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A GLOBAL SEMI-LAGRANGIAN SPECTRAL MODEL FOR THE REFORMULATED SHALLOW WATER EQUATIONS*

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Abstract. In this paper, we study the semi-Lagrangian spectral method for the shallow-water equations in a rotating, spherical geometry. With the reformulation of a vector calculus identity for spherical geometries, we are able to write the vorticity and divergence equations in advective form and directly apply the semi-Lagrangian, spectral method. The scalar vorticity and divergence equations are used to avoid the pole problems. Shape preserving interpolation is used for the calculation of departure point values for all fields. The results of the standard test set are presented showing accuracy, stability and regularity properties of the method for atmospheric flows.

1. **Introduction.** The shallow water equations modeling flow on a sphere are useful for the development and testing of numerical algorithms for atmospheric climate and weather models. In this paper a new formulation of the shallow water equations are derived which exhibits an advective form for the vorticity-divergence equations. This form is particularly well suited for numerical computations using a semi-Lagrangian spectral discretization [11]. In the right side, there are no first-order derivatives involved, which is good for the representation of the functions in the spectral space. A set of test problems [14], standard for the shallow water equations on a sphere, are solved and results compared with an Eulerian spectral model.

The semi-Lagrangian formulation of Williamson and Olson [15] and Bates *et al* [1] has the property that the metric terms of the advective form are treated discretely requiring a delicate spherical vector addition of terms at the departure point and arrival point. The spherical geometry associated with this combination is derived in Bates *et al* [1]. The formulation derived in this paper avoids this vector addition since the discreization is based entirely on a scalar, advective form of the momentum equations.

The scalar momentum equations are required for a spectral discretization unless the vector spherical harmonic transform is used. Swarztrauber [12] explores

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a variety of formulations and spectral discretization methods which artfully avoid problems at the poles due to unbounded terms and velocity representations with singularities. The spectral method [2, 3, 8], when combined with a Semi-Lagrangian Transport method [1, 4, 9, 11, 13] is particularly effective because it allows for long time steps avoiding the CFL restriction of Eulerian methods, while retaining accurate treatment of the spatial derivatives and a fast semi-implicit solution.

2. **Vorticity and Divergence Forms of the Shallow-Water Equations.** The shallow water equations in advective form [15] in spherical geometry are

$$\frac{d\mathbf{v}}{dt} = -f\vec{k} \times \mathbf{v} - \nabla\phi + \mathbf{S}_{\mathbf{v}}, \tag{1}$$

$$\frac{d\phi^*}{dt} = -\phi^*\nabla\cdot\mathbf{v} + S_\phi. \tag{2}$$

where the vector $\mathbf{v} = u\vec{i} + v\vec{j}$ is the horizontal vector velocity and is always orthogonal to the unit vector $\vec{k}$ in the radial direction; (It is assumed that $\bar{i}$ and $\bar{j}$ are two unit vectors in the increasing directions of longitude and latitude respectively.) the free surface geopotential is denoted by $\phi = \phi^* + \phi_s = g(h^* + h_s)$ and $\phi^* = \phi' + \bar{\phi}$, where g is the gravitational acceleration and h^* is the height of the free surface above the bottom height, h_s. The bottom surface height specifies orography as a time invariant function and $\bar{\phi}$ is the time invariant spatial mean. The Coriolis parameter $f = 2\Omega\sin\theta$ incorporates the effect of rotation at latitude θ of the sphere with angular velocity Ω. In the shallow water equations a source term $\mathbf{S}_{\mathbf{v}}$ and S_ϕ are included as a formality.

The equations require no boundary conditions but are posed with initial conditions on $\mathbf{v}$ and ϕ if we consider the global geometry. For simplicity, let the source terms $\mathbf{S}_{\mathbf{v}}$ and S_ϕ be identically zero.

2.1. **Notations and Identities.** In lon-lat coordinates (λ, θ) we recall the definition of Laplacian, denoted by ∇^2 as follows: (all functions are assumed to be sufficiently regular)

$$\nabla^2 A = \frac{1}{a^2\cos\theta}\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\cos\theta}\frac{\partial A}{\partial\lambda}\right) + \frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial A}{\partial\theta}\right)\right],$$

where A is a scalar function defined on the sphere.

