Compacton Solutions in a Class of Generalized Fifth Order Korteweg-de Vries Equations

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Solitons play a fundamental role in the evolution of general initial data for quasilinear dispersive partial differential equations, such as the Korteweg-de Vries (KdV), nonlinear Schrödinger and the Kadomtsev-Petviashvili equations. These integrable equations have linear dispersion and the solitons have infinite support. We have derived and investigate a new KdV-like Hamiltonian partial differential equation from a four parameter Lagrangian where the nonlinear dispersion gives rise to solitons with compact support (compactons). The new equation does not seem to be integrable and only mass, momentum, and energy seem to be conserved; yet, the solitons display almost the same modal decompositions and structural stability observed in integrable partial differential equations. The compactons form from arbitrary initial data, are nonlinearly self stabilizing and maintain their coherence after multiple collisions. The robustness of these compactons and the inapplicability of the inverse scattering tools, that worked so well for the KdV equation, makes it clear that there is a fundamental mechanism underlying the processes beyond integrability. We have found explicit formulas for multiple classes of compact traveling wave solutions. When there are more than one compacton solution for a particular set of parameters, the wider compacton is a minimum of a reduced Hamiltonian and is the only one that is stable.

I. INTRODUCTION

Solitary waves with compact support ("compactons") were recently found in various generalizations of the Korteweg-de Vries equation. The original class of equations studied by Rosenau and Hyman [1], [2] posessed solutions which were compact solitary waves with remarkable properties. Upon scattering they reemerged with almost the same coherent shape. The energy that was lost reappeared in the form of compact solitary waves (compactons and anticompactons). (An "anticompacton" is one with negative amplitude compacton travelling in the opposite direction to the compacton.) Initially localized packets upon evolution broke up into a series of the compactons. Thus the compactons were robust in that any arbitrary pulse eventually ended up as compactons.

One of our main interests is in the light that understanding the dynamics of the compactons will shed on the theory of solitons. These particle-like waves exhibit both elastic and nearly elastic collisions that are similar to the soliton interactions associated with completely integrable PDEs supporting an infinite number of conservation laws.

The equations investigated by Rosenau and Hyman [1], [2]

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \tag{1.1}$$

has several conservation laws, but since it was not derivable from a Lagrangian (except for the special case of the original KdV equation), it did not possess the usual conservation laws of Mass and Energy. We thus thought that by finding a similar class of equations derivable from a Lagrangian we might be able to prove integrability using one of the standard methods.

Because of certain scaling properties of the generalized KdV equation that we will derive below, there always exists a subclass of solutions that have the interesting property that the width of the solitary wave is independent of the amplitude. For the above equation when $1 < m = n \le 3$, where $\xi = x - ct$ and $-\pi/2 \le a\xi \le \pi/2$, the solutions of the form

$$A[\cos(d\xi)]^{2/(m-1)} \tag{1.2}$$

have this property.

Unlike classical solitons, the compactons are nonanalytic solutions. The points of nonanalyticity at the edge of the compacton correspond to points of genuine nonlinearly for the differential equation and introduce singularities in the associated dynamical system for the traveling waves. Li et al. [3], [4], [5] have shown the connection between nonlinear dispersion and the existence of these nonclassical solutions. They identify the compactons as pseudo-classical solutions and demonstrate how they can be characterized as the limiting case of a classical analytic solution.

By starting with the first order Lagrangian [6] [7]

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x\varphi_t + \frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2 \right) dx, \tag{1.3}$$

we derived and studied a generalized sequence of KdV equations of the form:

$$\begin{aligned} u_t + u^{l-2}u_x + \alpha(2u^p u_{xxx} + 4pu^{p-1}u_x u_{xx} \\ + p(p-1)u^{p-2}(u_x)^3) = 0, \end{aligned} \tag{1.4}$$

where the usual field $u(x,t)$ of the generalized KdV equation is defined by $u(x,t) = \varphi_x(x,t)$. For $0 < p \le 2$ and $l = p + 2$, these models admit compacton solutions for which the width is independent of the amplitude.

These equations have the same terms as the equations considered by Rosenau and Hyman, but the relative weights of the terms are different. Using the underlying Hamiltonian structure, one can show [8] using linear stability analysis as well as Lyapunov stability criterion that the compactons, whose width is independent of the amplitude, are stable under perturbations.

We have been unable to determine if there are examples within this class of equations which are integrable. In our previous work [6], we attempted to repeat the induction proof of the existence of an infinite number of conservation laws, by following the strategy used in the KdV equation of assuming the conservation laws obeyed a recursion relation. We assumed that the conserved Momentum P could be identified with a second Hamiltonian and Poisson Bracket structure. The method of generating all the conservation laws from a Bi-Hamiltonian structure is discussed in [9], [10], [11]. However, when applied to this problem the iteration method failed after a few iterations in generating further conservation laws except for the special case of the original KdV equation. Since integrability is connected with having true solitons, one of the purposes of this paper is to present numerical evidence that the systems described by our new Lagrangian are not integrable since the collision of the solitary waves leaves behind a small wake.

In this paper we further generalize both the usual KdV Lagrangian and our previous generalized KdV Lagrangian to preserve the invariance of the action under time, and space translations as well as the shift of the field by a constant ($\varphi \to \varphi + a$);

$$L(p,m,n,l) = \int dx\ [\frac{1}{2}\varphi_x\varphi_t + \alpha\frac{(\varphi_x)^{p+2}}{(p+1)(p+2)} - \beta(\varphi_x)^m(\varphi_{xx})^2 + \frac{\gamma}{2}\varphi_x^n\varphi_{xx}^l\varphi_{xxx}^2]. \tag{1.5}$$

Note that the generalization includes an extra term with higher derivatives. Recently, it has been shown in the context of higher order KdV equations which admit soliton solutions that the higher order derivative terms improve the range of stability of these soliton solutions [12,13]. In this paper, we will investigate the stability of the compacton solutions of the resulting equation for the Lagrangian (1.5)

$$u_t + \frac{\alpha}{p+1}\partial_x u^{p+1} - \beta m\partial_x(u^{m-1}u_x^2) + 2\beta\partial_{xx}(u^m u_x) + \frac{\gamma n}{2}\partial_x(u^{n-1}u_x^l u_{xx}^2) - \frac{\gamma l}{2}\partial_{xx}(u^n u_x^{l-1}u_{xx}^2) + \gamma\partial_{xxx}(u^n u_x^l u_{xx}) = 0. \tag{1.6}$$

This new four parameter family of nonintegrable PDEs preserves energy, momentum and mass by Noether's theorem. Most of the previously studied nonintegrable PDEs with soliton solutions are known to be near an integrable PDE. For most of the parameter range, Eq. (1.6) is far from any known integrable PDE. We investigate new compacton solutions of this higher order generalized KdV equation with special emphasis on the cases where the width of the compacton is independent of the amplitude (note that in the third-order generalized KdV case, these compactons have been shown to be stable). We also study the scattering properties of the compactons numerically to determine if they behave similarly to those found by Rosenau and Hyman and to see if they shed any light on the integrability of these equations.

While we do not have a rigorous proof, Equation (1.6) seems to have only a finite number of local conservation laws and yet it exhibits behavior similar to that usually associated with integrable equations with an infinite number of conservation laws. A wide initial pulse will break into a train of compactons all having the same width, but different amplitudes. The compactons almost remain coherent when they collide (but for the creation of a low amplitude oscillatory wave).

