Design of Serially Concatenated Trellis Coded Modulation

S. Benedetto, D. Divsalar, R. Garello, G. Montorsi, F. Pollara

S. Benedetto, R. Garello, and G. Montorsi are with Dipartimento di Elettronica, Politecnico di Torino, 10129 Torino, Italy.
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D. Divsalar and F. Pollara are with Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109.
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Abstract

Serial concatenation of an outer binary convolutional code with an inner TCM code over a multidimensional Euclidean constellation through an interleaver, allows to extend the extremely good performance of turbo codes to the case of high spectral efficiency. For constituent codes of concatenated schemes, input/output relationships of the encoders have a strong impact on performance; in this framework, Uniform Bit Error Property (UBEP) simplifies the design of good encoders. We introduce a class of TCM encoders, called *bit geometrically uniform* (BGU) encoders, for which the UBEP holds, and the bit error probability does not depend on the transmitted sequence. The properties of BGU encoders prove quite useful for the design of good serially concatenated trellis coded modulation. Performance of new concatenated codes with spectral efficiency 2 bps/Hz, constructed by this approach, are presented, in terms of simulation and analytical results. trellis code

I. Introduction

A Euclidean-space constellation has the Uniform Error Property (UEP) if the *symbol* error probability does not depend on the transmitted signal. For Geometrically Uniform (GU) constellations [1] the Voronoi (decision) regions of the signals are all congruent and the UEP holds. Most of the usual constellations are either GU (PSK, M-ary orthogonal constellations, Slepian constellations, infinite lattices), or approximately GU if we neglect boundary effects (PAM, QAM).

The notion of geometrical uniformity can be extended to Euclidean-space codes of finite or infinite length over GU constellations [1]. Also in this case, the Voronoi regions of GU code sequences are congruent and the symbol sequence error probability (often called *error event* probability) can be computed from any signal sequence.

For most transmission systems, the *bit* error probability (or the *frame* error probability) is a more important performance measure than the symbol sequence error probability, and can be computed by considering the binary information sequences that must be transmitted to the receiver. Association between binary information sequences and signal sequences is provided by *labelings* for uncoded constellations, and by *encoders* for codes.

In classical coding theory, encoders are often not considered, or somehow confused with the code; the main reason is that given a code, BER performance of various encoders generating it are usually almost the same, and differences are limited to fractions of dB. As a consequence, the attention in design is focused on the code and, usually, on its (Hamming or Euclidean) distance properties. In this case, UEP is of great utility, because it dramatically simplifies code performance analysis.

In some applications, however, a clear separation between codes and encoders is essential (e.g. the concept of rotational invariance [2]), or different encoders generating the same code lead to significantly different performance (e.g. constituent codes of turbo codes [3], [4]). In this case the design must be focused on the encoder/code pair.

We first extend the concept of UEP to encompass the binary information sequences, and look for the conditions under which also the bit error probability can be computed by considering any transmitted sequence, for example the all-zero sequence. While UEP applies to the code, this property, which will be called *uniform bit error property* (UBEP) in the following, applies to the encoder/code pair. As UEP for code design, UBEP proves very useful for encoder design because it highly simplifies performance analysis.

We derive some conditions under which an encoder satisfies UBEP, and apply this concept to the design of good serially concatenated TCM codes [5], a technique that provides very good performance at high spectral efficiency via iterative decoding strategies. As for all concatenated codes with interleavers, whose aim is the minimization of the bit error probability obtained acting on the multiplicities of near neighbours, the input/output relationship of constituent encoders has a strong impact on performance, and UBEP properties highly simplify the design of good TCM encoders.

In Section II we discuss the properties of labelings for uncoded constellations. A distance rule is introduced, and labelings satisfying it are called *bit geometrically uniform* (BGU) labelings. It is shown that a BGU labeling satisfies the UBEP, and that a 2^k-signal GU constellation admits a BGU labeling if and only if it has a generating group isomorphic to a generating group of the Hamming space $\mathbf{H}_k$.

In Section III codes with infinite length are considered; the distance rule is defined on the code trellis. Some necessary and sufficient conditions for BGU encoders are derived: they are simplified when abelian generating groups are involved.

In Section IV the principles of serially concatenated TCM (SCTCM) codes are briefly recalled. A main parameter for the design of these codes is the *effective free Euclidean distance* $d_{\mathrm{f,eff}}$ of the inner TCM encoder [5], defined as the minimum Euclidean distance between code sequences generated by information sequences that differ only by two bits. BGU encoder properties are applied to the construction of good constituent TCM encoders with large $d_{\mathrm{f,eff}}$. Performance of new good SCTCM codes are presented.

II. Bit geometrically uniform encoders for signal constellations

A Euclidean-space constellation $\mathcal{S} \subseteq \mathbf{R}^n$ is a set of signals (points) in the n-dimensional Euclidean space. Finite constellations will have cardinality $M = 2^k$ in this paper.