In contrast to the Laplacian operator on a scalar function, there are several possible definitions of the Laplacian operator on a vector function F. Here we set

$$\nabla^2 F = \nabla^2 F_1\vec{i} + \nabla^2 F_2\vec{j}$$

and use the covariant derivative $\nabla F = \{\nabla_{\vec{i}}F, \nabla_{\vec{j}}F\}$ of the vector function F

$$\nabla_{\vec{i}}F = \left(\frac{1}{a\cos\theta}\frac{\partial F_1}{\partial\lambda} - \frac{F_2\sin\theta}{a\cos\theta}\right)\vec{i} + \left(\frac{1}{a\cos\theta}\frac{\partial F_2}{\partial\lambda} + \frac{F_1\sin\theta}{a\cos\theta}\right)\vec{j},$$

$$\nabla_{\vec{j}}F = \frac{1}{a}\frac{\partial F_1}{\partial\theta}\vec{i} + \frac{1}{a}\frac{\partial F_2}{\partial\theta}\vec{j}.$$

Then we obtain

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = u\nabla_{\vec{i}}\mathbf{v} + v\nabla_{\vec{j}}\mathbf{v}.$$

To avoid problems with the vector representation of the velocity at the pole and to maintain compatibility with the scalar spectral transform method, we introduce the vorticity $\zeta \equiv \vec{k}\cdot\nabla\times\mathbf{v}$ and the divergence $\delta \equiv \nabla\cdot\mathbf{v}$.

Using the vector identity

$$(\mathbf{v}\cdot\nabla)\mathbf{v} \equiv \nabla(\frac{\mathbf{v}\cdot\mathbf{v}}{2}) + \zeta\vec{k}\times\mathbf{v},$$

the horizontal momentum equation can be expanded to the form

$$\frac{\partial\mathbf{v}}{\partial t} + \nabla\phi' = -(\zeta+f)\vec{k}\times\mathbf{v} - \nabla(\phi_s + \frac{\mathbf{v}\cdot\mathbf{v}}{2}). \tag{3}$$

Also, by direct computation and the definition of the Laplacian operator on a vector function, we obtain

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v} + \frac{1}{a^2\cos^2\theta}\mathbf{v} - \frac{2\sin\theta}{a^2\cos^2\theta}\vec{k}\times\frac{\partial\mathbf{v}}{\partial\lambda}$$

or without the first order derivatives, we rewrite it as

$$\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \frac{1}{\cos\theta}\nabla^2(\mathbf{v}\cos\theta) - \frac{2\sin\theta}{a\cos\theta}\mathcal{F} \tag{4}$$

where

$$\mathcal{F} = \begin{pmatrix} -\zeta \\ \delta \end{pmatrix}. \tag{5}$$

2.2. **Vorticity Form.** Applying $\vec{k}\cdot\nabla\times$ to (3) gives

$$\frac{\partial\zeta}{\partial t} = -\vec{k}\cdot\nabla\times[(\zeta+f)\vec{k}\times\mathbf{v}],$$

but

$$\begin{aligned}\vec{k}\cdot\nabla\times[(\zeta+f)\vec{k}\times\mathbf{v}] &= \nabla\cdot(\zeta+f)\mathbf{v}\\ &= (\zeta+f)\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla(\zeta+f)\\ &= \mathbf{v}\cdot\nabla\zeta + (\zeta+f)\delta + \mathbf{v}\cdot\nabla f.\end{aligned}$$

Then, we obtain

$$\frac{\partial\zeta}{\partial t} = -\mathbf{v}\cdot\nabla\zeta - (\zeta+f)\delta - \mathbf{v}\cdot\nabla f$$

i.e.