In section 2 we will discuss the Hamiltonian structure of these classes of theories and determine the equation satified by travelling wave. We use scaling arguments to show that there are solutions with the width of the solitary wave being independent of the amplitude when $p = m = n + l$.

In section 3 we obtain exact compacton solutions of the form $A\cos^r d(x - ct)$ and verify that the relations among the global variables obtained using the variational approach are exact. Often there are two different compacton solutions for a single set of parameters having this same generic form. When this is the case, the wider solution is found to be numerically stable. By studying a class of solutions with arbitrary width parameter d variationally, we find that the stable solution is the minimum of the reduced Hamiltonian. So it seems that one can use a simple variational method to check for stability without carrying out a complete stability analysis.

In section 4 we give numerical results for the scattering of two compactons, as well as for the breakup of an arbitrary wave packet into several compactons.

In the appendix we discuss a variational approach for obtaining approximate and exact solutions. We obtain the relationship between Energy and Momentum satisfied by the solitary waves and show that most of the exact solutions as well as the criteria for their stability could be obtained by assuming the exact compacton ansatz and minimizing the action on that class of functions. When there are multiple exact travelling wave solutions, we find that the narrow solutions are maxima of the effective Hamiltonian and are numerically unstable in our simulations.

II. GENERALIZED KDV EQUATION AND PROPERTIES

The generalized KdV equation, Eq.(1.6), has the conserved Hamiltonian:

$$H = \int dx[-\alpha\frac{u^{p+2}}{(p+1)(p+2)} + \beta u^m u_x^2 - \frac{\gamma}{2}u^n u_x^l u_{xx}^2], \tag{2.1}$$

where $u(x,t) = \varphi_x(x,t)$. We notice that the Lagrangian given by Eq. (1.5) is invariant under the transformations (i) $\varphi(x,t) \to \varphi(x,t) + c_1$; (ii)$x \to x + c_2$ and (iii) $t \to t + c_3$, where c_1, c_2 and c_3 are constants. By a direct application of Noether's theorem this leads to the three conservation laws of mass M, momentum P and energy H, where H is given by Eq. (2.1) and M and P are given by

$$M = \int u(x,t)dx; \qquad P = \frac{1}{2}\int u^2(x,t)dx. \tag{2.2}$$

The equation of motion Eq. (1.6) is also invariant under the transformations as given by

$$u \to ku; \quad x \to k^a x; \quad t \to k^b t$$

provided that

$$\frac{n+l-p}{l+4} = a = \frac{m-p}{2}; \quad b = a-p.$$

We thus obtain the following scaling relation between the speed, width and the amplitude of the travelling wave

$$u(x-ct) = (\lambda)^{\frac{1}{b-a}} u((\lambda)^{\frac{a}{b-a}}[x-\lambda ct]) . \tag{2.3}$$

In the special case of $m = p = n + l$, $a = 0$. It then follows that the soliton width is independent of the amplitude. As an aside we would like to point out that all the compactons so far discovered with width independent of amplitude have the property that the field equations are invariant under

$$u \to ku; \quad x \to x; \quad t \to k^b t$$

.

The canonical structure of these theories is similar to that found in [6] in that by postulating that the $u(x)$ satisfy the Poisson bracket structure [11]

$$\{u(x), u(y)\} = \partial_x \delta(x-y). \tag{2.4}$$

we obtain that

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\} \tag{2.5}$$

with H being given by Eq.(2.1). We also find that with our definition of P given by Eq.(2.2), that P is indeed the generator of the space translations:

$$\{u(x,t), P\} = \frac{\partial u}{\partial x}. \tag{2.6}$$

Since our equation is a generalization of the equation discussed in [6], one is also not able to show the existence of a bi-Hamiltonian structure using the conserved momentum as a possible second Hamiltonian as was done for the ordinary KdV equation [9], [10]. So on these grounds one expects that our general Lagrangian may not correspond to an exactly integrable system, except for the original KdV equation case. Our numerical results on scattering where there is some energy going into compacton pair production following scattering supports this expectation.

A. Equation for solitary waves

If we assume a travelling wave solution of (1.6), (1.6) in the form of a travelling wave:

$$u(x,t) = f(x-ct) \equiv f(y), \tag{2.7}$$

one obtains on integrating once

$$\begin{aligned} cf &= \frac{\alpha}{p+1} f^{p+1} - \beta m (f^{m-1} f_y^2) + 2\beta \partial_y (f^m f_y) \\ &+ \frac{\gamma n}{2} (f^{n-1} f_y^l f_{yy}^2) - \frac{\gamma l}{2} \partial_y (f^n f_y^{l-1} f_{yy}^2) \\ &+ \gamma \partial_{yy} (f^n f_y^l f_{yy}) + c_1, \end{aligned} \tag{2.8}$$

where c_1 is a constant of integration. This equation needs several more integrations before a solution in terms of quadrature is obtained, unlike the previous equation we studied where two integrations were sufficient. Thus an explicit solution in terms of quadratures is not available and one must use "educated" guess ansatze to find exact solutions. Some properties of the solitary waves can be obtained before obtaining exact solutions.

In obtaining exact solutions, we found some values of the parameter where analysis is much simplified. From eq.(2.3) we find that this occurs at the special values $p = m = n+l$, where the solitary wave solutions have compact support and the feature that their width is independent of the amplitude. By considering trial variational wave functions of a post-Gaussian type, as done in [6] and sketched in the appendix one finds that the solitary waves with $p = m = n + l$ obey relations of the form

$$H = -\frac{2c}{p+2} P . \tag{2.9}$$

The variational approach leads to solutions which correspond to $c_1 = 0$. When this relation is satisfied, the above connection between the conserved quantities is found to be true for the exact solution. For the special cases when $c_1 \neq 0$ this relation is no longer true. We also find that many of the solitary wave solutions found by the variational method are unstable in that they are stationary values of the Hamiltonian which are not minima as a function of the variational parameters. We have found numerically, that the exact solitary waves associated with these variational ones will be unstable whenever the variational ones are unstable. This suggests that stability in the subspace of the variational parameters is a useful guide for understanding the results of our numerical simulations.

Before proceeding with our study of solitary waves, we just remind the reader that at $\gamma = 0$ the Lagrangian we are studying reduces to the previous generalized KdV problem we studied [6] so that these equations include all the KdV solitons and compactons that we discussed earlier as a special case. In this paper, we will find exact solutions of Eq. (2.8) by inserting the ansatz:

$$f(y) = A \cos^r(dy) \tag{2.10}$$

and determining the parameters A, r, and d by a consistency argument. In the appendix we show that the stability of the solutions can be inferred (but not proven) by studying the behavior of the energy as a function of the parameter d.

III. EXACT COMPACTON SOLUTIONS

Our variational calculations described in the appendix suggest that at the particular case where $p = m = n + l \neq 0$. which corresponds to Compactons whose width is independent of amplitude, the analysis simplifies greatly, so we will restrict our attention to this case, which already has quite a rich structure. Assuming a solution of the form

$$\begin{aligned} u(x,t) &= A \cos^r[d(x-ct)]; \frac{-\pi}{2} \leq d(x-ct) \leq \frac{\pi}{2} \\ u(x,t) &= 0; \quad |d(x-ct)| > \frac{\pi}{2}, \end{aligned} \tag{3.1}$$

we look for consistent solutions for A, r,c and d in terms of α, β and γ and c_1 which are real. Having found these solutions we then check whether these solutions are maxima or minima of the effective Hamiltonian defined in the appendix as a function of the width parameter d. When the effective Hamiltonian is a maximum, then these narrower solutions turn out (by numerical investigation) to be unstable at the leading edge of the soliton. This numerical instability of the narrow solitons continues to be observed even after extensive grid refinement. Thus the effective Hamiltonian method presents a simple way of checking stability. It is also true that the more general variational approximations are either stable or unstable depending depending on whether the effective Hamiltonian is a minimum with respect to the variational parameters.