The Hamming space $\mathbf{H}_k$ is the set of all 2^k binary k-ples. Given a finite constellation $\mathcal{S}$, a binary labeling $E[\mathcal{S}, k]$ for $\mathcal{S}$ is a one-to-one function

$$E : \mathcal{S} \leftrightarrow \mathbf{H}_k$$

that associates a distinct k-bit information label $E(s)$ to each signal $s \in \mathcal{S}$.

The bit error probability with ML symbol decoding, when a signal s_i is transmitted is:

$$P(b|s_T = s_i) = \sum_{j \neq i} \frac{w_H(E(s_j) + E(s_i))}{k} P(s_R = s_j | s_T = s_i),$$

where s_T and s_R are the transmitted and the received signal. The average bit error probability over $\mathcal{S}$ with ML symbol decoding is:

$$P(b) = \frac{1}{M} \sum_{i=0}^{M-1} P(b|s_T = s_i)$$

Definition 1

A binary labeling $E[\mathcal{S}, k]$ is said to satisfy the uniform bit error property (UBEP) if the average bit error probability obtained with maximum likelihood (ML) symbol decoding does not depend on the transmitted signal, i.e., *$P(b|s_T = s_i)$ is the same for each signal $s_i \in \mathcal{S}$. It follows: $P(b) = P(b|s_T = s_i)$.* △

A symmetry ρ of a constellation $\mathcal{S}$ is an isometric permutation, *i.e.*, a permutation of the signals that preserves Euclidean distance. The symmetry group $\Gamma(\mathcal{S})$ is the group of all symmetries of $\mathcal{S}$. If $\Gamma(\mathcal{S})$ is transitive, *i.e.*, $\Gamma(\mathcal{S})(s_i) = \mathcal{S}, \forall s_i \in \mathcal{S}$, the constellation is called Geometrically Uniform (GU) [1]. A GU constellation has congruent Voronoi regions, and the UEP holds [1].

A GU constellation is a good starting point for a BGU labeling; when a signal s_i is transmitted, the relative position of all possible "wrong" decoded signals $s_j \neq s_i$ is the same for any s_i. When also the number of "wrong" bits associated by the labeling E to s_j only depends on the relative position of the pair (s_i, s_j), the labeling E satisfies the UBEP.

Example 1

The 8-PSK constellation with the natural binary labeling of Figure 1 a) does not satisfy the UBEP: if s_0 is transmitted, three bits are wrong if s_7 is received and one bit is wrong if s_1 is received; if s_1 is transmitted, two bits are wrong if s_2 is received and one bit is wrong if s_0 is received; ◇

Given a GU constellation $\mathcal{S}$, consider now a *generating group* $G \subseteq \Gamma(\mathcal{S})$, i.e., a 2^k-element transitive subgroup of $\Gamma(\mathcal{S})$; by definition [1], we have $G(s) = \mathcal{S}$, for each $s \in \mathcal{S}$. When we fix the zero signal s_0, there is a one-to-one correspondence between an element

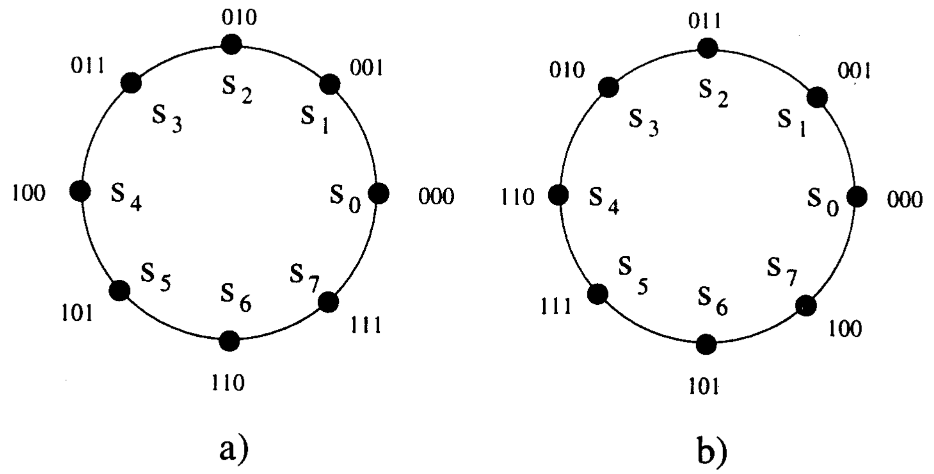

Fig. 1. 8-PSK: a) natural labeling and b) Gray labeling.

$g \in G$ and a signal $g(s_0) \in \mathcal{S}$, so that we will also write the labeling $E[\mathcal{S}, k]$ as $E[G, k]$ and the labeling function $E(g(s_0))$ as $E(g)$. We will also always assume $E(0) = \mathbf{0}$, where 0 is the identity of G and $\mathbf{0}$ is the k-bit all-zero label.