$$\frac{d\zeta}{dt} = -(\zeta+f)\delta - \mathbf{v}\cdot\nabla f. \tag{6}$$

By using $\eta = \zeta + f$, we can rewrite (6) as,

$$\frac{d\eta}{dt} = -\eta\delta \tag{7}$$

2.3. **Divergence Form.** Application of divergence operator to (3) gives

$$\frac{\partial\delta}{\partial t} + \nabla^2\phi' = -\nabla\cdot[(\zeta+f)\vec{k}\times\mathbf{v}] - \nabla^2(\phi_s + \frac{\mathbf{v}\cdot\mathbf{v}}{2}).$$

By using the vector identities, we obtain

$$\begin{aligned}\nabla\cdot[(\zeta+f)\vec{k}\times\mathbf{v}] &= -\vec{k}\cdot\nabla\times(\zeta+f)\mathbf{v}\\ &= -(\zeta+f)\vec{k}\cdot\nabla\times\mathbf{v} + \vec{k}\cdot\mathbf{v}\times\nabla(\zeta+f)\\ &= -(\zeta+f)\zeta + \vec{k}\cdot\mathbf{v}\times\nabla f + \vec{k}\cdot\mathbf{v}\times\nabla\zeta.\end{aligned}$$

Notice that

$$\nabla\zeta = \nabla(\vec{k}\cdot\nabla\times\mathbf{v}) = \vec{k}\times[\nabla\times(\nabla\times\mathbf{v})].$$

Thanks to (4), we obtain

$$\nabla\zeta = \vec{k}\times(\nabla\delta - \tilde{\mathcal{F}})$$

where

$$\tilde{\mathcal{F}} = \frac{1}{\cos\theta}\nabla^2(\mathbf{v}\cos\theta) + \frac{2\sin\theta}{a\cos\theta}\mathcal{F}, \tag{8}$$

and $\mathcal{F}$ is defined in (5). Then,

$$\begin{aligned}\vec{k}\cdot\mathbf{v}\times\nabla\zeta &= \vec{k}\cdot\mathbf{v}\times[\vec{k}\times(\nabla\delta-\tilde{\mathcal{F}})] \\ &= \vec{k}\cdot[\mathbf{v}\cdot(\nabla\delta-\tilde{\mathcal{F}})]\vec{k} - \vec{k}\cdot(\mathbf{v}\cdot\vec{k})(\nabla\delta-\tilde{\mathcal{F}}) \\ &= \mathbf{v}\cdot(\nabla\delta-\tilde{\mathcal{F}}).\end{aligned}$$

Combining terms, we obtain,

$$\frac{\partial\delta}{\partial t} + \nabla^2\phi' = (\zeta+f)\zeta - \vec{k}\cdot\mathbf{v}\times\nabla f - \mathbf{v}\cdot(\nabla\delta-\tilde{\mathcal{F}}) - \nabla^2(\phi_s + \frac{\mathbf{v}\cdot\mathbf{v}}{2})$$

i.e. with $\eta = \zeta + f$

$$\frac{d\delta}{dt} + \nabla^2\phi' = (\zeta+f)\zeta - \vec{k}\cdot\mathbf{v}\times\nabla f + \mathbf{v}\cdot\tilde{\mathcal{F}} - \nabla^2(\phi_s + \frac{\mathbf{v}\cdot\mathbf{v}}{2}). \tag{9}$$

Note that the equations (7) and (9) contain the substantial derivatives of two scalar functions. This form is reminiscent of the forms derived by Truesdell and reported in Serrin [10]. We can apply the semi-Lagrangian method directly to the vorticity and divergence equations to avoid the complications of dealing with operations of the vector functions on the sphere.

3. **Semi-Implicit, Semi-Lagrangian Formulation.** A three time level semi-Lagrangian method for the shallow water equations was described in Ritchie [9]. For meteorological models the form of the method is based on a division of terms in the equations between those involved in the fast moving gravity waves, and the slower moving Rossby waves. Let G denote the fast moving wave terms and R the slower wave terms. Then an advected scalar field F is represented by the equation

$$\frac{dF}{dt} + G = R. \tag{10}$$

Let subscripts $(\ )_A$, $(\ )_M$ and $(\ )_D$ represent quantities evaluated at the arrival point, midpoint and departure point, respectively. Let superscripts $\tau-1$, τ and $\tau+1$ represent the time levels at which quantities are evaluated.

