{(0)} =$ unitary):

$$D_{taste}^{-1}(\alpha_0) = \alpha_0^{-1} + Q^{(0)} \, D_{stag}^{-1} \, Q^{(0)\dagger}$$

$D_{taste}(\alpha_0)$ satisfies Ginsparg-Wilson relation for $m = 0$ and $\alpha_0 \neq \infty$.

$\Longrightarrow$ Modified Ginsparg-Wilson-Lüscher chiral symmetry.

Doublers removed by "GW mechanism" (compatible with taste symmetry).

Free Theory (warm up I)

We can start directly in taste basis

Blocked Dirac operator contains all the long-distance physics

$$\begin{aligned} D_n^{-1} &= \alpha_n^{-1} + Q^{(n)}\, D_{n-1}^{-1}\, Q^{(n)\dagger} \\ &= \left(\alpha_n^{-1} + (16\,\alpha_{n-1})^{-1} + \cdots + (16^{n-1}\alpha_1)^{-1}\right) + \qquad \text{zero if } \tilde{x} \neq \tilde{y} \\ &\quad + Q^{(n)}Q^{(n-1)} \cdots Q^{(1)}\, D_{taste}^{-1}\, Q^{(1)\dagger} \cdots Q^{(n-1)\dagger}Q^{(n)\dagger} \end{aligned}$$

$\Longrightarrow$ Original propagator between smeared sources!

$\Longrightarrow$ Correlation functions constrained by all the original lattice symmetries.

Continuum limit:

$$D_n \to [\tilde{D}_\infty \otimes \mathbf{1}]$$

$$\det{}^{1/4}(D_{taste}) \text{ [UV part removed]} \to \det(\tilde{D}_\infty)$$

Interacting theory: Master Plan

Blocked Dirac operator: $D_n = [\tilde{D}_{inv,n} \otimes \mathbf{1}] + \Delta_n$

Taste breaking comes from irrelevant Δ_n, should scale like $a_f/a_c^2 \times \log$.

Continuum limit: $\Delta_n \to 0$ for $a_f \to 0$, at fixed a_c.

Continuum-limit coarse-lattice theory with exact taste symmetry!

$$
\begin{aligned}
Z_\infty &= \int \mathcal{D}\mathcal{V} \, \exp\left[-\frac{F^2}{g_r^2(a_c)} - ****\right] \left\{ \det{}^{1/4}\Big([\tilde{D}_{inv,\infty} \otimes \mathbf{1}]\Big) + \#\#\# \right\} \\
&= \int \mathcal{D}\mathcal{V} \, \exp\left[-\frac{F^2}{g_r^2(a_c)} - ****\right] \left\{ \det\Big(\tilde{D}_{inv,\infty}\Big) + \#\#\# \right\} \\
&= \int \mathcal{D}\mathcal{V} dq d\overline{q} \, \exp\left[-\frac{F^2}{g_r^2(a_c)} - ****\right] \exp\left[-\overline{q}\, \tilde{D}_{inv,\infty}\, q - \#\#\#\right]
\end{aligned}
$$

Assume $m > 0$, hence $\det(D_n)$ is positive; take positive 4th root.

Continuum-limit [coarse-lattice] theory

- One-taste representation: quark fields $= q, \overline{q}$. Physical states only.

$$Z_\infty = \int \mathcal{D}\mathcal{V} dq d\overline{q} \; \exp\left[-\frac{F^2}{g_r^2(a_c)} - * * * * \right] \exp\left[-\overline{q}\, \tilde{D}_{inv,\infty}\, q - \#\#\# \right]$$

More Wilson loops: $* * **$ (technicality when a_c is small enough).
Multifermion interactions: $\#\#\#$ (same as above).

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- Fourth-root four-taste representation:
  - Replica rule at the level of the chiral effective theory.
  - Extended Hilbert space with unphysical (tasty) states.
  - Physical, unitary subspace (taste singlet sector) in the continuum limit.
  - No "paradoxes" based on symmetries (reply to Creutz, hep-lat/0603020).

What do we leave behind?