Lemma 1

Given a GU constellation $\mathcal{S}$ with generating group G, if a labeling $E[G, k]$ satisfies the following distance rule: [1]

$$d_H(E(g_i), E(g_j)) = w_H(E(-g_i + g_j)) \quad \forall g_i, g_j \in G, \tag{1}$$

the UBEP holds. This labeling will be called a *bit geometrically uniform* (BGU) labeling

∇

Proof: Given any signal $s_i \in \mathcal{S}$, for each signal pair $(s_i, s_j) = (g_i(s_0), g_j(s_0))$, with $d_H(E(g_i), E(g_j)) = x$, there exists a pair (s_0, s^*), such that $s^* = g^*(s_0) = (-g_i + g_j)(s_0)$
with $d_H(E(0), E(g^*)) = w_H(E(g^*)) = x$. We have:

$$P(b, s_j|s_i) = \frac{d_H(E(g_i), E(g_j))}{k} P(s_j|s_i) = \frac{w_H(E(g^*))}{k} P(s^*|s_0) = P(b, s^*|s_0),$$

where the symbol error probability $P(s_j|s_i) = P(s^*|s_0)$ for the GU property. It follows that
$P(b|s_i) = P(b|s_0) = P(b) \ \forall s_i \in \mathcal{S}$. ∎

[1] The additive notation will be used also for nonabelian groups. d_H is the Hamming distance, and w_H the Hamming weight.

The previous distance rule (1) is similar to the definition of signal sets matched to groups of [6].

Let's introduce the symmetry group $\Gamma(\mathbf{H}_k)$ of the Hamming space $\mathbf{H}_k$, defined as the set of all symmetries of $\mathbf{H}_k$, *i.e.*, all the isometric permutations $\rho : \mathbf{H}_2^k \rightarrow \mathbf{H}_k$ that preserve the Hamming distance. A generating group for $\mathbf{H}_k$ is a 2^k-element transitive subgroup of $\Gamma(\mathbf{H}_k)$. This lemma follows directly from Theorem 5 of [6]:

Lemma 2

A 2^k-signal GU constellation $\mathcal{S}$ with generating group G admits a BGU labeling $E[G,k]$ if and only if G is isomorphic to a generating group of $\mathbf{H}_k$. $\bigtriangledown$

Given $\Gamma(\mathbf{H}_k)$, the binary group $\mathbf{Z}_2^k$ is always isomorphic to a generating group of $\mathbf{H}_k$. However, BGU definition works for any generating group of $\mathbf{H}_k$: the additive group properties of the elements of $\mathbf{Z}_2^k$ are not directly involved in the BGU definition, which only concerns Hamming weights. This fact has consequences on the "linear" properties: in fact, also constellations without generating groups isomorphic to $\mathbf{Z}_2^k$ can admit BGU labelings.

Example 2

An 8-PSK constellation does not admit a generating group isomorphic to $\mathbf{Z}_2^3$, but only two generating groups isomorphic to $\mathbf{Z}_8$ and $\mathbf{D}_4$, the dihedral group of order eight [1]. However, it admits the BGU Gray labeling depicted in Figure 1 b). In this case, there is an 8-element transitive subgroup of $\Gamma(\mathbf{H}_3)$ isomorphic to $\mathbf{D}_4$, generated by the symmetries s_a and s_b, where:

- s_a is the order-two symmetry obtained by summing (001) and the permutation (2,1,3);
- s_b is the order-four symmetry obtained by summing (101) and the permutation (2,1,3).

$\diamond$

III. BGU encoders for infinitely long trellis codes

In this section we will consider infinitely long time-invariant Euclidean-space GU codes $\mathcal{C} \subseteq \mathcal{S}^{\mathbf{Z}}$ over a GU constellation $\mathcal{S}$, obtained by applying to the all-zero signal sequence an infinitely long time-invariant group code $C \subseteq G^{\mathbf{Z}}$ over a generating group G of $\mathcal{S}$.

Following the approach in [7], a time-invariant group code C is identified, for each $i \in \mathbf{Z}$, by a minimal state/output trellis section characterized by:

- the state group Σ (of order $|\Sigma|$);
- the input group F (of order 2^k in this paper), i.e., the group of elements labeling the edges exiting the zero state;
- the trellis section group $\mathcal{T}$, i.e., the group of all $|\Sigma| \cdot 2^k$ edges (σ_j, g, σ_j'); $\tilde{F}$ will denote the subgroup of $\mathcal{T}$ composed by the 2^k edges exiting the zero state, and $\tilde{F}_j$ the coset of $\tilde{F}$ composed by the edges exiting the state $\sigma_j \in \Sigma$.

