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The Quantum Vlasov Equation and its Markov Limit

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Abstract

The adiabatic particle number in mean field theory obeys a quantum Vlasov equation which is nonlocal in time. For weak, slowly varying electric fields this particle number can be identified with the single particle distribution function in phase space, and its time rate of change is the appropriate effective source term for the Boltzmann-Vlasov equation. By analyzing the evolution of the particle number we exhibit the time structure of the particle creation process in a constant electric field, and derive the local form of the source term due to pair creation. In order to capture the secular Schwinger creation rate, the source term requires an asymptotic expansion which is uniform in time, and whose longitudinal momentum dependence can be approximated by a delta function only on time scales much longer than $\sqrt{p_\perp^2 + m^2c^2}/eE$. The local Vlasov source term amounts to a kind of Markov limit of field theory, where information about quantum phase correlations in the created pairs is ignored and a reversible Hamiltonian evolution is replaced by an irreversible kinetic one. This replacement has a precise counterpart in the density matrix description, where it corresponds to disregarding the rapidly varying off-diagonal terms in the adiabatic number basis and treating the more slowly varying diagonal elements as the probabilities of creating pairs in a stochastic process. A numerical comparison between the quantum and local kinetic approaches to the dynamical backreaction problem shows remarkably good agreement, even in quite strong electric fields, $eE \simeq m^2c^3/\hbar$, over a large range of times.

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I. INTRODUCTION

In recent years there has been considerable interest in establishing the precise connection between quantum field theory and classical kinetic theory. This interest is motivated by the wide variety of problems in different fields of physics which require a consistent description of quantum many body phenomena far from equilibrium. Examples include chiral symmetry restoration and the quark-gluon plasma phase of QCD, soon to be probed by relativistic heavy-ion colliders, baryogenesis at the electroweak phase transition, and the formation and decay of topological defects or Bose condensates, whether in the hot, dense early universe, or a cryogenic laboratory environment.

At their root all these systems may be treated as field theories with well-defined Hamiltonian evolutions and (except for the case of explicit CP violation in the electroweak theory) microscopic time reversal invariance. Yet, a large body of experience confirms the macroscopically irreversible behavior of such systems far from equilibrium, so that it should be possible to approximate the unitary Hamiltonian evolution of such systems by an irreversible kinetic description, under suitable circumstances. In addition to the numerous potential applications, this raises the fundamental issue of the precise connection between microscopic reversibility and macroscopic irreversibility which lies at the heart of much of nonequilibrium statistical mechanics.

The nature of the relationship between quantum theory and transport theory has been a subject of discussion since the very early days of the quantum theory. Several important developments which laid out clearly the general principles required to derive transport equations from the Liouville equation appeared in the 1950's [1]. However, the first steps in the practical numerical solution of nonequilibrium problems in the context of quantum field theory have been taken only relatively recently [2–4]. With these developments and the increasing variety of applications requiring a proper field theoretic treatment, establishing the precise relationship between the field theory and kinetic theory approaches to nonequilibrium systems in situations of practical interest has taken on a new urgency.

As a practical matter the kinetic description is certainly the simpler one to formulate and implement numerically on a computer. However, the Boltzmann-Vlasov equation essentially describes classical point particles, and extensions to quantum collective phenomena, time-evolving mean fields and off-shell virtual processes, which are quite natural in field theory, present considerable difficulties for a purely kinetic approach. Also lost in the kinetic description from the very outset is a detailed understanding of how time reversible Hamiltonian evolution comes to be replaced by time irreversible dissipative behavior. For these reasons of both fundamental interest and practical application, our purpose in this paper is to expose the relationship between the two approaches in a concrete example.

In the interest of being as clear and specific as possible we focus our attention in this paper on charged particle creation in electric fields, a phenomenon which was discussed nearly seventy years ago by Klein and Sauter, and twenty years later by Schwinger as a prime example of the then newly developed theory of quantum electrodynamics [5]. The results of this Schwinger mechanism which are relevant for the present paper are reviewed in Section 4. Over the years there has developed an extensive literature on this topic [6–9], which has continued to attract interest up to the present time [10–18]. Several monographs summarizing this activity have also appeared [19]. Given this background it might be supposed that no aspect of particle creation in electric fields has been left unresolved. However, this is not quite the case since attempts to incorporate the real time evolution of particle creation into the transport desription by an effective source term in the Vlasov equation is relatively recent, and these have met with some problems.

In many treatments of particle creation, analytic continuation of the amplitudes to complex time have been employed. Though elegant and useful in other contexts, complex continuation methods cannot address directly the real time evolution of the particle creation event and thus cast little light on the source term for a kinetic description. A suggestion of how to incorporate the Schwinger pair creation mechanism in the context of kinetic theory was first made in 1979, based on an intuitively appealing picture of the instantaneous semiclassical creation event [11]. This mechanism has been a subject of renewed interest in the context of heavy ion collisions and QCD due to the suggestion that the receding ions might produce a strong chromoelectric flux tube between them which shorts itself out by the creation quark/anti-quark pairs (see *e.g.* [3] and references therein). The ansatz of ref. [11] has been taken over to the QCD flux tube model as well. Yet it should be clear from the outset that a delta function source term which requires that the charged particles be created at precisely zero momentum, at a definite instant of time can only be an approximation to the rapid but continuous evolution of wave amplitudes in the underlying quantum theory. Calculations of the backreaction of the charged particle pairs on the electric field in QED in a well defined continuous evolution were compared with the *ad hoc* kinetic theory, according to the ansatz of ref. [11]. Reasonable qualitative agreement between the mean field evolutions in the two approaches was found, although they certainly differ in quantitative detail, such as in the distributions of created particles [3]. In these numerical investigations the time structure of the individual creation events was not addressed, leaving open the question of the limit of validity of the delta function ansatz for the source term.

The Wigner function formalism has also been proposed [20,21] as a method for deriving relativistic transport equations from the underlying field theory. It has become increasingly clear however, that the covariant Wigner function does not readily lend itself to practical calculation, because covariance requires splitting the time variable in the Wigner transformation in parallel to the splitting of the spatial variable, with the consequence that the problem ceases to be well posed as an evolution from initial data. More recently, an alternative, noncovariant formalism, in which the time variable is not split has been suggested [14,15]. As has been emphasized in earlier work the lack of manifest covariance is not a problem since the initial value description of even a relativistic field theory in Hamiltonian terms is necessarily noncovariant in form, but the evolution equations are completely equivalent to those derived from a covariant action principle [4]. In any case, a firm conclusion about the source term has not been obtained by these investigations either. Finally, the general projection formalism of Zwanzig [22] has been advocated as a route to a transport description of particle creation [16–18], although the time structure of the creation process itself has not been investigated in detail in this approach, and the conditions of validity of the delta function approximation for a local source term in the Vlasov equation has remained obscure.

By revisiting the electrodynamic pair creation problem our purpose in this paper is to elucidate fully the precise connection between the field theoretic and kinetic treatments in this particular case. Application and extension of our methods of incorporating particle creation into a kinetic description for the other situations of interest will then become possible. Our first step will be to specify completely the adiabatic particle number basis in which particle creation can be described as a phase interference (or *dephasing*) phenomenon of the quantum theory from the effective Hamiltonian point of view [4]. Writing the explicit Bogoliubov transformation to this adiabatic particle basis then identifies a time dependent particle number whose total change recaptures the Schwinger formula in a constant, uniform electric field, and whose time derivative yields the appropriate source term for the Boltzmann-Vlasov equation. The adiabatic particle number obeys a nonlocal quantum Vlasov equation, and in this sense is completely consistent with the general approach advocated in refs. [16–18]. The relationship of our method to that of the projection formalism may be seen most clearly by considering the density matrix in the adiabatic particle number basis. However we have no need for the general projection formalism, since the source term for the Vlasov equation can be written in closed form in terms of the wave functions of the charged particle modes in the background constant electric field. In this way we derive for the first time a local form for the source term, which explicitly exhibits the relationship to the semiclassical picture of particles spontaneously appearing out of the vacuum in real time. The electromagnetic current of the charged particle pairs also has a simple form in this basis, corresponding to a clear physical interpretation in terms of a quasiclassical conduction current and the quantum polarization current of particle creation. The fact that the current grows linearly in time for a fixed external electric field and that therefore backreaction must eventually become important even for arbitrarily small coupling is also easy to see in the adiabatic particle basis. This will also serve to clarify the nature of the "time divergences" discussed in refs. [10].

The essential physical ingredient in passing from the quantum unitary evolution to the irreversible Vlasov description is the dephasing phenomenon, *i.e.* the near exact cancellation of the rapidly varying phases of the quantum mode functions contributing to the mean electric current of the created pairs. This cancellation depends in turn upon a clean separation of the time scales,

1. τ_{qu}, of the very rapidly oscillating modes of the microscopic quantum theory,
2. τ_{cl}, of the more slowly varying mean number of particles in the adiabatic number basis, and
3. τ_{pl}, of the collective plasma oscillations of the electric current and mean electric field produced by those particles.

In the limit $\tau_{qu} \ll \tau_{cl}$ quantum coherence between the created pairs can be neglected because of efficient dephasing and a (semi)classical local kinetic approximation to the underlying quantum theory becomes possible. In the limit $\tau_{cl} \ll \tau_{pl}$ the electric field may be treated as approximately *constant* over the interval of particle creation. Thus when both inequalities apply we can replace the true nonlocal source term which describes particle creation in field theory by one that depends only on the instantaneous value of the quasistationary electric field, at least over very long intervals of time.

The essential mathematical ingredient in the exploitation of this hierarchy of time scales is an asymptotic expansion of the wave functions and particle number for constant electric fields uniformly valid on the real time axis, so that secular particle creation effects (which are lost in the usual nonuniform WKB expansion) are retained. It is this precise sense of evaluating the effect of rapid degrees of freedom on slow degrees of freedom by treating the latter as constant in leading order of a uniform asymptotic expansion (which recalls the Born-Oppenheimer approximation in atomic and molecular physics) and by so doing, deriving a local effective source term for the change of adiabatic particle number, that we refer to as the Markov limit of the quantum Vlasov equation.

The importance of a *uniform* asymptotic expansion of the wave functions is that secular particle creation effects are retained in an expansion valid everywhere on the real time axis. The true wave functions exhibit a sharp change in amplitude, on the time scale τ_{cl}, at or near the time of the semiclassical creation event which is captured very well by a uniform asymptotic expansion in terms of Airy functions. As we shall see, if one is interested only in the collective phenemena on time scales of τ_{pl} or longer, then the details of the particle creation process on the time scale τ_{cl} are unimportant and one can replace the momentum distribution of the source term by one localized at zero kinetic momentum, as has been the practice in the earlier phenomenological approaches, provided only that the integrated distribution gives the correct total creation rate. This will clarify the precise conditions of validity of such instantaneous anätze for the first time.

Since an asymptotic (not a convergent) expansion is involved, the limit of the ratio of time scales $\tau_{qu}/\tau_{cl} \to 0$ for fixed t and the long time limit $t \to \infty$ of the evolution for fixed ratio τ_{qu}/τ_{cl} do *not* commute in general. Hence for any small but finite ratio τ_{qu}/τ_{cl} there can be eventually a very large but finite t at which the quantum phases reassemble and the irreversible local kinetic description breaks down. Up to this very long (typically exponential and possibly infinite) recurrence time the system behaves in many practical respects like an irreversible one, in which the quantum phase coherence between the created pairs appears to have been lost. In this way the apparent incongruity of an effectively irreversible time evolution emerging from a unitary Hamiltonian field theory is removed.

The paper is organized as follows. In the next section we review (scalar) QED mean field theory in the leading order of the large N expansion. By exhibiting explicitly the Hamiltonian structure of these equations we demonstrate that they are completely time reversible. In Section 3 we define the adiabatic particle number basis which is selected by the Hamiltonian evolution and derive the exact nonlocal form of the quantum Vlasov equation for this quantity. The quantum density matrix in this basis is also derived. In Section 4 we review the Schwinger mechanism and solve for the source term of the Vlasov equation in the limit of constant mean electric field, studying the pair creation process for this case in some detail. It is shown that particle creation in a fixed external field produces an electric current which grows linearly with time, so that any amount of particle creation (no matter how small) eventually requires a substantial backreaction on the field in any self-consistent treatment. In Section 5 the technique of uniform asymptotic expansions for the mode functions and adiabatic particle number is brought to bear. The source term for particle creation in a constant field is calculated to leading order in this asymptotic expansion in terms of Airy functions and yields an effectively Markovian source term for the *local* Vlasov equation describing pair creation in weak, slowly varying electric fields. The circumstances under which further approximation of the Airy function source term by an instantaneous delta function source term becomes permissable is also discussed. In Section 6 the dynamical backreaction problem for the charged particles whose current is self-consistently coupled to the mean electric field is compared to the two (Airy and delta function) local approximations for the Vlasov source term in the kinetic description, and relatively good agreement is obtained. We close with a summary of our results and some concluding remarks on possible generalizations of the analysis to other systems of interest. The derivation of the density matrix in the adiabatic particle number basis is relegated to an Appendix.

II. SCALAR QED IN THE LARGE N LIMIT

Let us begin by reviewing the equations of motion for scalar QED in a uniform electric field in the semiclassical limit in which the matter field is fully quantized and the electromagnetic field is treated classically. This limit can be obtained in a consistent way by taking the leading order of a large N expansion (where N is the number of identical copies of the charged matter field) [2,3]. We take the electric field spatially homogeneous, and express the vector potential in the gauge,

$$\mathbf{A} = A(t)\hat{\mathbf{z}}\,, \qquad \mathrm{A}_0 = 0\,, \tag{2.1}$$

so that the electric field is

$$\mathbf{E} = -\dot{A}\hat{\mathbf{z}} = E\hat{\mathbf{z}}\,. \tag{2.2}$$

The charged scalar field operator is expanded in Fourier modes in Fock space in the usual way,

$$\Phi(\mathbf{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\varphi_{\mathbf{k}}(t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\left\{e^{i\mathbf{k}\cdot\mathbf{x}}f_{\mathbf{k}}(t)a_{\mathbf{k}} + e^{-i\mathbf{k}\cdot\mathbf{x}}f^*_{-\mathbf{k}}(t)b^\dagger_{\mathbf{k}}\right\}\,. \tag{2.3}$$

The time-independent creation and destruction operators obey the commutation relations

$$[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = [b_{\mathbf{k}}, b^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \tag{2.4}$$

in the finite large volume V, and the Fourier components,

$$\varphi_{\mathbf{k}}(t) \equiv f_{\mathbf{k}}(t)a_{\mathbf{k}} + f^*_{\mathbf{k}}(t)b^\dagger_{-\mathbf{k}} \tag{2.5}$$

may be regarded as (complex) generalized coordinates of the field Φ for the purposes of the Hamiltonian description. The momentum canonically conjugate to this coordinate is

$$\pi_{\mathbf{k}}(t) = \dot{\varphi}^\dagger_{\mathbf{k}}(t) = \dot{f}^*_{\mathbf{k}}(t)a^\dagger_{\mathbf{k}} + \dot{f}_{\mathbf{k}}(t)b_{-\mathbf{k}}\,, \tag{2.6}$$

which obeys the canonical commutation relation,

$$[\varphi_{\mathbf{k}}, \pi_{\mathbf{k}'}] = i\hbar\delta_{\mathbf{k},\mathbf{k}'}\,, \tag{2.7}$$

provided that the mode functions satisfy the Wronskian condition,

$$f_{\mathbf{k}}\dot{f}^*_{\mathbf{k}} - \dot{f}_{\mathbf{k}}f^*_{\mathbf{k}} = i\hbar\,, \tag{2.8}$$

and (2.4) is used.

The time dependence in this basis is carried by the complex mode functions $f_{\mathbf{k}}(t)$ which satisfy the equations of motion,

$$\left(\frac{d^2}{dt^2} + \omega^2_{\mathbf{k}}(t)\right) f_{\mathbf{k}}(t) = 0 \tag{2.9}$$

where the time dependent frequency $\omega^2_{\mathbf{k}}(t)$ is given by

$$\omega^2_{\mathbf{k}}(t) = (\mathbf{k} - e\mathbf{A})^2 + m^2 = (k - eA(t))^2 + k^2_\perp + m^2\,. \tag{2.10}$$

Here k is the constant canonical momentum in the $\hat{\mathbf{z}}$ direction which should be clearly distinguished from the gauge-invariant but time dependent *kinetic* momentum,

$$p(t) = k - eA(t) \quad ; \qquad \dot{p} = -e\dot{A} = eE \tag{2.11}$$

which reflects the acceleration of the charged particle due to the electric field. In the directions transverse to the electric field the kinetic and canonical momenta are the same and do not need to be distinguished, *i.e.* we shall use the notation $p_\perp = k_\perp$ interchangeably. When expressed as a function of the kinetic momenta we use the notation, $\omega(p, p_\perp) = \sqrt{p^2 + p_\perp^2 + m^2}$, or simply ω.

The mean value of electromagnetic current in the $\hat{\mathbf{z}}$ direction is

$$j(t) = 2e \int [d\mathbf{k}]\, (k - eA(t))\, |f_\mathbf{k}(t)|^2 (1 + N_+(\mathbf{k}) + N_-(-\mathbf{k})) \tag{2.12}$$

where

$$\begin{aligned} N_+(\mathbf{k}) &\equiv \langle a^\dagger_\mathbf{k} a_\mathbf{k} \rangle \\ N_-(\mathbf{k}) &\equiv \langle b^\dagger_\mathbf{k} b_\mathbf{k} \rangle \end{aligned} \tag{2.13}$$

are the mean numbers of particles and antiparticles in the time independent basis and

$$\frac{1}{V} \sum_\mathbf{k} \rightarrow \int [d\mathbf{k}] \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tag{2.14}$$

in the infinite volume continuum limit. We make use of the freedom in defining the initial phases of the mode functions to set the correlation densities $\langle a_\mathbf{k} a_\mathbf{k} \rangle = \langle b_\mathbf{k} b_\mathbf{k} \rangle = 0$, without any loss of generality.

The mean charge density must vanish,

$$j^0(t) = e \int [d\mathbf{k}]\, [N_+(\mathbf{k}) - N_-(-\mathbf{k})] = 0\,, \tag{2.15}$$

by Gauss' Law for a spatially homogeneous electric field ($i.e. \nabla \cdot \mathbf{E} = 0$). We shall further restrict ourselves to the subspace of states for which

$$N_+(\mathbf{k}) = N_-(-\mathbf{k}) \equiv N_\mathbf{k} \tag{2.16}$$

for simplicity in what follows, although this is a stronger condition than is required by Eq. (2.15). Clearly the vacuum $N_+(\mathbf{k}) = N_-(-\mathbf{k}) = 0$ (as well as a thermal mixed state) belongs to this class of states.

Self-consistent evolution of the mean electric field requires coupling it to the expectation value of the current of the charged field by the only nontrivial Maxwell equation remaining in this homogeneous example, namely,

$$-\dot{E} = \ddot{A} = j = 2e \int [d\mathbf{k}]\, (k - eA(t))\, |f_\mathbf{k}(t)|^2 (1 + 2N_\mathbf{k})\,. \tag{2.17}$$

For the analysis of the source term in a constant electric field and its uniform expansion in the next three sections we will treat the electric field as fixed and nondynamical, returning to Eqn. (2.17) and the dynamical backreaction problem in Section VI.

By a slight change of notation it is possible to recast the mean field evolution equations (2.9) and (2.17) together with the quantum Wronskian condition (2.8) as Hamilton's equation for an effective classical Hamiltonian in which $\hbar$ appears as a parameter. Defining the real quantities,

$$\begin{aligned}\sigma_{\mathbf{k}} &\equiv 1 + N_+(\mathbf{k}) + N_-(-\mathbf{k}) = 1 + 2N_{\mathbf{k}}\,,\\ \xi^2_{\mathbf{k}}(t) &\equiv \sigma_{\mathbf{k}}|f_{\mathbf{k}}(t)|^2\,, \qquad \text{and}\\ \eta_{\mathbf{k}}(t) &\equiv \dot{\xi}_{\mathbf{k}}(t)\end{aligned} \tag{2.18}$$

we find that the mode equation (2.9) can be rewritten in the form,

$$\ddot{\xi}_{\mathbf{k}} = \dot{\eta}_{\mathbf{k}} = -\omega^2_{\mathbf{k}}\xi_{\mathbf{k}} + \frac{\hbar^2\sigma^2_{\mathbf{k}}}{4\xi^3_{\mathbf{k}}}\,, \tag{2.19}$$

when account is taken of (2.8). This last equation together with the Maxwell equation (2.17) will be recognized as Hamilton's equations for the Hamiltonian,

$$H_{eff}(A, p_A; \{\xi_{\mathbf{k}}\}, \{\eta_{\mathbf{k}}\}; \{\sigma_{\mathbf{k}}\}) = \frac{V}{2}E^2 + \sum_{\mathbf{k}}\left(\eta^2_{\mathbf{k}} + \omega^2_{\mathbf{k}}\xi^2_{\mathbf{k}} + \frac{\hbar^2\sigma^2_{\mathbf{k}}}{4\xi^2_{\mathbf{k}}}\right)\,, \tag{2.20}$$

where $p_A \equiv -E$ is the momentum conjugate to A and $\eta_{\mathbf{k}}$ is the momentum conjugate to $\xi_{\mathbf{k}}$.

Moreover, the quantum statistical density matrix of the charged scalar field corresponding to the mean field evolution can be written as a product of Gaussians in Fourier space, *viz.*

$$\langle\{\varphi'_{\mathbf{k}}\}|\rho|\{\varphi_{\mathbf{k}}\}\rangle = \prod_{\mathbf{k}}\langle\{\varphi'_{\mathbf{k}}\}|\rho(\xi_{\mathbf{k}}, \eta_{\mathbf{k}}; \sigma_{\mathbf{k}})|\{\varphi_{\mathbf{k}}\}\rangle \equiv \prod_{\mathbf{k}}\rho_{\mathbf{k}} \tag{2.21}$$

with

$$\rho_{\mathbf{k}} = (2\pi\xi^2_{\mathbf{k}})^{-\frac{1}{2}}\exp\Big\{-\frac{\sigma^2_{\mathbf{k}}+1}{4\xi^2_{\mathbf{k}}}\left[\varphi'^*_{\mathbf{k}}\varphi'_{\mathbf{k}} + \varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\right] + i\,\frac{\eta_{\mathbf{k}}}{\hbar\xi_{\mathbf{k}}}\left[\varphi'^*_{\mathbf{k}}\varphi'_{\mathbf{k}} - \varphi^*_{\mathbf{k}}\varphi_{\mathbf{k}}\right] + \frac{\sigma^2_{\mathbf{k}}-1}{4\xi^2_{\mathbf{k}}}\left[\varphi'^*_{\mathbf{k}}\varphi_{\mathbf{k}} + \varphi'_{\mathbf{k}}\varphi^*_{\mathbf{k}}\right]\Big\}\,, \tag{2.22}$$

and $\varphi_{\mathbf{k}}$ is the complex generalized coordinate of the classical field in Fourier space, defined by (2.5) (with $a_{\mathbf{k}}$ and $b_{-\mathbf{k}}$ treated as c-numbers). The Liouville equation for the evolution of this density matrix according to the quantum Hamiltonian of a free charged scalar field in a background electric potential,

$$\dot{\rho} = -i[H_{qu}, \rho]\;; \qquad H_{qu} = \frac{1}{2}\sum_{\mathbf{k}}\left(\pi_{\mathbf{k}}\pi^{\dagger}_{\mathbf{k}} + \omega^2_{\mathbf{k}}\varphi_{\mathbf{k}}\varphi^{\dagger}_{\mathbf{k}} + h.c.\right) \tag{2.23}$$

gives precisely the equations of motion (2.19) for the width parameters of the time dependent Gaussian. The effective classical Hamiltonian (2.20) is nothing else than the expectation value of the quantum Hamiltonian of scalar QED H_{qu} in the Gaussian density matrix ρ, *i.e.* $H_{eff} = \mathrm{Tr}(\rho H_{qu})$. Notice that in this Schrödinger representation of the time evolution all the equations are local in time, *i.e.*, they involve a single time argument, and there is no need to introduce Wigner functions with two time arguments, although these correlation functions at unequal times may be calculated easily enough from knowledge of the density matrix, if desired. In contrast to several earlier approaches to kinetic theory from field theory principles [20,21], we shall not require these unequal time correlators or Wigner functions.