Now let us look at various cases.

A. $p = m = n$ and $l = 0$ case

For the case $p = m = n$ and $l = 0$ it is possible to find a general class of solutions for arbitrary p. Inserting a trial solution of the form Eq.(2.10) into Eq.(2.8) we obtain the consistency equation:

$$\begin{aligned} 0 = & -3c_1 + 2Acx^r \\ & + A^{1+p} x^{-4+r+rp} (1-r)\, r d^4 \gamma \times \\ & \left(12 - 10r + 2r^2 - 11rp + 5r^2 p + 2r^2 p^2\right) \\ & + 2A^{1+p} x^{-2+r+rp} r d^2 (-2 + 2r + rp) \times \\ & \left(-\beta + 2d^2\gamma - 2rd^2\gamma + r^2 d^2 \gamma - rd^2\gamma p + 2r^2 d^2 \gamma p\right) \\ & + A^{1+p} r^2 d^2 (2+p) \left(2\beta - r^2 d^2 \gamma - 2r^2 d^2 \gamma p\right) x^{r+rp} \\ & - \frac{2}{1+p} A^{1+p}\, \alpha\, x^{r+rp} \end{aligned} \tag{3.2}$$

Here $x = \cos(dy)$. All the powers of x must have zero coefficient for the trial solution to be an actual solution. This leads to various conditions depending on the values of r and p. If $r(1 + p) = 4$ then there can also be solutions with c_1 being nonzero. First let us consider the case when $c_1 = 0$. In that case for consistency we need either $rp = 2$ or $rp = 4$.

1. rp=2

When $rp = 2$ Eq. 3.2 tells us that either $\gamma = 0$ or $r = 1$. When $\gamma = 0$, we get the solution we found in our earlier work [7]: Namely:

$$d^2 = \frac{\alpha\, p^2}{4\beta\, (1+p)\, (2+p)}$$

$$A^p = \left\{\frac{c\, (1+p)\, (2+p)}{2\,\alpha}\right\}$$

When $\gamma \neq 0$ we instead get the solution $r = 1$ ($p = 2$) and

$$A^2 = \frac{c}{2\,\beta\, d^2 - 6\, d^4\, \gamma}$$

and two possible solutions for the width:

$$d^2 = \frac{12\,\beta \pm \sqrt{144\,\beta^2 - 120\,\alpha\,\gamma}}{60\,\gamma}.$$

Which means we also can write the equation for A^2 as

$$c = \frac{A^2}{5}(\alpha - 2\beta d^2)$$

Thus when $\gamma \neq 0$ (and $pr = 2$) one only gets a solution when $p = 2$. A particular case of this solution is: $\alpha = 6$, $\gamma = 3$ and $\beta = 4$. Then there are two solutions. The first is

$$u = \sqrt{3c/2}\cos\left(\frac{x - ct}{\sqrt{3}}\right) \tag{3.3}$$

The conserved quantities for this solution are

$$M = 3\sqrt{2c};\ P = \frac{3}{8}\sqrt{3}c\pi;\ H = -\frac{3}{16}\sqrt{3}c^2\pi$$

which satisy the relationship Eq. (2.9).

The second solution :

$$u = \sqrt{\frac{25c}{22}}\cos\frac{y}{\sqrt{5}} \tag{3.4}$$

has as its conserved quantities:

$$M = 5\sqrt{\frac{10c}{11}};\ P = \frac{25\sqrt{5}c\pi}{88};\ H = -\frac{25\sqrt{5}c^2\pi}{176}.$$

Thus again the relationship Eq. (2.9) is satisfied.

We can check whether these solutions are maxima or minima of the reduced Hamiltonian discussed in the Appendix. Inserting the trial wave function:

$$u = A\cos[dy]; \quad -\frac{\pi}{2} \leq dy \leq \frac{\pi}{2}; \quad y = x - q(t) \tag{3.5}$$

into the action, and using the relationship

$$A^2 = \frac{4dP}{\pi}$$

and using the above values of α,β and γ we obtain for the effective Hamiltonian:

$$H = \frac{P^2}{\pi}(-9d^5 + 8d^3 - 3d) \tag{3.6}$$

The solitary wave solutions are the stationary points of the actions and are therefore also stationary values of the Hamiltonian. These stationary values are $d^2 = 1/5$ which is a minimum of the energy and $d^2 = 1/3$ which is a maximum of the energy. We have confirmed numerically that the narrower compacton with $d^2 = 1/3$ is unstable. We have also shown numerically (see below) that if we start with initial compact data which is wider than the compacton with $d^2 = 1/5$ it breaks up into a number of these compactons.

2. rp=4 case

Let us now consider the case $r = \frac{4}{p}$, with $c_1 = 0$. When $\gamma = 0$ one of the consistency conditions $Ac = 0$ can only be satisfied for static solutions. When $\gamma \neq 0$ the consistency conditions leads to :

$$\begin{aligned} d^2 &= \frac{\beta}{\gamma(r^2 + 6r - 2)}, \\ A^p &= \frac{c}{(r-1)\, r^2\, (5+r)\, d^4\, \gamma} \end{aligned} \tag{3.7}$$

with the parameters also obeying the constraint:

$$\alpha = \frac{(2+r)\ (4+r)\ \left(-4+4\,r+r^2\right)\ \beta^2}{\left(-2+6\,r+r^2\right)^2\,\gamma} \tag{3.8}$$

Let us look at 2 examples from this class of solutions:
(i) $p = m = n = 1; l = 0$
the solution is of the form:

$$u = A\cos^4(dy)$$

Choosing for example $\gamma = 1/38$ and $\beta = 1$, the above relations yields $d = 1$, $A = 19c/216$ and $\alpha = 672/19$. Thus the solution is

$$u = \frac{19c}{216}\cos^4(x-ct) \tag{3.9}$$

For this solution we have that the global quantities are:

$$M = \frac{19c\pi}{576};\quad P = \frac{12635c^2\pi}{11943936};\quad H = -\frac{12635c^3\pi}{17915904}$$

So that the relationship Eq. (2.9) is again exact. We show in the appendix that this solution can also be found by minimizing the effective action using the compacton ansatz. In that case one also gets another solution with $d^2 = \frac{77}{18}$, which is the maximum of the Hamiltonian, with $d^2 = 1$ being a minimum as a function of d.