Removed cutoff effects contained in $\det(H_k)$, all k, where

$$H_k = [\gamma_5 \otimes \xi_5]\Big(D_{k-1} + \alpha_k Q^{(k)\dagger} Q^{(k)}\Big)$$

Free theory: gap is $O(1/a_k)$ by construction $\Longrightarrow$ Only cutoff effects removed.

Effective action: $S_{eff}^{(k)} = -\mathrm{tr}\,\log H_k$

Ordinary staggered theory (before integrating over gauge fields):

$$Z = \int \mathcal{D}\mathcal{U}\, \mathcal{D}\mathcal{V}^{(0)}\, \mathcal{D}\mathcal{V}^{(1)} \cdots \mathcal{D}\mathcal{V}^{(n)}\ \exp\left(- S_g - \sum_{k=0}^{n} \mathcal{K}_B^{(k)} - \sum_{k=0}^{n} S_{eff}^{(k)} \right)$$
$$\times \int d\psi^{(n)} d\overline{\psi}^{(n)}\ \exp\left(- \overline{\psi}^{(n)} D_n\, \psi^{(n)} \right)$$

Mutatis mutandis: $1/H_k$ is short ranged $\Longrightarrow$ $S_{eff}^{(k)}$ local.

Mobility edge of H_k is $O(1/a_k)$ coarse-lattice theory is local!

Ordinary staggered theory (warm up **II**)

Recall $D_n = [\tilde{D}_{inv,n} \otimes \mathbf{1}] + \Delta_n$. Introduce re-weighted theories:

$$Z_{inv,n} \;=\; \int \mathcal{D}\mathcal{U}\,\mathcal{D}\mathcal{V}^{(0)}\,\mathcal{D}\mathcal{V}^{(1)} \cdots \mathcal{D}\mathcal{V}^{(n)}\;\exp\Big(- S_g - \sum_{k=0}^{n} \mathcal{K}_B^{(k)} - \sum_{k=0}^{n} S_{eff}^{(k)} \Big)$$
$$\times \int d\psi^{(n)} d\overline{\psi}^{(n)}\;\exp\Big(- \overline{\psi}^{(n)} [\tilde{D}_{inv,n} \otimes \mathbf{1}] \psi^{(n)} \Big)$$

- exact taste symmetry by construction
- local $+$ renormalizable

Scaling: $|\Delta_n| \sim 2^{-n-1}/a_c = a_f/a_c^2$ (up to logs)

Same continuum limit: $Z_\infty(J) = Z_{inv,\infty}(J)$ ($J =$ source)

$\Longrightarrow$ continuum-limit theory has exact taste symmetry.

Actually ....

Actually (gory details!) ...

- Need an IR bound: $\|1/D_n\| \leq 1/m_r(a_c)$

gives bound on difference between corresponding observables

$$\begin{aligned} \det(D_n) &= \det\Big([\tilde{D}_{inv,n} \otimes \mathbf{1}]\Big) \det\Big(1 + \Delta_n \, [\tilde{D}_{inv,n} \otimes \mathbf{1}]^{-1}\Big) \\ &= \det\Big([\tilde{D}_{inv,n} \otimes \mathbf{1}]\Big) \Big(1 + O(\epsilon_n^2)\Big) \end{aligned}$$

traceless on taste index

where

$$\epsilon_n = \frac{2^{-n-1}}{a_c \, m_r(a_c)} = \frac{a_f}{a_c^2 \, m_r(a_c)}$$

- Scaling of Δ_n

$\Longrightarrow$ Convergence of Taylor expansion ($\epsilon_n < 1$) for $n \geq n_0$.

$\Longrightarrow$ $\epsilon_n \to 0$ for $n \to \infty$.

- However, need $m_r(a_c) > 0$.

$\Longrightarrow$ limits $a \to 0$ and $m \to 0$ not always commute!

What did we use (ordinary staggered)?

What shall we use (4th-root staggered)?

- Power-counting renormalizability (with/out rooting)

$\Longrightarrow$ scaling of $g_r(a_k),\ m_r(a_k)$.

- Locality of S^k_{eff} (with/out rooting)

- Scaling of irrelevant operators: trust in local + renormalizable theories only!

$\Longrightarrow$ In 4th-root theory, rely on taste-breaking scaling in the reweighted theories only.

- Reweighted theories have physical Hilbert space, belong to the correct universality class.