A time-invariant, invertible, minimal binary encoder $\mathcal{E}[C,k]$ maps binary information sequences $\mathbf{u} \in (\mathbf{H}_k)^{\mathbf{Z}}$ and code sequences $\mathbf{c} \in C$. Every code sequence $\mathbf{c}$ is generated by a unique information sequence $\mathbf{u} = \mathcal{E}(\mathbf{c})$. At each $i \in \mathbf{Z}$, $\mathcal{E}$ is characterized by:

- a state space in a one-to-one correspondence with Σ;
- a binary input set $U = \mathbf{H}_k$;
- a mapping $E : \mathcal{T} \longrightarrow \mathbf{H}_k$, that can be viewed as the set of $|\Sigma|$ labelings $E_j[\tilde{F}_j, k]$ for $0 \leq j \leq |\Sigma| - 1$, *i.e.*, one-to-one functions

$$E_j : \tilde{F}_j \longleftrightarrow \mathbf{H}_k \quad .$$

Every code sequence $\mathbf{c} = \{g_{(i)}\}_{i\in\mathbf{Z}}$, corresponds to a unique edge sequence $\{(\sigma, g, \sigma')_{(i)}\}_{i\in\mathbf{Z}}$ through $\mathcal{T}$; as a consequence, the encoder is effectively invertible, and every code sequence $\mathbf{c} = \{g_{(i)}\}_{i\in\mathbf{Z}}$ corresponds to a unique information sequence $\mathbf{u} = \mathcal{E}(\mathbf{c})$, with $\mathbf{u} = \{u_{(i)}\}_{i\in\mathbf{Z}} = \{E(\sigma, g, \sigma')_{(i)}\}_{i\in\mathbf{Z}}$. A minimal encoder is never catastrophic [7].

The definition of the UBEP for the encoders is the same as that for the constellations, except that now infinitely long sequences are involved.

Definition 2

A binary encoder $\mathcal{E}[C,k]$ is said to satisfy the uniform bit error property (UBEP) if the average bit error probability obtained with maximum likelihood (ML) symbol sequence decoding does not depend on the transmitted sequence, i.e., *$P(b|\mathbf{c}_i)$ is the same for each $\mathbf{c}_i \in C$. It follows $P(b) = P(b|\mathbf{c}_i)$.* △

Testing the BGU Hamming distance rule for a given encoder $\mathcal{E}$ would involve all infinitely long code sequences. However, every code sequence corresponds to a unique edge sequence through $\mathcal{T}$; as a consequence, the distance rule can be tested directly on the $\mathcal{T}$ edges.

Lemma 3

If a binary encoder $\mathcal{E}[C,k]$ for a GU code $\mathcal{C}$ generated by C satisfies the following distance rule:

$$d_H(E(t_i), E(t_j)) = w_H(E(-t_i + t_j)) \quad \forall t_i, t_j \in \mathcal{T}. \tag{2}$$

the UBEP holds. This encoder will be called a *bit geometrically uniform* (BGU) encoder. ▽

Proof: Since every code sequence corresponds to a unique edge sequence, we have:

$$\mathbf{c}_1 \equiv \ldots, (\sigma_1, g_1, \sigma_1')_{(i)}, \ldots \qquad \mathbf{c}_2 \equiv \ldots, (\sigma_2, g_2, \sigma_2')_{(i)}, \ldots$$

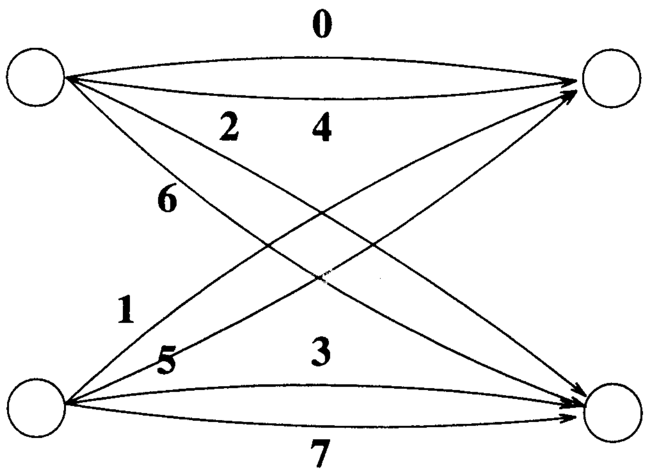

Fig. 2. Two-state group code over 8-PSK of Example 3.

and

$$-\mathbf{c}_1 + \mathbf{c}_2 \equiv \ldots, (-\sigma_1 + \sigma_2, -g_1 + g_2, -\sigma_1' + \sigma_2')_{(i)}, \ldots$$

As a consequence, the encoder satisfies the distance rule of Lemma 1, extended to infinite-length sequences:

$$d_H(\mathcal{E}(\mathbf{c}_1)\mathcal{E}(\mathbf{c}_2)) = d_H(\mathcal{E}(\mathbf{0}), \mathcal{E}(-\mathbf{c}_1 + \mathbf{c}_2)) \quad \forall \mathbf{c}_1, \mathbf{c}_2 \in C.$$

■

Example 3

Consider the 8-PSK constellation of Figure 1 with generating group $G = \mathbf{D}_4$ and the 2-state group code C over G depicted in Figure 2. The nonabelian dihedral group $G' = \mathbf{D}_4 = \{0,1,2,3,4,5,6,7\}$ has the following multiplication table:

·	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	7	6	5	4	3	2
2	2	3	4	5	6	7	0	1
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	3	2	1	0	7	6
6	6	7	0	1	2	3	4	5
7	7	6	5	4	3	2	1	0

We have:

$$\mathcal{T} = \{(0,0,0),(0,4,0),(0,2,1),(0,6,1),(1,1,0),(1,5,0),(1,3,1),(1,7,1)\}$$

$\tilde{F} = \{(0,0,0),(0,4,0),(0,2,1),(0,6,1)\}$ and $\tilde{F}_1 = (1,1,0) + \tilde{F}$.