(ii) $p = m = n = 2$,$l = 0$
In this case we obtain the conditions:

$$r = 2;\quad d^2 = \frac{\beta}{14\gamma};\quad A^2 = \frac{c}{28d^4\gamma} \tag{3.10}$$

Choosing $\beta = 1$, $\gamma = \frac{1}{14}$, $\alpha = \frac{96}{7}$, we obtain $c = 2A^2$ and the solution is

$$u = \sqrt{\frac{c}{2}}\cos^2(x-ct). \tag{3.11}$$

For this choice of parameters we obtain for the conserved quantities:

$$M = \frac{\pi\sqrt{c}}{2\sqrt{2}};\ P = \frac{3\pi c}{32}; H = -\frac{3\pi c^2}{64}$$

so that the relationship Eq. (2.9) is again exact. For these parameters, if we use the compacton form as the trial wave function then the effective Hamiltonian for the variational parameter d is

$$H = -\frac{8P^2}{9\pi}(10d - 5d^3 + d^5) \tag{3.12}$$

This Hamiltonian has 2 stationary points as a function of d, $d = 1$ (minimum) and $d = \sqrt{2}$ (maximum) The second solution is not a solution to the equation of motion.

Next, let us consider a particular special solution for the case $c_1 \neq 0$,$p = m = n = 1$,$l = 0$.
For this case we have $r = 2$. Assuming a solution of the form

$$u(y) = A\cos^2(dy)$$

we get the consistency equations:

$$c_1 = -6A^2d^4\gamma. \tag{3.13}$$

$$d^2 = \frac{1}{12\gamma}[\beta \pm (\beta^2 - \alpha\gamma)^{1/2}] \tag{3.14}$$

$$A = \frac{c}{8d^2(\beta - 8\gamma d^2)}. \tag{3.15}$$

For the special choice of $\alpha = 5$, $\beta = 3$ and $\gamma = 1$, there are two real solutions for d^2,corresponding to $d^2 = 1/12$ and $d^2 = 5/12$. The first solution which is stable and which we will discuss further in our section on numerical simulations is :

$$u = \frac{9c}{14}\cos^2(\frac{x-ct}{\sqrt{12}}). \tag{3.16}$$

This solution has for its constants of motion:

$$M = \frac{9\sqrt{3}\pi c}{14};\ P = \frac{243\pi c^2\sqrt{3}}{1568}; H = \frac{2349\,c^3\,\pi}{21952}.$$

Thus we obtain:

$$\frac{H}{P} = \frac{29c}{42.}$$

This shows a failure of the relationship 2.9.

The second solution, which we found to be numerically unstable, is

$$u = -\frac{9c}{10}\cos^2[\sqrt{\frac{5}{12}}(x-ct)];\ c > 0. \tag{3.17}$$

The conserved quantities H, M, P are given by

$$M = -\frac{9\pi c\sqrt{3}}{10\sqrt{5}};\ P = \frac{243\pi c^2\sqrt{3}}{800\sqrt{5}};\ H = -\frac{81\pi c^3\sqrt{3}}{1600\sqrt{5}}. \tag{3.18}$$

Thus we again find the breakdown of the relationship as given by Eq. (2.9). It may be noted here that the special solutions with $c_1 \neq 0$ are not obtainable from a variational calculation. In any case, these are a very restricted class of solutions.

B. Case $p = m = l, n = 0$

For this case inserting a trial solution of the form $u = A\cos^r dy$ usually leads to an overdetermined set of equations. For example for $p = 1, r = 2$ we obtain conditions which has only a trivial solution. For $p = 2$ the situation is simpler and one obtains when $r = 1$ the two relations:

$$A^2 = \frac{c}{2\,\beta\,d^2 + 5\,d^6\,\gamma}$$

and

$$0 = 18d^6\gamma + 12d^2\beta - \alpha$$

These equations can have 2 or even 3 positive solutions for d^2. One particular case is $\alpha = 216$, $\beta = 21$, $\gamma = -2$. In that case we get two positive solutions for d^2 namely $d = 1$ leading to : (i)

$$u = \frac{\sqrt{c/2}}{4}\cos(x-ct) \tag{3.19}$$

where the constants of motion are:

$$M = \frac{1}{2}\sqrt{c/2}; P = \frac{c\pi}{128}; H = -\frac{\pi c^2}{256}$$

and $d^2 = 2$ giving the solution: (ii)

$$u = \frac{\sqrt{c}}{2}\cos\sqrt{2}(x - ct) \tag{3.20}$$

and we obtain for the conserved quantities:

$$M = \sqrt{c/2}; P = \frac{c\pi}{16\sqrt{2}}; H = -\frac{\pi c^2}{32\sqrt{2}}$$

Thus both these solutions again obey the relationship as given by Eq. (2.9). The second solution with $d^2 = 2$ turns out to be numerically unstable.

For $p = 2$ and $r = 2$ we get relations:

$$\begin{aligned} &c_1 = 8A^3 d^6 \gamma \\ &c - -208A^2 d^6 \gamma \\ &-\alpha + 48\beta d^2 + 1152 d^6 \gamma = 0 \end{aligned} \tag{3.21}$$

as well as one constraints among the parameters

$$\beta = -48 d^4 \gamma$$

Eliminating the constraint, we obtain for the width:

$$d^6 = -\frac{\alpha}{1152\gamma} \tag{3.22}$$

As an example if we choose $\gamma = -3$ and $\alpha = 3456$ then we have $d = 1$ and for our solution:

$$u = \frac{1}{4}\left(\frac{c}{39}\right)^{1/2}\cos^2(x - ct)$$

We have not exhausted all possible solutions for this case, but the method for finding them should be clear to the reader by now.

C. Some Other More General Cases

When $l + n = p = m$, we instead have one parameter family of solutions depending on the velocity c. That is, for fixed α β and γ there is a solution of different amplitude for different velocities c. In some special cases there is the possibility for two different solutions with the same value of c. However, in general for a given α β and γ there is only one solution with a fixed velocity c. To illustrate this fact, let us consider the case

$$p = n; \; m = l = 0$$

Inserting a trial solution of the form

$$u = A\cos^{2/p}[d(x - ct)]; \; p \neq 2 \tag{3.23}$$

into Eq. (2.8) we obtain for example if $\gamma = 1$, $\beta = 1$, and $\alpha = -1/2$ the conditions:

$$d = \frac{p}{2}[(p+1)(2p^2 + 4p + 3)]^{-1/4}$$

$$A^p = \frac{8\beta}{(4 - p^2)}[(p+1)(2p^2 + 4p + 3)]^{1/2}$$

$$c = \frac{2\beta(p^2 + 8p + 4)}{(4 - p^2)\sqrt{(p+1)(2p^2 + 4p + 3)}}. \tag{3.24}$$

Choosing for example $p = n = 1$, $l = m = 0$, $\gamma = 1$ and $\alpha = -1/2$ we get the single solution:

$$u = 8\sqrt{2}\cos^2\left[\frac{(x - \frac{13\sqrt{2}}{9}t)}{2(18)^{1/4}}\right] \tag{3.25}$$

The constants of motion for this case are:

$$M = 8\pi(2)^{3/4}\sqrt{3}; P = 48\pi(2)^{1/4}\sqrt{3}; H = -\frac{400\pi(2)^{3/4}}{3\sqrt{3}}$$

So that $H/P = -\frac{25c}{39}$ which not unexpectedly does not obey relation (2.9). Note that in this case, the width of the compacton solution is not independent of its amplitude.

