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A Restricted Matching Problem on Bipartite Graphs *

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Abstract—We pose a restricted matching problem on bipartite graphs, and present a solution for it. This matching problem is motivated by an engineering problem where signal interference in interference-limited wireless networks is desired to be controlled for an increased system capacity, while some quality of service guarantees (such as delay and loss guarantees) are sought to be delivered via the system. An interesting aspect of this matching problem is that the Max-Flow Min-Cut Theorem does not seem to be applicable to find a solution, although for another problem with very similar formulation it has been applicable. Finally, the solution presented here could be viewed as a generalization of König's Minimax Theorem.

1 Introduction

In this study, we pose a restricted matching problem on bipartite graphs, and present a solution for it. This matching problem is motivated by an engineering problem where signal interference in interference-limited wireless networks is desired to be controlled for an increased system capacity, while some quality of service guarantees (such as delay and loss guarantees) are sought to be delivered via the system. The underlying engineering problem will be elaborated more clearly after we have posed the problem in section 3.

An application of the main result of this study, which is theorem 1 in section 4, is similar to that of the main result in [1]. Some applications of the result in [1] could be followed in [2] and [3].

An interesting aspect of the matching problem that we pose here is that the celebrated Max-Flow Min-Cut Theorem [4, 5] does not seem to be applicable to find a solution for this problem—this will be pointed out more clearly in section 5. This is in contrast to the fact that it has been applicable previously to a similar problem solved in [1], whose solution has a very similar form as the solution that we have here.

The similarity between the two problems, however, could be misleading. There are many examples in the literature that minor changes in some discrete mathematics problems could often turn the problem into a much harder one. An example for this, which is closely related to the problems that we study here (and in [1]) is the problem examined in [6]. That problem is what we call the *E-partition Constrained Matching Problem* on bipartite graphs, see [1] or [2] for explanations. In that problem, the edges of a bipartite graph is partitioned into some sets, and later the cardinality of a maximum matching is asked such that the number of edges included in a matching in each part of the partition is not allowed to exceed a certain integer assigned to that part. This problem is shown to be NP-Complete [6], although a solution for one of its counterparts (*V-partition Constrained Matching Problem* on bipartite graphs) is presented in [1].

Various applications of the Max-Flow Min-Cut Theorem in matching problems could be found, for example, in [7].

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We also would like to note that the theorem 1, as well as the theorem in [1], could be viewed as a generalization of König's Minimax Theorem (see [7], for example).

With that, the rest of the paper is organized as follows: Section 2 provides a background material. Section 3 poses the new problem. Section 4 presents a solution. And finally, section 5 provides some concluding remarks and discussions.

2 Background and Convention

We give a brief terminology needed for mathematical clarity; first for *sets*, and then for *graphs*.

The *set difference* $A - B$ of set A from set B is the set of all the elements in A but not in B. The *symmetric set difference* of two sets A and B is the set of all the elements that are in either one of the sets, but not in both, i.e., the set $(A - B) \cup (B - A)$. The *complement* A^c of a set A with respect to a universal set U is the set $A - U$. In taking the complement of a set, the universal set is understood to be the largest possible set in the context in which the complement is taken. The *cardinality* of a set A is the total number of elements in A, denoted by $|A|$. Two sets A and B are said to be *disjoint* if they have no elements in common, i.e., $A \cap B = \emptyset$. A *partition* of a set A is a set $\{A_1, A_2, \ldots\}$ of non-empty and disjoint sets whose union is A. Each set A_i in a partition $\{A_1, A_2, \ldots\}$, where $A = \bigcup_i A_i$, is called a *part* of the set A. The *power set* of a set A is the set of all the subsets of A, denoted by 2^A.