Integrating the equation (10) along the trajectory between the departure point and the arrival point with the application of the trapezoidal rule for G and a mid-point rule for R with the spatial average at the midpoint, we obtain the semi-implicit formulation:

$$\frac{F_A^{\tau+1} - F_D^{\tau-1}}{2\delta t} + \frac{G_A^{\tau+1} + G_D^{\tau-1}}{2} = \frac{R_A^\tau + R_D^\tau}{2}. \tag{11}$$

The assignment of terms of the shallow water equations to G and R results in different numerical algorithms. If we take $G \equiv 0$, then the algorithm is *explicit* in time. The explicit method requires small timesteps to stabilize the gravity wave components of the solution. A *semi-implicit* method results by assigning selected terms to G, terms that give rise to the gravity waves. In the momentum equation (3) the gravity wave component is identified with the $\nabla\phi'$ term and the Coriolis term is identified with the R. In this paper we take

$$F = \begin{pmatrix} \eta \\ \delta \\ \phi' \end{pmatrix}, \qquad G = \begin{pmatrix} 0 \\ \nabla^2\phi' \\ \bar{\phi}\delta \end{pmatrix}, \qquad R = \begin{pmatrix} -\eta\delta \\ \Psi \\ -\phi'\delta \end{pmatrix}, \tag{12}$$

where

$$\Psi = \eta(\eta - f) - \vec{k} \cdot \mathbf{v} \times \nabla f + \mathbf{v} \cdot \tilde{\mathcal{F}} - \nabla^2(\phi_s + \frac{\mathbf{v} \cdot \mathbf{v}}{2})$$

and $\phi' + \bar{\phi} + \phi_s$ is the free surface geopotential and $\bar{\phi}$ is a constant reference value; $\tilde{\mathcal{F}}$ is defined in (8).

For simplicity, the prime in ϕ' will be dropped from the equations.

Substitute (12) into (11), we obtain the semi-Lagrangian scheme for the shallow-water model in vorticity-divergence form

$$\begin{cases} \eta_A^{\tau+1} & = N_\eta \\ \delta_A^{\tau+1} + \delta t[\nabla^2\phi]_A^{\tau+1} & = N_\delta \\ \phi_A^{\tau+1} + \delta t[\bar{\phi}\delta]_A^{\tau+1} & = N_\phi, \end{cases} \tag{13}$$

where $N = (N_\eta, N_\delta, N_\phi)^T$ and $N = N_A + N_D$, the arrival and departure components of N, is defined by

$$N_A = \begin{pmatrix} \delta t[-\eta\delta]_A^\tau \\ \delta t[\Psi]_A^\tau \\ \delta t[-\phi\delta]_A^\tau \end{pmatrix}, \qquad N_D = \begin{pmatrix} \eta_D^{\tau-1} + \delta t[-\eta\delta]_D^\tau \\ \delta_D^{\tau-1} + \delta t[\Psi]_D^\tau - \delta t[\nabla^2\phi]_D^{\tau-1} \\ \phi_D^{\tau-1} + \delta t[-\phi\delta]_D^\tau - \delta t[\bar{\phi}\delta]_D^{\tau-1} \end{pmatrix}. \tag{14}$$

There are no first order derivatives involving unknowns, and the only second-order operator is the Laplacian. When the spectral transformation is used to discretize the equations, the properties of Laplacian operator will simplify the computation of the solution.

4. **Spectral Semi-Implicit Formulation.**

The combination of the semi-Lagrangian method with the spectral transform method has proven to be a powerful technique, providing fast solution of the semi-implicit equations, high spatial accuracy, and effectively double the resolution of a pure spectral transform method with the same cost, for details the reader is referred to Hortal's paper [6]. Williamson and Rasch [13] observed that the semi-Lagrangian treatment of the advection eliminates the nonlinear aliasing of terms in the spectral representation. This allows use of more terms of the spectral expansion but using the same physical grid and effectively doubling the resolution at little extra cost.