The constant parameters $\sigma_{\mathbf{k}} = 1 + 2N_{\mathbf{k}} \geq 1$ measure the extent that the quantum state is a mixed state. If $N_{\mathbf{k}} = 0$, $\sigma_{\mathbf{k}} = 1$ and the state is pure, as is evident from the vanishing of the last term in (2.22), so that the density matrix becomes a simple product $|\psi\rangle\langle\psi|$. In either the pure or more general mixed state the density matrix (2.22) possesses a $U(1)$ symmetry under

$$\begin{aligned} \varphi_{\mathbf{k}} &\to \varphi_{\mathbf{k}} \exp(i\zeta_{\mathbf{k}}) \\ \varphi'_{\mathbf{k}} &\to \varphi'_{\mathbf{k}} \exp(i\zeta_{\mathbf{k}}) \end{aligned} \tag{2.24}$$

for each $\mathbf{k}$. This is a reflection of the fact that the generator of the local $U(1)$ gauge transformation of electrodynamics for a spatially uniform electric field is the charge density (2.15), and we have restricted ourselves to charge symmetric states obeying (2.16), so that the density matrix has this $U(1)$ invariance in each Fourier mode independently.

In this leading order of the large N expansion the density matrix of the electric field is also a Gaussian and multiplies the matter field Gaussian above, so that the evolution of the closed system with the backreaction Eqn. (2.17) is also Hamiltonian. Clearly the Hamiltonian evolution equations (2.9), with or without the Maxwell Eqn. (2.17) are completely time reversible upon reversing the signs of all the momenta.

Forgetting for the moment the Maxwell equation of backreaction on the electric field we see that the mean field evolution is equivalent to a set of time-dependent harmonic oscillators, with a different time-dependent frequency $\omega_{\mathbf{k}}(t)$ for each Fourier mode $f_{\mathbf{k}}$. Treating these frequencies as arbitrary, slowly varying functions of time we may write down the Hamilton-Jacobi equation corresponding to the effective classical Hamiltonian H_{eff}, namely,

$$\left(\frac{dW_{\mathbf{k}}}{d\xi_{\mathbf{k}}}\right)^2 + \omega_{\mathbf{k}}^2\xi_{\mathbf{k}}^2 + \frac{\hbar^2\sigma_{\mathbf{k}}^2}{4\xi_{\mathbf{k}}^2} = \epsilon_{\mathbf{k}}\,, \tag{2.25}$$

and find that the Hamilton principal function $W_{\mathbf{k}}$ evaluated over one full period,

$$\frac{W_{\mathbf{k}}}{2\pi\hbar} = \frac{1}{2\pi\hbar}\oint d\xi_{\mathbf{k}}\sqrt{2\epsilon_{\mathbf{k}} - \omega_{\mathbf{k}}^2\xi_{\mathbf{k}}^2 - \frac{\hbar^2\sigma_{\mathbf{k}}^2}{4\xi_{\mathbf{k}}^2}} = \frac{\epsilon_{\mathbf{k}}}{2\hbar\omega_{\mathbf{k}}} - \frac{\sigma_{\mathbf{k}}}{2} \tag{2.26}$$

is an adiabatic invariant of the periodic motion. Since $\sigma_{\mathbf{k}}$ is strictly a constant for all $\mathbf{k}$, this implies that

$$\frac{\epsilon_{\mathbf{k}}(t)}{\hbar\omega_{\mathbf{k}}(t)} \equiv 2\mathcal{N}_{\mathbf{k}}(t) + 1 \tag{2.27}$$

is an adiabatic invariant of the motion for slowly varying $\omega_{\mathbf{k}}(t)$. It is this adiabatic invariant that defines a time-dependent particle number basis which becomes the appropriate one for making contact with the Boltzmann-Vlasov kinetic description of particle creation.

III. THE ADIABATIC NUMBER BASIS

From the field theory development of the last section we note that N_+, N_- and k, appear quite naturally in either the time independent (Heisenberg) or time dependent (Schrödinger) descriptions as constants of motion under the Hamiltonian evolution. However, kinetic theory is expressed in terms of time-evolving quantities $\mathcal{N}_+(t), \mathcal{N}_-(t)$ and $p(t)$ which must be clearly distinguished from the analogous time independent quantities above. The difference between the canonical and kinetic momenta k and $p(t)$ in (2.11) is clear enough on basic kinematic grounds. The specification of the time dependent particle numbers $\mathcal{N}_+(t)$ and $\mathcal{N}_-(t)$ may not be quite as obvious, but as they provide the essential connection between the field theory and kinetic descriptions we must take special care to be equally clear and explicit about their definition. This requires that we introduce a Bogoliubov transformation from the time independent to a time-dependent (but adiabatic) number basis.

The observation underlying the introduction of this basis is that the mode equation (2.9) generally posseses time dependent solutions which have no clear *a priori* physical meaning in terms of particles or antiparticles. The familiar notion that positive energy solutions to the wave equation correspond to particles while negative energy solutions correspond to antiparticles is quite meaningless in time dependent background fields where the energy of individual particle/antiparticle modes is not conserved, and no such neat invariant separation into positive and negative energy solutions of the wave equation is possible. This is just a reflection of the fact that physical particle number does not correspond to a sharp operator which commutes with the Hamiltonian, *i.e.* particle/antiparticle pairs are created or destroyed, and physical particle number is not conserved in time dependent background fields.

Given this fact, one possible point of view is to forget completely about particle number in time-dependent backgrounds and deal only with conserved physical currents like $j(t)$ in (2.12). Indeed, in arbitrarily strong and rapidly time-varying fields this is the only possible point of view, since all notion of even an approximately conserved particle number disappears, and there is no possibility whatsoever of a classical kinetic description in such extreme situations. One must rely then exclusively on the field theoretic framework.

When the fields are not quite so strong and/or so rapidly varying in time we would expect to be able to define a particle number which varies slowly enough for the comparison to an effective semiclassical kinetic description to be meaningful. Clearly this physical slowly varying particle number is *not* the $N_{\mathbf{k}}$ of the time independent Heisenberg basis defined by (2.13) above, since this $N_{\mathbf{k}}$ is part of the initial data, a strict constant of the equations of motion, no matter how strong or rapidly varying the electric field is. The physical particle number at time t must be defined instead with respect to a time dependent basis (the adiabatic number basis) which permits a semiclassical correspondence limit to ordinary positive energy plane wave solutions in the limit of slowly varying $\omega_{\mathbf{k}}(t)$, and which is related to the Heisenberg basis by a time dependent Bogoliubov transformation. Since the mode equation (2.9) is the equation of motion of a (complex) harmonic oscillator with time varying frequency $\omega_{\mathbf{k}}(t)$, governed by the effective classical Hamiltonian H_{eff} of (2.20) classical Hamilton-Jacobi theory informs us that there is an adiabatic invariant proportional to the energy of the oscillator divided by its frequency, given by (2.27). It is this quantity which defines the adiabatic particle number and allows us to make the connection with classical kinetic theory. Corresponding to this slowly varying action variable there is a conjugate angle variable which is rapidly varying, of which classical kinetic theory takes no account.

The adiabatic basis is defined by first constructing the adiabatic mode functions,

$$\tilde{f}_{\mathbf{k}}(t) \equiv \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}(t)}} \exp\left(-i\int^t \omega_{\mathbf{k}}(t')dt'\right) . \tag{3.1}$$

We will make use of the shorthand notation for the phase,

$$\Theta_{\mathbf{k}}(t) \equiv \int^t \omega_{\mathbf{k}}(t')dt' , \tag{3.2}$$

suppressing the explicit dependence on t (and occasionally also the momentum index $\mathbf{k}$) except when needed for clarity in most of the following. The lower limit of the integral in (3.2) and therefore also the absolute phase of the mode function $\tilde{f}$ are left arbitary for the moment, to be fixed in a convenient way in the next section. In the limit of arbitrarily weak electric fields $\omega_{\mathbf{k}}(t)$ becomes nearly independent of time and can be removed from the integral in (3.2). In that limit the adiabatic mode function becomes the usual positive energy plane wave solution with respect to which the usual definition of particle number is taken. Otherwise the adiabatic mode functions (3.1) will not be exact solutions of the mode equation (2.9), but we are still free to specify a basis with respect to them, provided only that $\omega_{\mathbf{k}}(t)$ remains real and positive for all $\mathbf{k}$ and t.

The transformation to this basis from the original one is specified by the two linear relations,

$$\begin{aligned} f_{\mathbf{k}}(t) &= \alpha_{\mathbf{k}}(t)\tilde{f}_{\mathbf{k}}(t) + \beta_{\mathbf{k}}(t)\tilde{f}^*_{\mathbf{k}}(t) \\ \dot{f}_{\mathbf{k}}(t) &= -i\omega_{\mathbf{k}}\alpha_{\mathbf{k}}(t)\tilde{f}_{\mathbf{k}}(t) + i\omega_{\mathbf{k}}\beta_{\mathbf{k}}(t)\tilde{f}^*_{\mathbf{k}}(t) \end{aligned} \tag{3.3}$$

between the exact and adiabatic mode functions. When the phase of $\tilde{f}$ is fixed these relations completely fix the complex coefficients $\alpha_{\mathbf{k}}(t)$ and $\beta_{\mathbf{k}}(t)$. It is straightforward to solve for the Bogoliubov coefficients directly in the form,

$$\begin{aligned} \alpha_{\mathbf{k}} &= i(\dot{f}_{\mathbf{k}} - i\omega_{\mathbf{k}} f_{\mathbf{k}})\tilde{f}^*_{\mathbf{k}} \qquad \text{and} \\ \beta_{\mathbf{k}} &= -i(\dot{f}_{\mathbf{k}} + i\omega_{\mathbf{k}} f_{\mathbf{k}})\tilde{f}_{\mathbf{k}} . \end{aligned} \tag{3.4}$$

An equivalent form of this Bogoliubov transformation in the Fock space of creation and destruction operators is

$$\begin{aligned} a_{\mathbf{k}} &= \alpha^*_{\mathbf{k}}(t)\tilde{a}_{\mathbf{k}}(t) - \beta^*_{\mathbf{k}}(t)\tilde{b}^\dagger_{-\mathbf{k}}(t) \\ b^\dagger_{-\mathbf{k}} &= \alpha_{\mathbf{k}}(t)\tilde{b}^\dagger_{-\mathbf{k}}(t) - \beta_{\mathbf{k}}(t)\tilde{a}_{\mathbf{k}}(t) , \end{aligned} \tag{3.5}$$

so that the field coordinate $\varphi_{\mathbf{k}}(t)$ may be expressed equally well in the time independent basis by (2.5) or in the time dependent basis by

$$\varphi_{\mathbf{k}}(t) = \tilde{f}_{\mathbf{k}}(t)\tilde{a}_{\mathbf{k}}(t) + \tilde{f}^*_{\mathbf{k}}(t)\tilde{b}^\dagger_{-\mathbf{k}}(t) , \tag{3.6}$$

and likewise the field momentum variable is given either by (2.6) or by

$$\pi_{\mathbf{k}}(t) = -i\omega_{\mathbf{k}}(t)\tilde{f}^*_{\mathbf{k}}(t)\tilde{a}^\dagger_{\mathbf{k}}(t) + i\omega_{\mathbf{k}}(t)\tilde{f}_{\mathbf{k}}(t)\tilde{b}_{-\mathbf{k}}(t) . \tag{3.7}$$

In this basis the Hamiltonian of the set of time dependent oscillators in (2.23) becomes diagonal,

$$H_{qu} = \frac{\hbar}{2}\sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\tilde{a}^{\dagger}_{\mathbf{k}}\tilde{a}_{\mathbf{k}} + \tilde{a}_{\mathbf{k}}\tilde{a}^{\dagger}_{\mathbf{k}} + \tilde{b}_{-\mathbf{k}}\tilde{b}^{\dagger}_{-\mathbf{k}} + \tilde{b}^{\dagger}_{-\mathbf{k}}\tilde{b}_{-\mathbf{k}}\right) . \tag{3.8}$$

The transformation from the time independent $(a_{\mathbf{k}}, b^{\dagger}_{-\mathbf{k}})$ basis to the time dependent adiabatic basis $(\tilde{a}_{\mathbf{k}}, \tilde{b}^{\dagger}_{-\mathbf{k}})$ requires two independent relations (3.3) or (3.5), corresponding to a canonical transformation in a two dimensional (complex) phase space, for which

$$|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1 \tag{3.9}$$

for each $\mathbf{k}$. It is easily verified that (3.4) satisfies this relation when the Wronskian condition (2.8) is used. Because of (3.9) the magnitude of the Bogoliubov transformation to the adiabatic number basis $\gamma_{\mathbf{k}}(t)$ may be specified by

$$\begin{aligned} |\alpha_{\mathbf{k}}(t)| &= \cosh\gamma_{\mathbf{k}}(t)\, , \\ |\beta_{\mathbf{k}}(t)| &= \sinh\gamma_{\mathbf{k}}(t)\, . \end{aligned} \tag{3.10}$$

We now define the adiabatic particle number to be

$$\begin{aligned} \mathcal{N}_{\mathbf{k}}(t) &\equiv \langle \tilde{a}^{\dagger}_{\mathbf{k}}(t)\tilde{a}_{\mathbf{k}}(t)\rangle = \langle \tilde{b}^{\dagger}_{-\mathbf{k}}(t)\tilde{b}_{-\mathbf{k}}(t)\rangle \\ &= |\alpha_{\mathbf{k}}|^2\langle a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}}\rangle + |\beta_{\mathbf{k}}|^2\langle b_{-\mathbf{k}}b^{\dagger}_{-\mathbf{k}}\rangle \\ &= \left(1 + |\beta_{\mathbf{k}}|^2\right) N_{+}(\mathbf{k}) + |\beta_{\mathbf{k}}|^2 \left(1 + N_{-}(-\mathbf{k})\right) \\ &= N_{\mathbf{k}} + (1 + 2N_{\mathbf{k}})\ |\beta_{\mathbf{k}}(t)|^2 \\ &= N_{\mathbf{k}} + (1 + 2N_{\mathbf{k}})\ \sinh^2\gamma_{\mathbf{k}}(t)\, . \end{aligned} \tag{3.11}$$

The second of the relations (3.3) is essential to define the adiabatic basis in which particle number is given by the ratio of energy to frequency. In fact,

$$\begin{aligned} \frac{\epsilon_{\mathbf{k}}(t)}{\hbar\omega_{\mathbf{k}}(t)} &= (1 + 2N_{\mathbf{k}})\, \frac{\left(|\dot{f}_{\mathbf{k}}|^2 + \omega^2_{\mathbf{k}}|f_{\mathbf{k}}|^2\right)}{\hbar\omega_{\mathbf{k}}} \\ &= (1 + 2N_{\mathbf{k}})\left(1 + 2|\beta_{\mathbf{k}}|^2\right) \\ &= 1 + 2\mathcal{N}_{\mathbf{k}}(t)\, . \end{aligned} \tag{3.12}$$

Hence the particle number $\mathcal{N}_{\mathbf{k}}(t)$, though time dependent, is an adiabatic invariant of the motion. Consequently, it is the natural candidate for a particle density in phase space for a kinetic description, becoming the ordinary asymptotic constant particle number in the limit of slowly varying $\omega_{\mathbf{k}}(t)$. This adiabatic definition of particle number which diagonalizes the time dependent Hamiltonian has been considered before, most recently in the context of particle creation in curved space backgrounds [23,24]. Although this choice of basis is not unique, since we could have chosen a different condition on $\dot{f}_{\mathbf{k}}$ in (3.3), it is the only basis where the ratio $\varepsilon_{\mathbf{k}}/\omega_{\mathbf{k}}$ is simply related to particle number (without the appearance of $\dot{\omega}_{\mathbf{k}}$ or higher derivative terms, for example) which is the standard adiabatic invariant of the harmonic oscillator with time dependent frequency as in (2.27). In different contexts (such as particle creation in external gravitational fields where the even the Hamiltonian is not uniquely defined) it may be appropriate to consider a somewhat different definition of the adiabatic number basis, depending on the application.

Now that we have completely specified the time dependent particle number basis it is straightforward to derive the equation of motion which it obeys. We note that from the explicit representation (3.4) by differentiation and use of the mode equation (2.9) we have

$$\dot{\alpha}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\beta_{\mathbf{k}}\exp(2i\Theta_{\mathbf{k}})\,, \qquad \text{and}$$
$$\dot{\beta}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\alpha_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}})\,. \tag{3.13}$$

These two first order differential equations are entirely equivalent to the second order mode equation in Hamiltonian form. We now obtain by differentiating (3.11),

$$\begin{aligned}\frac{d}{dt}\mathcal{N}_{\mathbf{k}} &= 2\,(1+2N_{\mathbf{k}})\,\mathrm{Re}\,(\beta^*_{\mathbf{k}}\dot{\beta}_{\mathbf{k}})\\ &= \frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}\,(1+2N_{\mathbf{k}})\;\mathrm{Re}\,\{\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}})\}\\ &= \frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}\,\mathrm{Re}\,\{\mathcal{C}_{\mathbf{k}}\exp(-2i\Theta_{\mathbf{k}})\}\;,\end{aligned} \tag{3.14}$$

where we have defined the time dependent pair correlation function,

$$\begin{aligned}\mathcal{C}_{\mathbf{k}}(t) &\equiv \langle \tilde{a}_{\mathbf{k}}(t)\tilde{b}_{-\mathbf{k}}(t)\rangle\\ &= (1+2N_{\mathbf{k}})\,\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}\,.\end{aligned} \tag{3.15}$$

Thus the time derivative of the adiabatically slowly varying particle number involves the pair correlation function $\mathcal{C}_{\mathbf{k}}(t)$ which is itself very rapidly varying, since the time dependent phases on the right side of (3.15) *add* rather than cancel, although the phases do nearly cancel in the final combination of (3.14). The time derivative of the pair correlation function,

$$\begin{aligned}\frac{d}{dt}\mathcal{C}_{\mathbf{k}} &= \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\,(1+2N_{\mathbf{k}})\exp(2i\Theta_{\mathbf{k}})\;\left(1+2|\beta_{\mathbf{k}}|^2\right)\\ &= \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\,(1+2\mathcal{N}_{\mathbf{k}})\exp(2i\Theta_{\mathbf{k}})\,,\end{aligned} \tag{3.16}$$

brings us back again to $\mathcal{N}_{\mathbf{k}}$. This last equation may be solved formally for $\mathcal{C}_{\mathbf{k}}$ and subsituted into (3.14) to obtain

$$\frac{d}{dt}\mathcal{N}_{\mathbf{k}} = \frac{\dot{\omega}_{\mathbf{k}}}{2\omega_{\mathbf{k}}}\int_{t_0}^{t}dt'\,\left\{\frac{\dot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}}(t')\,(1+2\mathcal{N}_{\mathbf{k}}(t'))\cos\left[2\Theta_{\mathbf{k}}(t)-2\Theta_{\mathbf{k}}(t')\right]\right\}\,, \tag{3.17}$$

where we have assumed that $\mathcal{C}_{\mathbf{k}}$ vanishes at some $t=t_0$ (which could be taken to $-\infty$).

Equation (3.17) may be be called a "quantum Vlasov equation," in the sense that it gives the quantum creation rate of particle number in an arbitary time varying mean field. Let us remark that the Bose enhancement factor $(1+2\mathcal{N}_{\mathbf{k}})$ appears in (3.17), so that both spontaneous and induced particle creation are included automatically in the quantum treatment. The most important feature of Eqn. (3.17) for our present purpose is that it is nonlocal in time, the particle creation rate depending on the entire previous history of the system. In that sense the particle creation process is certainly non-Markovian in general [16,18]. Eqn. (3.17) becomes exact in the limit in which the electric field can be treated classically, *i.e.* the large N limit in which real and virtual photon emission is neglected, and there is no scattering. Inclusion of scattering processes lead to collision terms on the right side of (3.17) which are also nonlocal in general. This nonlocality is essential to the quantum description in which phase information is retained for all times. The phase oscillations in the cosine term are a result of the quantum coherence between the created pairs, which must be present in principle in any unitary evolution. However, precisely because these phase oscillations are so rapid it is clear that the integral in (3.17) receives most of its contribution from t' close to t, which suggests that some local approximation to the integral should be possible, provided that we are not interested in resolving the short time structure or measuring the phase coherence effects. The time scale for these quantum phase coherence effects to wash out is the time scale of several oscillations of the phase factor $\Theta_{\mathbf{k}}(t) - \Theta_{\mathbf{k}}(t')$, which is of order $\tau_{qu} = 2\pi/\omega_{\mathbf{k}} = 2\pi\hbar/\epsilon_{\mathbf{k}}$, where $\epsilon_{\mathbf{k}}$ is the single particle energy.

The steps we have just performed to arrive at (3.17) are a special case of the general projection formalism of Zwanzig [22], where some subset of fast dynamical variables deemed "irrelevant" (in this case $\mathcal{C}_{\mathbf{k}}$) are eliminated in favor of slow variables deemed "relevant" (in this case $\mathcal{N}_{\mathbf{k}}$). Because the two variables are coupled by the underlying Hamiltonian equations of motion the result of solving for some variables in terms of others is generally nonlocal in time. The nonlocal form (3.17) is still completely equivalent to the mode equation (2.9) and absolutely nothing has been lost (or gained) by this rewriting. In other words, the projection method is essentially free of any physical content, until and unless one makes further approximations that replace the nonlocal relations satisfied by the relevant observables by *local* ones. It is at this point that great care must be exercised, since the precise form of the local approximation made will determine the usefulness and range of validity of the resulting truncation.

A natural suggestion might be to replace the Bose enhancement factor $1+2\mathcal{N}_{\mathbf{k}}(t')$ by $1+2\mathcal{N}_{\mathbf{k}}(t)$ and remove it from the integral, on the basis that it is slowly varying function for real t', and attempt to perform the remaining integral over the rapidly varying phase by the method of stationary phase. However, the phase becomes stationary at $\dot{\Theta}_{\mathbf{k}} = \omega_{\mathbf{k}} = 0$ which is precisely where the integrand has a pole in the complex t' plane. At such a turning point the adiabatic (WKB) approximation certainly breaks down. Hence the stationary phase method is somewhat less straightforward in this case, and a naive application of the method results in the correct exponential factor but the *in*correct prefactor [7]. More seriously, the stationary phase method is of no general utility unless one already possesses detailed knowledge of the analytic structure of the integrand in Eqn. (3.17) in the complex t' plane, and in particular, the location of the turning points where $\omega_{\mathbf{k}}$ vanishes.

The importance of the complex turning point(s) for determing the asymptotic mixing between particle and antiparticle modes as $t \to \pm\infty$ has been emphasized by Marinov and Popov in refs. [8]. In their method the analytic continuation of the solutions of the mode equation around the Stokes' lines emanating from the turning point in the complex time plane determines the subdominant component of the wave function with the opposite sign of the frequency on the real axis. The amplitude of this exponentially subdominant component of antiparticle waves in the wavefunction is the Schwinger particle creation effect. However, the method outlined by these authors does not seem to be applicable to the integral in (3.17) directly, since it is designed for calculating the particle creation asymptotically over infinite time, not for determining the evolution of the particle creation process in finite real time t, which is what we require for the transport description.