A minimal encoder for this code can be represented by this tabular form that describes the mapping $E : \mathcal{T} \rightarrow \mathbf{H}_k$, via the two labelings $E_0 : \tilde{F} \leftrightarrow \mathbf{H}_k$ and $E_1 : \tilde{F}_1 \leftrightarrow \mathbf{H}_k$.

	$\mathcal{T}$	
$\mathbf{H}_2$	$\tilde{F}$	$\tilde{F}_1$
00	0,0,0	1,1,0
11	0,4,0	1,5,0
01	0,2,1	1,3,1
10	0,6,1	1,7,1

The labeling functions are determined by the row correspondences. For example we have $E_0(0,2,1) = 01$, and $E_1(1,5,0) = 11$. This encoder is BGU because Condition 2 of Lemma 3 is satisfied. $\diamond$

The previous BGU encoder definition has an obvious consequence on the signals labeling the edges exiting the zero-state:

Lemma 4

For a BGU encoder, the set of signals $\mathcal{S}_0 = F(s_0)$, labeling the edges exiting the zero state, is a GU constellation generated by the input group F that admits a BGU labeling $E[F,k]$; F is isomorphic to a generating group of $\mathbf{H}_k$. ∇

Any other set $\mathcal{S}_j$, composed by all signals labeling the edges exiting σ_j, is generated by a coset of F and is congruent to $\mathcal{S}_0$. This suggests that it must have a binary labeling "congruent" to that of $\mathcal{S}_0$. For any $\sigma_j \in \Sigma$, denote by $a_j \in \tilde{F}_j$ the edge exiting from σ_j and labeled by the all-zero k-bit label: $E_j(a_j) = \mathbf{0}$. Let A be the set of all these $|\Sigma|$ edges. The following lemma yields necessary and sufficient conditions for BGU encoders:

Lemma 5

An encoder $\mathcal{E}$ is BGU if and only if these conditions are satisfied:

1. E_0 is a BGU labeling $E_0[F,k]$ for $\mathcal{S}_0$;
2. $E_j(\tilde{f} + a_j) = E_0(\tilde{f})$, for all $\tilde{f} \in \tilde{F}$, and for all $a_j \in A$.
3. A is a subgroup of $\mathcal{T}$;
4. for all $a_j \in A$, $w_H(E_0(\tilde{f})) = w_H(E_0(\tilde{f}'))$, with $\tilde{f}' = -a_j + \tilde{f} + a_j$.

Condition 4 is automatically verified when $\mathcal{T}$ is the direct product $\mathcal{T} = \tilde{F} \times A$, *i.e.*, when G is abelian, or, when G is not abelian and A is a normal subgroup of $\mathcal{T}$. ∇

Conditions of Lemma 5 allow a simplified test on the trellis section to determine if an encoder is BGU or not.

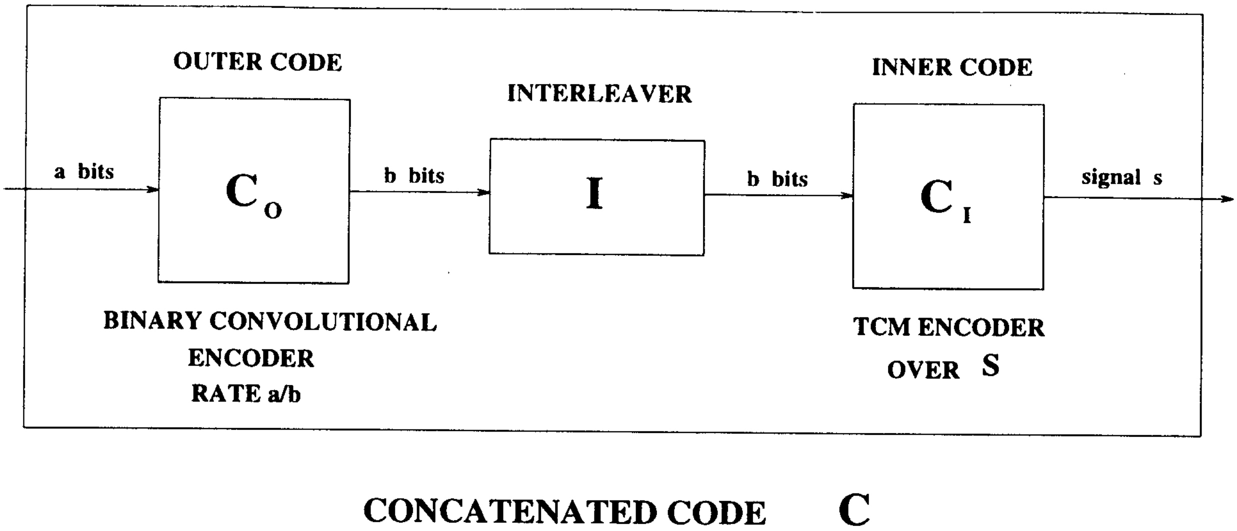

Fig. 3. Serially concatenated trellis coded modulation.