The spectral transform, or more correctly, the spherical harmonic transform, is based on the representation of scalar fields as a linear combination of spherical harmonics as

$$\xi(\lambda, \mu) = \sum_{m=-M}^{M} \sum_{n=|m|}^{N(m)} \xi_n^m P_n^m(\mu) e^{\mathbf{i}m\lambda}, \tag{15}$$

and

$$\xi_n^m = \int_{-1}^{1} \frac{1}{2\pi} \int_0^{2\pi} \xi(\lambda, \mu) e^{-\mathbf{i}m\lambda} d\lambda P_n^m(\mu) d\mu, \tag{16}$$

where P_n^m are the normalized associated Legendre functions. Once spectral coefficients are known, the transformation from spectral to physical space is accomplished using this formula and usually referred to as harmonic synthesis. The transform from physical space to spectral space (harmonic analysis) uses a discrete version of the continuous transform,

$$\xi^m(\mu) = \frac{1}{2\pi}\int_0^{2\pi} \xi(\lambda,\mu)e^{-\mathbf{i}m\lambda}d\lambda = \frac{1}{I}\sum_{i=1}^{I}\xi(\lambda_i,\mu)e^{-\mathbf{i}m\lambda_i}, \tag{17}$$

and

$$\xi_n^m = \int_{-1}^{+1}\xi^m(\mu)P_n^m(\mu)d\mu = \sum_{j=1}^{J}\xi^m(\mu_j)P_n^m(\mu_j)w_j, \tag{18}$$

where $\lambda_i = 2\pi i/I$; μ_j denote the Gaussian latitudes; I is the number of gridpoints in the east-west direction; J the number of Gaussian latitudes from pole to pole; w_j the Gaussian weight at latitude μ_j; The Gaussian latitudes μ_j are determinated from the roots of the Legendre polynomial $P_J(\mu)$.

First the Laplacian (∇^2) operator is easily computed in spectral space using the relationship,

$$\nabla^2 P_n^m(\mu)e^{im\lambda} = \frac{-n(n+1)}{a^2}P_n^m(\mu)e^{im\lambda}.$$

Then applying the transformation given in (16) to each term of equations (13), we will obtain the spectral formulation on the spectral space. Only two types of terms are involved: the terms with constant coefficients and the terms with a Laplacian operator. For the simplicity we drop the index $\tau+1$ and the subscript A to obtain the spectral form of the governing equations,

$$\eta_n^m = \{N_\eta\}_n^m, \tag{19}$$

$$\delta_n^m + \delta t C(n)\phi_n^m = \{N_\delta\}_n^m, \quad C(n) = -\frac{n(n+1)}{a^2}, \tag{20}$$

$$\phi_n^m + \delta t\bar{\phi}\delta_n^m = \{N_\phi\}_n^m, \tag{21}$$

where a is the radius of the earth; η_n^m is directly calculated by (19) and the last two equations, (20) and (21), can be solved together for the advanced time level values in spectral space for δ_n^m and ϕ_n^m. For each mode (m,n) a 2×2 system of equations needs to be solved,

$$\begin{bmatrix} 1 & \delta t C(n) \\ \delta t\bar{\phi} & 1 \end{bmatrix}\begin{bmatrix} \delta_n^m \\ \phi_n^m \end{bmatrix} = \begin{bmatrix} \{N_\delta\}_n^m \\ \{N_\phi\}_n^m \end{bmatrix}. \tag{22}$$