If one takes no account of the stationary phase point in the complex t' plane but attempts to approximate the integral in (3.17) entirely in real time, for example by integrating the rapidly varying cosine function by parts any number of times, it is easy to see that an asymptotic series is generated in which the exponentially small subdominant solution can *never* appear after any finite number of such steps. Any asymptotic expansion of the wavefunction on the real axis which discards the exponentially small antiparticle component will miss the Schwinger creation effect at late times.

From this discussion we see that the essential difficulty with Eqn. (3.17) is that the point(s) in the complex t' plane where the phase $\Theta_{\mathbf{k}}$ is stationary must play the critical role in determining the particle creation for asymptotically late times, but we cannot evaluate the contribution to the integral of these stationary phase points where $\omega_{\mathbf{k}}$ vanishes without in effect knowing the full $\mathcal{N}_{\mathbf{k}}, \omega_{\mathbf{k}}$ and $\Theta_{\mathbf{k}}$ as analytic functions in the entire complex t plane before we even begin. If we were in possession of these analytic functions we would already have the full solution to our dynamical problem, without any need to make any approximations to the integral. This is clearly impossible except for a small number of special cases where the complete analytic structure is known *a priori*. Thus the nonlocal form of the quantum Vlasov equation (3.17) makes it difficult to extract any useful information about a source term for a kinetic description in general.

Consideration of this difficulty immediately suggests a different approach. Instead of trying to work with the nonlocal equation (3.17), in the next section we evaluate the spontaneous pair creation rate $\frac{d}{dt}\mathcal{N}_{\mathbf{k}}(t)$ for a *constant* electric field analytically and directly in real time, thereby assuring agreement with the Schwinger result in both its exponential and nonexponential factors. This is one of special cases where $\mathcal{N}_{\mathbf{k}}$ and its time derivative can be evaluated analytically in local form, directly from the definition (3.11) without any need for the nonlocal integral representation (3.17). Then by making use of an asymptotic expansion of the exact analytic result for constant fields, uniformly valid everywhere on the real time axis, we obtain a useful *local* approximation to the spontaneous pair creation rate for the slowly varying electric fields, without any need for analytic continuation or stationary phase methods in complex time. By such an approach we shall bypass completely the difficulties of dealing with the nonlocal integral equation (3.17) resulting from the projection method.

The transformation to the adiabatic number basis and elimination of the rapid variables $\mathcal{C}_{\mathbf{k}}$ in favor of the slow variables $\mathcal{N}_{\mathbf{k}}$ by Eqns. (3.15)-(3.17) has its counterpart in the density matrix description as well. It is shown in the Appendix that the density matrix (2.22) may be transformed to the adiabatic number basis, with the general form of the nonvanishing matrix elements given by (A27). In the pure state case $\sigma_{\mathbf{k}} = 1$ the only nonvanishing matrix elements of ρ are in uncharged pair states with equal numbers of positive and negative charges, $\ell_{\mathbf{k}} = n^{(+)}_{\mathbf{k}} = n^{(-)}_{\mathbf{k}}$, with $\ell_{\mathbf{k}}$ the number of pairs in the mode $\mathbf{k}$, *viz.*

$$\langle 2\ell'_{\mathbf{k}}|\rho|2\ell_{\mathbf{k}}\rangle\Big|_{\sigma=1} = e^{i(\ell_{\mathbf{k}}'-\ell_{\mathbf{k}})\vartheta_{\mathbf{k}}(t)}\ \mathrm{sech}^2\gamma_{\mathbf{k}}(t)\ \ (\tanh\gamma_{\mathbf{k}}(t))^{\ell_{\mathbf{k}}'+\ell_{\mathbf{k}}} \tag{3.18}$$

where the magnitude of the Bogoliubov transformation, $\gamma_{\mathbf{k}}(t)$ is defined by (3.10) and its phase, $\vartheta_{\mathbf{k}}(t)$ is specified by

$$\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}e^{-2i\Theta_{\mathbf{k}}} = -\sinh\gamma_{\mathbf{k}}\ \cosh\gamma_{\mathbf{k}}\ e^{i\vartheta_{\mathbf{k}}}\,. \tag{3.19}$$

Hence the off-diagonal matrix elements $\ell' \neq \ell$ of ρ are rapidly varying on the time scale τ_{qu} of the quantum mode functions, while the diagonal matrix elements $\ell' = \ell$ depend only on the adiabatic invariant average particle number via

$$\langle 2\ell_{\mathbf{k}}|\rho|2\ell_{\mathbf{k}}\rangle\Big|_{\sigma=1} \equiv \rho_{2\ell_{\mathbf{k}}} = \mathrm{sech}^2\gamma_{\mathbf{k}}\tanh^{2\ell_{\mathbf{k}}}\gamma_{\mathbf{k}} = \frac{|\beta_{\mathbf{k}}|^{2\ell_{\mathbf{k}}}}{(1+|\beta_{\mathbf{k}}|^2)^{\ell_{\mathbf{k}}+1}} = \frac{\mathcal{N}_{\mathbf{k}}^{\ell_{\mathbf{k}}}}{(1+\mathcal{N}_{\mathbf{k}})^{\ell_{\mathbf{k}}+1}}\Big|_{\sigma=1}, \tag{3.20}$$

and are therefore much more slowly varying functions of time. The average number of positively charged particles (or negatively charged antiparticles) in this basis is given of course by

$$\sum_{\ell_{\mathbf{k}}=0}^{\infty} \ell_{\mathbf{k}}\rho_{2\ell_{\mathbf{k}}} = \mathcal{N}_{\mathbf{k}}\,. \tag{3.21}$$

Thus the diagonal and off-diagonal elements of the density matrix in the adiabatic particle number basis stand in precisely the same relationship to each other and contain the same information as the particle number $\mathcal{N}_{\mathbf{k}}$ and pair correlation $\mathcal{C}_{\mathbf{k}}$ respectively.

Using the representation (3.18) or (3.20) we can understand how entropy can increase and the evolution become time irreversible if we replace the exact nonlocal quantum Vlasov equation (3.17) by a local expression in which the rapid phase variables $\mathcal{C}_{\mathbf{k}}$, $\vartheta_{\mathbf{k}}$, or the off-diagonal matrix elements of ρ no longer appear. Time reversal in the field theory requires that both the slow and fast variables be time reversed, which involves the full density matrix ρ. If we restrict attention to only the diagonal matrix elements of ρ in the adiabatic particle number basis without any account of the phase information present in the rapidly varying off-diagonal elements, then time reversal no longer holds. In the effective density matrix (3.20) the diagonal elements $\rho_{2\ell_{\mathbf{k}}}$ may be interpreted (for $\sigma_{\mathbf{k}} = 1$) as the independent probabilities of creating $\ell_{\mathbf{k}}$ pairs of charged particles with canonical momentum $\mathbf{k}$ from the vacuum. This corresponds to disregarding the intricate quantum phase correlations between the created pairs in the unitary Hamiltonian evolution, and treating the creation events as essentially independent in a stochastic Markovian processes. Thus the Markov approximation to the field theory arises quite naturally when the quantum density matrix is expressed in the adiabatic particle number basis.

Such an approximation is known to be quite accurate for long intervals of time in the backreaction of the current on the electric field producing the pairs, for the simple reason that the phase information in the pair correlations cancels very efficiently when one considers the sum over all the $\mathbf{k}$ modes in the current (2.12). It is for this reason that for practical purposes one can approximate the full Gaussian density matrix over large time intervals by its diagonal elements only, in this basis. Naturally this truncation of the unitary Hamiltonian evolution according to (2.23) leads to a nonunitary irreversible evolution in which the *effective* von Neumann entropy of the diagonal density matrix (3.20),

$$S_{eff}(t) = -\mathrm{Tr}\,\rho_{eff}\ln\rho_{eff} = -\sum_{\mathbf{k}}\sum_{\ell_{\mathbf{k}}=0}^{\infty}\rho_{2\ell_{\mathbf{k}}}\ln\rho_{2\ell_{\mathbf{k}}} \tag{3.22}$$

can increase with time. In fact, upon subsituting (3.20), the sums over $\ell_{\mathbf{k}}$ are geometric series which are easily performed, with the result that the von Neumann entropy of this truncated density matrix,

$$S_{eff}(t)\Big|_{\sigma=1} = \sum_{\mathbf{k}}\{(1+\mathcal{N}_{\mathbf{k}})\ln(1+\mathcal{N}_{\mathbf{k}}) - \mathcal{N}_{\mathbf{k}}\ln\mathcal{N}_{\mathbf{k}}\} \tag{3.23}$$

is precisely equal to the Boltzmann entropy of the single particle distribution function $\mathcal{N}_{\mathbf{k}}(t)$. Hence

$$\frac{d}{dt}S_{eff} = \sum_{\mathbf{k}} \ln\left(\frac{1+\mathcal{N}_{\mathbf{k}}}{\mathcal{N}_{\mathbf{k}}}\right)\frac{d}{dt}\mathcal{N}_{\mathbf{k}} \tag{3.24}$$

increases if the mean particle number increases. This is always the case *on average* if one starts with vacuum initial conditions, $\sigma_{\mathbf{k}} = 1$, since $|\beta_{\mathbf{k}}|^2$ is necessarily nonnegative and can only increase if it is zero initially [25]. Locally, or once particles are present in the initial state, there is no reason why particle number or the entropy (3.24) must continue to increase monotonically in time, and indeed small temporary decreases are observed in backreaction simulations [4]. Hence there is no Boltzmann H-theorem for the effective entropy (3.24) without introducing some explicit time averaging and/or further assumptions into the scheme.

Before closing this section we wish to take note of one additional especially simple property of the adiabatic particle number basis. Inserting the Bogoliubov transformation of the mode functions (3.3) into the expression for the current (2.12) we obtain

$$j(t) = e\int [d\mathbf{k}]\frac{(k-eA(t))}{\omega_{\mathbf{k}}(t)}(1+2|\beta_{\mathbf{k}}(t)|^2 + 2\mathrm{Re}\{\alpha_{\mathbf{k}}\beta^*_{\mathbf{k}}e^{-2i\Theta_{\mathbf{k}}(t)}\})(1+2N_{\mathbf{k}})\,. \tag{3.25}$$

We note that the vacuum term in this expression, $\int [d\mathbf{k}]\frac{(k-eA(t))}{\omega_{\mathbf{k}}(t)}$ vanishes by charge conjugation symmetry, when proper gauge invariant integration boundaries are chosen. Using the mean value of particles in the adiabatic number basis (3.11), its time derivative and the equations of motion (3.14), we can rewrite the current (3.25) as

$$\begin{aligned} j(t) &= 2e\int [d\mathbf{k}]\frac{(k-eA(t))}{\omega_{\mathbf{k}}(t)}\mathcal{N}_{\mathbf{k}}(t) + \frac{2}{E}\int [d\mathbf{k}]\omega_{\mathbf{k}}(t)\dot{\mathcal{N}}_{\mathbf{k}}(t) \\ &= j_{cond} + j_{pol}\,. \end{aligned} \tag{3.26}$$

On the other hand, from a classical point of view if the particle distribution $\mathcal{N}_{\mathbf{k}}$ is coupled to a uniform electric field the energy density and its time derivative are given by

$$\varepsilon = \frac{E^2}{2} + 2\int [d\mathbf{k}]\omega_{\mathbf{k}}\mathcal{N}_{\mathbf{k}} \tag{3.27}$$

$$\dot{\varepsilon} = \dot{E}E + +2\int [d\mathbf{k}]\left(eE\frac{(k-eA)}{\omega_{\mathbf{k}}}\mathcal{N}_{\mathbf{k}} + \omega_{\mathbf{k}}\dot{\mathcal{N}}_{\mathbf{k}}\right) = 0\,. \tag{3.28}$$

Using the Maxwell equation $-\dot{E} = j$ this last relation is precisely the *same* as the mean value of the quantum current in (3.26). Hence we may identify the adiabatic particle number $\mathcal{N}_{\mathbf{k}}(t)$ with the (quasi)classical single particle distribution. Other definitions of time varying particle number, such as that used in our own earlier work [2] do not have this property or admit this simple quasiclassical interpretation. This exercise also demonstrates that the two terms in the mean current (3.26) should indeed be interpreted as the conduction and polarization terms of the earlier phenomenological descriptions.

IV. CONSTANT ELECTRIC FIELD

In order to derive the source term due to particle creation in a slowly varying electric field, we first analyze the time structure of the creation process in a constant, uniform electric field, for which

$$A(t) = -Et. \tag{4.1}$$

It is useful to define the rescaled dimensionless variables

$$u \equiv \epsilon \frac{k + eEt}{\sqrt{|eE|}} = \frac{\epsilon p(t)}{\sqrt{|eE|}} \qquad \text{and} \qquad \lambda \equiv \frac{k_\perp^2 + m^2}{|eE|} > 0\,, \tag{4.2}$$

where $\epsilon = \epsilon(eE) = \pm 1$ is the sign of eE. Then the mode equation (2.9) may be put into the form,

$$\left(\frac{d^2}{du^2} + u^2 + \lambda\right) f = 0 \tag{4.3}$$

whose solutions are parabolic cylinder (Weber) functions [26,27]. In fact, the two complex conjugate pairs of solutions,

$$\begin{aligned} f_{(+)}(u) = f^*_{(-)}(u) &\propto D_{-\frac{1}{2}+i\frac{\lambda}{2}}\left(-(1-i)u\right) \qquad \text{and} \\ f^{(+)}(u) = f^{(-)*}(u) &\propto D_{-\frac{1}{2}-i\frac{\lambda}{2}}\left((1+i)u\right) \end{aligned} \tag{4.4}$$

each comprise complete sets of basis functions in which to expand the scalar charged field Φ. Normalizing these solutions according to the Wronskian condition (2.8) and defining the phase,

$$\psi \equiv \frac{\lambda}{4} - \frac{\lambda}{4}\ln\lambda + \frac{\lambda}{4}\ln 2 - \frac{\pi}{8} \tag{4.5}$$

we can write the properly normalized positive frequency mode functions in the form,

$$\begin{aligned} f_{(+)\mathbf{k}}(t) &= |2eE|^{-\frac{1}{4}} e^{-\frac{\pi\lambda}{8}} e^{i\psi}\, D_{-\frac{1}{2}+i\frac{\lambda}{2}}\left(-(1-i)u\right) \quad \text{and} \\ f^{(+)}_{\mathbf{k}}(t) &= |2eE|^{-\frac{1}{4}} e^{-\frac{\pi\lambda}{8}} e^{-i\psi}\, D_{-\frac{1}{2}-i\frac{\lambda}{2}}\left((1+i)u\right) \end{aligned} \tag{4.6}$$

which approach the adiabatic functions $\tilde{f}_{\mathbf{k}}(t)$ in the asymptotic limits $t \to -\infty$ and $t \to \infty$ respectively. Notice that with u defined including the ϵ function as in (4.2) these limits are equivalent to $u \to -\infty$ and $u \to \infty$ respectively, independently of the sign of eE. The complex conjugates of these solutions are the corresponding negative frequency mode functions and are denoted by $f_{(-)\mathbf{k}}$ or $f^{(-)}_{\mathbf{k}}$ respectively. The phase ψ has been defined in such a way that the phase of the exact mode functions $f_{\mathbf{k}}$ agrees with the adiabatic mode functions (3.1) with phase $\Theta_{\mathbf{k}}$ measured from the symmetric point $u = 0$, *i.e.*

$$\begin{aligned} \Theta_{\mathbf{k}}(t) &= \int_{u=0}^{t} dt'\omega_{\mathbf{k}}(t') = \int_0^u du'\sqrt{u'^2 + \lambda} \\ &= \frac{1}{2}u\sqrt{u^2+\lambda} + \frac{\lambda}{2}\ln\left(\frac{u + \sqrt{u^2+\lambda}}{\sqrt{\lambda}}\right)\,. \end{aligned} \tag{4.7}$$

It may seem surprising at first sight that the exact mode functions approach the adiabatic ones in the infinite past and infinite future even though the electric field E is constant and never vanishes in these limits. The reason for this is that the corrections to the lowest-order adiabatic mode functions involve

$$\frac{\delta\omega_{\mathbf{k}}^2(t)}{\omega_{\mathbf{k}}^2(t)} = \frac{1}{2}\frac{\ddot{\omega}_{\mathbf{k}}}{\omega_{\mathbf{k}}^3} - \frac{3}{4}\frac{\dot{\omega}_{\mathbf{k}}^2}{\omega_{\mathbf{k}}^4} = \frac{(-3u^2 + 2\lambda)}{4(u^2+\lambda)^3} \tag{4.8}$$

which goes to zero like $|t|^{-4}$ as $t \to \pm\infty$.

If the state of the system is the vacuum *in* state then the mode function $f_{\mathbf{k}}$ to be used in (3.4) is the $f_{(+)\mathbf{k}}$ of (4.6) and the effective source term for the creation of particles from the vacuum is

$$\frac{d}{dt}\mathcal{N}_{\mathbf{k}}\Big|_{N_{\mathbf{k}}=0} = \frac{d}{dt}|\beta_{\mathbf{k}}|^2 = |8eE|^{-\frac{1}{2}} e^{-\frac{\pi\lambda}{4}}\,\frac{\partial}{\partial t}\left\{\frac{1}{\omega_{\mathbf{k}}}\left|\left(\frac{\partial}{\partial t} + i\omega_{\mathbf{k}}\right) D_{-\frac{1}{2}+i\frac{\lambda}{2}}\left(-(1-i)u\right)\right|^2\right\}. \tag{4.9}$$

We note that for a strictly constant electric field this is an *exact* result for the rate of adiabatic particle number change starting from vacuum initial conditions at $t = -\infty$. Phase correlation information for this particular initial state has not been discarded, although the pair correlation function does not appear explicitly in (4.9), which is *local* in time.

Now since the two pairs of complex functions $f_{(\pm)\mathbf{k}}$ and $f_{\mathbf{k}}^{(\pm)}$ both satisfy the same second order wave equation there exist linear relations between them. Indeed it follows from the properties of the Weber parabolic cylinder functions that

$$f_{(+)\mathbf{k}} = \bar{\alpha} f_{\mathbf{k}}^{(+)} + \bar{\beta} f_{\mathbf{k}}^{(-)} \tag{4.10}$$

with [26,27]

$$\begin{aligned} \bar{\alpha} &= \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1-i\lambda}{2}\right)} e^{2i\psi + \frac{i\pi}{4}} e^{-\frac{\pi\lambda}{4}} \qquad \text{and} \\ \bar{\beta} &= -ie^{-\frac{\pi\lambda}{2}} \end{aligned} \tag{4.11}$$

The fact that $\bar{\beta} \neq 0$ is the statement that the Bogoliubov transformation between the two basis pairs is nontrivial and the adiabatic vacuum state in the infinite past contains particle–antiparticle pairs with respect to the adiabatic vacuum state in the infinite future. The magnitude of this total Bogoliubov transformation from $t = -\infty$ to $t = +\infty$ is finite and given by

$$|\bar{\beta}|^2 \equiv \sinh^2\bar{\gamma} = e^{-\pi\lambda}, \tag{4.12}$$

which is independent of k in the direction of the electric field.

By transforming the Gaussian density matrix corresponding to the evolution of the charged scalar field in a background electric field one can show that (4.12) is also the mean number of particles in the final state with respect to the *out* vacuum, assuming that the field was prepared in the *in* vacuum. The details of this transformation are given in the Appendix. From the result, (3.20) with $\gamma_{\mathbf{k}}$ replaced by $\bar{\gamma}$ and the discussion of the previous section discarding the rapidly varying off-diagonal elements of ρ, we may interpret the diagonal elements as the probability of finding ℓ pairs at late times if none were present initially. Hence the $\ell = 0$ matrix element,

$$\text{sech}^2\bar{\gamma} = (1 + e^{-\pi\lambda})^{-1} \tag{4.13}$$

is the probability of creating no pairs in the given mode, and the probability that the vacuum remains the vacuum in the future is given by the product over all modes,

$$\prod_{\mathbf{k}}(1+e^{-\pi\lambda})^{-1} = \exp\left(-\sum_{\mathbf{k}} \ln(1+e^{-\pi\lambda})\right) . \tag{4.14}$$

Taking the infinite volume limit this can be expressed as $\exp(-VT\Gamma)$ where the rate of vacuum decay per unit volume is

$$\Gamma = \frac{1}{T}\int [d\mathbf{k}] \ln(1+e^{-\pi\lambda}) . \tag{4.15}$$

Since the kinetic momentum of the created charged particles in the direction of the electric field is $k + eEt$, the longitudinal integration element dk can be replaced by eET as $T \to \infty$ and the vacuum decay rate becomes

$$\begin{aligned}\Gamma &= \frac{eE}{(2\pi)^3}\int d^2\mathbf{k}_\perp \ln(1+e^{-\pi\lambda}) \\ &= \frac{eE}{(2\pi)^3}\int d^2\mathbf{k}_\perp \sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n}e^{-\pi n\lambda} \\ &= \frac{(eE)^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{n^2}e^{-\frac{\pi n m^2}{\hbar|eE|}} \end{aligned} \tag{4.16}$$

which is Schwinger's result for scalar QED.

One should note that the replacement of the longitudinal momentum integral over k in (4.15) by eET in the large T limit can be justified only if one understands the time evolution of the pair creation event, for otherwise the expression (4.15) is formally meaningless. This replacement of $\int dk$ by eET and the resulting finite expression (4.16), which can be obtained by quite different methods imply that only those k in a linearly growing window in time actually contribute to the rate, although the mixing coefficient $\bar{\beta}$ over all time is independent of k. It is the time dependent evolution of $\beta_{\mathbf{k}}(t)$ which we can investigate in detail with our definition of the time dependent adiabatic number basis in the next section. This definition smoothly interpolates between the *in* and *out* vacuum states specified respectively by the two wave functions in (4.6), so that $\beta_{\mathbf{k}}(t)$ starts at zero as $t \to -\infty$ and approaches $\bar{\beta}$ as $t \to +\infty$. The wave functions depend on k and t only through the variable u defined in (4.2), and the potential $u^2 + \lambda$ is even in u. Hence we should expect each k mode to go through its creation event at a different time t according to $k + eEt \approx 0$, *i.e.* for the particles to be created with kinetic momenta near zero. We shall see that this is indeed the case and that therefore the range of k which have gone through the creation process at time t depends linearly on t, which justifies the passage from (4.15) to (4.16).

Omitting the integration over $\mathbf{k}_\perp$ and the phase space factor $1/(2\pi)^3$ in (4.16), one obtains the probability per unit time per unit volume to produce pairs with transverse momentum $\mathbf{k}_\perp$ [6,11]. This result has been interpreted as the rate at which pairs are created [12] and used as a source term in Vlasov equation, which involves particle production [13]. However, a necessary condition for this interpretation to be correct is that the time integration over the rate of particle production

$$eE\int dt \ln(1+e^{-\pi\lambda}) \tag{4.17}$$

be identical to the total number of particles produced per unit volume with transverse momentum $\mathbf{k}_\perp$, which is given by integration over k of Eq. (4.11)

$$\int dk\ e^{-\pi\lambda} = eE \int\ dt\ e^{-\pi\lambda}\,. \tag{4.18}$$

Expressions (4.17) and (4.18) are *not* equivalent, because the probability rate of particle production differs from the production rate of the mean value of particles. They become equal only in the limit of large λ when both the probability and mean number of produced particles become very small. It is clear that the source term for the mean rate of particle production in the Vlasov equation should involve the latter quantity (4.18) in principle, without the appearance of any logarithm in the final answer.