IV. Serially concatenated trellis coded modulation

Serial concatenation of an outer binary convolutional encoder with an inner trellis encoder over a multidimensional Euclidean constellation through an interleaver, and a suitable iterative decoding algorithm were proposed in [5] with some examples of codes with very good performance. The basic structure of serially concatenated trellis coded modulation (SCTCM) is shown in Fig. 3. The outer binary convolutional encoder has rate $a/(a+1)$; the output bits are interleaved and passed to a TCM encoder defined over a $2L$-dimensional constellation with a spectral efficiency of $(a+1)/L$ bps/Hz (with ideal Nyquist pulse shaping). The spectral efficiency of the overall SCTCM is then a/L bps/Hz.

A global design of the overall SCTCM is (and is likely to remain) an open problem. In [5], extending the approach introduced in [4] for binary encoders, the design approach was based on the assumption of a large uniform interleaver, and led to the requirement of maximizing the *effective free Euclidean distance* of the inner TCM code $\mathcal{C}$, defined as the minimum distance between code sequences generated by information sequences that differ only by two bits:

$$d_{\mathrm{f,eff}} = \min_{\mathbf{c}_1,\mathbf{c}_2} d_E(\mathbf{c}_1,\mathbf{c}_2) \quad \text{for all} \quad \mathbf{c}_1,\mathbf{c}_2 \in \mathcal{C} \quad \text{with} \quad d_H(E(\mathbf{c}_1),E(\mathbf{c}_2)) = 2 \tag{3}$$

where d_E means Euclidean distance. The inner TCM encoder must also be *recursive*, *i.e.*, no finite-weight code sequence must be generated by an input sequence of weight one.

According to the definition, the computation of $d_{\mathrm{f,eff}}$ requires in general testing of all possible pairs $(\mathbf{c}_1,\mathbf{c}_2)$. However, if the inner TCM encoder is BGU, we can choose as input sequence the all-zero sequence and compute $d_{\mathrm{f,eff}}$ as

$$d_{\mathrm{f,eff}} = \min_{\mathbf{c}} w_E(\mathbf{c}) \quad \text{for all} \quad \mathbf{c} \in \mathcal{C} \quad \text{with} \quad w_H(E(\mathbf{c})) = 2 \tag{4}$$

where by $w_E(\mathbf{c})$ we denote the Euclidean distance between the code sequence $\mathbf{c}$ and the all-zero sequence. The great simplification involved in passing from (3) to (4) is apparent. Moreover, since the search for good SCTCM codes also involved maximization of minimum Euclidean distance for pairs of input sequences with Hamming distances larger than 2, typically up to 6, the reduction of the computational burden involved in the search becomes dramatic.

Moreover, analytical upper bounds to the ML bit error probability performance for BGU SCTCM codes can be obtained as a straightforward extension of the technique developed in [4].

A. A search for good constituent TCM encoders for the construction of SCTCM

In [5] an SCTCM with spectral efficiency of 2 bps/Hz was obtained using a 2-state TCM inner encoder defined on a 2×8-PSK constellation with spectral efficiency 2.5 bps/Hz. The construction of the inner TCM encoder was done by "hand", based on heuristic considerations and on a partial search based on the maximization of the minimum Euclidean distance of code sequences generated by information sequences with Hamming distance 2, *i.e.*, by applying definition (3). The best 2-state encoder found had $d_{\mathrm{f,eff}} = 1.76$.

We have repeated this search, starting from the best 2-state 2.5 bps/Hz GU TCM codes over 2×8-PSK constellation of [8] (the same code used in [5]), and have constructed all possible BGU recursive encoders for this code. For each encoder we have computed $d_{\mathrm{f,eff}}$ (this computation is highly simplified for encoders possessing the BGU properties because definition (4) can be applied), and have selected the best one that yielded a $d_{\mathrm{f,eff}} = 3.76$, much larger than the one [5].

As for the search complexity, for this 2-state code with 32 edges leaving each state, we have $(32!)^2$ different minimal encoders. Restricting the search to BGU encoders, we first have to assign the encoder function to the zero-state, *i.e.*, look for a BGU mapper $E_0(F,k)$ of the input group. For this code, we have the 32-element group $F = G_{32}^{(1)}$, that is described in [8] as $G_{32}^{(1)} = Dih(\mathbf{Z}_4 \times \mathbf{Z}_4)$. We are interested in all possible BGU labeling for this group (or, more precisely, for the 32-element subset of 2×8-PSK generated by F), *i.e.*, all different transitive subgroups of $\Gamma(\mathbf{H}_5)$ isomorphic to $G_{32}^{(1)}$. This can be easily done by considering subgroups of F of increasing length and assigning a BGU labeling to them.