This is the same system found in the semi-implicit spectral method of Hack and Jakob [5]. The solution can be expressed by an application of Cramer's rule. With η_n^m, δ_n^m and ϕ_n^m at the arrival points at the new time level, the synthesis (inverse spherical harmonic transform) is used to obtain the values of ϕ and $\mathbf{v}$ in physical space. Notice that the ζ_n^m can be directly calculated from η_n^m at the new time level with the relationship $\eta = \zeta + f$. The components of $\mathbf{v}$ are obtained through the diagnostic relationship in terms of η_n^m and δ_n^m,

$$U(\lambda_i,\mu_j) = -\sum_{m=-M}^{M}\sum_{n=|m|}^{N(m)}\frac{a}{n(n+1)}[im\delta_n^m P_n^m(\mu_j) - \zeta_n^m H_n^m(\mu_j)]e^{im\lambda_i} \tag{23}$$

and

$$V(\lambda_i,\mu_j) = -\sum_{m=-M}^{M}\sum_{n=|m|}^{N(m)}\frac{a}{n(n+1)}[im\zeta_n^m P_n^m(\mu_j) + \delta_n^m H_n^m(\mu_j)]e^{im\lambda_i} \tag{24}$$

where

$$H_n^m(\mu) \equiv (1-\mu^2)\frac{dP_n^m(\mu)}{d\mu}. \tag{25}$$

The Laplacian operator is given by

$$\Delta\phi = \sum_{m=-M}^{M}\sum_{n=|m|}^{N(m)} \frac{-n(n+1)}{a^2}\phi_n^m P_n^m(\mu)e^{im\lambda}. \tag{26}$$

Finally, the velocity field is calculated on physical grid points with

$$u(\lambda_i,\theta_j,t) = U(\lambda_i,\mu_j)/\cos\theta_j$$
$$v(\lambda_i,\theta_j,t) = V(\lambda_i,\mu_j)/\cos\theta_j$$

and η, δ and ϕ are recovered on physical grid point with formula (15).

5. **Numerical Experiments and Results.** In this section, we present the numerical results of two standard test cases, which are described in Williamson *et al* [14]. For more details of other test cases, the reader is referred to Drake [4]. Those test cases can be used to characterize the numerical properties of a method and their appropriateness for numerical weather prediction and climate modeling. They provide a means of addressing accuracy as well as quality of solution in terms of conserved quantities. All results are based on the standard T42 mesh.

The discussion will focus on the properties of the solution and the errors in comparison with the T42 Eulerian spectral model. There are four sources of error:

- the truncation error of the time stepping method,
- the error associated with approximation of spatial derivatives,
- the interpolation error of the semi-Lagrangian scheme, and
- the error associated with computing trajectories in the semi-Lagrangian scheme.

It will not always be possible to determine which is the dominant error. Information is gained by performing a number of experiments not reported here. For example, the test can be run with several values for the timestep Δt to determine if the truncation error is dominant.

5.1. **Steady, Zonal Flow Test.** In this case, we let $\alpha = \pi - 0.05$. The global zonal geostrophic flow case is a trivial problem for the spectral transform method because the steady solution is exactly representable with a low order spherical harmonic. The initial height field is shown in Figure 1a. The final height field after one rotation (5 days or 360 time steps of 1200 seconds) is shown in Figure 1b. Comparing with the initial height field, it is slightly diffused. Figure 1c shows the difference between the analytic solution and the final height field after one rotation (5 days). The difference plot in Figure 1c illustrates the overall structures of the final height field. Errors in the numerical solution from the semi-Lagrangian model reflect the accumulation of rounding errors associated with the SLT cubic interpolation. The l_1, l_2, and l_∞ error measures are shown in Figure 1d. These errors show the same behavior as in Jakob *et al* [7], i.e. errors are small and smooth, but monotonically increasing in time.