V. UNIFORM ASYMPTOTIC EXPANSION OF THE SOURCE TERM

Eqn. (4.9) is the source term due to particle creation, with a specific choice of initial conditions and phase correlations in the initial state (namely none). Since all quantities in (4.9) are local functions of time, specified in terms of the mode functions (4.6) there is no need to resort to the nonlocal integral equation (3.17), and the time evolving phase correlation $\mathcal{C}_{\mathbf{k}}$ need not be considered explicitly. Because of Eqns. (4.10) and (4.11) the Schwinger pair creation amplitude is certainly contained in (4.9). Since our objective is the derivation of an effective Markovian source term for the Boltzmann-Vlasov equation fields which are slowly varying in time we now introduce the second important ingredient in our approach, *i.e.* the uniform asymptotic expansion of (4.9) for weak and slowly varying electric fields.

In order to motivate the introduction of this asymptotic expansion observe that for a constant electric field each time derivative of the mode function $f_{\mathbf{k}}$ brings with it a factor of $1/\lambda$. This can be made explicit by introducing a rescaled variable v which is independent of the strength of the electric field, $u \equiv v\sqrt{\lambda}$ and rewriting the wave equation (4.3) in the form,

$$\left(\frac{1}{\lambda^2}\frac{d^2}{dv^2} + v^2 + 1\right) f = 0\,. \tag{5.1}$$

Next, when we allow the electric field to vary in time we can consider the standard adiabatic expansion for the mode function in the time-varying field [2,3],

$$\begin{aligned} f_{\mathbf{k}} &\equiv \sqrt{\frac{\hbar}{2\Omega_{\mathbf{k}}}}\exp\left(-i\int^t dt'\,\Omega_{\mathbf{k}}(t')\right)\,, \\ \Omega^2 &= \omega^2 - \frac{\ddot{\Omega}}{2\Omega} + \frac{3}{4}\left(\frac{\dot{\Omega}}{\Omega}\right)^2 \\ &= \omega^2 - \frac{\ddot{\omega}}{2\omega} + \frac{3}{4}\left(\frac{\dot{\omega}}{\omega}\right)^2 + \dots \end{aligned} \tag{5.2}$$

for which (3.1) is the lowest order term, corresponding to no derivatives in (5.2) and order λ^0 in the constant field case. The quasistationary or adiabatic approximation to the mode equation is obtained by treating the derivative terms as small compared to the leading order term, *i.e.*

$$\frac{\ddot{\omega}}{\omega^3} \ll 1 \qquad \text{and} \qquad \frac{\dot{\omega}}{\omega^2} \ll 1\,. \tag{5.3}$$

In the case of a general electric field this implies that the field is both slowly varying and weak. For a constant electric field the adiabatic condition reduces to $\lambda \gg 1$.

Iterating the expansion to adiabatic order q, terms with q time derivatives of the adiabatic frequency $\omega_{\mathbf{k}}$ in the general time varying field will appear together with terms with q powers of $1/\lambda$ in the constant field case. Thus there is a precise correspondence between the terms appearing in the asymptotic expansion of $|\beta_{\mathbf{k}}|^2$ and $\frac{d}{dt}|\beta_{\mathbf{k}}|^2$ to a given power in $1/\lambda$ in a constant electric field background to the terms appearing in the local adiabatic expansion of the current (3.25) in higher time derivatives of the electric field in a general time varying electric field. The transport approximation amounts to a truncation of this expansion at the lowest order required for a consistent backreaction dynamics. This is determined by the order of the backreaction equation, $j = -\dot{E} = \ddot{A}$, which is second order in time. Thus we should expand the particle number $|\beta|^2$ only to second order *i.e.* $1/\lambda^2$, in order to match the asymptotic expansion of the current to the order of the backreaction equation for a weak, slowly varying electric field, self-consistently determined by solving the Maxwell-Vlasov system. To retain higher orders than this in the current would also involve higher derivatives of E in the general time varying electric field, and such terms can never be calculated correctly by the constant E approximation of (4.9). At adiabatic order 2 the only effect of approximating the source term for a slowly varying electric field by (4.9), evaluated in a constant field is the absence of the $\dot{E}$ term in the current (2.12) generated by the adiabatic expansion (5.2). This term is responsible for charge renormalization in the mean field theory [2,3]. Hence for comparison between the mean field evolution and that of the Vlasov-Maxwell system one must specify the scale of the renormalized charge of the mean field theory by some other criterion, or it will differ in general from the classical charge appearing in the Vlasov equation by a finite renormalization. This precise correspondence we fix by a linear response analysis in Section 6.

Even if we could calculate higher order terms (by calculating the source term in some other time varying background, for example), to include them would change the order of the Maxwell equation $\dot{E} = -j$ by making j a function of higher derivatives of E. This would introduce unphysical high frequency runaway solutions, not present in the underlying microscopic quantum field theory, in a manner similar to the higher derivative Lorentz radiation reaction force. Thus the order of the backreaction equation for time-varying electric fields determines the order of the asymptotic expansion we should use for the current, in the limit of weak, very slowly varying electric fields, which is the only limit in which such a replacement in the current is justified. The fact that the leading order asymptotic expansion of the constant field adiabatic particle number is already $1/\lambda^2$ (as we shall see shortly) which is the highest order we need to go in the expansion, justifies the use of the constant field expression (4.9), evaluated to this asymptotic order, for the local source term in the Markov limit of the quantum Vlasov equation. If the higher order terms in the expansion are numerically significant, then that is the signal that we must abandon the Boltzmann-Vlasov description entirely and return to the underlying field theory without the possibility of making any simple transport approximation to the self-consistent backreaction problem.

The key point is that we require an asymptotic expansion of the mode functions and adiabatic particle number source term in (4.9) in powers of $1/\lambda$ that is *uniformly* valid in time (and longitudinal momentum k), in order that the exponentially small Schwinger amplitude $\bar{\beta}$ which is the only secular effect of particle creation which survives as $t \to \infty$ will not be lost in the expansion. This condition is *not* satisfied by the naive asymptotic expansion of f in simple exponential functions such as (5.2). This failure of the usual adiabatic expansion to capture exponentially small (but secular) particle creation effects is due to the nonuniformity of the naive asymptotic expansion with respect to the limits $t \to \pm\infty$. This limitation can be removed by an asymptotic approximation uniformly valid everywhere on the real time axis, in a manner analogous to the uniform asymptotic approximation of the WKB turning point formulae [28].

In the case at hand, the asymptotic expansion of the solutions of Eqn. (4.3) uniformly valid everywhere on the real time axis have been given by Olver [29]. Converting to the notations of the present paper, Olver's result may be written in the form,

$$f_{(+)\mathbf{k}}(t) \simeq e^{-\frac{\pi\lambda}{4}} \sqrt{\frac{2\pi}{\omega_{\mathbf{k}}}} \left[1 + \sum_{s=1} \gamma_s \frac{2^{s-1}}{(i\lambda)^s}\right] \left\{ z^{\frac{1}{4}} \mathrm{Ai}(z) \sum_{s=0} \frac{\mathcal{P}_{2s}}{(i\lambda)^{2s}} + z^{-\frac{1}{4}} \mathrm{Ai}'(z) \sum_{s=0} \frac{\mathcal{Q}_{2s+1}}{(i\lambda)^{2s+1}} \right\}, \qquad (5.4)$$

where the coefficients $\mathcal{P}_{2s}$ and $\mathcal{Q}_{2s+1}$ are certain functions of $v = \frac{u}{\sqrt{\lambda}}$ given by

$$
\begin{aligned}
\mathcal{P}_0(v) &= 1\,,\\
\mathcal{P}_2(v) &= -\frac{(9v^4+249v^2-145)}{1152(v^2+1)^3} - \frac{7\;v\,(v^2+6)}{1728\xi(v^2+1)^{\frac{3}{2}}} + \frac{455}{10368\,\xi^2}\,,\\
\mathcal{Q}_1(v) &= -\frac{i}{24}\left[\frac{v(v^2+6)}{(v^2+1)^{\frac{3}{2}}} - \frac{5}{3\,\xi}\right]\,, \qquad etc.
\end{aligned}
\tag{5.5}
$$

the complex variables $\xi(v)$, w and z are defined by

$$
\begin{aligned}
\xi(v) &\equiv \frac{\Theta_{\mathbf{k}}}{\lambda} + \frac{i\pi}{4}\\
&= \frac{1}{2}v\sqrt{v^2+1} + \frac{1}{2}\ln\left(v+\sqrt{v^2+1}\right) + \frac{i\pi}{4} \qquad \text{and}\\
w &\equiv -\lambda\xi \equiv \frac{2i}{3}z^{\frac{3}{2}}\,,
\end{aligned}
\tag{5.6}
$$

where z is defined in the plane cut along the positive imaginary axis by

$$
z = e^{-i\pi}\left[\frac{3}{2}\left(\Theta_{\mathbf{k}} + \frac{i\pi\lambda}{4}\right)\right]^{\frac{2}{3}}
\tag{5.7}
$$

and the γ_s are the numerical constants,

$$
\gamma_1 = -\frac{1}{24}\,, \qquad \gamma_2 = \frac{1}{1152} \qquad \text{etc.}
\tag{5.8}
$$

With the definition of the phase of z according to (5.7) the complex argument of the Airy functions in (5.4) varies along the contour depicted in Fig. 1, as the real time t or u ranges from $-\infty$ to $+\infty$.

The terms we have written here explicitly determine the uniform asymptotic expansion of $f_{(+)\mathbf{k}}$ up to order $1/\lambda^2$. Since we are interested only in the lowest nonvanishing order in the expansion we could retain only the lowest order term in (5.4), substitute it into (4.9) to obtain the lowest order source term in the Vlasov equation directly. Some care is required in this procedure since the argument of the Airy functions depends on λ through (5.7) and the equations of motion (3.13) will not be satisfied unless both sides of the equation are expanded consistently to the same order in $1/\lambda$. For this reason it is useful to retain one higher order in the asymptotic expansion than would seem necessary at first sight, in order to have a nontrivial check on the algebra via the equations of motion.

The corresponding asymptotic expansion for the time derivative of the mode functions uniformly valid on the real axis is:

$$
\dot{f}_{(+)\mathbf{k}}(t) \simeq i\,e^{-\frac{\pi\lambda}{4}}\sqrt{2\pi\omega_{\mathbf{k}}}\left[1+\frac{1}{2}\sum_{s=1}\gamma_s\frac{2^s}{(i\lambda)^s}\right]\left\{z^{\frac{1}{4}}\mathrm{Ai}(z)\sum_{s=0}\frac{\mathcal{P}_{2s+1}}{(i\lambda)^{2s+1}} + z^{-\frac{1}{4}}\mathrm{Ai}'(z)\sum_{s=0}\frac{\mathcal{Q}_{2s}}{(i\lambda)^{2s}}\right\}\,,
\tag{5.9}
$$

where the coefficient functions are

$$
\begin{aligned}
\mathcal{Q}_0(v) &= 1\,,\\
\mathcal{Q}_2(v) &= \frac{(15v^4+327v^2-143)}{1152\;(v^2+1)^3} + \frac{5\;v\,(v^2-6)}{1728\xi(v^2+1)^{\frac{3}{2}}} - \frac{385}{10368\,\xi^2}\,,\\
\mathcal{P}_1(v) &= -\frac{i}{24}\left[\frac{v\,(v^2-6)}{(v^2+1)^{\frac{3}{2}}} + \frac{7}{3\,\xi}\right] \qquad etc.
\end{aligned}
\tag{5.10}
$$

From these expressions the uniform asymptotic expansion for the time dependent Bogoliubov coefficient $\beta_{\mathbf{k}}(t)$ is easily computed from its definition in (3.4), namely,

$$\beta_{\mathbf{k}} \simeq \sqrt{\pi} e^{-\frac{\pi\lambda}{4}} e^{-i\Theta_{\mathbf{k}}} \left[1 + \frac{1}{2}\sum_{s=1} \gamma_s \frac{2^s}{(i\lambda)^s}\right] \left\{ z^{\frac{1}{4}} \mathrm{Ai}(z) \sum_{s=0} \frac{\mathcal{P}_s}{(i\lambda)^s} + z^{-\frac{1}{4}} \mathrm{Ai}'(z) \sum_{s=0} \frac{\mathcal{Q}_s}{(i\lambda)^s} \right\} . \tag{5.11}$$

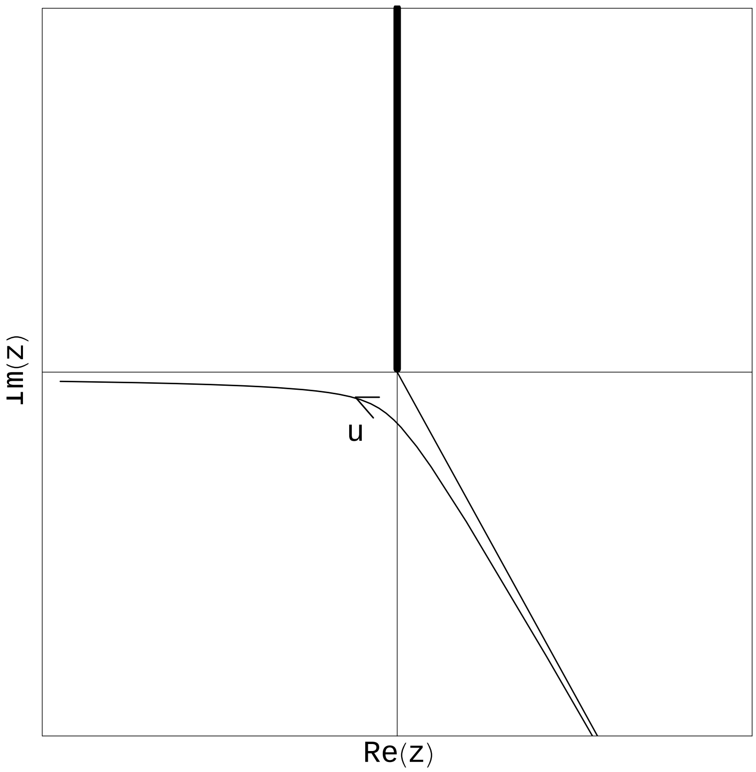

FIG. 1. The contour in the complex z plane along which the argument of the Airy functions in (5.4), (5.9), (5.11), and (5.12) are evaluated. The cut of the $\frac{2}{3}$ root appearing in (5.7) is taken along the positive imaginary z axis from 0 to $i\infty$. Following (5.6) the corresponding contour in the complex w plane is a straight horizontal line displaced from the real w axis into the lower half w plane by $\frac{\pi\lambda}{4}$, with Re w decreasing as t increases.

Since we have shown by Eqn. (3.12) that the particle number $|\beta_{\mathbf{k}}|^2$ is an adiabatic invariant to leading order in the time derivatives of the background, the lowest order λ^0 term in the asymptotic expansion must be absent from the particular linear combination in (5.11). Indeed with $s = 0$, $\mathcal{P}_0 = \mathcal{Q}_0 = 1$ and the symmetric linear combination of Airy functions, $z^{\frac{1}{4}}\mathrm{Ai}(z) + z^{-\frac{1}{4}}\mathrm{Ai}'(z)$, is of order λ^{-1}, as is verified explicitly in relations (5.20) and (5.21) below, by using (5.6) and the further asymptotic expansion of these functions for $|z| \sim \lambda^{\frac{2}{3}} \to \infty$. Any other linear combination of the same functions, and in particular the antisymmetric combination, $z^{\frac{1}{4}}\mathrm{Ai}(z) - z^{-\frac{1}{4}}\mathrm{Ai}'(z)$ is of order λ^0. Anticipating this result and substituting (5.5) and (5.10) into (5.11), we obtain simply,

$$\beta_{\mathbf{k}} \simeq \sqrt{\pi} e^{-\frac{\pi\lambda}{4}} e^{-i\Theta_{\mathbf{k}}} \left\{ \left[z^{\frac{1}{4}}\mathrm{Ai}(z) + z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] + \frac{i}{4}\left[z^{\frac{1}{4}}\mathrm{Ai}(z) - z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] \left[\frac{u}{(u^2+\lambda)^{\frac{3}{2}}} + \frac{1}{3w}\right] \right\}$$

$$+\mathcal{O}(\lambda^{-2})\,, \tag{5.12}$$

correct to the leading nonvanishing order, λ^{-1}. Squaring (5.12) and taking its time derivative gives the asymptotic approximation to the effective source term defined in (4.9). Since w and z are functions of u and λ (equivalently, v and λ) which depend only on the *kinetic* momenta $p(t)$ and $p_\perp$ through (4.2), the effective source term (for vacuum intitial conditions at $t = -\infty$) may be written in the form,

$$S_{vac}(p, p_\perp; E) = eE\frac{\partial}{\partial p}|\beta(p, p_\perp)|^2\,, \tag{5.13}$$

where

$$|\beta(p, p_\perp)|^2 = \pi e^{-\frac{\pi\lambda}{2}}\left|\left[z^{\frac{1}{4}}\mathrm{Ai}(z) + z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] + \frac{i}{4}\left[z^{\frac{1}{4}}\mathrm{Ai}(z) - z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right]\left[\frac{u}{(u^2+\lambda)^{\frac{3}{2}}} + \frac{1}{3w}\right]\right|^2 \tag{5.14}$$

is that function of p and $p_\perp$ determined by the subsitutions (4.2) together with the definitions (5.6) and (5.7). Using the relations,

$$\begin{aligned} eE\frac{\partial}{\partial p}\left[z^{\frac{1}{4}}\mathrm{Ai}(z) \pm z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] &= \pm i\omega\left[z^{\frac{1}{4}}\mathrm{Ai}(z) \pm z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] \\ &\quad + \frac{i\omega}{4z^{\frac{3}{2}}}\left[z^{\frac{1}{4}}\mathrm{Ai}(z) \mp z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right] \end{aligned} \tag{5.15}$$

which follow from the definition (5.7) and $\mathrm{Ai}''(z) = z\mathrm{Ai}(z)$, the differentiation in (5.13) can be carried out explicitly with the result,

$$\begin{aligned} S_{vac}(p, p_\perp; E) &= \pi|eE|e^{-\frac{\pi\lambda}{2}}\frac{u}{u^2+\lambda}\Bigg\{|z^{\frac{1}{4}}\mathrm{Ai}(z)|^2 - |z^{-\frac{1}{4}}\mathrm{Ai}'(z)|^2 \\ &+ \frac{1}{6}\mathrm{Im}\left[\left(z^{\frac{1}{4}}\mathrm{Ai}(z)\right)^* z^{-\frac{1}{4}}\mathrm{Ai}'(z)\right]\left(\frac{\Theta_{\mathbf{k}}}{3|w|^2} + \frac{u^3}{\lambda(u^2+\lambda)^{\frac{3}{2}}}\right) - \frac{1}{48}|z^{\frac{1}{4}}\mathrm{Ai}(z) - z^{-\frac{1}{4}}\mathrm{Ai}'(z)|^2 \\ &\times\left[\frac{\pi\lambda}{|w|^2} + \frac{6(2u^2-\lambda)}{(u^2+\lambda)^3} + \frac{u^3\Theta_{\mathbf{k}}}{3\lambda|w|^2(u^2+\lambda)^{\frac{3}{2}}} + \frac{35}{18|w|^4}\left(\frac{\pi^2\lambda^2}{16} - \Theta_{\mathbf{k}}^2\right) + \frac{1}{18|w|^2}\right]\Bigg\}, \end{aligned} \tag{5.16}$$

where we have neglected all terms of order λ^{-3} and higher within the curly brackets.

The expressions (5.13), (5.14) and (5.16) are the main results of this paper. In order to understand the physics of particle creation that is captured in these expressions we compare the lowest order asymptotic expression for the adiabatic particle number (5.14) with the analogous exact expression in terms of parabolic cylinder functions for constant external electric field. The results are plotted in Figs. 2 through 4 for $\lambda = 1, 2$ and 10 respectively. We see that the asymptotic expansion in terms of Airy functions reproduces the behavior of the adiabatic particle number quite accurately, even for moderately small λ of order one. The other important feature to notice about these figures is the relatively sharp increase in particle number right around $u = p = 0$. The transients after this particle creation event then settle down to the value $|\bar{\beta}|^2 = \exp(-\pi\lambda)$ which is independent of the initial longitudinal momentum.

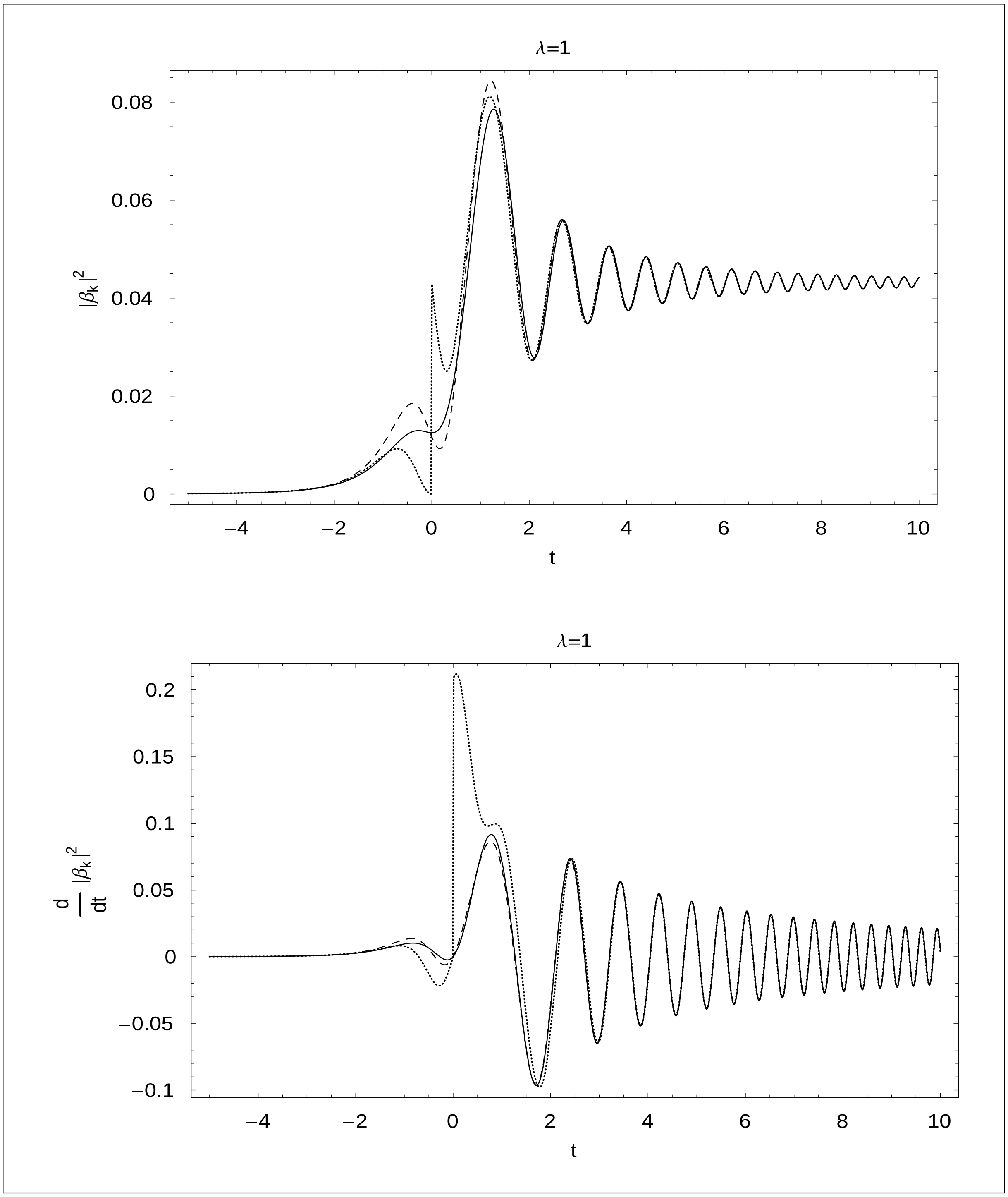

FIG. 2. The exact (solid curve), uniform (dashed curve) and adiabatic step function (dotted curve) asymptotic expansions of the adiabatic particle number and its time derivative for a constant electric field with $\lambda = 1$ and $k = 0$. The particle numbers approach the same value $e^{-\pi} = 0.0432$ as $t \to \infty$, although each $\mathcal{N}_k$ experiences a sharp rise at a different time, *viz.* near zero kinetic momentum, $p = k + eEt \simeq 0$. The delta function at $t = 0$ in the dotted curve of the second figure obtained from differentiating (5.27) is not shown.