Let V be a finite set, and E be a set of its 2-element subsets. The ordered pair (V, E) is called a *simple* and *undirected* ***graph*** G, and usually denoted as $G = (V, E)$. The elements of V are called the *nodes*, and those of E are called the *edges*. The two nodes composing an edge are called its *end-nodes*, and are said to be *joined* by the edge. A graph is said to be *simple* if it has no *loop* (i.e., the end-nodes of no edge are the same) and no *multi-edge* (i.e., the end-nodes of no two edges are the same), and *undirected* if none of its edges is an ordered pair—as these could also be easily inferred from the definition of a graph given here. A graph is called a *null graph* if it has no node (hence, no edge). A node in a graph is called an *isolated node*, if it is not an end-node of an edge in the graph. A graph is called ***bipartite*** if its nodes could be partitioned into two sets M and W such that all the edges have one end-node in M and the other in W, in which case the *bipartite graph* is usually denoted as $G = (M \cup W, E)$. The sets M and W are called the *parts* of the bipartite graph G. A ***matching*** in a graph is a set of edges where no two edges have a common end-node. A matching is said to be *maximum* if no other matching in the graph has a larger cardinality. A node is said to be *matched* in a matching if it is an end-node of an edge in the matching, and it is said to be *unmatched* otherwise. Finally, an edge is said to be *included* in a matching if it is an element of the matching, and *excluded* otherwise.

3 A Matching Problem

Problem 1 *A bipartite graph $G_i = (M_i \cup W_i, E_i)$ for each i in $\{1, 2, 3\}$, where M_i's and W_i's are all disjoint, and three functions f_M, f_t, and f_W are given. The function f_M is defined from $\{M_1, M_2, M_3\}$ to the positive integers, and the function f_t is defined from $\bigcup_i W_i$ to the integers. Define a set $V_{i,n}$ for each i in $\{1, 2\}$ and n in the range of f_t, as*

$$V_{i,n} = \{v : \ v \in (W_i \cup W_{i+1}), \ f_t(v) = n\} .$$

The function f_W is defined from $\{V_{i,n} : \ i \in \{1, 2\}, \ n \text{ in the range of } f_t\}$ to the positive integers. Find the cardinality of a maximum matching in bipartite graph $G = (M \cup W, E)$, where $M = \bigcup_i M_i$, $W = \bigcup_i W_i$, and $E = \bigcup_i E_i$, such that the number of nodes matched in any set M_i or $V_{i,n}$ does not exceed the integer to which it is being mapped.

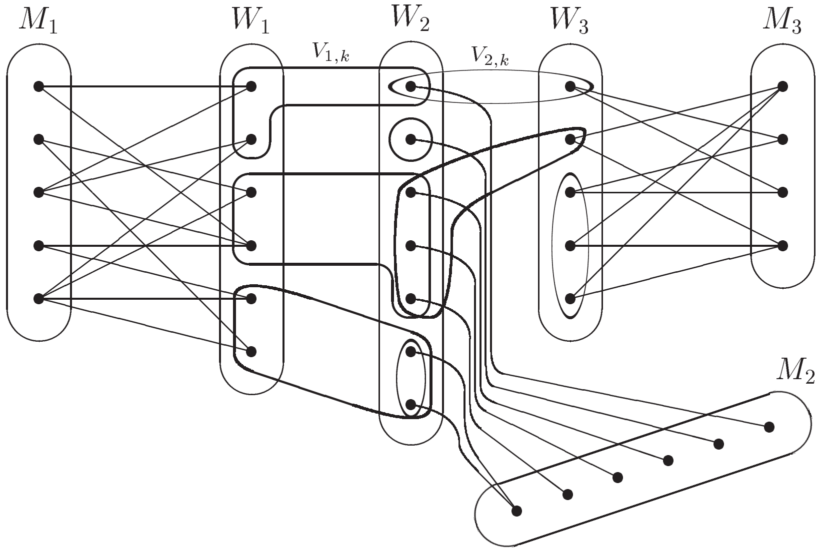

Figure 1: A graph for depicting a realization of problem 1.

In problem 1, we could assume without loss of generality that there is no isolated node in G, since they neither affect a matching nor truly alter a constraint as described in the problem. Also, since the solution of problem 1 is bound to matchings satisfying the constraints as described in the body of the problem, let any matching in G be understood as such whenever we refer to one, unless otherwise noted from here on.

