University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 8

1. (Taylor and Wheeler problem 27)
The clock paradox, version 1.
On their twenty-first birthday, Peter leaves his twin Paul behind on the earth and goes off in the x direction for seven years of *his* time at 24/25 the speed of light, then reverses direction and in another seven years of *his* time returns at the same speed. [In this most elementary version of the problem, we assume that the necessary periods of acceleration are infinitesimal in duration, requiring Peter's acceleration to be infinite. Nonetheless, our plucky twin remains uninjured.]
(a.)
Make a spacetime diagram (ct vs. x) showing Peter's motion. Indicate on it the x and ct coordinates of the turn-around point and the point of reunion. For simplicity idealize the earth as an inertial frame, adopt this inertial frame in the construction of the diagram, and take the origin to be the event of departure.
(b.)
How old is Paul at the moment of reunion?

Solution:
On a spacetime (ct vs. x) diagram in Paul's (unprimed) frame, Peter begins at (0,0) and proceeds with slope $\beta^{-1} = \frac{25}{24}$ for a time interval

$$\begin{aligned} c\Delta t &= \gamma c\Delta t' + \gamma\beta(\Delta x' = 0) \\ &= \gamma c\Delta t' \\ &= \sqrt{\frac{1}{1-\left(\frac{24}{25}\right)^2}}\, c\Delta t' \\ &= \frac{25}{7}\, c\Delta t' \\ &= 25 \text{ lightyr} . \end{aligned}$$

At Peter's point of maximum excursion, ($ct = 25$, $x = 24$) lightyr. Peter then returns with slope $\beta^{-1} = -\frac{25}{24}$, reaching $x = 0$ at $ct = 50$ lightyr where he reunites with Paul. Peter has aged only 14 years, while Paul has aged 50 years (and has reached the age of 71).

2. Prove that

$$\tanh(\eta_1 + \eta_2) = \frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2} .$$

Using this relation, deduce Einstein's law for the addition of velocities.

Solution:

$$\begin{aligned} &\tanh\eta = \\ &= \frac{\exp(2\eta) - 1}{\exp(2\eta) + 1} \\ &\tanh\eta_1 + \tanh\eta_2 = \\ &= \frac{\exp(2\eta_1) - 1}{\exp(2\eta_1) + 1} + \frac{\exp(2\eta_2) - 1}{\exp(2\eta_2) + 1} \\ &= \frac{2\exp(2\eta_1 + 2\eta_2) - 2}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &\tanh\eta_1 \tanh\eta_2 = \\ &= \frac{\exp(2\eta_1 + 2\eta_2) + 1 - \exp(2\eta_1) - \exp(2\eta_2)}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &1 + \tanh\eta_1 \tanh\eta_2 = \\ &= \frac{2\exp(2\eta_1 + 2\eta_2) + 2}{\exp(2\eta_1 + 2\eta_2) + 1 + \exp(2\eta_1) + \exp(2\eta_2)} \\ &\frac{\tanh\eta_1 + \tanh\eta_2}{1 + \tanh\eta_1 \tanh\eta_2} = \frac{2\exp(2\eta_1 + 2\eta_2) - 2}{2\exp(2\eta_1 + 2\eta_2) + 2} \\ &= \tanh(\eta_1 + \eta_2) . \end{aligned}$$

Suppose that all velocities are in the x direction. Take the velocity of frame $\mathcal{S}_1$ with respect to frame $\mathcal{S}$ to be $\beta_1 c$; of frame $\mathcal{S}_2$ with respect to frame $\mathcal{S}_1$ to be $\beta_2 c$; and of frame $\mathcal{S}_2$ with respect to frame $\mathcal{S}$ to be $\beta_3 c$. $\beta_{1,2,3}$ correspond to boost parameters (or rapidities) $\eta_{1,2,3}$ according to the relation

$$\beta_{1,2,3} = \tanh\eta_{1,2,3} .$$

The boost parameters have the unique property that they are additive, i.e. a boost of η_1 followed by a boost of η_2 is equivalent to a boost of $\eta_1+\eta_2$. So, with the above definitions,

$$\begin{aligned}
\eta_3 &= \eta_1+\eta_2\\
\beta_3 &= \tanh\eta_3\\
&= \tanh(\eta_1+\eta_2)\\
&= \frac{\tanh\eta_1+\tanh\eta_2}{1+\tanh\eta_1\tanh\eta_2}\\
&= \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\ .
\end{aligned}$$

This is Einstein's law for the addition of velocities.