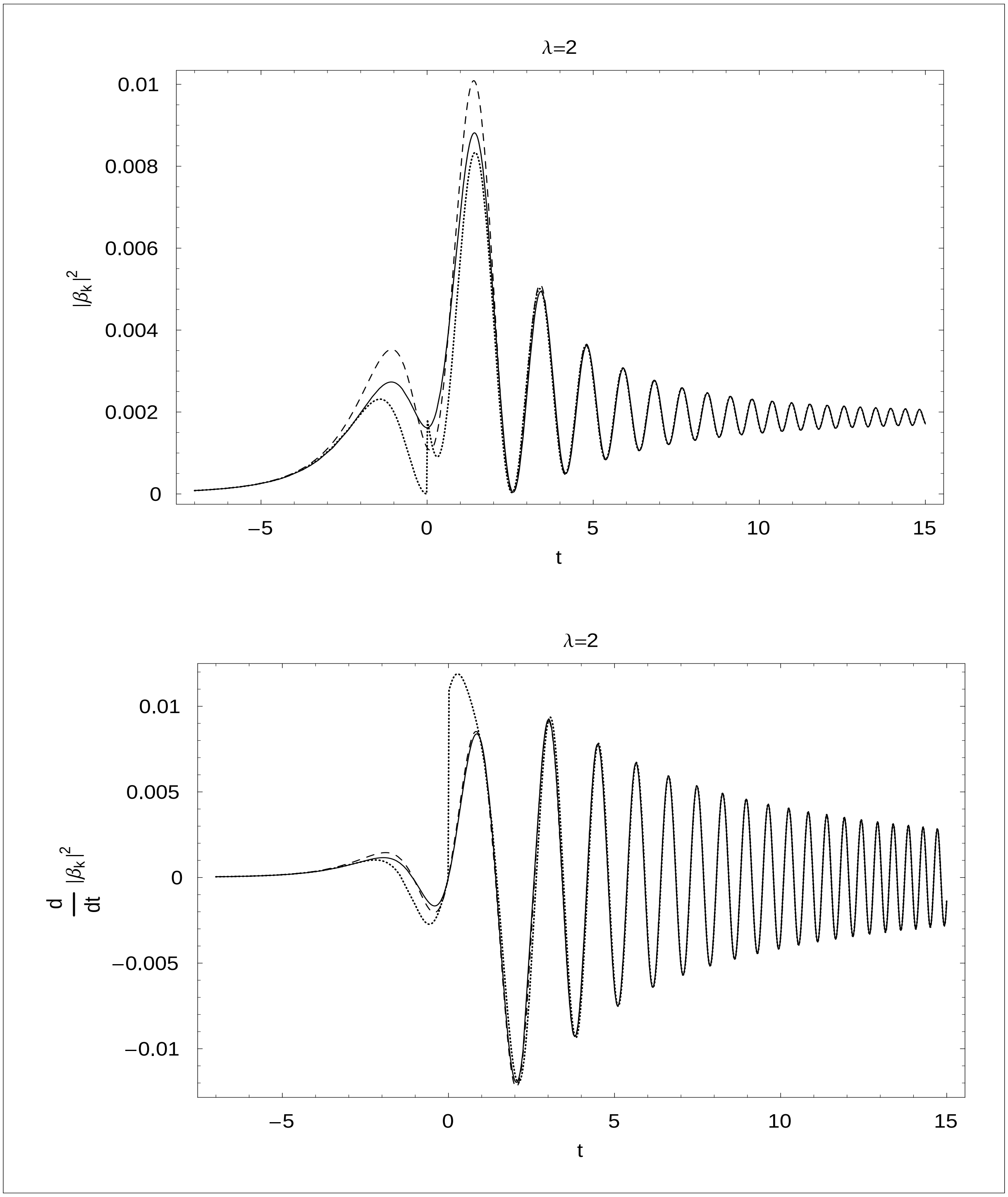

FIG. 3. Same as Fig. 2 but for $\lambda = 2$. The particle numbers approach the same value $e^{-2\pi} = 0.00187$ as $t \to \infty$, although each $\mathcal{N}_k$ experiences a sharp rise at a different time, *viz.* near zero kinetic momentum, $p = k + eEt \simeq 0$.

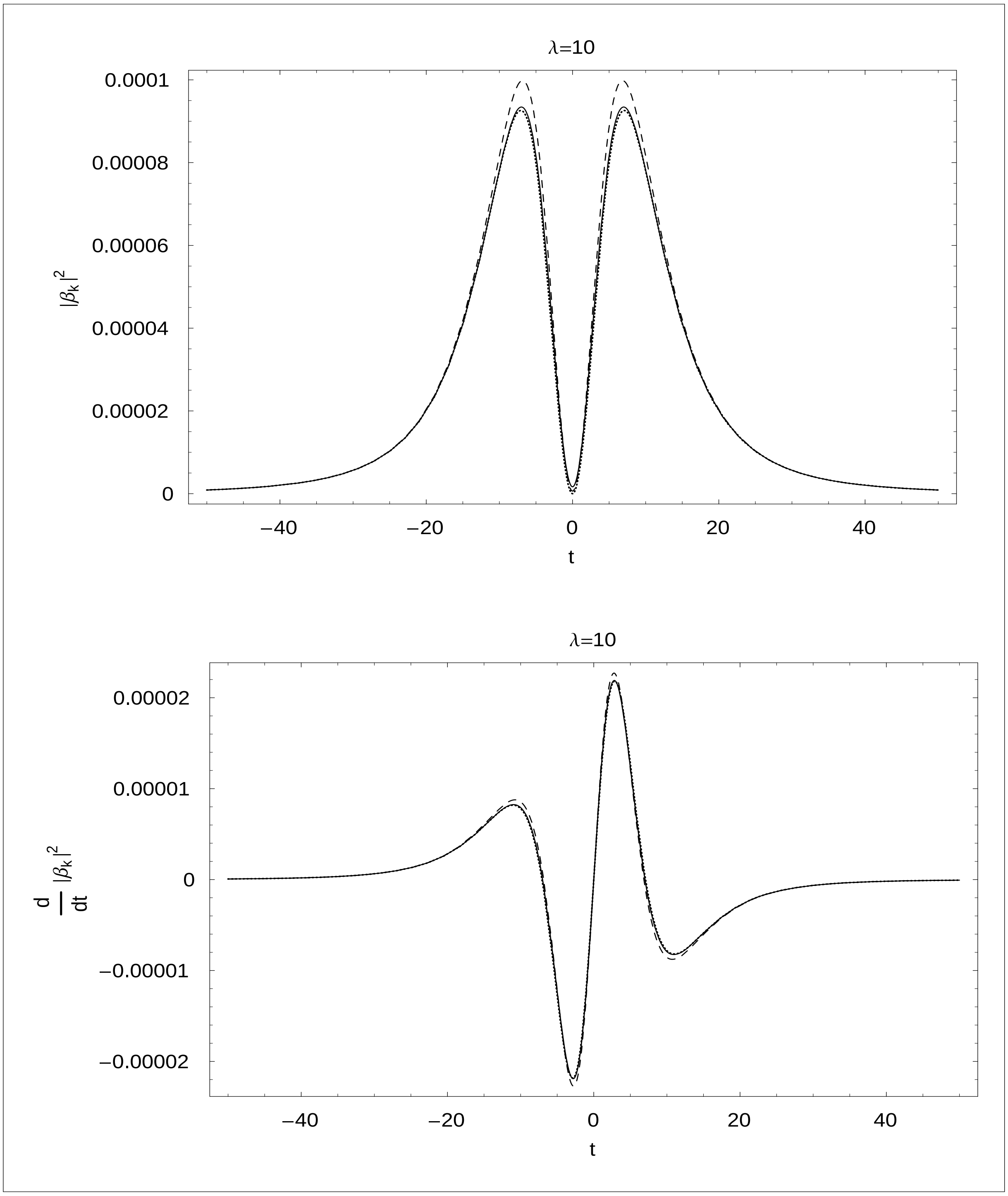

FIG. 4. Same as Figs. 2 and 3 but for $\lambda = 10$. In this case the magnitude of the step at late times, $e^{-10\pi} = 2.27 \times 10^{-14}$ is much smaller than the transient effects visible in the plot, and all three curves are very nearly (anti)symmetric around $t = 0$, showing that a nearly equal number of particles is created and destroyed. As in the previous figures the delta function at $t = 0$ in the dotted curve of the second figure is not shown.

Thus, the exponentially small Schwinger particle creation effect is captured very well by the *leading* order term in the uniform asymptotic expansion of $|\beta|^2$. Notice that the uniform asymptotic expansion for the source term works very well even at the expected limit of its validity at $\lambda = 1$. As a mathematical aside we remark that the exponentially small contribution to an adiabatic invariant quantity such as the particle number $\mathcal{N}_{\mathbf{k}}$ has been studied by various authors and bounds obtained in the general case [30]. However, for this particular case of constant electric field and mode equation (4.3) leading to Weber parabolic cylinder functions, it has apparently not been noticed that the asymptotic expansion of the solutions of this equation, uniformly valid on the real time axis, allows one to calculate the exponentially small secular change in the adiabatic invariant $\mathcal{N}_{\mathbf{k}}$ analytically. The same observation could clearly be generalized to other differential equations for which uniform asymptotic expansions are known.

The sharpness of the creation event at $u = 0$ is clearly determined by the wave equation (4.3) to be $\Delta u \sim \lambda^{\frac{1}{2}}$ or

$$\Delta t \sim \frac{\sqrt{p_\perp^2 + m^2c^2}}{eE} \equiv \tau_{cl} \tag{5.17}$$

which is the time scale for the growth of a sizable fraction of the final antiparticle amplitude in the quantum wave function. This time scale (which is also the time scale for the classical acceleration by the electric field to bring a charged particle to relativistic velocities) must be long compared to the quantum phase coherence time τ_{qu}, in order for the creation process to be described by a local approximation to the nonlocal Vlasov equation (3.17), *i.e.*

$$\tau_{cl} \sim \lambda \tau_{qu} \gg \tau_{qu}\,. \tag{5.18}$$

Hence the Markov limit of the Vlasov equation requires weak electric fields $\lambda \gg 1$ which is what we have assumed in the uniform adiabatic expansion of the source term. Conversely, if we consider the opposite limit where the electric field is strong, so many particles are created so rapidly in time that the individual particle creation events cannot be distinguished one from another during the quantum coherence time τ_{qu}. It is clear that in this case significant wave amplitude coherence during the creation process can be expected and we cannot hope to approximate the effects of such copious and coherent particle creation by a Boltzmann-Vlasov source term local in time, which takes no account of the prior time history. Indeed in this strong field limit these "particles" are not particles at all in the usual sense but are more accurately to be thought of as coherent wave amplitudes which lie outside of any classical or semiclassical kinetic particle description.

Restricting ourselves then to weak fields these coherence effects do not need to be considered explicitly and are built into the initial conditions of the vacuum at $t = -\infty$ once and for all. However in analyzing the particle creation process in a constant field and deriving (5.16) we have also assumed that the electric field does not vary over the typical time of the variation of $\mathcal{N}_{\mathbf{k}}$. Thus in order to use (5.16) in situations involving a time evolving electric field we also require that its time scale of variation τ_{pl} be much larger than the time scale of the creation event, *i.e.*,

$$\tau_{pl} \gg \tau_{cl}\,. \tag{5.19}$$

If this second inequality holds then it should be possible to coarsen our time resolution still further by not attempting to resolve the time scale τ_{cl}. On these still longer time scales it becomes reasonable to approximate the sharp growth of the antiparticle amplitude near $u = p_z = 0$ as a step function, provided only that we account for the integrated value of the step from $-\infty$ to $+\infty$. This is what we wish to explain next.

Let us first reiterate that the uniform asymptotic expansion in terms of Airy functions is indeed essential to capturing the step explicitly in Figs. 2-4 and that the Schwinger effect is lost completely if a naive WKB expansion in powers of $1/\lambda$ is used instead. This may be seen explicitly by taking the large λ asymptotics of the Airy functions in (5.12). To this end we note the Airy functions may be represented in terms of Hankel functions of the first kind,

$$
\begin{aligned}
z^{\frac{1}{4}}\mathrm{Ai}(z) &= \frac{1}{2\sqrt{2}}e^{\frac{5\pi i}{12}}\, w^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(w)\\
z^{-\frac{1}{4}}\mathrm{Ai}'(z) &= \frac{1}{2\sqrt{2}}e^{-\frac{5\pi i}{12}}\, w^{\frac{1}{2}}H^{(1)}_{\frac{2}{3}}(w)
\end{aligned}
\tag{5.20}
$$

with the branch cut of $w^{\frac{1}{2}}$ along the negative w axis, and w ranging from $+\infty - \frac{i\pi\lambda}{4}$ to $-\infty - \frac{i\pi\lambda}{4}$ along the horizontal contour displaced by $-\frac{i\pi\lambda}{4}$ from the real axis, as u ranges from $-\infty$ to $+\infty$, according to (5.6). Taking the large λ limit is equivalent to taking the large $|w|$ limit of the Hankel functions, which depends critically on the phase of w. This phase depends in turn on the sign of u from (5.6). When $u < 0$, then $|\arg w| < \pi$ and we can use the standard asymptotic expansion of the Hankel functions,

$$
w^{\frac{1}{2}}H^{(1)}_{\nu}(w) \simeq \frac{2}{\sqrt{\pi}}\exp\left(iw - i\frac{\pi}{2}\nu - i\frac{\pi}{4}\right)\left\{1 - \frac{1}{2iw}\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu-\frac{1}{2})} + \dots\right\}\,, \qquad -\pi < \arg w < \pi\,,
\tag{5.21}
$$

for large $|w|$ to find

$$
\beta_{\mathbf{k}} \sim \frac{e^{iw}}{w} \to 0 \qquad \text{as} \qquad t \to -\infty\,.
\tag{5.22}
$$

Since $w = -\lambda\xi$ is linear in λ this shows that the linear combination of Airy or Hankel functions in $\beta_{\mathbf{k}}$ is of order λ^{-1} and contains no λ^0 term, as stated above.

Thus if $t \to -\infty$ with eE fixed or if $\lambda \to \infty$ with $k + eEt < 0$ fixed, the adiabatic particle number vanishes. On the other hand if $u \to +\infty$ we cannot use (5.21) directly because arg $w \to -\pi$ in this limit, and the condition on the phase is not satisfied. Instead we must first use the connection formula,

$$
H^{(1)}_{\nu}(w) = \frac{\sin 2\pi\nu}{\sin \pi\nu}H^{(1)}_{\nu}(e^{i\pi}w) + e^{-i\pi\nu}H^{(2)}_{\nu}(e^{i\pi}w)
\tag{5.23}
$$

to bring the phase of $w' = e^{i\pi}w$ into the proper range in order to apply (5.21). Then we find that the e^{iw} terms from $H^{(1)}_{\nu}$ again vanish like $e^{iw}|w|^{-1}$ for large $|w|$, but that now there remains in addition the opposite frequency e^{-iw} term which gives

$$
\beta_{\mathbf{k}}(t) \to -ie^{-\frac{\pi\lambda}{4}}e^{-iw}e^{-i\Theta_{\mathbf{k}}} = -ie^{-\frac{\pi\lambda}{2}} = \bar{\beta}
\tag{5.24}
$$

which is finite as $t \to +\infty$ with eE fixed. As $\lambda \to \infty$ with $k + eEt > 0$ fixed this term is exponentially small compared to the ordinary λ^{-1} contribution.

In this way the uniform asymptotic expansion in terms of Airy or Hankel functions which contains the exponentially small Schwinger particle creation becomes non-uniform in time, depending on the sign of $k + eEt$, if the further asymptotic expansion of these functions in terms of exponentials $\exp(\pm iw)$ is taken. Only the uniform expansion in (5.4) and (5.9) can capture the particle creation event, and Figs. 2-4 show that this it does quite accurately even at the lowest nonvanishing order of the expansion. This exercise in asymptotic expansions as well as the explicit behavior in time of the adiabatic particle number in Figs. 2-4 does shows that we might try the simple adiabatic expansion of $\beta_{\mathbf{k}}$ according to (5.2), but that we must then add back *by hand* the exponentially small step $\bar{\beta}$ in the vicinity of the creation event at $k + eEt \simeq 0$, *i.e.*

$$\beta_{\mathbf{k}} \approx \beta_{\mathbf{k}}^{adb} + \theta(u)\bar{\beta}\,, \tag{5.25}$$

where

$$\beta_{\mathbf{k}}^{adb} \simeq \frac{ie^{-2i\Theta_{\mathbf{k}}}u}{4(u^2+\lambda)^{\frac{3}{2}}} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \tag{5.26}$$

is the result obtained by substituting the lowest order of the standard adiabatic approximation for the mode functions (5.2), rather than Olver's uniform expansion in terms of Airy functions. The Heaviside step function could be replaced by any smooth function with the correct limits at $t \to \pm\infty$. The point is that if the second inequality (5.19) holds, then in multiparticle collective quantities such as the mean current, integrations over large ranges of kinetic momenta are involved, and it makes little difference whether the continuous rise in each individual mode's particle number on the momentum scale τ_{cl}/eE is taken into account, provided only that the integral over all momenta accurately describes which modes have gone through the creation process. It is only this fact and the second inequality involving the collective time scale of the plasma that can justify replacing the continuous rise of $\mathcal{N}_{\mathbf{k}}$ by a step function.

In this admittedly rather crude approximation the function $\beta(p, p_\perp)$ of (5.14) in terms of Airy or Hankel functions is replaced by

$$|\beta(p,p_\perp)|^2 \approx \frac{(eE)^2p^2}{16\omega^6} - \frac{eEp}{2\omega^3}\exp\left(-\frac{\pi\lambda}{2}\right)\theta(u)\cos(2\Theta) + \exp\left(-\pi\lambda\right)\theta(u) \tag{5.27}$$

in terms of elementary functions. This approximation to (5.14) is compared to the uniform asymptotic expansion in the dotted curves of Figs. 2-4, where it is observed that it works better than might have been expected, except for the region near the creation event $u \approx 0$ where it is clearly inaccurate. The delta function obtained by differentiating the last two terms of (5.27) is not shown in the second halves of Figs. 2-4. Notice that the oscillations in these figures are well represented by the $\cos(2\Theta)$ term in (5.27), which may be interpreted as the interference between the usual adiabatic phase oscillations and the relatively sudden particle creation event. Thus we see that for numerical purposes it is probably sufficient to use the approximate form of $|\beta(p,p_\perp)|^2$ in (5.27) for all p, except those in a band of size several units of $\sqrt{p_\perp^2 + m^2}$ centered at the origin where the sharp (but continuous) rise of particle number takes place. When one is integrating over a region of p or t that is large compared to the time scale τ_{cl} over which the rise in particle number takes place, the crude approximation of this rise by a step function and its derivative by a delta function may be sufficient, provided only that their coefficient is fixed by the Schwinger formula, as in (5.27). On the other hand, in the region of $p \simeq 0$ the true behavior is certainly not discontinuous on the scale τ_{cl} and the more accurate form (5.14) in terms of Airy or Hankel functions should be used for moderately strong electric fields.

We have now succeeded in our main purpose, namely to analyze the time structure of the quantum particle creation process in the adiabatic number basis, and to capture that particle creation event by means of a uniform asymptotic expansion of the exact wave functions of the constant electric field background, without any need to analytically continue or approximate the nonlocal integral in (3.17). Because of the reasoning earlier in this section we can proceed to identify the time rate of change of the adiabatic particle number in the lowest order of this uniform asymptotic expansion given by (5.16) or the time derivative of (5.27) as the effective source term in the Vlasov equation which describes quantum particle creation in slowly varying electric fields, starting from vacuum initial conditions at $t = -\infty$.

One point that still requires some discussion is the effect of changing the initial conditions from vacuum at $t = -\infty$ to those at some finite time t_0. Indeed, the comparison of the effective source term in the Vlasov description with the mean field evolution in the next section requires that the initial conditions be specified at a finite initial time t_0, not at $-\infty$. This means that the effective source term which is given by Eqns. (5.16) will differ from the actual source term in its dependence on the initial data and the correlations (or lack of them) in the initial state. To the extent that a Markovian approximation to the source term is justified and dephasing is efficient we expect that the memory effects of the initial conditions will be washed out on the time scale of significant particle creation, and therefore that the initial conditions will affect only the transient behavior of the evolution for times close to t_0. This can be checked in more detail.

To examine the transient effects of the initial conditions let us consider arbitrary initial data on the mode functions $f_{\mathbf{k}}(t_0)$ and $\dot{f}_{\mathbf{k}}(t_0)$, subject only to the Wronskian condition (2.8) and finite initial energy density. The general solution of $f_{\mathbf{k}}(t)$ in a constant electric field is a linear combination of $f_{(\pm)\mathbf{k}}$,

$$f_{\mathbf{k}}(t) = A_{\mathbf{k}}(t_0) f_{(+)\mathbf{k}}(t) + B_{\mathbf{k}}(t_0) f^*_{(+)\mathbf{k}}(t)\,. \tag{5.28}$$

By using the Wronskian condition on the mode functions we can solve for the coefficients, $A_{\mathbf{k}}(t_0)$ and $B_{\mathbf{k}}(t_0)$ in terms of the initial conditions on the mode functions in the form,

$$\begin{aligned} A_{\mathbf{k}}(t_0) &= i(\dot{f}_{\mathbf{k}}(t_0) f^*_{(+)\mathbf{k}}(t_0) - f_{\mathbf{k}}(t_0)\dot{f}^*_{(+)\mathbf{k}}(t_0)) \\ B_{\mathbf{k}}(t_0) &= i(f_{\mathbf{k}}(t_0)\dot{f}_{(+)\mathbf{k}}(t_0) - \dot{f}_{\mathbf{k}}(t_0) f_{(+)\mathbf{k}}(t_0))\,. \end{aligned} \tag{5.29}$$

A specific example of initial data with finite energy density is provided by the adiabatic vacuum initial conditions at $t = t_0$, *i.e.*,

$$\begin{aligned} f_{\mathbf{k}}(t_0) &= \tilde{f}_{\mathbf{k}}(t_0) = \sqrt{\frac{\hbar}{2\omega_{\mathbf{k}}(t_0)}} \\ \dot{f}_{\mathbf{k}}(t_0) &= \dot{\tilde{f}}_{\mathbf{k}}(t_0) = \left[-i\omega_{\mathbf{k}}(t_0) + \frac{\dot{\omega}_{\mathbf{k}}(t_0)}{2\omega_{\mathbf{k}}(t_0)}\right] f_{\mathbf{k}}(t_0)\,. \end{aligned} \tag{5.30}$$

The second term in the time derivative of the mode function is essential to insure finite initial energy density and is nonzero for finite electric field at initial time t_0. It means that a definite nonzero value of the pair correlation $\mathcal{C}_{\mathbf{k}}$ is being assumed in the initial adiabatic vacuum state.