For the special case when $p = n = 2$,one obtains

$$u = A\cos dy$$

When $m = l = 0$ we find

$$d^4 = -\frac{\alpha}{30\gamma}; A^2 = \frac{150\gamma}{\alpha^2}\left(c + \frac{\alpha\beta}{15\gamma}\right).$$

Choosing further $\beta = \alpha = 1$; $\gamma = -1/30$, we obtain

$$u = \sqrt{10 - c}\; \cos y.$$

So again for this special case we get a continuous family of solutions as long as $c < 10$. The constants of motion are now:

$$M = 2\sqrt{10 - c}; P = \frac{\pi}{4}(10 - c); H = \frac{\pi}{40}(10 - c)(30 - c)$$

and relation (2.9) is again not obeyed.

For any particular value of p, m, n, l one can always find the consistency condition on the parameters α, β, γ, A, d, q and c so that solitary wave solutions of the form $A\cos^q[dy]$ exist. However we do not have a simple expression for these parameters for all p, m, n, l, and have instead in the above looked at some simple cases.

IV. NUMERICAL STUDY OF THE GENERALIZED KDV EQUATION

A. Numerical Method

In our calculations, we approximated the spatial derivatives with a pseudo-spectral method [14] using the discrete Fourier transform (DFT) [15]. The equations were integrated in time with a variable order, variable timestep Adams-Bashford-Moulton method using the method of lines approach as described in detail by Schiesser [16]. The numerical errors were monitored by varying the number of discrete Fourier modes between 128 and 512 and varying the estimated time error per unit step between 10^{-6} and 10^{-9} to insure that the solutions were well converged to within 10^{-4} in the L_2 norm. Also, the mass and momentum was conserved to an accuracy of at least 10^{-4} and the Hamiltonian was conserved to an accuracy of better than 10^{-2} in all of the calculations.

The numerical approximation must respect the delicate balance between the nonlinear numerical dispersion terms in the equation. For example, when the third term in Eq.(1.6) is expanded, it has a diffusion-like term $2\beta m(u^{m-1}u_x)u_{xx}$. On the trailing edge of the compacton $u_x > 0$ and this term acts like a destabilizing backward diffusion operator. The solution would be unstable if it were not for the stabilizing nonlinear dispersion. This balance is easily lost in numerical approximation if the aliasing, due to the nonlinearities, is not handled carefully. The loss of this delicate balance in very steep fronts may be the reason that the numerical simulations break down for very narrow initial data.

To identify numerical artifacts due to aliasing and other discrete affects, we solved the equations with the nonlinear terms expanded in different formulations. We compared the solutions of Eq.(1.6) when they were differenced in divergence form and when the derivatives in the nonlinear terms were expanded. Although the numerical solutions in all of these formulations were qualitatively similar, in very long integrations we found that integrating Eq.(1.6) in divergence form was more stable and preserved the conservation laws better than the approximations where the nonlinear terms were expanded.

The lack of smoothness at the edge of the compacton introduces high frequency dispersive errors into the calculation. These dispersive errors can destroy the accuracy of the simulation unless they are explicitly damped. To reduce these errors while preserving as much accuracy as we could for the lower frequency modes in the solution, we explicitly added an artificial dissipation (hyperviscosity) term $\delta \Delta x F u_{xx}$ to the right side of Eq.(1.6). The high pass filter F was defined in Fourier space to eliminate the lowest 1/3 Fourier modes, leave the highest 1/3 modes unchanged and was a linear transition between the two regions. Thus the dissipation has no direct effect on the lower 1/3 of the Fourier modes of the solution and only introduces dissipation into the higher modes. We also experimented in solving the equation with other artificial dissipation terms based on fourth spatial derivatives and mixed space time derivatives. All of the solutions behaved qualitatively the same, but the filtered hyperviscosity approach preserved the conserved quantities better than any other approach we tried and was the most stable for the widest range of problems. This is the same hyperviscosity term used in the original Rosenau-Hyman calculations. The modified PDE with the artificial dissipation no longer preserves the Hamiltonian and we chose the parameter δ experimentally to minimize the aliasing errors, while conservering the Hamiltonian to within 1%. The momentum L_2 norm of the solution was also monitored and converged within 0.01%.

We used a local average of $-u_t/u_x$ to estimate the local travelling wave velocity and computed the quantity $(u^{1/r})_{xx}/u^{1/r}$, with $r = 1, 2$ and 4 to verify the shape of the solution. Both of these quantities are constant for traveling wave solutions of the form Eq. (2.10). This approach allowed us to estimate the velocity of the wave and the parameters in Eq. (2.10) within 0.1%.

In the simulations shown here we used $\delta = 10$, solved the divergence form of the equations with 128 DFT modes, and specified a time error of 10^{-8} per unit time.

B. Numerical Investigations

For the original Rosenau and Hyman compacton equations numerical investigations showed some remarkable properties, namely that whatever initial compact data was given, it eventually evolved into compactons. When two compactons scattered any energy not in the original pair of compactons emerged as compacton-anticompacton pairs. We will find that the compactons of this fifth order generalized KdV equation have similar properties to those previously found in the studies of Rosenau and Hyman on their third order generalized KdV equation. However, unlike the Rosenau-Hyman equation with $m = n = 2$, the momentum is conserved in Eq.(1.6). Therefore, when a compacton-anticompacton pair is created in a collision the compactons must necessarily leave momentum behind and the collision cannot be elastic.

The first generic feature of these equations is that arbitrary initial compact data, as long as the width of the packet is larger than that of the compacton evolves into several compactons with the number depending on the initial energy. We remark that when the initial pulse was much narrower than the compacton that minimizes the reduced Hamiltonian, then our numerical solutions were unstable at the leading edge of the pulse. This instability was independent of the number of Fourier modes used in the numerical simulation.

We show the decomposition of a wide intial pulse for two different cases. The first case is related to the compacton of Eq.(3.16). We start of with an initial pulse which is four times the width of the compacton and watch it evolve. This is shown in Fig. 1.

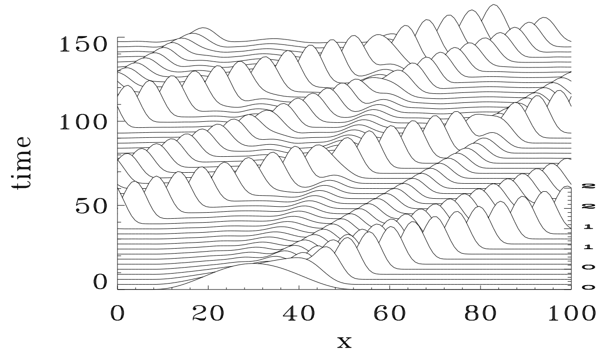

Fig. 1. Pulse with an initial width four times that of the compacton of Eq.(3.16) pertaining to the parameters $p = m = n = 1, l = 0$ and $\alpha = 5, \beta = 3, \gamma = 1$ namely: $u_0 = \frac{9}{14}\cos^2(\frac{x-30}{4\sqrt{12}})$. The initial wide pulse breaks into compactons that collide nearly elastically. Note the phase shift of the slower pulse after colliding with a faster, higher compacton.

In figures 2 and 3 we show the same phenomena for the compacton system described by Eq.(3.4), again starting from initial data wide compared to the compacton solution.