This problem is motivated by an engineering problem where signal interference in interference-limited wireless networks is desired to be controlled for an increased system capacity, while some quality of service guarantees (such as delay and loss guarantees) are sought to be delivered via the system. That engineering problem could be seen by the following interpretation of problem 1:

1. For each i in $\{1, 2, 3\}$,
   (a) the nodes in M_i represent the packet arrivals for a packet flow i,
   (b) the nodes in W_i represent the capacity "arrivals" (i.e., the available transmission capacity) for a wireless link through which flow i is to be transmitted,
   (c) the edges E_i take into account some delay requirements for each packet in flow i,
   (d) the constraint on M_i implemented by the function f_M takes into account some loss requirement for the entire flow i (refer to [2] or [3], for example, to see how this could be utilized).
2. The wireless links whose capacities are represented by the nodes in W_i's as mentioned in item 1-(b), are given to be interfering with each other. The transmissions in links 1 and 2 interfere with the transmissions in link 3, and the transmissions in links 2 and 3 interfere with the transmissions in link 1, in each time slot.
3. The function f_t simply labels the capacity arrivals in time slots, and is used to identify the set of transmissions (i.e., the sets $V_{i,n}$'s) in each time slot that would cause interference to a link as explained in item 2 above.
4. The function f_W implements a constraint on the maximum number of allowable transmissions in each set of interference causing transmissions, as explained in item 3 above, in order to have an acceptable signal reception for reliable communication.

5. A matching satisfying the constraints in graph G corresponds to a scheduling of packets facilitating a viable communication as described so far. A such maximum matching is sought in order to utilize the system resources most efficiently.

4 A Solution to Problem 1

We give the following definitions after which the solution to problem 1 will follow. The first two definitions are general, considering a graph $G = (V, E)$:

1. The **image** $\Gamma_G(A)$ of a set of nodes A in G is the set of all the nodes to which a node in A is joined in G.[1] The image $\Gamma_G(A)$ is read as "the image of A in G".

2. The **principal** $\Pi_G(A)$ of a set of nodes A in G is the set of all the nodes whose image is a subset of A. The principal $\Pi_G(A)$ is read as "the principals of A in G".

Whenever there is not any ambiguity as to which graph G that the image or the principal of a set A is considered, we drop the subscripts in $\Gamma_G(A)$ and $\Pi_G(A)$, and write them simply as $\Gamma(A)$ and $\Pi(A)$.

The next two definitions are specific to problem 1:

- The function σ_M, called the **permission-in-*M***, is defined on the power set of M, as

$$\sigma_M(A) = \sum_{i \in \{1,2,3\}} \min\{f_M(M_i), |A \cap M_i|\}.$$

  The value $\sigma_M(A)$ is read as "the permission of A in M".

- The function σ_W, called the **permission-in-*W***, is defined on the power set of W, as

$$\begin{aligned}\sigma_W(A) = \sum_n \min\Big\{ & f_W(V_{1,n}) + \min\{f_W(V_{2,n}), |A \cap V_{2,n} \cap V_{1,n}^c|\}, \\ & f_W(V_{2,n}) + \min\{f_W(V_{1,n}), |A \cap V_{1,n} \cap V_{2,n}^c|\}, \\ & \min\{f_W(V_{1,n}), |A \cap V_{1,n} \cap V_{2,n}^c|\} + \min\{f_W(V_{2,n}), |A \cap V_{2,n} \cap V_{1,n}^c|\} + |V_{1,n} \cap V_{2,n}| \Big\}.\end{aligned}$$

  The value $\sigma_W(A)$ is read as "the permission of A in W".

The name of the functions σ_M and σ_W follow from the observation that $\sigma_M(A)$ and $\sigma_W(A)$ are the maximum number of nodes allowable to be matched in A, in G as stated in problem 1. This is easier to be seen for σ_M. As for σ_W, we give the following lemma.

Although the summand in $\sigma_W(A)$ looks intimidatingly long, it is actually just equal to the maximum number of nodes allowable to be matched in the portion of A which falls in $V_{1,n} \cup V_{2,n}$, i.e., in $A \cap (V_{1,n} \cup V_{2,n})$. This is shown by lemma 1.

Lemma 1 *The maximum number of nodes allowable to be matched in A, as stated in problem 1, is equal to $\sigma_W(A)$ in G.*

[1] *Image* is also referred to as *neighborhood*, in graph theory.