Only a representative encoder must be retained into the class of "equivalent" labelings, i.e. that differ only for coordinate permutations. The labeling of the second state must be congruent to that of the zero state, and the element labeled by the zero label must have order two, so that all possible BGU encoders can be easily constructed from the BGU labeling of F, by verifying the conditions of Lemma 5.

Overall, we have found about three hundred distinct BGU recursive encoders for this code. Among those encoders, we have selected the one for which the pairs (d_i, N_i) are optimized (d_i maximized and N_i minimized), from $i = 2$ up to $i = 10$. The distance d_i is defined as $d_i = \min_{\mathbf{c}_1,\mathbf{c}_2} d_E(\mathbf{c}_1, \mathbf{c}_2)$ for all $\mathbf{c}_1, \mathbf{c}_2 \in C$ with $d_H(E(\mathbf{c}_1), E(\mathbf{c}_2)) = i$. The best BGU encoder found can be implemented by the linear shift register structure depicted in Figure 4, followed by a natural mapping (see Fig. 1 a) of the two binary triplets x_0, x_1, x_2 and y_0, y_1, y_2 onto the 2×8-PSK signals.

The analytical upper bounds to the bit error probability, evaluated through an extension of the technique described in [4] for four SCTCMs of spectral efficiency 2 bps/Hz employing as outer code a binary convolutional code of rate 4/5, 2-states, and minimum distance 2, a uniform interleaver [4] with length $N = 125$, or $N = 1250$, and as inner encoders four different TCM encoders of spectral efficiency 2.5 bps/Hz over a 2×8-PSK are shown in Fig. 5. Curve A referes to the best encoder found through the search, that sequentially optimizes (d_i, N_i), for $i = 2, ..., 10$, and is described in Fig. 4; curve B represents the performance of the code of [5]; curve C pertains to a second encoder found that sequentially maximizes d_i, for $i = 2, ..., 10$, without considering the multiplicities; curve D referes to a third encoder obtained through the search whose distance and multiplicities seem a compromise between code A and C. From the curves, we argue that code B of [5] is the worst among the 4 codes, whereas code A and C have essentially the same performance.

The simulation results for the 4 codes, obtained through the iterative decoding algorithm described in [5], with an interleaver length $N = 1250$ corresponding to an input block of 1000 bits and 8 iterations of the algorithm, are reported in Fig. 6. Code A is the best, and no errors are found for E_b/N_0 above 5.5 dB. The poorest behavior is the one yielded by code B of [5], whereas the other two codes stay in between. Both analytical and simulation results strongly support the practical relevance of the systematic code search toward the "ad hoc" design procedure adopted in [5] to obtain code B. The obtained result are at 2.6 dB from capacity of 8-PSK (2 bit/s per signal) with a relatively small input delay (1000 bits) and very low complexity (two 2-state encoders).

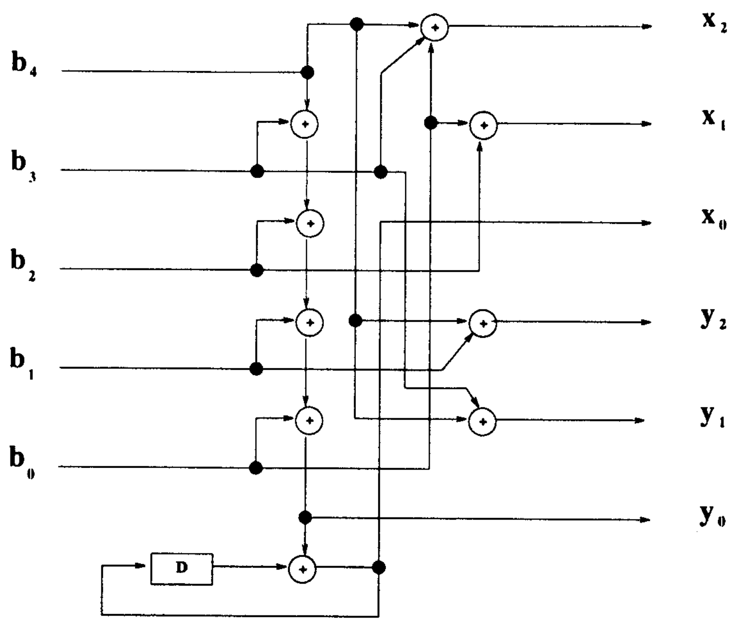

Fig. 4. Shift register realization of the best 2-state BGU TCM encoder.

V. Conclusion

The input/output encoder relationships between information and coded sequences are crucial for some applications. For BGU encoders, introduced and characterized in this paper, the bit error probability does not depend on the transmitted sequence, a property that can be useful for the analysis and design. As an example, we have shown how the BGU encoder construction can be applied to the design of serially concatenated trellis coded modulation.