3. The thermonuclear "deuterium-tritium" reactions are:

$$\begin{aligned}
{}^2\mathrm{H}+{}^2\mathrm{H} &\to {}^3\mathrm{He}+n\\
{}^2\mathrm{H}+{}^2\mathrm{H} &\to {}^3\mathrm{H}+p\\
{}^2\mathrm{H}+{}^3\mathrm{H} &\to {}^4\mathrm{He}+n\ .
\end{aligned}$$

These sum to

$$5({}^2\mathrm{H}) \to {}^3\mathrm{He}+{}^4\mathrm{He}+p+2n.$$

Using the following masses in AMU,

(proton) p	1.007825
(neutron) n	1.008665
(deuteron) ^{2}H	2.014102
(helium 3) ^{3}He	3.016030
(triton) ^{3}H	3.016050
(alpha particle) ^{4}He	4.002603 ,

calculate (to 5%) the kinetic energy released when one liter of heavy water $({}^2\mathrm{H})_2\mathrm{O}$ undergoes deuterium-tritium fusion in an H-bomb. Express your answer in terms of tons of TNT (1 ton of TNT = 4.2×10^9 J of explosive energy).

Solution:
The energy released in the summed reaction corresponds to a mass deficit equal to Δm = 5(2.014102) − 3.016030 − 4.002603 − 1.007825 − 2(1.008665) = 0.026722 amu. Heavy water has a density about $\frac{20}{18}$ times that of ordinary water, due to the extra two neutrons. Thus one liter of heavy water weighs 1.11 kg and corresponds to $\frac{1.11}{20}\times10^3 = 55.5$ moles. It contains $55.5\times N_{\mathrm{Avo}} = 55.5\times6.023\times10^{23} = 3.35\times10^{25}$ molecules of heavy water. Each of the summed reactions requires five deuterium nuclei, or 2.5 molecules of heavy water, so $3.35\times10^{25}/2.5 = 1.34\times10^{25}$ summed reactions take place. The mass energy in one amu is equivalent to $mc^2 = 0.9315\times10^9$ eV, or, with 1 eV = 1.6×10^{-19} J, 1.49×10^{-10} J. Therefore the energy released is $0.026722\times1.34\times10^{25}\times1.49\times10^{-10} = 5.33\times10^{13}$ J. This is equivalent to the energy released by the explosion of 12.7 kilotons of TNT.

A corollary is that about 80 liters – one Jeep gasoline tank – of $({}^2\mathrm{H})_2\mathrm{O}$ are needed to make a one megaton H-bomb. This sets a lower limit on the degree to which an H-bomb can be miniaturized, irrespective of any espionage.

4. The universe is filled with old cold photons that are remnants of the big bang. Typically their energy is $\approx 6.6\times10^{-4}$ eV.

A cosmonaut who is accelerated at 1 g for 10 years in her own rest frame attains a boost (= arctanh β) of 10.34. As seen by her, what is the typical energy of these photons?

Solution:
We know that the energy-momentum four-vector $(E/c,\mathbf{p})$ satisfies the same Lorentz transformation equations as the spacetime four-vector $(ct,\mathbf{r})$:

$$\begin{aligned}
E'/c &= \gamma E/c-\gamma\beta p_x\\
p'_x &= -\gamma\beta E/c+\gamma p_x\\
p'_y &= p_y\\
p'_z &= p_z\ ,
\end{aligned}$$

where the cosmonaut (in the primed frame) is assumed to be travelling with respect to the big bang's (unprimed) frame with a velocity $\beta c = \tanh 10.34$ in the x direction. [With respect to this large velocity, here we are neglecting the much smaller speed of 370 km/sec with which the solar system moves with respect to the big bang radiation; this was first measured by a Berkeley group, including Profs. Smoot and Muller, in the 1970s.]

On average, $\langle p_x \rangle = 0$ for the big bang photons in the big bang's frame; thus

$$\begin{aligned}
\langle E'/c \rangle &= \gamma \langle E/c \rangle \\
&= \sqrt{\frac{1}{1-\beta^2}}\, \langle E/c \rangle \\
&= \sqrt{\frac{1}{1-\tanh^2 \eta}}\, \langle E/c \rangle \\
&= \langle E/c \rangle \cosh \eta \\
&= 6.6 \times 10^{-4} \text{ eV/c } \times \cosh 10.34 \\
E' &= 10.2 \text{ eV} .
\end{aligned}$$

Therefore, while the cosmic background radiation is in the far infrared as seen in the solar system, on average it is boosted to the ultraviolet as seen in the frame of the cosmonaut.

5. Prove that an isolated photon (zero mass) cannot split into two photons which do not both continue in the original direction.