Proof: First, note that the sets $(V_{1,n} \cup V_{2,n})$'s, which are not empty, over all n (in the range of f_t) form a partition of W. Thus, the maximum number of nodes allowable to be matched in A is equal to the summation of the maximum number of nodes allowable to be matched in $A \cap (V_{1,n} \cup V_{2,n})$ over all n.

Next, for any n, it is easy to find the maximum number of nodes allowable to be matched in $A \cap (V_{1,n} \cup V_{2,n})$. For compactness, we adopt the following notations

$$\begin{aligned} V_1 &\equiv A \cap V_{1,n} \qquad & m_1 &\equiv f_W(V_{1,n}) \\ V_2 &\equiv A \cap V_{2,n} \qquad & m_2 &\equiv f_W(V_{2,n}) \,. \end{aligned} \tag{1}$$

It is clear that we try to find the maximum number of nodes allowable to be matched in $V_1 \cup V_2$, with the constraint that at most m_i nodes could be matched in each V_i for i equals to both 1 and 2, refer to Figure 2.

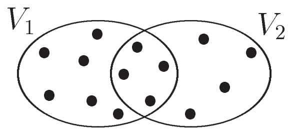

Figure 2: A venn diagram showing V_1 and V_2.

Note that if a node in $V_1 \cap V_2$ is decided to be matched in G, that would reduce the budget of the number of nodes allowable to matched in each set V_i by 1. Therefore, in order to find the maximum number of nodes allowable to be matched in $V_1 \cup V_2$, we would first need to exhaust the nodes in $V_1 - V_2$ and $V_2 - V_1$ (i.e., in the symmetric set difference).

Notice that the maximum number of nodes allowable to be matched in $V_1 - V_2$ and $V_2 - V_1$ are respectively equal to

$$\min\{m_1, |V_1 - V_2|\} \qquad \text{and} \qquad \min\{m_2, |V_2 - V_1|\}. \tag{2}$$

Hence, after exhausting these nodes, the maximum number of nodes allowable to be matched in $V_1 \cap V_2$ becomes equal to

$$\min\Big\{m_1 - \min\{m_1, |V_1 - V_2|\},\ m_2 - \min\{m_2, |V_2 - V_1|\},\ |V_1 \cap V_2|\Big\}. \tag{3}$$

Summing the terms in (2) and in (3), we get the maximum number of nodes allowable to be matched in $V_1 \cup V_2$ as

$$\begin{aligned} &= \min\{m_1, |V_1 - V_2|\} + \min\{m_2, |V_2 - V_1|\} + \\ &\quad \min\Big\{m_1 - \min\{m_1, |V_1 - V_2|\},\ m_2 - \min\{m_2, |V_2 - V_1|\},\ |V_1 \cap V_2|\Big\} \\ &= \min\Big\{m_1 + \min\{m_2, |V_2 - V_1|\},\ m_2 + \min\{m_1, |V_1 - V_2|\}, \\ &\qquad \min\{m_1, |V_1 - V_2|\} + \min\{m_2, |V_2 - V_1|\} + |V_1 \cap V_2|\Big\}. \end{aligned}$$

After revoking the substitutions given by the identities in (1), to the last line above, and comparing it to the summand of $\sigma_W(A)$, we observe that the quantity in last line above is equal to the summand of $\sigma_W(A)$. This concludes the proof. ∎

A solution to problem 1 is given by the following theorem.

Theorem 1 *The cardinality of a maximum matching satisfying the constraints as described in problem 1 is equal to*

$$\min_{A \subseteq W} \left\{ \sigma_M \left(M - \Pi(A) \right) + \sigma_W \left(A \right) \right\}. \tag{4}$$

Proof: Let μ be a maximum matching as described in problem 1. And let A be any subset of W. Now, let x be the set of all the edges in μ, which have an end-node in A. Clearly,

$$|x| \leqslant \sigma_W \left(A \right) \,, \tag{5}$$

since μ is a matching satisfying the constraints as described in problem 1. Similarly, the following inequality also holds;

$$|\mu - x| \leqslant \sigma_M \left(M - \Pi(A) \right) \,, \tag{6}$$

since (i) μ is a matching satisfying the constraints as described in problem 1, (ii) $\mu - x$ is a subset of $M - \Pi(A)$, and (iii) σ_M is a monotonic function (i.e., $A \subseteq B$ implies $\sigma_M(A) \leqslant \sigma_M(B)$). Adding the inequalities (5) and (6) side by side, we get

$$|\mu| \leqslant \sigma_M \left(M - \Pi(A) \right) + \sigma_W \left(A \right) \,.$$

Since the above inequality holds for any A, this concludes that the cardinality of a maximum matching (i.e., $|\mu|$) is less than or equal to the minimum (4) given in the theorem.