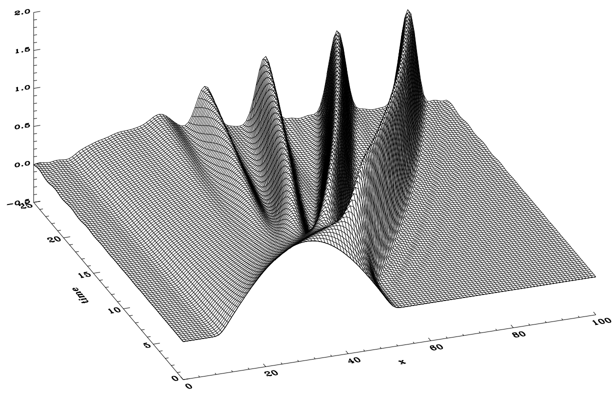

Fig. 2. The decomposition of an initial pulse for the parameters $p = m = n = 2, l = 0$ and $\alpha = 6, \beta = 4, \gamma = 3$ relevant to Eq.(3.4). An initial compact wave (solid line) $u_0 = \sqrt{\frac{25}{22}}\cos(\frac{x-30}{6})$ wider than a compacton with breaks into a string of compactons with the shape $A\cos(\frac{x-ct}{\sqrt{5}})$.

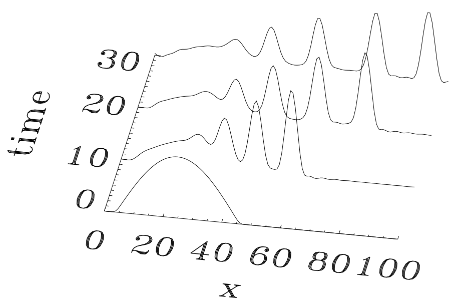

Fig. 3. A different graphical view of break up of the initial conditions shown in Fig.2 decomposing into compactons.

In figures 4, and 5 we show similar features of the breakup of a compact wave for the compacton Eq.(3.9).

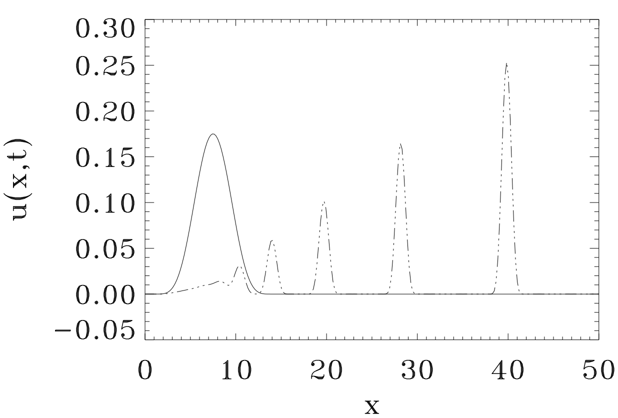

Fig. 4 Break up of a compact wave with four times the width of the compacton: $p = m = n = 1, l = 0$ and $\alpha = \frac{672}{19}, \beta = 1, \gamma = \frac{1}{38}$. An initial compact wave (solid line) $u_0 = 2\frac{19}{216}\cos^4(\frac{x-7.5}{4})$ breaks into a string of compactons with the shape $A\cos^4(x-ct)$ by time $t = 10$ (dashed line).

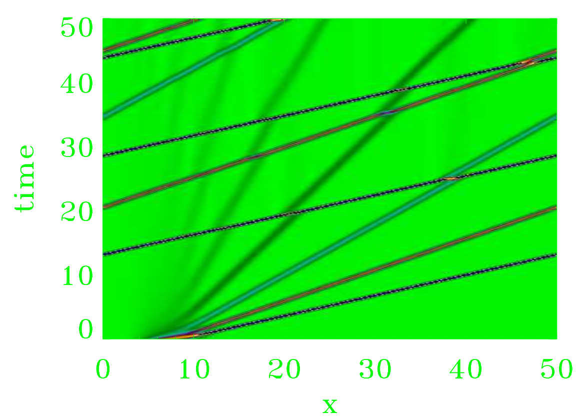

Fig. 5 Gray scale contour plot of the evolution of the compactons in Figs. 4.

Then next generic feature is what happens when two compactons of different speeds collide. The compactons remain coherent and experience a phase shift. This is shown in Fig. 6 for the compactons described by Eq.(3.16).
There are significant differences between the collision dynamics of the compactons and the solitons in integrable equations. The main difference is that in an integrable system, the infinite number of conservation laws only allow for time delays and the final product of scattering does not change the shape at all. Here, the resulting solitary waves have slightly reduced amplitude, but otherwise maintain their shape after scattering. Unlike the soliton collisions in an integrable system, the point where two compactons collide is marked by the creation of a low amplitude ($< 5\%$) compact oscillatory ripple. This ripple does not disperse but becomes compacton anticompacton pairs. These resultant compactons also have exactly the same shape (apart from their amplitude) as the initial ones. After the collision, the original compactons emerge intact, just like classical, integrable solitons, indicating that the remarkable stability of solitary waves lies deeper than mere integrability. The first collision shown in Fig. 6 creates a ripple, shown in Fig. 7, that decomposes into compacton-anticompacton pairs.

In an earlier numerical study of nonlinear wave phenomena [17] in generalized KdV equations it was also found that following the scattering of solitary waves a ripple or wake was left behind. However these authors did not have enough resolution to show if this wake turned into solitary waves or whether the wake dispersed.

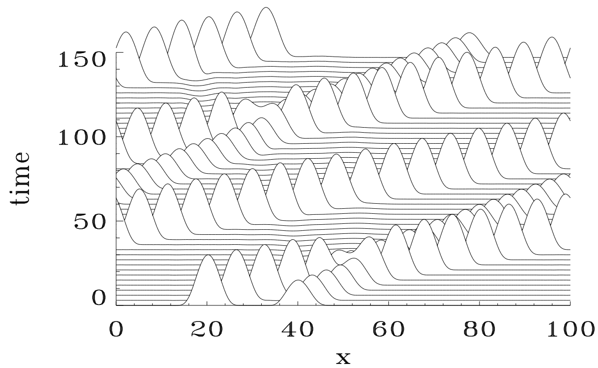

Fig. 6. Collision of two solitons for the case $p = m = n = 1, l = 0$ and $\alpha = 5, \beta = 3, \gamma = 1$. Two compactons described by Eq.(3.16) collide. One has speed $c = 2$; and the other has speed $c = 1$; Note the phase shift in the slower compacton after the collision. These compactons remained coherent, even after dozens more collisions.

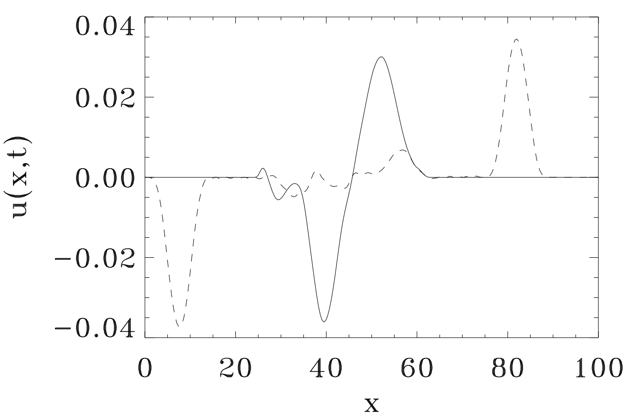

Fig.7. A ripple (solid line) is created when the compactons first collide in Fig. 6. This ripple was extracted from the solution at $t = 25$ and used as an initial condition. By time $t = 500$, the ripple (dashed line) has separated into compactons traveling in opposite directions. These compactons have a shape proportional to $A\cos^2(\frac{y}{\sqrt{12}})$.