Our previous choice of vacuum initial conditions is recovered if we let $t_0 \to -\infty$ so that $A_{\mathbf{k}} \to 1$, $B_{\mathbf{k}} \to 0$ and $f_{\mathbf{k}}(t) \to f_{(+)\mathbf{k}}(t)$. Retaining t_0 finite means that the general expression for $\beta_{\mathbf{k}}(t)$ in (3.4) with the mode functions given by (5.28) should be used so that

$$\beta_{\mathbf{k}}(t,t_0) = -iA_{\mathbf{k}}(t_0)\tilde{f}_{\mathbf{k}}(t)(\dot{f}_{(+)\mathbf{k}} + i\omega_{\mathbf{k}} f_{(+)\mathbf{k}}) - iB_{\mathbf{k}}(t_0)\tilde{f}_{\mathbf{k}}(t)(\dot{f}^*_{(+)\mathbf{k}} + i\omega_{\mathbf{k}} f^*_{(+)\mathbf{k}})\,. \tag{5.31}$$

Since $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are also given in terms of $f_{(+)\mathbf{k}}$ by Eqns. (5.29) one can develop the uniform asymptotic expansion for this $\beta_{\mathbf{k}}(t,t_0)$ using (5.4) and (5.9), repeating the steps leading to Eqn. (5.14) keeping t_0 finite.

The resulting rather complicated expression for the source term will depend on the electric field value at the initial time t_0. This expression would incorporate the initial data of the actual mean field evolution problem starting at t_0 more accurately than the simple choice of initial conditions, $A_{\mathbf{k}} = 1$ and $B_{\mathbf{k}} = 0$ which we have used in the source term (5.16). A good probe of the effect of these transient terms is the electric current which is plotted in Fig. 5 for $\lambda = 1$. The early oscillations observed in the exact current are the effect of the initial conditions (5.30). However the linear growth with t at late times can be understood from the simple approximation to the particle creation (5.27) by a step function.

For if we start at $t = t_0$ (rather than at $t = -\infty$) with no initial particles present, then the actual current integrated over all longitudinal momenta at time t is dominated by the conduction current j_{cond} in (3.26) and becomes

$$2e \exp(-\pi\lambda) \int \frac{dk}{2\pi}\frac{k+eEt}{\omega_k}\,\theta(k+eEt)\theta(-k-eEt_0) \to \frac{e^2E}{\pi}\exp(-\pi\lambda)\ (t-t_0) \tag{5.32}$$

in one spatial dimension at late times, which grows linearly with the elapsed time $T = t - t_0$ since the initial vacuum state was prepared. This is precisely the slope which is observed in all three curves in Fig. 5 at late times. The second step function involving t_0 is necessary because only modes with initially negative kinetic momentum can go through a creation event at $p(t) \simeq 0$ since $p(t) = k + eEt$ is a monotonically increasing function of t for constant positive E. It is in fact present in the $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ of (5.29) through $f_{\mathbf{k}}(t_0)$ and $\dot{f}_{\mathbf{k}}(t_0)$ which involve the same parabolic cylinder mode functions and the similar behavior near $k + eEt_0 \simeq 0$ as observed in Figs. 2-4. It is this dependence on the initial data which provides just the momentum window in (5.32) which we need to justify the replacement of the longitudinal momentum integration dk by the total elapsed time eET in (4.15), and which led to Schwinger's result for the decay rate. Such an understanding of the linear time divergence is possible only with a detailed description of the time structure of the particle creation process as given here.

The fact that the current in a constant electric field grows linearly with time is important for another reason. For it shows that backreaction must eventually be taken into account, and that simple perturbation theory must break down at late enough times for any nonzero eE, no matter how small. These backreaction effects can be taken into account only by a systematic resummation of perturbation theory, such as the large N expansion advocated in refs. [2]- [4], or by the solution of the Vlasov-Maxwell system of equations, valid when the inequalities of time scales $\tau_{qu} \ll \tau_{cl} \ll \tau_{pl}$ hold.

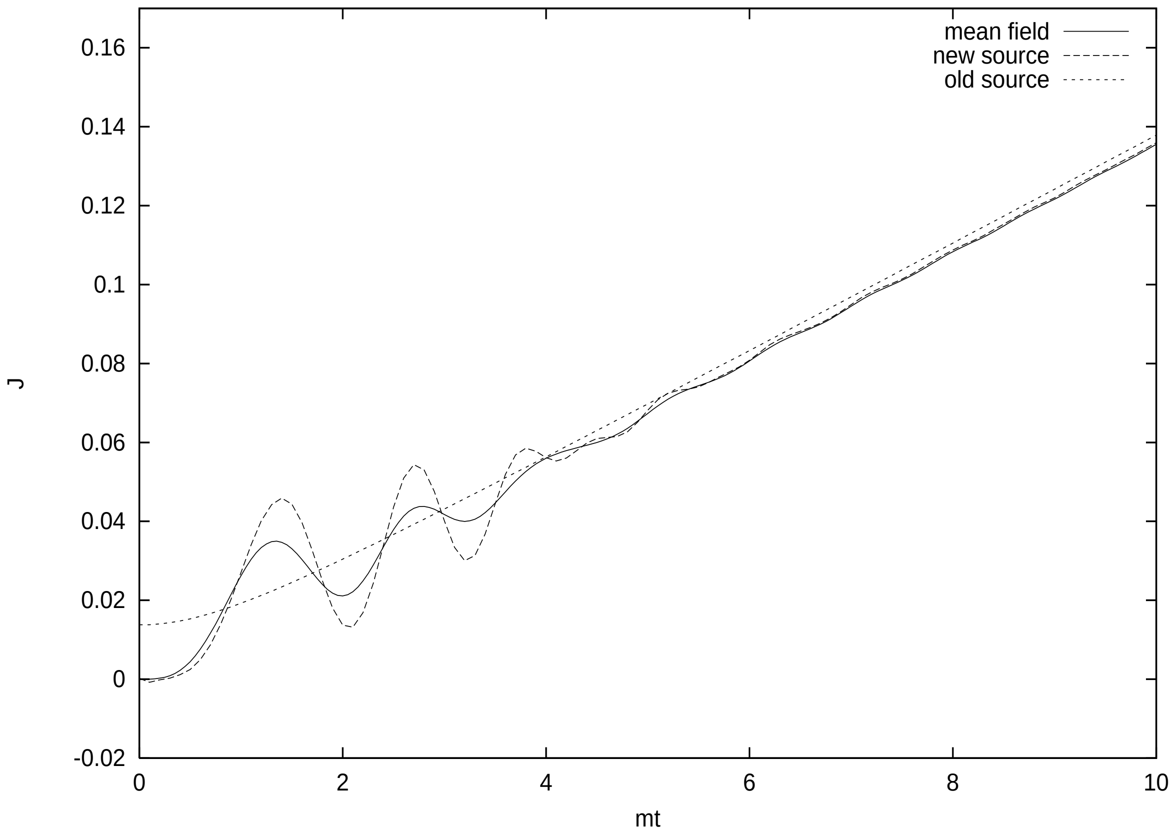

FIG. 5. The linear growth of the electric current with time in the case of fixed constant background electric field $E = 1$ and $e = 1$. The three curves shown are the current of the exact mode functions, the uniform Airy approximation to them with initial conditions at $t_0 = 0$ according to (5.28-5.30), and the simple step function ansatz of Eqn. (5.27).

When the electric fields are very weak fields ($eE \ll m^2c^3/\hbar$), particle creation is negligible, the linear slope in Fig. 5 is very small and even in backreaction the electric field will hardly change at all with time. In this case essentially all the effects on moderate time scales will be transient effects and one should reatin the initial condition information in $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$. In moderately strong electric fields ($eE \simeq m^2c^3/\hbar$) where particle creation is significant Fig. 5 shows that the transient effects of the initial data become unimportant before long and one might just as well use the simpler expression for the source term with $A_{\mathbf{k}} = 1$ and $B_{\mathbf{k}} = 0$, derived previously. This is equivalent to replacing the electric field value the particles feel at the actual time of creation by one assumed to have been constant for times long before the creation takes place. In that case the source term does not depend on the value of the electric field at the initial time t_0, which again is reasonable provided $\tau_{pl} \gg \tau_{cl}$. It is the quasistationary, Markov approximation for the source term in (5.16) or (5.27) $A_{\mathbf{k}} = 1$ and $B_{\mathbf{k}} = 0$ that we compare to the actual backreaction evolution of mean field theory in the next section.

We conclude this section by remarking on the relationship between the local source term (4.9) or its asymptotic expansion, (5.16) and the general nonlocal form (3.17) derived in Section II. For a constant electric field starting from vacuum initial conditions (4.9) and (3.17) must be identical of course. If, following Rau [16] one neglects the Bose enhancement factor $1 + 2\mathcal{N}_{\mathbf{k}}$ and changes variables from t' to $\lambda x \equiv 2\Theta_{\mathbf{k}}(t') - 2\Theta_{\mathbf{k}}(t)$ then the integral in (3.17) may be rewritten in the form,

$$\frac{d}{dt}\mathcal{N}_{\mathbf{k}} = \frac{eEp}{4\omega^2}\int_{-\infty}^{0} dx\, \frac{\sinh\varphi(x)}{\cosh^3\varphi(x)}\cos(\lambda x) \tag{5.33}$$

where

$$u' = \epsilon\frac{k + eEt'}{\sqrt{|eE|}} = \sqrt{\lambda}\,\sinh\varphi(x) \tag{5.34}$$

is given implicitly as a function of x by the relation,

$$\sinh\varphi(x)\cosh\varphi(x) + \varphi(x) = x + \frac{p\omega}{p_\perp^2 + m^2} + \sinh^{-1}\left(\frac{p}{\sqrt{p_\perp^2 + m^2}}\right)\,, \tag{5.35}$$

for constant electric field.

This is similar in form to Eqns. (25) and (26) of [16], the additional $\sinh\varphi$ in the numerator of (5.33) being due to the fact that we have treated charged scalars rather than fermions in this paper. Thus the form of the source term plotted in the second halves of Figs. 2-4 is qualitatively similar to those presented in [16] by numerical evaluation of an integral similar to (5.33). However, the neglect of the quantum statistical enhancement (or Pauli blocking) factor $1 \pm 2\mathcal{N}_{\mathbf{k}}$ in the integrand of (3.17) is valid *only* in the weak field limit $\lambda \gg 1$. Since that has already been assumed in writing (5.33) one should then properly evaluate the integral in the same limit. As already remarked in Section 2 there is no straightforward method of performing an asymptotic expansion of this integral in real time without losing the exponentially small Schwinger effect: integrating the $\cos(\lambda x)$ term successively by parts will generate the simple adiabatic expansion which contains no $\exp(-\pi\lambda)$ term or step function. In the case of weak fields $a^{-1} = \lambda \gg 1$ this effect is exponentially small in any case, so if one simply evaluates (5.33) or its equivalent for fermions numerically as in Fig. 1 of ref. [16] or Fig. 4 of this work, most of the numerical contribution to what is plotted is contained in the *first* (pure $\beta_{\mathbf{k}}^{adb}$) term of (5.27), which scales like $1/\lambda^3$, and *not* the last term which gives rise to the exponentially small delta function source of [11]. Hence multiplication by the factor $\exp(\pi\lambda/2)$ in Eqns. (24) and (25) of ref. [16] for weak fields is nugatory, while for strong fields $a^{-1} = \lambda < 1$, the neglect of the factor $1 \pm 2\mathcal{N}_{\mathbf{k}}$ in (5.33) or Eqn. (25) of [16] is not justified.

VI. BACKREACTION

The source term we have derived in (5.16) for vacuum initial conditions at $t_0 = -\infty$ must be modified to include induced creation when there are particles present in the initial state. Since

$$1 + 2\mathcal{N}_{\mathbf{k}} = (1 + 2N_{\mathbf{k}})(1 + 2|\beta_{\mathbf{k}}|^2) \tag{6.1}$$

for $N_{\mathbf{k}}$ particles in the initial state, the correct modified source term in constant electric field is

$$\dot{\mathcal{N}_{\mathbf{k}}} = (1 + 2N_{\mathbf{k}}) S_{vac}(p, p_\perp; E)\,. \tag{6.2}$$

In the backreaction problem the electric field will vary with time. Now the local Markov approximation to the nonlocal Vlasov equation (3.17) consists of using the source term (6.2) with the constant E replaced by $E(t)$ and the constant $N_{\mathbf{k}}$ by $\mathcal{N}_{\mathbf{k}}(t)$ at the *local* time of interest. The replacement of E by $E(t)$ is justified if the electric field is slowly varying (the quasistationary approximation), while the replacement of $N_{\mathbf{k}}$ by $\mathcal{N}_{\mathbf{k}}(t)$ is justified if the electric field is weak ($\lambda \gg 1$), since from (6.1) the difference between the two is proportional to $|\beta_{\mathbf{k}}|^2$ which is of order λ^{-2} and gives rise to corrections which are higher order than the terms we have retained in the asymptotic expansion of the source term. In this way the statistical factor $1 \pm 2\mathcal{N}_{\mathbf{k}}$ has effectively been removed from the nonlocal integral kernel (3.17) and we have obtained the final form of the source term for use in backreaction.

Converting the independent variables of the number distribution function from canonical momenta $\mathbf{k}$ and t to kinetic momenta $\mathbf{p}$ and t by the definition,

$$\mathcal{N}_{\mathbf{k}}(t) \equiv \mathcal{N}(\mathbf{p} = \mathbf{k} - e\mathbf{A}; t)\,, \tag{6.3}$$

we obtain the local Vlasov equation,

$$\frac{\partial}{\partial t}\mathcal{N}(p, p_\perp; t) + eE(t)\frac{\partial}{\partial p}\mathcal{N}(p, p_\perp; t) = S(p, p_\perp; E) = (1 + 2\mathcal{N}(p, p_\perp; t))\, S_{vac}(p, p_\perp; E)\,, \tag{6.4}$$

for spatially homogeneous fields. Spatial dependence in the distribution function could be included on the left side of (6.4) in the standard manner, provided it is also slowly varying in space compared to $c\tau_{cl}$. Together with the Maxwell equation,

$$\ddot{A}(t) = 2e\int [d\mathbf{p}]\ \frac{p}{\omega}\mathcal{N}(p, p_\perp; t) + \frac{2}{E}\int [d\mathbf{p}]\ \omega\,(1 + 2\mathcal{N}(p, p_\perp; t))\, S_{vac}(p, p_\perp; E) \tag{6.5}$$

(5.16), and the defining relations (4.2), (5.6) and (5.7) this constitutes the local kinetic approximation to the mean field equations.

In order to understand the time scale associated with the variation of the electric field and therefore the validity of the quasistationary approximation to the source term by that for a constant electric field consider first the Vlasov-Maxwell system ignoring particle creation. With the source term set to zero, (6.4) can be solved in closed form, *viz.*,

$$\mathcal{N}(p, p_\perp; t) = \mathcal{N}(p + eA(t), p_\perp; 0)\,. \tag{6.6}$$

Substituting this solution into (6.5) and linearizing in $A(t)$ gives

$$\delta\ddot{A}(t) - 2e^2\delta A(t)\int [d\mathbf{p}]\, \frac{p}{\omega}\,\frac{\partial}{\partial p}\mathcal{N}(p, p_\perp; 0) = 0\,. \tag{6.7}$$

Integrating the latter expression by parts demonstrates that the potential (and therefore also the electric field) will oscillate with a frequency,

$$\omega_{pl}^2 = 2e^2 \int [d\mathbf{p}]\, \mathcal{N}(p, p_\perp; 0)\, \frac{\partial}{\partial p}\left(\frac{p}{\omega(p, p_\perp)}\right) , \tag{6.8}$$

which is the relativistic plasma frequency. In the nonrelativistic limit $\omega(p, p_\perp)$ can be replaced by m and the integral, $2\int [d\mathbf{p}]\, \mathcal{N}(p, p_\perp; 0) = n$ simply gives the total number density of particles present in the initial state at $t = 0$. Then we recover the familiar expression $\omega_{pl}^2 \to e^2 n/m$ for the classical plasma oscillation frequency.

This classical plasma frequency may be obtained as well from a linear response analysis of the quantum mean field equations as follows. We perturb the vacuum solution for the mode functions,

$$\bar{f}_{\mathbf{k}}(t) = \sqrt{\frac{\hbar}{2\bar{\omega}_{\mathbf{k}}}} e^{-i\bar{\omega}_{\mathbf{k}} t} , \qquad \bar{\omega}_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2} \tag{6.9}$$

with zero electric field by writing

$$f_{\mathbf{k}}(t) = \bar{f}_{\mathbf{k}}(t) + \delta f_{\mathbf{k}}(t) \tag{6.10}$$

and expand the equations of motion to first order in $\delta f_{\mathbf{k}}$, A and $\dot{A}$. The linearized mode equation,

$$\left[\frac{d^2}{dt^2} + \bar{\omega}_{\mathbf{k}}^2\right] \delta f_{\mathbf{k}} = 2ke\, A\, \bar{f}_{\mathbf{k}} \tag{6.11}$$

can be solved by making use of the free retarded Green's function,

$$G_R(t - t'; \mathbf{k}) = \frac{\sin[\bar{\omega}_{\mathbf{k}}(t - t')]}{\bar{\omega}_{\mathbf{k}}} \theta(t - t') , \tag{6.12}$$

in the form,

$$\delta f_{\mathbf{k}}(t) = 2ek \int_0^t dt'\, G_R(t - t'; \mathbf{k}) A(t') \bar{f}_{\mathbf{k}}(t') + A_{\mathbf{k}} \bar{f}_{\mathbf{k}}(t) + B_{\mathbf{k}} \bar{f}_{\mathbf{k}}^*(t) , \tag{6.13}$$

where $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ are constants of integration and Re $A_{\mathbf{k}} = 0$ in order to preserve the Wronskian condition (2.8) under the perturbation. The corresponding linearized Maxwell equation is

$$\begin{aligned} \ddot{A} &= e \int [d\mathbf{k}] \left\{ 4k\, (1 + 2N_{\mathbf{k}})\, \mathrm{Re}(\bar{f}_{\mathbf{k}}^* \delta f_{\mathbf{k}}) - 2eA\, \frac{N_{\mathbf{k}}}{\bar{\omega}_{\mathbf{k}}} - eA\, \frac{k^2}{\bar{\omega}_{\mathbf{k}}^3} \right\} \\ &= 2e^2 \int_0^t dt'\, A(t') \int [d\mathbf{k}]\, \frac{k^2}{\bar{\omega}_{\mathbf{k}}^2}\, (1 + 2N_{\mathbf{k}})\, \sin[2\bar{\omega}_{\mathbf{k}}(t - t')] - e^2 A(t) \int [d\mathbf{k}] \left(\frac{k^2}{\bar{\omega}_{\mathbf{k}}^3} + \frac{2N_{\mathbf{k}}}{\bar{\omega}_{\mathbf{k}}}\right) + B(t) , \end{aligned} \tag{6.14}$$

where

$$B(t) \equiv 2e \int [d\mathbf{k}]\, \frac{k}{\bar{\omega}_{\mathbf{k}}}\, (1 + 2N_{\mathbf{k}})\, \mathrm{Re}(B_{\mathbf{k}} e^{-2i\bar{\omega}_{\mathbf{k}} t}) \tag{6.15}$$

is given by the initial perturbation away from the vacuum solution.

The most direct method of solving a linear integral equation such as (6.14) is to make use of the Laplace transform,

$$\tilde{A}(s) \equiv \int_0^\infty dt\, e^{-st} A(t)\,. \tag{6.16}$$

After some regrouping of terms the Laplace transform of (6.14) may be put into the form,

$$\tilde{A}(s)\left\{s^2\left[1+\frac{e^2}{4}\int[d\mathbf{k}]\,\frac{k^2}{\bar{\omega}_{\mathbf{k}}^5}\,(1+2N_{\mathbf{k}})\right]+2e^2\int[d\mathbf{k}]\,N_{\mathbf{k}}\,\frac{\partial}{\partial k}\left(\frac{k}{\bar{\omega}_{\mathbf{k}}}\right)\right.$$
$$\left.-\frac{e^2s^4}{4}\int[d\mathbf{k}]\,\frac{k^2}{\bar{\omega}_{\mathbf{k}}^5}\,\frac{(1+2N_{\mathbf{k}})}{(s^2+4\bar{\omega}_{\mathbf{k}}^2)}\right\} = sA(0)+\dot{A}(0)+\tilde{B}(s)\,, \tag{6.17}$$

where the right hand side depends only upon the initial data. We notice in (6.17) the presence of the two-particle threshold at $s^2=-4\bar{\omega}_{\mathbf{k}}^2$ for the creation of a pair of charged particles which would give rise to an imaginary part and damping in the linear Maxwell equation. Since the particles are massive this imaginary part is zero if we find an oscillatory solution of the equation with $s=\pm i\omega_{pl}$ and $\omega_{pl}\ll 2m$. Such a solution is easily found by setting the expression in curly brackets in (6.17) to zero and neglecting the s^4 term:

$$\omega_{pl}^2 = 2e_{R,N}^2\int[d\mathbf{k}]\,N_{\mathbf{k}}\frac{\partial}{\partial k}\left(\frac{k}{\bar{\omega}_{\mathbf{k}}}\right)\,, \tag{6.18}$$

where

$$\frac{1}{e_{R,N}^2} = \frac{1}{e^2}+\frac{1}{4}\int[d\mathbf{k}]\,\frac{k^2}{\bar{\omega}_{\mathbf{k}}^5}\,(1+2N_{\mathbf{k}})\,. \tag{6.19}$$

In $3+1$ dimensions the combination in (6.19) is independent of the ultraviolet cut-off and the renormalized value of the charge depends in general on the distribution $N_{\mathbf{k}}$. The only requirement on the distribution is that ω_{pl} in (6.18) must be much smaller than $2m$, in which limit there is no particle creation at all and the time independent $N_{\mathbf{k}}$ in (6.18) may be identified with the particle density in phase space $\mathcal{N}(\mathbf{p})$.

Thus the linear response analysis of the quantum mean field theory gives exactly the same result for the plasma frequency, provided that the classical charge e appearing in (6.8) is identified with the renormalized charge of the quantum theory according to (6.19). This provides a consistency check with the classical Vlasov transport description of the plasma, valid in the adiabatic or infrared limit of slowly varying mean fields, and identifies the proper correspondence limit of the classical coupling with that in the underlying quantum description.

It is ω_{pl} that sets the time scale of the variation of the electric field in the backreaction problem, *i.e.* $\tau_{pl}\simeq 2\pi/\omega_{pl}$. Hence the local, quaistationary approximation to the source term in (6.4) requires that the three time scales obey

$$\omega_{pl}\tau_{qu} \ll \omega_{pl}\tau_{cl} \ll 1\,. \tag{6.20}$$

If one starts the evolution with zero initial particles then the distribution function $\mathcal{N}$ changes from its classical value $\mathcal{N}(p+eA(t),p_\perp;0)$ due to the particle creation effect embodied in the source term. Since the second inequality requires $\tau_{cl}/\tau_{qu}\sim\lambda\gg 1$ and the source term is exponentially small in λ, the number of created particles n and therefore the plasma oscillation frequency will also be exponentially small in λ. Hence the time for enough particles to be produced to significantly influence the electric field will be exponentially long and the second inequality in (6.20) will also be satisfied automatically. Thus our local, quasistationary approximation scheme for the source term is valid *a posteriori*, and we would expect even the cruder approximation of the source term by (5.27) to be not far from correct.