For the other direction (i.e., $|\mu|$ is greater than or equal to the minimum (4)), we give an induction argument on the cardinality of W. The induction is carried out over all the realizations of problem 1, where moving from one inductive step to another is controlled by incrementing $|W|$.

For $|W| = 1$, note that any realization of problem 1 is such that two of the graphs G_1, G_2, and G_3 should be null. This is by the construction of problem 1, as W_i's are given to be disjoint. Hence, in this case, we are essentially considering a singe bipartite graph with appropriate matching constraints as described in problem 1. Notice that, in this case, there are two elements of the set of which the minimum is taken in (4). It is easy to find that the term corresponding to $A = W$ is 1 and the other corresponding to $A = \emptyset$ is greater than or equal to 1. Thus, the minimum (4) is equal to 1. It is clear that the cardinality of a maximum matching in any realization of problem 1 is also equal to 1, since any edge in G simply constitutes a such maximum matching. Hence, this establishes the basis for our induction argument. (The $|W| = 0$ case also establishes a basis for our induction, however—although it is shorter to do so—it might look confusing.)

Suppose that our induction hypothesis (i.e., $|\mu|$ is greater than or equal to the minimum (4)) holds for all realizations of problem 1 where $|W| \leqslant n$ that $n \geqslant 1$.

Consider a realization of problem 1 on a graph $G = (M \cup W, E)$ where $|W| = n + 1$. Let us refer to any set M_i or $V_{i,k}$ in problem 1 as a *part*.

Now, if all the parts $V_{1,k}$ and $V_{2,k}$ are either disjoint or identical, then problem 1 reduces to a special case of the problem solved in [1, 2] which shows that the induction hypothesis for this realization of problem 1 where $|W| = n + 1$ holds.

Therefore, suppose that there exists at least one pair of parts $V_{1,k}$ and $V_{2,k}$ for some k that neither the intersection nor the symmetric set difference of these parts are empty.

In this case, pick any edge $\{b, g\}$ such that g belongs to a symmetric set difference of two parts $V_{1,k}$ and $V_{2,k}$ for some k. Note that a such edge exists in G by what we have just supposed. Also note that g belongs to W, and b belongs to M.

Now, tear apart the nodes b and g from G, and obtain a new realization of problem 1 on a graph $G' = (M' \cup W', E')$, as follows:

1. Delete the nodes b and g.
2. Also delete all the nodes in the part to which b or g belongs, if the integer to which the part is mapped is 1.
3. Remove all the edges which have an end-node in the set of deleted nodes in items 1 and 2.
4. Delete all the isolated nodes as a result of item 3.
5. Update the constraint functions f_M and f_W, by decreasing the integers to which the parts, where b or g belongs to, is mapped in f_M or f_W by 1; if these parts have not been already deleted in item 2.
6. Also update the function f_t appropriately by considering the deletions in items 1, 2, and 4.

Let us denote (i) the nodes that are deleted above in items 1, 2, and 4 by D, (ii) the edges that are removed in item 3 by R, and (iii) the updated functions in items 5 and 6 by $f_{M'}$, $f_{t'}$, and $f_{W'}$. With these explanations, the graph G' and the new realization of problem 1 should be clear.

Notice that the inductions hypothesis holds for the new realization of problem 1 on G', since $|W'| \leqslant n$. Thus, the cardinality of a maximum matching μ' (satisfying the newly constructed constraints) in G' is given by

$$\min_{A \subseteq W'} \left\{ \sigma_{M'} \left(M' - \Pi_{G'}(A) \right) + \sigma_{W'} \left(A \right) \right\}$$

where the permission functions $\sigma_{M'}$ and $\sigma_{W'}$ are defined appropriately via the updated functions $f_{M'}$, $f_{t'}$, and $f_{W'}$.