Solution:
Assume that a photon decays into two other photons a and b. The photons have energy-momentum four-vectors denoted by $(E/c, \mathbf{p})$, $(E_a/c, \mathbf{p}_a)$, and $(E_b/c, \mathbf{p}_b)$, repectively. Both energy and momentum must be conserved in the decay. We can express this requirement in a single four-component equation:

$$(E/c, \mathbf{p}) = (E_a/c, \mathbf{p}_a) + (E_b/c, \mathbf{p})_b) .$$

To save writing we will use the shorthand notation $p \equiv (E/c, \mathbf{p})$; similarly for p_a and p_b. Rewriting the above equation in this shorthand notation, and taking the inner product of each side with itself,

$$\begin{aligned}
p &= p_a + p_b \\
p \cdot p &= (p_a + p_b) \cdot (p_a + p_b) \\
&= p_a \cdot p_a + p_b \cdot p_b + 2 p_a \cdot p_b .
\end{aligned}$$

In the above, the symbol "$\cdot$" refers to the *four-vector* inner product, i.e. $p \cdot p \equiv E^2/c^2 - \mathbf{p} \cdot \mathbf{p}$. Since the inner product of any two four-vectors has the same value in any Lorentz frame, it is easiest to evaluate $p \cdot p$ in the rest frame of the particle; there one finds that

$$p \cdot p = E^2/c^2 - |\mathbf{p}|^2 = m^2 c^2 ,$$

where m is the particle's rest mass. This is the *fundamental equation* for solving relativistic kinematics problems. The fundamental equation tells us that $p \cdot p = 0$ and $|\mathbf{p}| = E/c$ for any massless particle like the photon. Returning to the problem,

$$\begin{aligned}
p \cdot p &= p_a \cdot p_a + p_b \cdot p_b + 2 p_a \cdot p_b \\
0 &= 0 + 0 + 2 E_a E_b / c^2 - 2 \mathbf{p}_a \cdot \mathbf{p}_b \\
&= 2 E_a E_b / c^2 (1 - \cos \theta_{ab}) \\
1 &= \cos \theta_{ab} ,
\end{aligned}$$

where θ_{ab} is the opening angle between the two photons. The last equation tells us that photons a and b must be travelling in the same direction (they are "collinear"); by conservation of momentum, that must be the direction of the initial photon.

[For the case in which photons a and b do travel in the direction of the initial photon, which is allowed by the above kinematic calculation, the decay nevertheless is prevented by conservation of angular momentum. Angular momentum nonconservation in the collinear decay arises from the photon's internal angular momentum ("spin").]

[Also, we note that the (electrically neutral) photon couples to electric charge, so, to lowest order, no electromagnetic interaction occurs when three photons meet at a common vertex. This is not the case for the strong force carriers (gluons), which also are massless; gluons both carry and couple to a different kind of charge called "color".]

[How could we express the above solution in words? "If the two decay photons are not collinear, their combined invariant mass must be greater than zero. Since the initial state has invariant mass equal to zero, this violates energy-momentum conservation."]

6. The now retired Bevatron at Berkeley Lab is famous for having produced the first observed antiprotons (you may have glimpsed white-maned Nobelist Owen Chamberlain, one of the first observers, being helped to his seat at Physics Department colloquia). An economical reaction for producing antiprotons is

$$p + p \to p + p + p + \bar{p} \ ,$$

where the first proton is part of a beam, the second is at rest in a target, and $\bar{p}$ is an antiproton. Because of the CPT theorem, both p and $\bar{p}$ must have the same mass ($= 0.94 \times 10^9$ eV).

At threshold, all four final state particles have essentially zero velocity *with respect to each other*. What is the beam energy in that case? (The actual Bevatron beam energy was 6×10^9 eV).

Solution:
We shall use the notation of the previous solution. Denote by p_a and p_b the four-momenta of the incident and target protons, each of which has mass m. At threshold, we are told that the four final-state particles are at rest with respect to each other. Therefore, for kinematic purposes, they are equivalent to a single particle of mass $4m$. Denote by p_c the four-momentum of this four-particle state. Energy-momentum conservation demands

$$\begin{aligned}
p_a + p_b &= p_c \\
(p_a + p_b) \cdot (p_a + p_b) &= p_c \cdot p_c \\
p_a \cdot p_a + p_b \cdot p_b + 2p_a \cdot p_b &= p_c \cdot p_c \\
m^2c^2 + m^2c^2 + 2p_a \cdot p_b &= (4m)^2c^2 \\
7m^2c^2 &= p_a \cdot p_b \\
&= (E_a, \mathbf{p}_a) \cdot (m, \mathbf{0}) \\
&= E_a m \\
E_a &= 7mc^2 \\
&= 7 \times 0.94 \times 10^9 \text{ eV} \\
&= 6.58 \times 10^9 \text{ eV} \ .
\end{aligned}$$

This is $\sim 10\%$ more proton beam energy than the Bevatron ($= 6 \times 10^9$ GeV) was able to supply!