Glossary of local theories

Interpolating theories for ordinary staggered fermions

$$Z_{inter,n}(t) \;=\; \int \mathcal{D}\mathcal{U}\,\mathcal{D}\mathcal{V}^{(0)}\,\mathcal{D}\mathcal{V}^{(1)}\cdots\mathcal{D}\mathcal{V}^{(n)}\;\exp\Big(-S_g-\sum_{k=0}^{n}\mathcal{K}_B^{(k)}-\sum_{k=0}^{n}S_{eff}^{(k)}\Big)$$
$$\times\int d\psi^{(n)}d\overline{\psi}^{(n)}\;\exp\Big[-\overline{\psi}^{(n)}\Big([\tilde{D}_{inv,n}\otimes\mathbf{1}]+t\Delta_n\Big)\psi^{(n)}\Big]$$

Re-weighted theories for 4th-root staggered fermions

$$Z_{inv,n}^{root} \;=\; \int \mathcal{D}\mathcal{U}\,\mathcal{D}\mathcal{V}^{(0)}\,\mathcal{D}\mathcal{V}^{(1)}\cdots\mathcal{D}\mathcal{V}^{(n)}\;\exp\Big(-S_g-\sum_{k=0}^{n}\mathcal{K}_B^{(k)}-\frac{1}{4}\sum_{k=0}^{n}S_{eff}^{(k)}\Big)$$
$$\times\int dq^{(n)}d\overline{q}^{(n)}\;\exp\Big(-\overline{q}^{(n)}\tilde{D}_{inv,n}\,q^{(n)}\Big)$$

4th-Root Theory (the real thing!)

- Use scaling of Δ_n in re-weighted theories. Obtain:

$$\left\langle \mathcal{O}^{(n)} \right\rangle_{\text{4th root}} = \left\langle \mathcal{O}^{(n)} \ \exp\left[\frac{1}{4}\mathrm{tr}\log\left(1 + \Delta_n\,[\tilde{D}_{inv,n} \otimes \mathbf{1}]^{-1}\right)\right] \right\rangle_{\text{re-weighted}}$$

$$= \left\langle \mathcal{O}^{(n)} \right\rangle_{\text{re-weighted}} \left(1 + O(\epsilon_n^2)\right).$$

Again, same continuum limit: $Z_\infty^{root}(J) = Z_{inv,\infty}^{root}(J)$

- Restrict to taste singlet sources $J = [\tilde{J} \otimes \mathbf{1}]$, obtain

$$Z_\infty(\tilde{J}) = \int \mathcal{D}\mathcal{V} dq d\overline{q} \ \exp\left[-\frac{F^2}{g_r^2(a_c)} - ***\right] \exp\left[-\overline{q}\,\tilde{D}_{inv,\infty}\,q - \overline{q}\,\tilde{J}\,q - \#\#\right]$$

$\Longrightarrow$ Continuum-limit theory is local + renormalizable

$\Longrightarrow$ Continuum-limit theory in the correct universality class

Conclusion

• 4th-root theory is valid in the continuum limit, under plausible assumptions.

• Actual taste-breaking scaling in low-energy physics $= O(a_f \Lambda_{QCD}^2)$, much better than assumed $O(a_f/a_c^2)$.

• That's why it works in practice.

• Works in principle, but fails (badly!) in practice for non-zero density: root of complex-det needed! (B. Svetitsky, M. Golterman, YS, hep-lat/0602026).

• Need effective low energy theory.

C. Bernard: staggered chiral perturbation theory + replica trick (plausible assumptions) .

• Re-derive from underlying theory (work in progress).

Summary

– Ordinary and fourth-root staggered are power-counting renormalizable –
progress towards a rigorous proof (J. Giedt, hep-lat/0606003)

– Effective action obtained by integrating out UV modes is local –
homework: mobility-edge picture requires numerical confirmation.

– Scaling of irrelevant operators in reweighted theories –
homework: set up perturbation theory, compute (and confirm) the scaling.

$\Longrightarrow$ Reweighted theories are in the correct universality class.

$\Longrightarrow$ 4th-root and reweighted theories have the same continuum limit.

$\Longrightarrow$ 4th-root theory is valid in the continuum limit;
once homework done, true under plausible/conventional assumptions.