Also note that by the construction of the new realization of problem 1 on G', the matching μ' appended by the edge $\{b, g\}$, i.e., $\mu' \cup \{b, g\}$, also satisfies the constraints in the original realization of problem 1 from which the new realization is obtained. Thus, we have

$$|\mu| \geqslant |\mu'| + 1 \,,$$

since μ is a maximum matching.

Next, we will show that

$$|\mu'| + 1 \geqslant \min_{A \subseteq W} \left\{ \sigma_M \left(M - \Pi_G(A) \right) + \sigma_W \left(A \right) \right\}, \tag{7}$$

hence conclude the proof.

First, consider any strict subset A of W' (i.e., $A \neq W'$). Let V be the set of all the nodes in the part to which g belongs, that are deleted in items 1 and 2 in the list where we have explained how to obtain the new realization of problem 1. Observe that

$$\Pi_{G'}(A) \subseteq \Pi_G(A \cup V) - (M - M') \,. \tag{8}$$

Figure 3 provides a symbolic graph to help illustrate the arguments from here on. In figure 3, A stands for a subset of W', and the sets I_M and I_W stand for the sets of isolated nodes obtained by removing edges in item 3, which fall in M and W, respectively.

Note that in (8) we have not used the set $D \cap W$ instead of V, i.e., have not included the nodes in $(D \cap W) - V$ into our consideration in finding $\Pi_{G'}(A)$ as a result of deletions in obtaining G'. This is because of the fact that the nodes in $(D \cap W) - V$ could be seen only by the nodes deleted in items 1 and 2 in the part to which b belongs (i.e., in the image of these nodes), hence need not to be considered in finding $\Pi_{G'}(A)$. This could be seen more clearly if we pay a closer attention

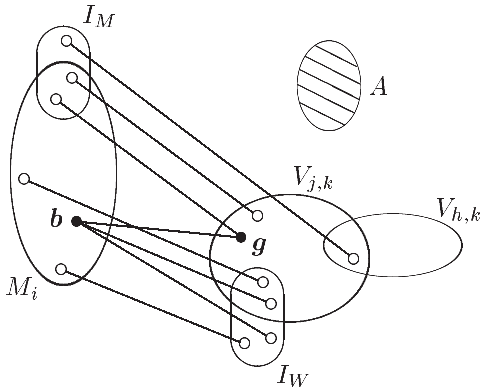

Figure 3: A symbolic graph to help illustrate some of the arguments in the proof.

to items 3 and 4. Note that the only other deletions of nodes in W is due to having some isolated nodes in W by removing the edges in item 3. In other words, the nodes in $(D \cap W) - V$ were deleted because they were seen only by the nodes deleted in items 1 and 2 in the part to which b belongs, otherwise they would not have become isolated by removing the edges in item 3.

Also note that the principals of A under G', i.e., $\Pi_{G'}(A)$, could not include the nodes in $M - M'$ as they have been already deleted. This is why we have excluded them in (8).

Now, note the following manipulations of (8);

$$\begin{aligned}
\Pi_{G'}(A) &\subseteq \Pi_G(A \cup V) - (M - M') \\
M - \Pi_{G'}(A) &\supseteq M - \Big(\Pi_G(A \cup V) - (M - M')\Big) \\
\Big(M' \cup (M - M')\Big) - \Pi_{G'}(A) &\supseteq M - \Big(\Pi_G(A \cup V) - (M - M')\Big) \\
\Big(M' - \Pi_{G'}(A)\Big) \cup (M - M') &\supseteq M - \Big(\Pi_G(A \cup V) - (M - M')\Big)
\end{aligned}$$

note that for any three sets X, Y, and Z, we have[2] $X - (Y - Z) = (X - Y) \cup (X \cap Z)$, hence we get

$$\begin{aligned}
\Big(M' - \Pi_{G'}(A)\Big) \cup (M - M') &\supseteq \Big(M - \Pi_G(A \cup V)\Big) \cup (M - M') \\
M' - \Pi_{G'}(A) &\supseteq M - \Pi_G(A \cup V).
\end{aligned}$$