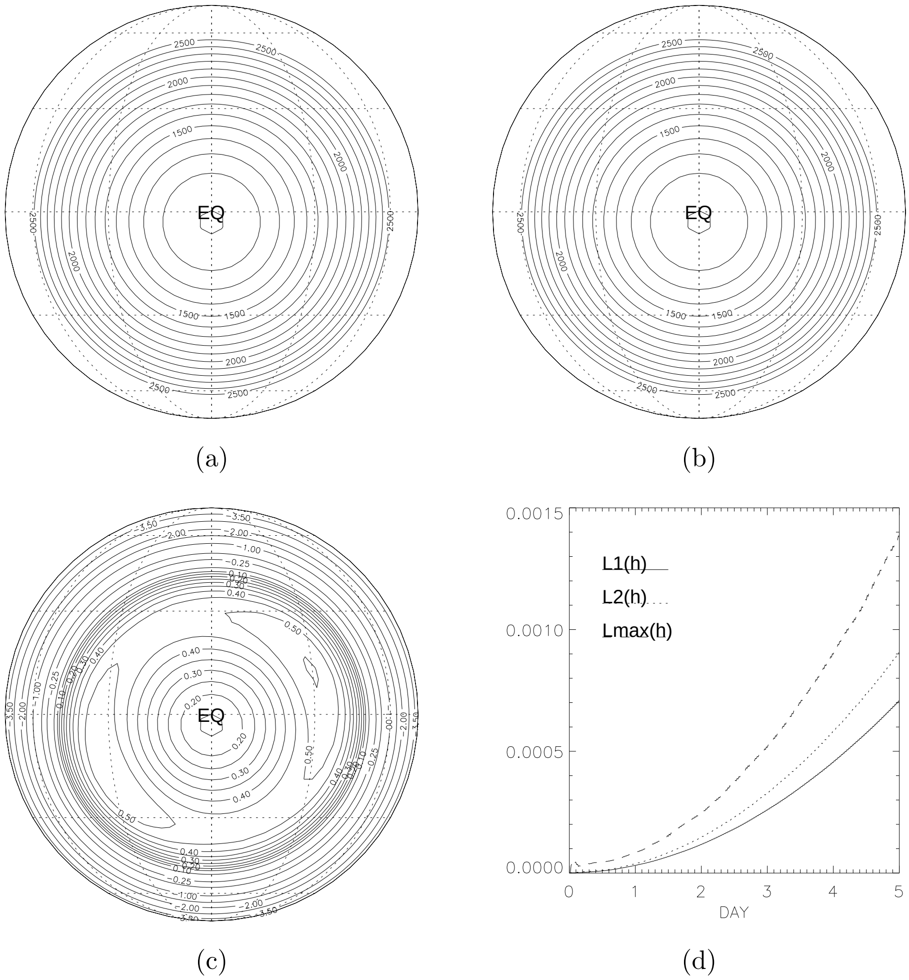
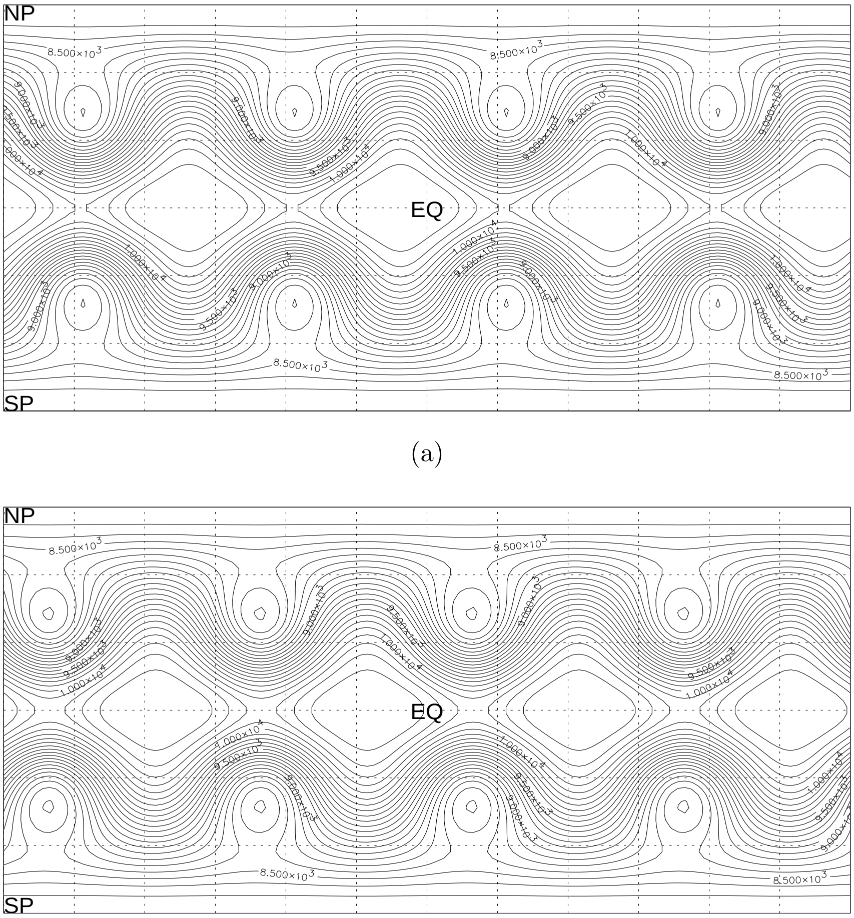