How then were Chamberlain, Segrè, Wiegand, and Ypsilantis able to discover the antiproton at the Berkeley Bevatron in 1956? They took advantage of the fact that protons confined inside the atomic nucleus have a significant (~ 200 MeV/c) rms momentum as a result of Heisenberg's uncertainty principle. This is called "Fermi momentum". When the target proton's Fermi momentum is directed *against* the incoming beam proton, the energy available for the interaction can be augmented up to $\approx 20\%$.

7. Using Eqs. 1.33 in the lecture notes, prove that $E^2 - B^2$, where E (B) is the magnitude of the electric (magnetic) field, is a Lorentz invariant.

Solution:
The equations for Lorentz transformation of the electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ may be derived from three facts:

$$\begin{aligned}
(\phi, \mathbf{A}) &= \text{a four vector} \\
\mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\
\mathbf{B} &= \nabla \times \mathbf{A} \ ,
\end{aligned}$$

where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. The result of the derivation is Eq. 1.33 in the distributed relativity notes:

$$\begin{aligned}
\mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \boldsymbol{\beta} \times \mathbf{E}_\perp) \\
E'_\parallel &= E_\parallel \\
B'_\parallel &= B_\parallel \ ,
\end{aligned}$$

where $\boldsymbol{\beta}c$ is the velocity of frame $\mathcal{S}'$ relative to $\mathcal{S}$, the subscript $_\perp$ refers to the component perpendicular to $\boldsymbol{\beta}$, and the subscript $_\parallel$ refers to the component parallel to $\boldsymbol{\beta}$. Note that, in the first two equations, the subscript $_\perp$ may be dropped from the last term, since taking the cross product with $\boldsymbol{\beta}$ automatically picks out the perpendicular part. Using the first two equations,

$$\begin{aligned}
\gamma^{-2}(E'_\perp)^2 &= E_\perp^2 + \beta^2 B_\perp^2 + 2\mathbf{E}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\gamma^{-2}(B'_\perp)^2 &= B_\perp^2 + \beta^2 E_\perp^2 - 2\mathbf{B}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{E}_\perp) \ .
\end{aligned}$$

We rearrange the last line using the invariance under cyclic permutation of the triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \ ,$$

a relation which may be found on the inside cover of Griffiths (distributed in class), or, more physically, may be understood from the fact that the triple product describes the (invariant) volume of a parallelopiped with sides **a**, **b**, and **c**. Cyclically permuting the triple product in the last line,

$$\begin{aligned}
\gamma^{-2}(B'_\perp)^2 &= B_\perp^2 + \beta^2 E_\perp^2 - 2\mathbf{E}_\perp \cdot (\mathbf{B}_\perp \times \boldsymbol{\beta}) \\
&= B_\perp^2 + \beta^2 E_\perp^2 + 2\mathbf{E}_\perp \cdot (\boldsymbol{\beta} \times \mathbf{B}_\perp) \\
\frac{E'^2_\perp - B'^2_\perp}{\gamma^2} &= E_\perp^2 + \beta^2 B_\perp^2 - B_\perp^2 - \beta^2 E_\perp^2 \\
&= (1-\beta^2)(E_\perp^2 - B_\perp^2) \\
E'^2_\perp - B'^2_\perp &= E_\perp^2 - B_\perp^2 \ .
\end{aligned}$$

This demonstrates that $E_\perp^2 - B_\perp^2$ is conserved; $E_\parallel^2 - B_\parallel^2$ is conserved automatically since $\mathbf{E}_\parallel$ and $\mathbf{B}_\parallel$ are invariant under the transformation. Finally, $E^2 = E_\perp^2 + E_\parallel^2$, etc., because the dot product in the cross term vanishes.

8. You shine a one-watt beam of photons on a crow, who absorbs them. Calculate the force (in N) on the crow.

Solution:
Suppose that a photon in the flashlight beam has an energy E. Then it must have momentum $p = E/c$. The beam power ($= 1$ W) is $P = N\langle E\rangle$, where N is the number of photons emitted per second and $\langle E\rangle$ is their average energy. If the photons are totally absorbed by the crow, the momentum absorbed by the crow per second is

$$F = N\langle p\rangle = N\langle E\rangle/c = P/c \ .$$

Therefore the force F on the crow is

$$F = \frac{1\ \mathrm{W}}{c} = 3.3 \times 10^{-9}\ \mathrm{N} \ .$$