Taking the permission-in-M (i.e., σ_M) of both sides above, we get (since σ_M is monotonic)

$$\sigma_M\left(M' - \Pi_{G'}(A)\right) \geqslant \sigma_M\left(M - \Pi_G(A \cup V)\right).$$

Also note that we have

$$\sigma_M\left(M' - \Pi_{G'}(A)\right) \leqslant \sigma_{M'}\left(M' - \Pi_{G'}(A)\right) + 1 \qquad (9)$$

[2]For any three sets X, Y, and Z, we have

$$\begin{aligned}
X - (Y - Z) &= X \cap (Y - Z)^c \\
&= X \cap (Y \cap Z^c)^c \\
&= X \cap (Y^c \cup Z) \\
&= (X \cap Y^c) \cup (X \cap Z) \\
&= (X - Y) \cup (X \cap Z).
\end{aligned}$$

since the updated constraint function $f_{M'}$ is obtained by decreasing the permission of only one part in M, see item 5 in the list where we have explained how to obtain the new realization of problem 1 on G'. The only part in M' where the constraint function f_M is changed is the part to which b belongs. In order to see the claimed inequality (9) more clearly, consider the portion of $M' - \Pi_{G'}(A)$ which falls in M_i' where $M_i' = M_i - D$ and $b \in M_i$, and observe the relations given by the following table.

$f_M(M_i)$	$\lvert M_i' \cap (M' - \Pi_{G'}(A))\rvert$	$\sigma_M\left(M_i' \cap (M' - \Pi_{G'}(A))\right)$	$\sigma_{M'}\left(M_i' \cap (M' - \Pi_{G'}(A))\right)$
> 1	$\geqslant f_M(M_i)$	$f_M(M_i)$	$f_M(M_i) - 1$
> 1	$< f_M(M_i')$	$\lvert M_i' \cap (M' - \Pi_{G'}(A))\rvert$	$\lvert M_i' \cap (M' - \Pi_{G'}(A))\rvert$
1	> 0	1	0
1	0	0	0

It becomes clear by the above table that the inequality (9) holds.

Next, note that

$$\begin{aligned}\sigma_W(A \cup V) &= \sigma_W(V) + \sigma_{W'}(A) \\ &= 1 + \sigma_{W'}(A)\,. \end{aligned} \tag{10}$$

The above equalities hold since (i) A and V are disjoint, and (ii) the permission of V in W (i.e., $\sigma_W(V)$) is clearly 1 by the construction of the new realization of problem 1 on G'.

Thus, we have shown that for any strict subset A of W', we have

$$\begin{aligned}\sigma_{M'}\left(M' - \Pi_{G'}(A)\right) + \sigma_{W'}(A) &= \sigma_{M'}\left(M' - \Pi_{G'}(A)\right) + 1 + \sigma_{W'}(A) - 1 \\ &\geqslant \sigma_M\left(M - \Pi_G(A \cup V)\right) + \sigma_W(A \cup V)\,. \end{aligned} \tag{11}$$

Secondly, note that for $A = W'$, we have $\Pi_{G'}(A) = M'$; hence,

$$\sigma_{M'}(M' - \Pi_{G'}(A)) = 0\,.$$

Thus, we get for $A = W'$

$$\begin{aligned}\sigma_{M'}\left(M' - \Pi_{G'}(A)\right) + \sigma_{W'}(A) &= 0 + \sigma_{W'}(A) \\ &= \sigma_W(A \cup V) - 1 \qquad \text{by (10)} \\ &= \sigma_W(W' \cup V) - 1\,. \end{aligned} \tag{12}$$

Finally, combining the results (11) and (12) shows that the inequality (7) holds. This concludes the proof. ■

5 Discussions and Conclusion

We have posed a restricted matching problem on bipartite graphs (i.e., problem 1), and presented a solution for it (i.e., theorem 1). This matching problem is motivated by an engineering problem where signal interference in interference-limited wireless networks is desired to be controlled for an increased system capacity, while some quality of service guarantees (such as delay and loss guarantees) are sought to be delivered via the system—see the explanations given at the end of section 3.