FIGURE 1. **Case 2** Contour plots of the height field for $\alpha = \pi/2 - 0.05$ and $\Delta t = 1200$ seconds. (a) Initial field; (b) after one rotation (5 days); (c) Height field error; (d) l_1 (solid), l_2 (short dash), and l_∞ (long dash) errors. Contour interval is 100m for the field and 0.05m for the error.

5.2. **Rossby-Haurwitz Wave.** In this test case, the wavenumber of Rossby-Haurwitz waves is four. Rossby-Haurwitz waves are analytic solutions to the barotropic vorticity equation. The relation to solutions of the nonlinear shallow water equations is not quite clear. For a stable wave pattern like wavenumber 4, the test can indicate if there are other modes of systematic error that a scheme introduces. In Figure 2a-d, the basic wave pattern is well maintained through the entire simulation. The spectral Eulerian solution requires introduction of artificial dissipation (for example a biharmonic term) to control the tail of the energy spectrum. The semi-Lagrangian



(a)

(b)

FIGURE 2. **Case 6** Contour plots of the height field for $\Delta t = 1200$ seconds at (a) day 7 and (b) day 14.

method apparently has no need for this as enough dissipation is introduced by the method.

The change in global quantities of mass, enstrophy, and total energy from the initial conditions as a function of time are showed in Figure 3. For this case, the total mass almost has no change, but the enstrophy and the total energy is decreasing (eventually), with a total change of -1.2% by 15 days.

6. **Conclusions.** The semi-Lagrangian transport (SLT) algorithm is an important component of global atmospheric circulation models, and will continue to be attractive for the next generation of models because of its accuracy and ability to take long timesteps. A new advective formulation of the shallow water equations in a spherical geometry allows a natural application of the SLT algorithm. The method presented here combines SLT with a spectral, semi-implicit algorithm. The results

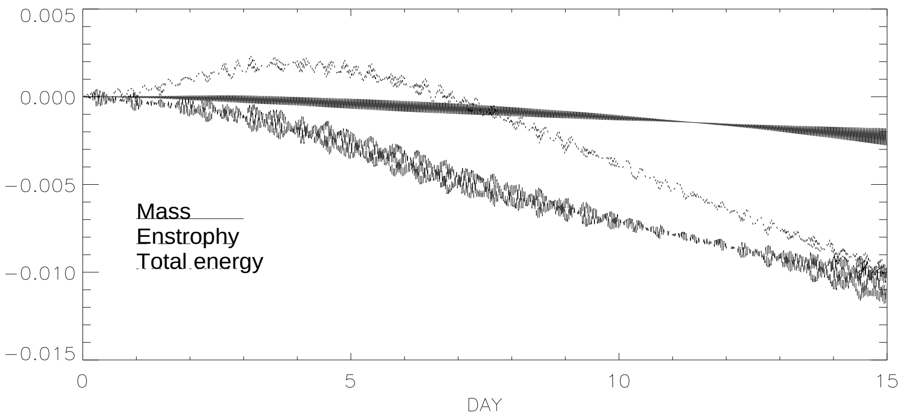


FIGURE 3. **Case 6** (cont.) change in global mass (solid), total energy (short dash), and enstrophy (long dash).

compare favorably in terms of accuracy, stability and smoothness of solution with a spectral Eulerian shallow water equation model. Future work will extend this formulation to the full baroclinic equations and explore solutions using two time-level semi-Lagrangian methods as well as other gridding systems.

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