Indeed in previous work we have shown that solving a Vlasov system with a phenomenological source term of the form

$$(1 + 2\mathcal{N}(p, p_\perp; t))|eE| \ln(1 + e^{-\pi\lambda})\, \delta(p)\,, \tag{6.21}$$

reproduces results qualititatively similar to the mean field theory calculation of charged matter field coupled to a classical electric field [3]. In the present work we have shown that no logarithm should be present in the source term, *i.e.* $\ln(1+e^{-\pi\lambda})$ in (6.21) should be replaced by simply $e^{-\pi\lambda}$, and that in fact, the particle creation event is continuous and can only be crudely approximated as a sharp step function in (5.27) with some loss of information about the true time structure of the event, as demonstrated in Figs. 2-4. However if the second inequality in (6.20) is valid then the evolution of the mean electric field on the time scale of ω_{pl}^{-1} should be affected but little by the further approximation of the source term by a delta function.

In order to test the validity of this approximation we present numerical results for the Maxwell-Vlasov system of equations (6.4) and (6.5) with both the new source term (5.16) and the old delta function source term (but with no logarithm), and compare the results to the exact solution of the mean field evolution of the mode functions (2.9) coupled to the Maxwell equation, (2.17) for scalar QED. The electric field evolution is plotted in Fig. 6 for the three cases. We observe that the corrected delta function source term gives qualitatively correct results, but the new source term (6.2) does a better overall job of tracking the mean field evolution, particularly at late times, where the old source term begins to drift out of phase. The new source term (6.2) also drifts out of phase eventually, but at a much slower rate, or in other words it more accurately estimates the plasma frequency of the collective motion. We deliberately chose moderately large values of the coupling $e = 1$ and the initial electric field $E = 1$ in order to amplify the small discrepancy between the mean field evolution and that of the new source term. For evolutions at $e = 0.1$ such as in earlier work [3], the discrepancy is negligible on the scale of the plot.

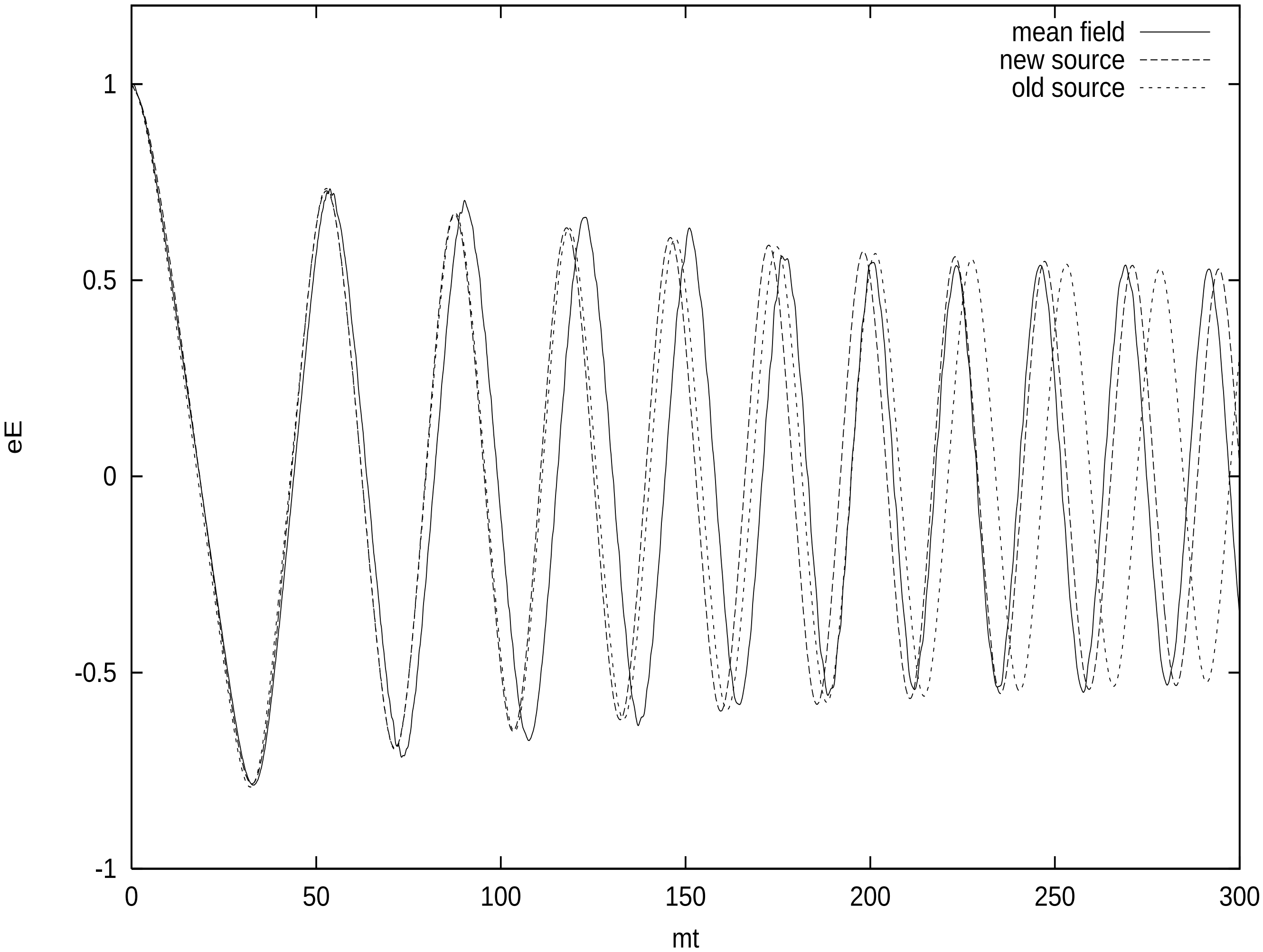

FIG. 6. The evolution of the electric field in one space dimension, according to the exact mean field Eqns. (2.17), the new source term (6.2) derived in this paper, and the old source term used previously (6.21), but with no logarithm, for initial electric field $eE = e^2 = m^2 = 1$, and no particles present in the initial state. The new source term tracks the mean field evolution more accurately than the delta function source term, which gives a too small plasma oscillation frequency at late times.

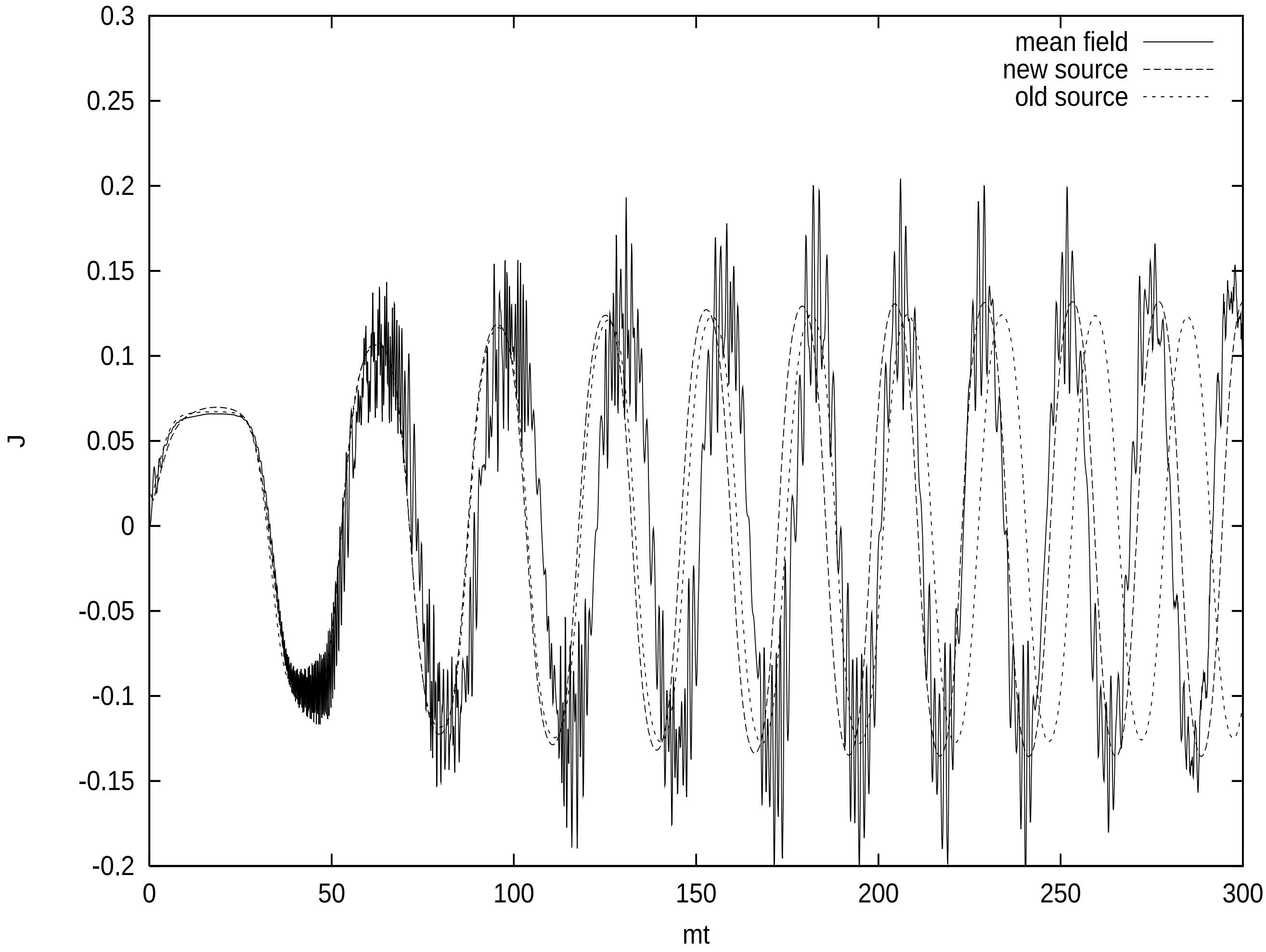

FIG. 7. The evolution of the current in one space dimension, for the same initial conditions as Fig. 6. Both the source terms for the Vlasov equation neglect the oscillations of the current on the time scale $\tau_{qu} \sim 1$.

In Fig. 7 we display the electric current (3.26) for the three evolutions. The new feature observed here are the rapid oscillations of the quantum mean field evolution on the time scale τ_{qu} and their complete absence from the evolutions with the two local Vlasov source terms which follow the value of the current averaged over this rapid time scale. This is in accord with our previous discussion of the neglect of such quantum coherence effects in any local transport description. The particle distribution function $\mathcal{N}$ is plotted as a function of k for the mean field and Vlasov evolutions at a particular value of t in Fig. 8. We observe the same quantum coherence effects in the mean field evolution here in the rapid oscillations of $\mathcal{N}$ in momentum space on a scale $\Delta k \sim 1/c\tau_{qu}$ as one observes as a function of time. Again these oscillations in the particle distribution are absent in the Vlasov evolutions. We note also the slightly negative value of the distribution function in the case of the new Airy source term. This is a transient effect due to our setting $A_{\mathbf{k}}(t_0)$ and $B_{\mathbf{k}}(t_0)$ to one and zero respectively. With the more accurate source term computed from (5.31) which takes account of the initial conditions this artificial negative region is much smaller. It also grows less and less pronounced as time progresses, and may be eliminated entirely by binning the distribution in momentum bins. Some small discrepancy of this kind is to be expected in any truncation of the unitary field theory evolution by a local Vlasov source term, unless that source term is always and everywhere positive, corresponding to a strictly monotonic increase of total particle number and entropy, according to (3.24). It may be regarded as a rough estimate of the systematic error induced by the Markov approximation in the source term (6.2).

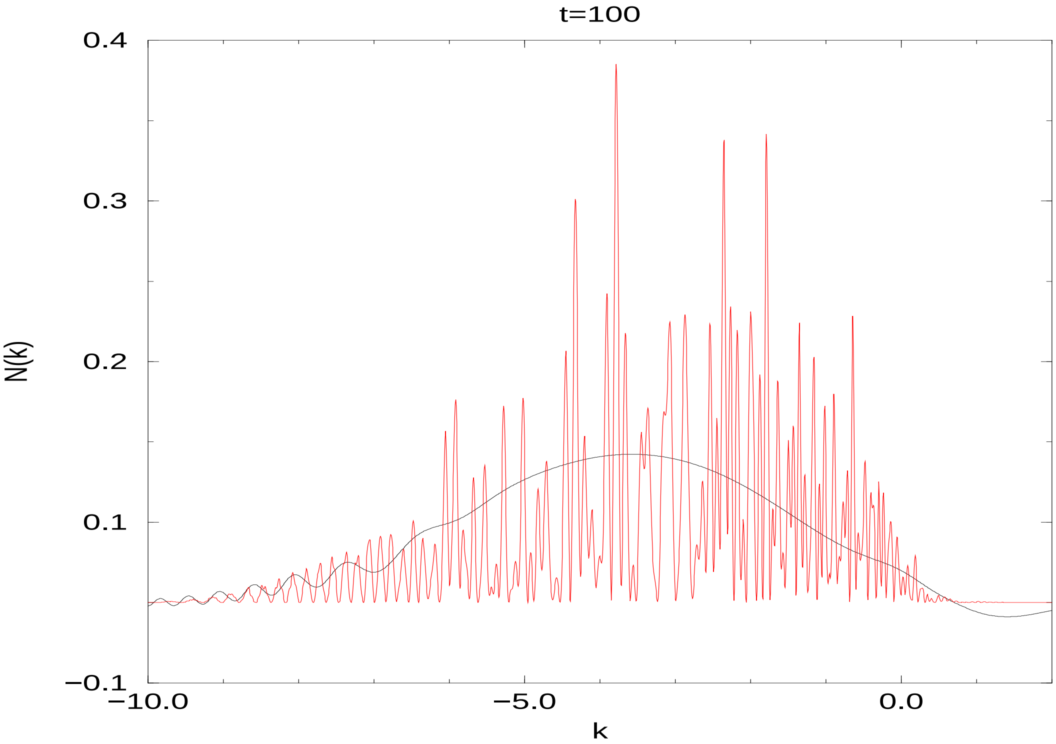

FIG. 8. The particle distribution $\mathcal{N}$ as a function of canonical momentum k at a fixed time $t = 100$ for the mean field evolution (jagged curve) and the new source term (smooth curve) derived in this paper, for the same initial conditions as Figs. 6 and 7. The Vlasov equation with the new source term is approximately a smooth average of the actual mean field evolution on spatial scales of order $c\tau_{qu} \sim 1$. The slight dip into negative values of the smooth curve becomes less and less prominent at later times.

In these numerical evolutions the renormalized charge of the mean field theory was chosen to be $e_R = 1$ in order to compare to the Vlasov evolution with unit classical charge $e = 1$, according to (6.18) and (6.19). In $1 + 1$ dimensions where the simulations were performed the charge renormalization is finite and in the vacuum is given by

$$\frac{m^2}{e^2_{R,N=0}} = \frac{m^2}{e^2} + \frac{1}{12\pi}\,. \tag{6.22}$$

so that the finite renormalization effects for $e = m = 1$ are of order $1/12\pi \simeq 0.026$ or a few percent in the range of the simulations shown in the figures. In the extreme weak coupling limit $\frac{m^2}{e_R^2} \gg 1$ where the Vlasov approximation becomes more and more accurate, this finite renormalization effect is completely negligible.

VII. SUMMARY AND OUTLOOK

Based on the Hamiltonian description of mean field theory and the existence of an adiabatic invariant of this evolution, we identified the (lowest order) adiabatic particle number as the most suitable analog for the single particle distribution function of semiclassical transport theory. Although not unique, this definition of particle number involves the fewest number of derivatives (namely zero) of the frequency $\omega_{\mathbf{k}}$, and hence its time rate of change is most appropriate for identification as the source term for the Boltzmann-Vlasov equation, which is first order in time derivatives. Confirming this identification, the electric current in this basis has an intuitively appealing and simple quasiclassical form, (3.26). Since $\mathcal{N}_{\mathbf{k}}$ is already adiabatic order two with this definition of particle number, including higher order adiabatic corrections in $\mathcal{N}_{\mathbf{k}}$ would be inconsistent with the use of the source term in backreaction as well, since Maxwell's equations are second order in time.

Analyzing the time dependence of the mean particle number in a constant electric field we derived the rate of pair creation of charged scalar particles, and clarified the time scales involved in the particle production phenomenon. Although formally equivalent to the quantum Vlasov equation (3.17) and consistent with the general projection method of Zwanzig applied to the density matrix in the adiabatic number basis, our approach bypasses the mathematical difficulties inherent in the nonlocal integral representation, and does not require explicit use of the projection formalism. Unlike any direct formal manipulation of the nonlocal form (3.17) or simple WKB expansions, we used a uniform asymptotic expansion for the local source term, which retains the Schwinger creation effect at the *lowest* order of the expansion. This local source term is not obtained by putting to zero the phase correlations in the pair creation process, but rather by the assumption that the actual correlations in a time varying field can be replaced by those present in a constant field at at $t = -\infty$. This can only be approximately valid when the electric field is very slowly varying in time, so that any actual phase correlations in the initial state are no longer important.

Given the hierarchy of time scales (6.20) we showed that a simple modification of the usual expansion in terms of exponential functions is nearly adequate for most analysis of the collective plasma effects in scalar QED. The asymptotic expansion in terms of the elementary exponential functions modified by the step function, which leads to the ansatz (5.27), demonstrates in a simple way the origin of the linear growth in time in the current which makes backreaction essential at late times, for *any* nonzero coupling no matter how weak. It also shows why taking the pair production source term to be proportional to $\delta(p)$ was a reasonably good proposal after all, although the use of the logarithm in the source term of [12,13] and subsequent references seems to have been due to a confusion between the rate of particle creation and the vacuum persistence probability. Using this ansatz in conjunction with (6.1) explains the origin of the Bose enhancement source term, which was incorporated in the phenomenological source term (6.21) for physical reasons. The source term obtained in explicit form from the mode functions in constant electric field (6.2) is in better agreement with the mean field evolution than the phenomenological source term (6.21), even for quite large electric fields, although the difference between the two is not dramatic.

The methods employed in this paper can be readily extended to other situations of interest, such as fermions, chromoelectric fields in QCD, or the creation of massive particles by strong gravitational fields in an early universe context. The limit in which such processes can be described by a semiclassical source term in a transport approach should be clear from the present work: one requires a clean separation of the three time scales, τ_{qu} associated with the rapid quantum phase oscillations, τ_{cl} associated with one particle creation amplitudes, and τ_{pl} associated with the collective motion of the mean field(s). Conversely, it should also be clear that when such a clean separation does not exist the methods of this paper cannot be applied, and very likely, no semiclassical transport approach is appropriate or possible. Unfortunately, this includes the cases of most interest in QCD, relativistic heavy-ion physics and early universe cosmology, where light or strictly massless degrees of freedom play an important role. If $m = 0$ then the low momentum modes will never behave like classical particles admitting a Blotzmann-Vlasov description. Even pions are light enough to cause the hierarchy of time or momentum scales in (6.20) to break down in heavy-ion collisions. Indeed in the formation of disoriented chiral condensates the infrared instability of the low momentum modes and growth of a large condensate field by coherence effects (*not* incoherent particle emission) is precisely the point. In cases such as these where Bose condensation plays a central role, the frequencies $\omega_{\mathbf{k}}$ become small or even imaginary, the turning point(s) of the adiabatic particle number approach or reach the real time axis, τ_{qu} becomes large, and no simple quasiclassical particle interpretation within the Boltzmann framework is possible. Complementary coherent classical field methods can be developed in this regime, matched to a transport description of the hard modes on a case by case basis, but only the full field theoretic approach of mean field theory and its higher order $1/N$ corrections is powerful enough to encompass all the various cases of interest in a comprehensive fashion.

Acknowlegement Just as this work was being completed one of the authors (J. M. E.) fell seriously ill and passed away a few weeks later. Y. K. and E. M. wish to express their profound sadness at this premature loss of a dear teacher, colleague and friend, and to dedicate this work to his memory.

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REFERENCES

[1] N. G. van Kampen, *Physica* **20** 603 (1954);
L. van Hove, *Physica* **21** 517 (1955);
R. W. Zwanzig, *Lectures in Theoretical Physics*, Summer Institute for Theoretical Physics, University of Colorado, 1960, Vol. 3, W. E. Brittin, B. W. Downs and J. Downs, eds. (Interscience Publishers, New York 1961).

[2] F. Cooper and E. Mottola, *Phys. Rev. D* **36**, 3114 (1987); **40** 456, (1989).

[3] Y. Kluger, J. M. Eisenberg, B. Svetitsky, F. Cooper, and E. Mottola, *Phys. Rev. Lett.* **67** 2427 (1991); *Phys. Rev. D* **45**, 4659 (1992); **48** 190 (1993);
Y. Kluger, J. M. Eisenberg, and B. Svetitsky, *Int. J. Mod. Phys. E* **2** 333, (1993).

[4] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and, P. R. Anderson, *Phys. Rev. D* **50** 2848 (1994);
S. Habib, Y. Kluger, E. Mottola, and J. P. Paz, *Phys. Rev. Lett.* **76** 4660 (1996);
F. Cooper, S. Habib, Y. Kluger, and E. Mottola, *Phys. Rev. D* **55** (1997) 6471.

[5] O. Klein, *Z. Phys.* **53**, 157 (1929);
F. Sauter, *Z. Phys.* **69**, 742 (1931); **73**, 547 (1931);
J. Schwinger, *Phys. Rev.* **82**, 664 (1951).

[6] N. B. Narozhnyi, *Zh. Eksp. Theor. Fiz.* **54**, 676 (1968) [*Sov. Phys. JETP* **27**, 360 (1968)];
A. I. Nikishov, *Zh. Eksp. Theor. Fiz.* **57**, 1210 (1969) [*Sov. Phys. JETP* **30**, 660 (1970)];
N. B. Narozhnyi and A. I. Nikishov, *Yad. Fiz.* **11** (1970) 1072 [*Sov. J. Nucl. Phys.* **11**, 596 (1970)]; *Teor. Mat. Fiz.* **26**, 16 (1976) [*Theor. Math. Phys.* **26**, 9 (1976)];
A. I. Nikishov, *Proc. P. N. Lebedev Phys. Inst.* **111**, 153 (1979).

[7] E. Brezin and C. Itzykson, *Phys. Rev. D* **2**, 1191 (1970).

[8] V. L. Pokrovskii and I. M. Khalatnikov, *Zh. Eksp. Theor. Fiz.* **40**, 1713 (1961) [*Sov. Phys. JETP* **13**, 1207 (1961)];
M. S. Marinov and V. S. Popov, *Yad. Fiz.* **15**, 1271 (1972); **16**, 809 (1972) [*Sov. J. Nucl. Phys.* **15**, 702 (1972)]; **16**, 449 (1973)]; *Zh. Eksp. Theor. Fiz.* **65**, 2141 (1973) [Sov. Phys. JETP 38 (1974) 1069]; *Fortsch. Phys.* **25**, 373 (1977);
V. S. Popov, *Zh. Eksp. Theor. Fiz.* **62**, 1248 (1972); **63**, 1586 (1972) [*Sov. Phys. JETP* **35**, 659 (1972); **36**, 840 (1973)]; *Yad. Fiz.*, **19**, 1140 (1974) [*Sov. J. Nucl. Phys.* **19**, 584 (1974)].

[9] A. M. Perelomov, *Teor. Mat. Fiz.* **16**, 303 (1973); **19**, 83 (1974) [*Theor. Math. Phys.* **16**, 852 (1973); **19**, 368 (1974)].
V. M. Mostepanenko and V. M. Frolov, *Yad. Fiz.* **19**, 885 (1974) [*Sov. J. Nucl. Phys.* **19**, 451 (1974)];
V. I. Ritus, *Zh. Eksp. Theor. Fiz.* **80**, 1288 (1981) [*Sov. Phys. JETP* **53**, 659 (1981) 659].