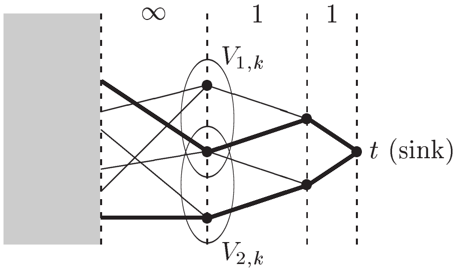

Figure 4: A counterexample to illustrate that the Max-Flow Min-Cut Theorem does not apply to solve problem 1.

An application of the main result of this study, which is theorem 1, is similar to that of the main result in [1]. Applications of the result in [1] could be followed in [2] and [3].

An interesting aspect of the matching problem that we pose here is that the Max-Flow Min-Cut Theorem [4, 5] does not seem to be applicable to find a solution for this problem. This is in contrast to the fact that it has been previously applicable to a similar problem solved in [1] whose solution has a very similar form as the solution that we have here.

For the problem in [1], the Max-Flow Min-Cut Theorem provides a solution by constructing a *flow-network* around the bipartite graph. For clarity, we copy that problem below.

Problem 2 *A bipartite graph $G = (M \cup W, E)$, a partition $\{V_i : i \in S\}$ of its nodes, where each part V_i is a subset of either M or W, and a mapping f of the partition into the positive integers are given. Find the cardinality of a maximum matching such that the number of nodes matched in any part of the partition does not exceed the integer to which it is being mapped.*

A solution to problem 2 becomes apparent once the flow-network, that we have mentioned above, is constructed around G as follows: Introduce a source node s and a sink node t, and a node v_i for each part V_i. Join the source node s to each node v_i for which the part V_i is a subset of M, and let the capacity of the edge $\{s, v_i\}$ be $f(V_i)$. Repeat the same procedure by replacing s with t, and M with W. Join every node v_i to the nodes in V_i, and let their capacity be 1. Finally, let the capacity of any edge in E be "very large", say ∞. Clearly, a *maximum $s-t$ flow* in this construction corresponds to a maximum matching (as described in problem 2) in G. Hence, the Max-Flow Min-Cut Theorem provides a solution for problem 2 easily, as we have also found in [1].

One might suggest that constructing a flow-network similar to the one above would provide a solution for problem 1 as well. However, this would not work since $V_{i,n}$'s are in general not disjoint. As a counterexample, consider two such parts (i.e., $V_{i,n}$'s) where each part has two nodes, and their intersection has exactly one node. Let none of the nodes in these parts be isolated (i.e., have at least one edge attached to each node). Also let the function f_W map each one of these parts to 1. Now, although a flow which might pick two edges where each is attached to a node in any one of these parts is a valid flow in such a construction, the corresponding matching does not satisfy the constraints described in problem 1. Refer to Figure 4 for further clarification, where (1) a section of the flow-network is illustrated, (2) the capacities of all the edges between any two consecutive dashed lines have the indicated number, (3)-the highlighted edges show a portion of a valid flow.

To this end, we also would like to note that the solution that we have provided here (i.e., theorem 1) does not hold for a more general formulation of problem 1, where M_i's and W_i's are not disjoint among themselves. A simple counterexample is shown in Figure 5. However, we would like to note that we are not sure about an engineering application of a such formulation in the context that we have posed problem 1.

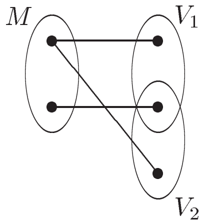

Figure 5: A counterexample to illustrate that theorem 1 does not hold for a more general formulation of problem 1, where M_i's and W_i's are not disjoint among themselves.

An algorithm to find a such maximum matching in problem 1, whose cardinality is given by theorem 1, could be devised by a generalized *augmenting path*[3] argument, similarly as the algorithm suggested in the proof of the result in [1]. By such an algorithm, another proof to theorem 1 could also be constructed, similar to the proof given in [1]. However, since a such algorithm is more involved, the proof as a result might not look very coherent.

Finally, we would like to note that theorem 1, as well as the theorem in [1], could be viewed as a generalization of König's Minimax Theorem (see [7], for example). Future work includes examining the problem 1 in a probabilistic setting.

References

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[3] See [7], for example, for a definition of *augmenting path*.