When a compacton and an anticompacton collide, as well as when one starts with initial data which is narrower than the width of the stable compacton one finds numerically blow up at late times. The numerical simulations converge up to the blowup time as the grid is refined. However, although we believe the blow up is a property of the differential equation, it is not certain that it is not a numerical artifact. This effect is shown in Fig. 8 for the same compactons as in Figs. 6 and 7.

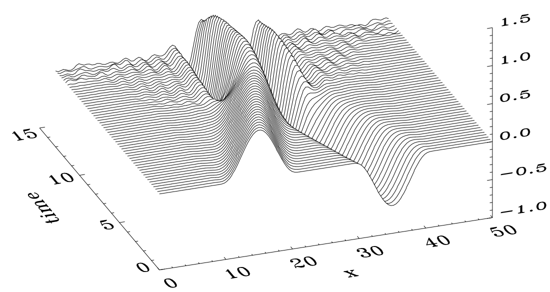

Fig. 8. Possible blowup after a compacton anticompacton collision. $p = m = n = 1, l = 0$ and $\alpha = 5, \beta = 3, \gamma = 1$. Two compactons described by Eq.(3.16) with speed $c = +1$ and $c = -1$ collide. The numerical solution breaks down slightly after $t = 15$. It is not clear if the break down is due to the steep gradients in the solution, or because there is a true singularity that develops in the equations.

V. SUMMARY

We have generalized the Lagrangian for the KdV equation to one that supports a wide class of KdV-like equations and preserves the invariance of the action under time, and space translations as well as the shift of the field by a constant. We have derived explicit formulas for the traveling waves for this equation and demonstrated in numerical experiments that the traveling waves exhibit soliton-like behavior. Even though the equation is most likely not integrable and satisfies only a handful of conservation laws, our numerical experiments indicate that the compactons for these equations play the role of nonlinear local basis functions. Positive compact initial data (wider than a compacton) decomposes into a train of nonlinearly stable compactons. The robustness of these compactons interaction and the inapplicability of the inverse scattering tools that worked so well for integrable equations.

Following the initial research presented in this paper, Dey [18] considered a slightly different version of the 5th order KdV-like equation and obtained compacton solutions which are similar to solutions described here. Our future study of these nonlinear PDEs will aim at understanding the nonlinear mechanism that causes these structures to be so robust. Research into this mechanism has the potential opening new doors in our understanding of the central role of solitons in nonlinear dispersion.

APPENDIX- VARIATIONAL APPROACH

Our time-dependent variational approach for studying solitary waves is based on the principle of least Action. In previous work [6] [19] [20] [21], we introduced a post-Gaussian variational approximation, a continuous family of trial variational functions more general than Gaussians, which can still be treated analytically. We assumed a variational ansatz of the form

$$u_v(x,t) = A(t)\exp\left[-b(t)\,|x-q(t)|^{2s}\right]$$

The variational parameters have a simple interpretation in terms of expectation values with respect to the "probability" $\bar{P}$

$$\bar{P}(x,t) = \frac{[u_v(x,t)]^2}{2P}, \tag{5.1}$$

where the conserved momentum P is defined as above

$$P \equiv \frac{1}{2}\int [u_v(x,t)]^2\,dx. \tag{5.2}$$

We have that $q(t) = \langle x \rangle$,

$$G_{2s} \equiv \langle |x-q(t)|^{2s}\rangle = \frac{1}{4sb}. \tag{5.3}$$

and

$$A(t) = \frac{P^{1/2}(2b)^{1/4s}}{\left[\Gamma\left(\frac{1}{2s}+1\right)\right]^{1/2}}. \tag{5.4}$$

Extremizing the effective action for the trial wave function u_v leads to Lagrange's equations for the variational parameters. We find that for all values of the parameters (l,m,n,p) the dynamics of the variational parameters lead to solitary waves moving with constant velocity and constant width. However the solutions found are often maxima or saddle points of the action. We find from our numerical experiments that when this happens, our computer simulation of the exact solitary wave solution is unstable. In particular, the numerical simulation of the exact narrow travelling wave, and in fact any compact initial conditions that are significantly narrower than the stable wider compacton, blow up at the leading edge of the pulse. We do not know if this blowup is due to a numerical instability or to an inherent instability in the equations.

For the special case of $p = m = n + l$ a simple variational calculation using these trial wave functions, similar to that found in [6] tells us that the width of the soliton is independent of the amplitude and velocity. For that case we also obtain the relationship:

$$H = -\frac{2c}{p+2}P\,. \tag{5.5}$$

The exact solitary wave solutions satisfy this relationship as long as the integration constant c_1 of Eq. (2.8)is zero.

The starting point for the variational calculation is the action $\Gamma = \int L dt$, where L is given by (1.3). Inserting the trial wave function u_v we obtain:

$$\Gamma(q,\beta,P,s) = \int dt[-P\dot{q} - H_{eff}]; \tag{5.6}$$

where H_{eff} is the Hamiltonian evaluated using the variational wave function u_v. The effective Hamiltonian is a function of the variational paramers P, b, s. The parameters b and s are determined by finding the stationary points of the action. This leads either to H being either a minimum, maximum or saddle point as a function of these variables. Only the approximate solitary waves corresponding to a minimum of the effective Hamiltonian turn out to correspond to stable exact solitary wave solutions of the compacton variety.

Exact Variational Ansatz

Now we would like to ask the question to what extent we could recover from the variational ansatz the exact solitary wave solutions we have discovered earlier by trial and error. That is if we assume solutions of the form

$$A(t)\cos^r[b(t)(x-q(t))];\ \frac{-\pi}{2} \le b(x-q(t)) \le \frac{\pi}{2}$$

for compact solitary waves and

$$A(t)\mathrm{sech}^r[b(t)(x-q(t))]$$

for ordinary solitary waves would we recover all the exact solutions? We also want to make the suggestion, that the stability of the solitary wave solutions found in this manner is determined by whether these solutions are minima of the effective Hamiltonian as a function of the parameter $b(t)$.

First let us show that for the KdV equation and for the generalized KdV equation that we investigated earlier, we indeed obtain the exact solution. The Lagrangian for the KdV equation is

$$L = \int dx\ [\frac{1}{2}\varphi_x\varphi_t - (\varphi_x)^3 - \frac{1}{2}(\varphi_{xx})^2].$$

The conserved Hamiltonian is given by

$$H = \int dx\ [(\varphi_x)^3 + \frac{1}{2}(\varphi_{xx})^2].$$

Assuming the trial wave function:

$$\varphi_x = u(x,t) = A(t)\mathrm{sech}^2[b(t)(x-q(t)] \tag{5.7}$$

we find that the reduced action is

$$\Gamma = -P\dot{q} - H[A(P),b], \tag{5.8}$$

where

$$P = \int \frac{1}{2}u^2 dx = \frac{2A^2(t)}{3b} \tag{5.9}$$

and

$$H = \frac{8}{15}A^2(\frac{2A}{b} + b) \tag{5.10}$$

We can rewrite H in terms of A as follows:

$$H = \frac{4}{5}P(\pm(6bP)^{1/2} + b^2) \tag{5.11}$$

where we have used the two possible solutions:

$$A = \pm(\frac{3bP}{2})^{1/2}$$

Since the Hamiltonian is independent of q, we have that P is conserved. b is a variable of constraint and is eliminated by the equation:

$$\frac{\partial H}{\partial b} = 0 = \pm(\frac{6P}{b})^{1/2} + 2b \tag{5.12}$$

Only the negative choice for A in terms of P yields a positive solution for b namely:

$$b = \frac{(6P)^{1/3}}{4^{2/3}} \tag{5.13}$$

Eliminating b the reduced action is

$$\Gamma = -P\dot{q} + \frac{(3P)^{5/3}}{5} \tag{5.14}$$

Varying the action we find the velocity is a constant:

$$\dot{q} = (3P)^{2/3} = c$$

Thus $A = -\frac{c}{2}$ and we get the usual exact answer:

$$u(x,t) = -\frac{c}{2}\mathrm{sech}^2[\frac{c^{1/2}}{2}(x-ct)] \tag{5.15}$$

We also find that

$$H = \frac{4}{5}P(-(6bP)^{1/2} + b^2) \tag{5.16}$$

has a minimum at the exact value of b for fixed P.