[10] V. G. Bagrov, D. M. Gitman, and S. M. Shvartsman, *Zh. Eksp. Theor. Fiz.* **68**, 392 (1975) [*Sov. Phys. JETP* **41**, 191 (1975)];
D. M. Gitman, *J. Phys. A* **10**, 2007 (1977);
S. P. Gavrilov and D. M. Gitman, *Phys. Rev. D* **53**, 7162 (1996).

[11] A. Casher, H. Neuberger, and S. Nussinov, *Phys. Rev. D* **28**, 179 (1979).

[12] N. Glendenning and T. Matsui, *Phys. Lett. B* **141**, 419 (1984);
G. Gatoff, A. K. Kerman, and T. Matsui, *Phys. Rev. D* **36**, 114 (1987).

[13] A. Białas and W. Czyż, *Acta Phys. Pol. B* **17**, 635 (1986);
A. Białas, W. Czyż, A. Dyrek, and W. Florkowski, *Nucl. Phys. B* **296**, 611 (1988).

[14] I. Bialynicki-Birula, P. Górnicki, and J. Rafelski, Phys. Rev. D 44 (1991) 1825.

[15] C. Best, P. Górnicki, and W. Greiner, *Ann. Phys.* **225**, 169 (1993);
C. Best and J. M. Eisenberg, *Phys. Rev. D* **47**, 4639 (1993).

[16] J. Rau, *Phys. Rev. D* **50**, 6911 (1994).

[17] J. Rau and B. Mueller, *Phys. Rep.* **272**, 1 (1996).

[18] S. A. Smolyansky, G. Roepke, S. Schmidt, D. Blaschke, V. D. Toneev, A. V. Prozorkevich, e-print archive, hep-ph/9712377 (unpublished).

[19] W. Greiner, B. Müller, and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer-Verlag, Berlin, 1985);
Issues in Intense Field Quantum Electrodynamics, Proc. Lebedev Phys. Inst. Sci. USSR, Vol. 168, V. L. Ginzburg ed. (Nova Science, Commack, 1987);
A. A. Grib, S. G. Mamaev, and V. M. Mostepanenko, *Vacuum Quantum Effects in Strong Fields* (Atomizdat, Moscow, 1988; Friedmann Laboratory Publishing, St. Petersburg, 1994);
E. S. Fradkin, D. M. Gitman, and S. M. Shvartsman, *Quantum Electrodynamics with Unstable Vacuum* (Springer-Verlag, Berlin, 1991);
Quantum Electrodynamics with Unstable Vacuum, Proc. Lebedev Phys. Inst. Sci. Rus., Vol. 220, V. L. Ginzburg ed. (Nova Science, Commack, 1995).

[20] S. R. deGroot, W. A. van Leeuwen, and C. G. van Weert, *Relativistic Kinetic Theory* (North-Holland, Amsterdam, 1980).

[21] H.-T. Elze and U. Heinz, *Phys. Rep.* **183**, 81 (1989) and references therein.

[22] R. W. Zwanzig, *Physica* **30**, 1109 (1964).

[23] A. A. Grib and S. G. Mamaev, *Yad. Fiz.* **10** 1276 (1969), **14** 800 (1971) [*Sov. J. Nucl. Phys.* **10** 722 (1970), **14** 450 (1972)];
B. L. Hu, S. A. Fulling and L. Parker, *Phys. Rev. D* **8** 2377 (1973).

[24] See also M. Morikawa and M. Sasaki, *Prog. Theo. Phys.* **72** 782 (1984).

[25] B. L. Hu and D. Pavon, *Phys. Lett.* **180B**, 329 (1986);
H. E. Kandrup, *Phys. Rev. D* **37**, 3305 (1988); **38**, 1773 (1988).

[26] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York 1980, 1994).

[27] *Higher Transcendental Functions* (Bateman Manuscript Project), A. Erdelyi *et. al.* eds. (McGraw-Hill, New York 1953), Vols. 1 and 2.

[28] See *e.g.* C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York 1978).

[29] F. W. J. Olver, *J. Res. Nat. Bur. Stds. B*, **63**, 131 (1959).

[30] R. E. Meyer, *J. Appl. Math. Phys.* **24**, 293 (1973); **24**, 517 (1973);
See also H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA 1980).

APPENDIX A: DENSITY MATRIX IN THE ADIABATIC PARTICLE BASIS

In this Appendix we derive the form of the Gaussian density matrix (2.22) in the adiabatic number basis. Since in the case of a spatially homogeneous mean electric field the density matrix is a product of Gaussians for each wave number $\mathbf{k}$, we consider a single wave number and drop the subscript $\mathbf{k}$ in the derivation in order to simplify the notation of this Appendix.

For each wave number we have positively and negatively charged modes obeying the time dependent harmonic oscillator equation (2.9). Because of this and using Eqns. (3.6) and (3.7) the adiabatic particle basis is that which diagonalizes the Hamiltonian of the two-dimensional harmonic oscillator,

$$H_{osc} = \frac{1}{2}\left(\pi^\dagger \pi + \omega^2 \varphi^\dagger \varphi + h.c.\right) = \frac{\omega}{2}\left(\tilde{a}^\dagger \tilde{a} + \tilde{a}\tilde{a}^\dagger + \tilde{b}^\dagger \tilde{b} + \tilde{b}\tilde{b}^\dagger\right) \tag{A1}$$

in the complex representation. The states which diagonalize this Hamiltonian are labelled by two quantum numbers n_+ and n_- with energy $\omega(n_+ + n_- + 1)$. Expressed in polar coordinates,

$$\varphi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2) \equiv \frac{1}{\sqrt{2}} r e^{i\theta} \tag{A2}$$

we can label the states by the radial quantum number $n = n_+ + n_-$ and the angular quantum number $m = n_+ - n_-$ corresponding to the eigenmodes of the two-dimensional harmonic oscillator,

$$\left(-\frac{1}{2r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{1}{2r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{2}\omega^2 r^2\right)\langle r\,\theta | n\, m\rangle = \omega(n+1)\langle r\,\theta|n\,m\rangle \tag{A3}$$

in real polar coordinates. As is well known these wavefunctions are given in terms of the associated Laguerre polynomials $L^\alpha_\nu(x)$ in the form,

$$\langle r\,\theta | n\, m\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{im\theta} e^{-\omega r^2/2} (\sqrt{\omega} r)^m \left[\frac{\left(\frac{n-m}{2}\right)!}{\left(\frac{n+m}{2}\right)!}\right]^{\frac{1}{2}} L^m_{\frac{n-m}{2}}(\omega r^2)\,. \tag{A4}$$

The normalized eigenstates themselves may be written in the form,

$$|n\, m\rangle = \frac{(\tilde{a}^\dagger)^{\frac{n+m}{2}} (\tilde{b}^\dagger)^{\frac{n-m}{2}}}{\left[\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!\right]^{\frac{1}{2}}}|0\rangle \tag{A5}$$

with m taking on the values $-n+2k$, $k = 0, 1, \ldots, n$, so that $n \pm m$ is an even integer.

In order to transform the density matrix from the coordinate basis to the adiabatic number basis it is easiest first to define the coherent states,

$$\begin{aligned} |s\,\chi\rangle &\equiv \exp\left(i\tilde{a}^\dagger s e^{-i\chi} - i\tilde{b}^\dagger s e^{i\chi}\right)|0\rangle \\ &= \sum_{n=0}^{\infty} {\sum_{m=-n}^{n}}' \frac{s^n e^{-im\chi}}{\left[\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!\right]^{\frac{1}{2}}}|n\,m\rangle\,, \end{aligned} \tag{A6}$$

where the prime on the sum over m denotes that m is incremented by even integers. Upon subsituting the explicit wave functions (A4) we find the wave function of these coherent states can be expressed in the form,

$$\langle r\,\theta|s\,\chi\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\omega r^2/2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n}{}' s^n e^{im(\theta-\chi)} \frac{(\sqrt{\omega}r)^m}{\left(\frac{n+m}{2}\right)!} L^m_{\frac{n-m}{2}}(\omega r^2)\,. \tag{A7}$$

The sums in this expression may be performed in closed form by first switching the orders of the n and $m = -n + 2k$ sums, and making use of the summation formula [26],

$$\sum_{n=k}^{\infty} z^n L^{-n+2k}_{n-k}(x) = z^k \sum_{n=0}^{\infty} z^n L^{k-n}_n(x) = z^k(1+z)^k e^{-xz}\,. \tag{A8}$$

The remaining sum over k from 0 to infinity is then a pure exponential and easily performed with the result,

$$\langle r\,\theta|s\,\chi\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{\omega}{2}r^2 + 2irs\sqrt{\omega}\cos(\chi-\theta) + s^2\right\}\,, \tag{A9}$$

or in two-component vector notation,

$$\langle\vec{r}\,|\vec{s}\,\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \exp\left\{-\frac{\omega}{2}\vec{r}^{\,2} + 2i\sqrt{\omega}\,\vec{r}\cdot\vec{s} + \vec{s}^{\,2}\right\}\,. \tag{A10}$$

This $U(1)$ invariant exponential form may be verified also as the solution of the differential equation,

$$\langle r\,\theta|H_{osc}|s\,\chi\rangle = \left(-\frac{1}{2r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - -\frac{1}{2r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\omega^2 r^2\right)\langle r\,\theta|s\,\chi\rangle = \omega\left(s\frac{\partial}{\partial s}+1\right)\langle r\,\theta|s\,\chi\rangle \tag{A11}$$

obeying the initial condition,

$$\langle r\,\theta|s=0\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} e^{-\frac{\omega}{2}r^2} \tag{A12}$$

which follows from the Schrödinger equation (A3) and the definition of the coherent states (A6).

The utility of the coherent state basis is apparent from the simple exponential form of (A9) or (A10), since the transformation of the density matrix from the original coordinate basis,

$$\langle\vec{r}\,'|\rho|\vec{r}\,\rangle = \frac{1}{2\pi\xi^2}\exp\left\{-\frac{(\sigma^2+1)}{8\xi^2}(\vec{r}\,'^2+\vec{r}^{\,2}) + \frac{i\eta}{2\xi}(\vec{r}\,'^2-\vec{r}^{\,2}) + \frac{(\sigma^2-1)}{4\xi^2}\vec{r}\,'\cdot\vec{r}\right\} \tag{A13}$$

to the coherent state basis becomes a straightforward exercise in the integration of a product of Gaussians, *viz.*,

$$\begin{aligned}\langle\vec{s}\,'|\rho|\vec{s}\,\rangle &= \int d^2\vec{r}\,' \int d^2\vec{r}\,\langle\vec{s}\,'|\vec{r}\,'\rangle\,\langle\vec{r}\,'|\rho|\vec{r}\,\rangle\,\langle\vec{r}\,|\vec{s}\,\rangle \\ &= \frac{2\omega\xi^2}{B}\exp\left\{\frac{A}{B}e^{-i\vartheta}\vec{s}^{\,2} + \frac{A}{B}e^{i\vartheta}\vec{s}\,'^2 + \frac{C}{B}\vec{s}\cdot\vec{s}\,'\right\}\end{aligned} \tag{A14}$$

where the real coefficients A, B, C, and ϑ are given by

$$\begin{aligned} A\cos\vartheta &= -\omega^2\xi^4 + \eta^2\xi^2 + \frac{\sigma^2}{4} \\ A\sin\vartheta &= -2\omega\eta\xi^2 \\ B &= \omega^2\xi^4 + \frac{(\sigma^2+1)}{2}\omega\xi^2 + \eta^2\xi^2 + \frac{\sigma^2}{4} \\ C &= (\sigma^2 - 1)\omega\xi^2 \,. \end{aligned} \tag{A15}$$

With this result in hand all that remains to be done is to expand the coherent state density matrix (A14) in powers of s and s' to identify the matrix elements of ρ in the adiabatic particle number basis via

$$\langle s'\,\chi'|\rho|s\,\chi\rangle = \sum_{n',n=0}^{\infty}{\sum_m}' \frac{s'^{n'}\,s^n e^{im(\chi'-\chi)}}{\left[\left(\frac{n'+m}{2}\right)!\left(\frac{n'-m}{2}\right)!\left(\frac{n+m}{2}\right)!\left(\frac{n-m}{2}\right)!\right]^{\frac{1}{2}}}\,\langle n'\,m|\rho|n\,m\rangle\,. \tag{A16}$$

The fact that the coherent state density matrix is a function of only $\chi' - \chi$ and hence only $m' = m$ matrix elements of ρ appear in the sum is a result of the $U(1)$ invariance of the density matrix. We also note that in the pure state case, $\sigma = 1$, $C = 0$ and the last dot product cross term in (A14) vanishes, and with it all dependence on $\chi' - \chi$. In that case only $m = 0$ and even n and n' appear in the expansion, and hence the only nonvanishing matrix elements of the density matrix are between uncharged states with $n_+ = n_-$. Conversely, if $\sigma > 1$ this is no longer the case and ρ has nonvanishing matrix elements also with charged particle states with $m \neq 0$.

We first expand the exponential of the dot product,

$$\begin{aligned} \exp\left(\frac{C}{B}\vec{s}\cdot\vec{s}'\right) &= \sum_{m=-\infty}^{\infty} i^m J_m\left(-i\frac{C}{B}ss'\right)e^{im(\chi'-\chi)} \\ &= \sum_{m=-\infty}^{\infty} i^m e^{im(\chi'-\chi)} \sum_{p=0}^{\infty}\frac{(-)^p}{p!\Gamma(p+m+1)}\left(-i\frac{C}{2B}ss'\right)^{m+2p}\,. \end{aligned} \tag{A17}$$

Multiplying this by the expansion of the exponentials of s^2,

$$\exp\left(\frac{A}{B}e^{-i\vartheta}s^2\right) = \sum_{l=0}^{\infty}\frac{1}{\Gamma(l+1)}\left(\frac{A}{B}\right)^l e^{-il\vartheta}s^{2l}\,, \tag{A18}$$

and likewise for s'^2 yields a fourfold sum over l, l', m and p. Collecting the powers of s and s' by defining new summation variables, $n \equiv 2l + m + 2p$ and $n' \equiv 2l' + m + 2p$ we observe that $l + p = \frac{n-m}{2} \geq 0$ and $l' + p = \frac{n'-m}{2} \geq 0$ so that $m \leq n$ and $m \leq n'$. Also from the presence of the Γ function in the denominator of (A17) we observe that $p + m$ must be nonnegative, which implies $\frac{n'+m}{2} \geq 0$ and $\frac{n'+m}{2} \geq 0$. Hence $m \geq -n$ and $m \geq -n'$ as well, and we can write

$$\exp\left\{\frac{A}{B}e^{-i\vartheta}\vec{s}^2 + +\frac{A}{B}e^{i\vartheta}\vec{s}'^2 + \frac{C}{B}\vec{s}\cdot\vec{s}'\right\} =$$

$$\sum_{n,n'=0}^{\infty}\sum_{m=-M}^{M\prime} e^{i\vartheta(n'-n)/2}e^{im(\chi-\chi')}s'^{n'}s^n\sum_{p=0}^{\infty}\frac{i^m(-)^p}{p!\Gamma(p+m+1)}\frac{\left(-\frac{iC}{2B}\right)^{m+2p}}{\Gamma(\frac{n-m}{2}-p+1)}\frac{\left(\frac{A}{B}\right)^{\frac{n+n'}{2}-m-2p}}{\Gamma(\frac{n'-m}{2}-p+1)}\,, \tag{A19}$$

where $M = min(n, n')$. We also note that $\frac{n \pm m}{2}$ and $\frac{n' \pm m}{2}$ are necessarily integers in this expression.

Because of the Γ functions in the denominator the final sum over p in (A19) terminates at $p = min(n, n')$. However it is convenient to retain the formal infinite range of p and make use of the relation for the Γ function,

$$\frac{1}{\Gamma(1-z)} = \Gamma(z) \frac{\sin(\pi z)}{\pi} \tag{A20}$$

for $z = p - \frac{n-m}{2}, p - \frac{n'-m}{2}, -\frac{n-m}{2}$ and $-\frac{n'-m}{2}$, temporarily continuing $\frac{n-m}{2}$ and $\frac{n'-m}{2}$ to noninteger values to avoid the appearance of divergences in the intermediate steps. In this way the sum over p is recognized as the expansion for the hypergeometric function ${}_2F_1 \equiv F$,

$$\begin{aligned}
&\sum_{p=0}^{\infty} \frac{i^m (-)^p}{p! \Gamma(p+m+1)} \frac{\left(-\frac{iC}{2B}\right)^{m+2p}}{\Gamma(\frac{n-m}{2} - p + 1)} \frac{\left(\frac{A}{B}\right)^{-m-2p}}{\Gamma(\frac{n'-m}{2} - p + 1)} \\
&= \sum_{p=0}^{\infty} \left(\frac{C}{2A}\right)^{m+2p} \frac{\Gamma(p - \frac{n-m}{2}) \Gamma(p - \frac{n'-m}{2}) \Gamma(m+1)}{\Gamma(-\frac{n-m}{2}) \Gamma(-\frac{n'-m}{2}) p! \Gamma(p+m+1)} \frac{1}{m! \left(\frac{n-m}{2}\right)! \left(\frac{n'-m}{2}\right)!} \\
&= \left(\frac{C}{2A}\right)^m \frac{1}{m! \left(\frac{n-m}{2}\right)! \left(\frac{n'-m}{2}\right)!} F\left(\frac{m-n}{2}, \frac{m-n'}{2}; m+1; \frac{C^2}{4A^2}\right) .
\end{aligned} \tag{A21}$$

and we secure

$$\begin{aligned}
&\langle s' \, \chi' | \rho | s \, \chi \rangle = \frac{2\omega \xi^2}{B} \times \\
&\sum_{n', n=0}^{\infty} {\sum_{m=-M}^{M}}' \frac{s'^{n'} \, s^n e^{im(\chi' - \chi)}}{m! \left(\frac{n-m}{2}\right)! \left(\frac{n'-m}{2}\right)!} \left(\frac{A}{B}\right)^{\frac{n+n'}{2}} \left(\frac{C}{2A}\right)^m e^{i\vartheta(n'-n)/2} F\left(\frac{m-n}{2}, \frac{m-n'}{2}; m+1; \frac{C^2}{4A^2}\right) .
\end{aligned} \tag{A22}$$

The finite sum represented by the hypergeometric function with integral indices may be expressed in terms of Jacobi polynomials $P_\nu^{(\alpha,\beta)}$ if desired, through the relation [27]

$$P_{\frac{n-m}{2}}^{(\frac{n'-n}{2}, m)} \left(\frac{1+z}{1-z}\right) = \frac{\left(\frac{n+m}{2}\right)!}{m! \left(\frac{n-m}{2}\right)!} \left(\frac{z}{1-z}\right)^{\frac{n-m}{2}} F\left(\frac{m-n}{2}, \frac{m-n'}{2}; m+1; z\right) . \tag{A23}$$

Comparing (A22) to the general form (A16) we may identify the matrix elements of the density matrix in the adiabatic particle basis to be

$$\begin{aligned}
&\langle n' \, m | \rho | n \, m \rangle = \\
&\frac{2\omega\xi^2}{B} \left(\frac{A}{B}\right)^{\frac{n+n'}{2}} \left(\frac{C}{2A}\right)^m \frac{e^{i\vartheta(n'-n)/2}}{m!} \left[\frac{\left(\frac{n+m}{2}\right)! \left(\frac{n'+m}{2}\right)!}{\left(\frac{n-m}{2}\right)! \left(\frac{n'-m}{2}\right)!}\right]^{\frac{1}{2}} F\left(\frac{m-n}{2}, \frac{m-n'}{2}; m+1; \frac{C^2}{4A^2}\right) .
\end{aligned} \tag{A24}$$

This is the desired result. It may be expressed in terms of the magnitude and phase of the Bogoliubov transformation from the Heisenberg basis to the time dependent adiabatic particle basis introduced in the text. In fact, making use of the definitions (2.18), (3.3) and (3.19) we have

$$
\begin{aligned}
2\omega\xi^2 &= \sigma(\cosh 2\gamma - \sinh 2\gamma \cos\vartheta) \qquad \text{and} \\
2\xi\eta &= -\sigma \sinh 2\gamma \sin\vartheta\,,
\end{aligned} \tag{A25}
$$

and from (A15) we obtain

$$
\begin{aligned}
A &= \omega\xi^2\sigma\sinh 2\gamma \qquad \text{and} \\
B &= 2\omega\xi^2\left[\sigma\cosh^2\gamma + \left(\frac{\sigma-1}{2}\right)^2\right],
\end{aligned} \tag{A26}
$$

so that finally,

$$
\begin{aligned}
\langle n'\, m|\rho|n\, m\rangle = {} & \frac{(\sigma\sinh\gamma\cosh\gamma)^{\frac{n+n'}{2}-m}}{\left[\sigma\cosh^2\gamma + \left(\frac{\sigma-1}{2}\right)^2\right]^{\frac{n+n'}{2}+1}}\left(\frac{\sigma^2-1}{4\sigma}\right)^m e^{i\vartheta(n'-n)/2}\times \\
& \frac{1}{m!}\left[\frac{\left(\frac{n+m}{2}\right)!\left(\frac{n'+m}{2}\right)!}{\left(\frac{n-m}{2}\right)!\left(\frac{n'-m}{2}\right)!}\right]^{\frac{1}{2}} F\left(\frac{m-n}{2}, \frac{m-n'}{2}; m+1; \frac{(\sigma^2-1)}{4\sigma^2\sinh^2 2\gamma}\right).
\end{aligned} \tag{A27}
$$

Since ρ is symmetric under charge conjugation we have

$$
\langle n',\, -m|\rho|n,\, -m\rangle = \langle n'\, m|\rho|n\, m\rangle\,, \tag{A28}
$$

which implies that the mean charge $\sum_m m\langle n'\, m|\rho|n\, m\rangle = 0$. The fact that ρ has nonvanishing matrix elements with states of nonzero m implies that the fluctuations of the charge about its mean value is nonzero in the general case of $\sigma \neq 1$. Otherwise the most important feature of the general result (A27) for the density matrix in the adiabatic particle basis for the purposes of the discussion in the text is that all the off diagonal matrix elements for $n \neq n'$ are rapidly varying functions of time because of the appearance of the phase ϑ. Since all the phase correlation information of the function $\mathcal{C}$ of (3.15) resides in these off diagonal elements, while the average adiabatic particle number $\mathcal{N}$ is sensitive only to the diagonal matrix elements of ρ, the Markov limit of quantum Vlasov equation corresponds to replacement of the density matrix by only its diagonal matrix elements in this basis.

In the pure state case $\sigma = 1$, $F = 1$, and the only nonvanishing matrix elements of ρ have $m = 0$ and $n = 2\ell, n' = 2\ell'$ both even. The general result (A27) simplies considerably in this case to

$$
\langle 2\ell'\, m=0|\rho|2\ell\, m=0\rangle\Big|_{\sigma=1} = \mathrm{sech}^2\gamma(\tanh\gamma)^{\ell+\ell'}e^{i\vartheta(\ell'-\ell)}\,, \tag{A29}
$$

which yields the result (3.20) quoted in the text. We should note that this expression differs from that used in previous work [4], since in the present derivation the distinguishability of the positive and negative charged particles was taken into account, leading to a two-dimensional harmonic oscillator problem with a $U(1)$ invariance, while the expression Eqn. (15) of the second of refs. [4] or Eqn. (5.24) and the entire Appendix of the third of refs. [4] was based on a single scalar particle species. This is appropriate for the real uncharged Φ^4 theory considered in the last of [4], whereas (A29) is the correct expression for the charged particle case.