Next we consider the class of exact compact solitary waves that we found for the generalized KdV equation of ref. [7] . In this case the Lagrangian is

$$L = \int dx\ [\frac{1}{2}\varphi_x\varphi_t + \alpha\frac{1}{p(p+1)}(\varphi_x)^{p+2} - \beta\varphi_x^p(\varphi_{xx})^2].$$

and the Hamiltonian is:

$$H = \int dx[-\frac{\alpha}{p(p+1)}(\varphi_x)^{p+2} + \beta\varphi_x^p(\varphi_{xx})^2].$$

Now we assume a solution of the form ($r = 2/p$)

$$\varphi_x = u(x,t) = A\cos^{2/p}[d(t)(x-q(t))] \tag{5.17}$$

We obtain for the reduced action

$$\Gamma = -P\dot{q} - H[A,d] \tag{5.18}$$

where now

$$P = \frac{A^2\sqrt{\pi}\Gamma(1/2+2/p)}{2d(t)\Gamma(1+2/p)} \tag{5.19}$$

$$H = \frac{A^{2+p}\{4\beta d^2(2+3p+p^2) - \alpha(4p+p^2)\}\sqrt{\pi}\Gamma(\frac{1}{2}+\frac{2}{p})}{2dp^2\,(1+p)\,(2+p)\,\Gamma(2+\frac{2}{p})} \tag{5.20}$$

On using the relation as given by Eq. (5.19) we obtain:

$$H = (2d)^{\frac{p}{2}}\,P^{1+\frac{p}{2}}\,\frac{\{4\beta d^2(2+3p+p^2) - \alpha(4p+p^2)\}\,\Gamma^{\frac{p}{2}}(1+\frac{2}{p})}{(2\,p+5\,p^2+4\,p^3+p^4)\ \pi^{\frac{p}{4}}\,\Gamma^{\frac{p}{2}}(\frac{1}{2}+\frac{2}{p})} \tag{5.21}$$

We determine the constraint variable d by $\frac{\partial H}{\partial d} = 0$, we obtain

$$d^2 = \frac{\alpha p^2}{4\beta(p+1)(p+2)} \tag{5.22}$$

Lagrange's equations gives:

$$\dot{q} = c = -\frac{\partial H}{\partial P} = -\frac{p+2}{2}\{\frac{H}{P}\} \tag{5.23}$$

We then have that

$$A^p = \frac{c(p+1)(p+2)}{2\alpha}$$

and recover our previous exact result [7]:

$$u(x,t) = [\frac{c(p+1)(p+2)}{2\alpha}]^{1/p}\cos^{2/p}\{\frac{p(x-ct)}{[4\alpha(p+1)(p+2)]^{1/2}}\} \tag{5.24}$$

As a function of d for fixed P, H is a minimum at the constraint equation value of d. As an example, when $p=1$,$P=1$ and $\beta = 1/2$,$\alpha = 1$ one obtains for $H[d]$:

$$H = \frac{2}{9}(\frac{d}{3\pi})^{1/2}(-5+12d^2)$$

which has a minimum at $d^2 = 1/12$.

Now let us look at our generalized equation when $\gamma \neq 0$. For the special case $p=m=n; l=0$ considered in this paper we have that the Lagrangian is:

$$L(p=m=n;l=0) = \int dx\ [\frac{1}{2}\varphi_x\varphi_t + \alpha\frac{(\varphi_x)^{p+2}}{(p+1)(p+2)} - \beta(\varphi_x)^p(\varphi_{xx})^2 + \frac{\gamma}{2}\varphi_x^p\varphi_{xxx}^2], \tag{5.25}$$

Introducing a trial variational function of the form:

$$u = A\cos^r d(t)(x-q(t)) \tag{5.26}$$

with $r = 4/p$ and the constraint:

$$\alpha = \frac{(2+r)(4+r)(-4+4r+r^2)\beta^2}{\gamma(-2+r^2+6r)^2} \tag{5.27}$$

Using the fact that

$$P = A^2\sqrt{\pi}\frac{\Gamma(1/2+r)}{2d\ \Gamma(1+r)}$$

to eliminate A in favor of P, we again find we can write the reduced action as

$$\int dt\,\{-P\dot{q} - H[P,d]\} \tag{5.28}$$

where

$$\begin{aligned} H =& (2d)^{2/r}P^{1+2/r}r^2\Gamma[1+r]^{1+2/r}\times \\ &(12\beta^2 + 8\beta d^2\gamma - 32d^4\gamma^2 + 20\beta^2 r - 32\beta d^2\gamma r + 176d^4\gamma^2 r \\ &-19\beta^2r^2 - 32\beta d^2\gamma r^2 - 172d^4\gamma^2r^2 - 24\beta^2r^3 \\ &+152\beta d^2\gamma r^3 - 152d^4\gamma^2r^3 - 4\beta^2r^4 + 50\beta d^2\gamma r^4 \\ &-152d^4\gamma^2r^4 + 4\beta d^2\gamma r^5 - 40d^4\gamma^2r^5 - 3d^4\gamma^2r^6)\times \\ &(4\gamma\pi^{1/r}\left(-2+6r+r^2\right)^2\Gamma[3+r]\Gamma[1/2+r]^{2/r})^{-1} \end{aligned} \tag{5.29}$$

From the equation that eliminates the constraint variable d, $\frac{\partial H}{\partial d} = 0$, we find there are two solutions for d^2. One solution,

$$d^2 = \frac{\beta}{\gamma(r^2+6r-2)} \tag{5.30}$$

is a minimum of $H[d]$ for fixed P and is an exact solution of the generalized KdV equation. The other solution for d^2

$$d^2 = \frac{\beta\,(3+2\,r)\,\left(-4+4\,r+r^2\right)}{\gamma\,(-2+6\,r+r^2)\,(8+4\,r+3\,r^2)} \tag{5.31}$$

is a maximum of the energy $H[d]$ for fixed P and is not a solution of the equation of motion. An example discussed earlier is the case $p = r = 2$ with $\beta = 1$, $\gamma = 1/14$, and $\alpha = 96/7$. In that case we have

$$H[d] = -\frac{8P^2}{9\pi}(10d - 5d^3 + d^5), \tag{5.32}$$

with two extrema: $d = 1$ which is a minimum of H and yields an exact solution, $u = \sqrt{c/2}\cos^2(x - ct)$, and $d = \sqrt{2}$ which is a maximum and leads to $u = \sqrt{3c/2}\cos^2\sqrt{2}(x - ct)$ which is not a solution of the original generalized KdV equation.

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