References

[1] G.D. Forney, Jr., "Geometrically uniform codes", *IEEE Transactions on Information Theory*, vol. IT-37, pp. 1241–1260, Sept. 1991.

[2] M.D. Trott, S. Benedetto, R. Garello, and M. Mondin, "Rotational invariance of trellis codes—Part I: Encoders and precoders", *IEEE Transactions on Information Theory*, vol. IT-42, pp. 751–765, May 1996.

[3] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon Limit Error-Correcting Coding and Decoding: Turbo-Codes", in *Proceedings of ICC'93*, Geneve, Switzerland, May 1993, pp. 1064–1070.

[4] S. Benedetto and G. Montorsi, "Unveiling turbo-codes: some results on parallel concatenated coding schemes", *IEEE Transactions on Information Theory*, vol. IT-42, no. 2, pp. 409–429, Mar. 1996.

[5] S. Benedetto, D. Divsalar, G. Montorsi, and F. Pollara, "Serial concatenated trellis coded modulation with iterative decoding: design and performance", in *Proceedings of CTMC97 (in conjunction with Globecom 97)*, Phoenix, USA, Nov. 1997, pp. 38–43.

[6] H.A. Loeliger, "Signal sets matched to groups", *IEEE Transactions on Information Theory*, vol. IT-37, pp. 1675–1682, Nov. 1991.

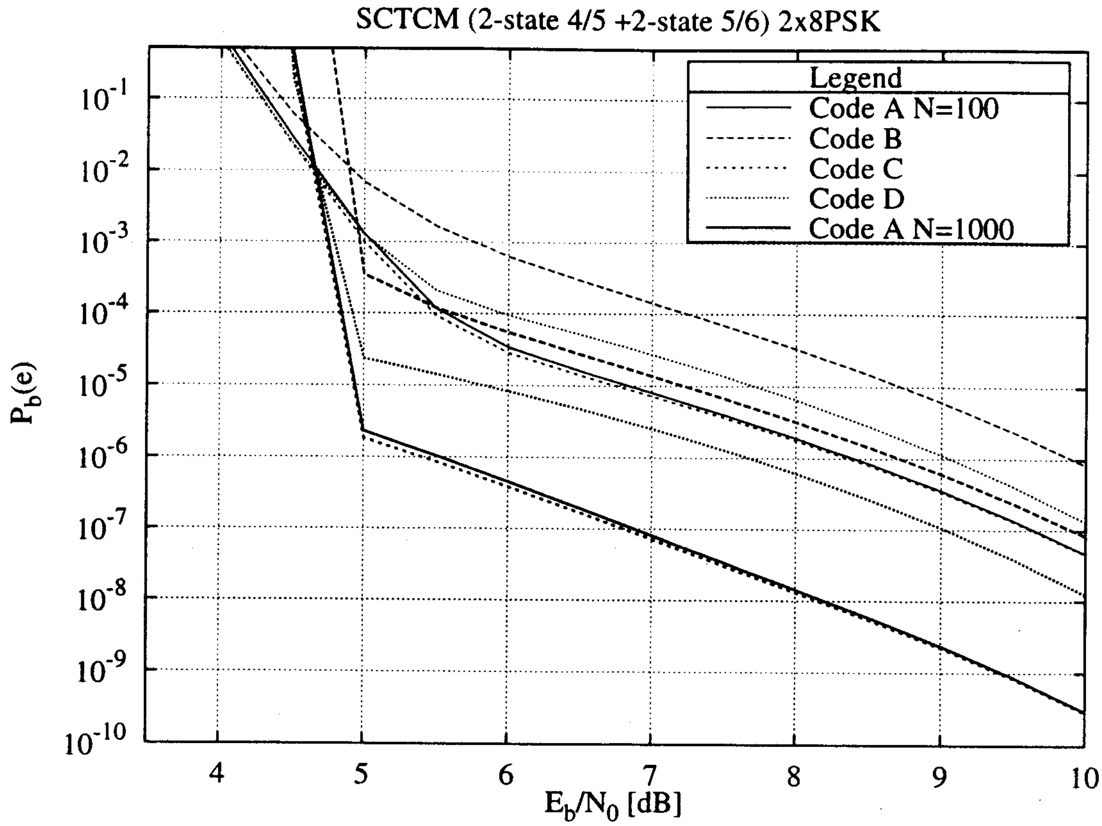

Fig. 5. Upper bounds to the bit error probability for four SCTCM codes of spectral efficiency 2 bps/Hz employing a uniform interleaver yielding an input delay $N = 100$, or $N = 1000$.

[7] G.D. Forney, Jr. and M.D. Trott, "The dynamics of group codes: State spaces, trellis diagrams, and canonical encoders", *IEEE Transactions on Information Theory*, vol. IT-39, pp. 1491–1513, Sept. 1993.

[8] S. Benedetto, R. Garello, M. Mondin, and G. Montorsi, "Geometrically uniform partitions of L×MPSK constellations and related binary trellis codes", *IEEE Transactions on Information Theory*, vol. IT-39, pp. 1773–1798, Nov. 1993.

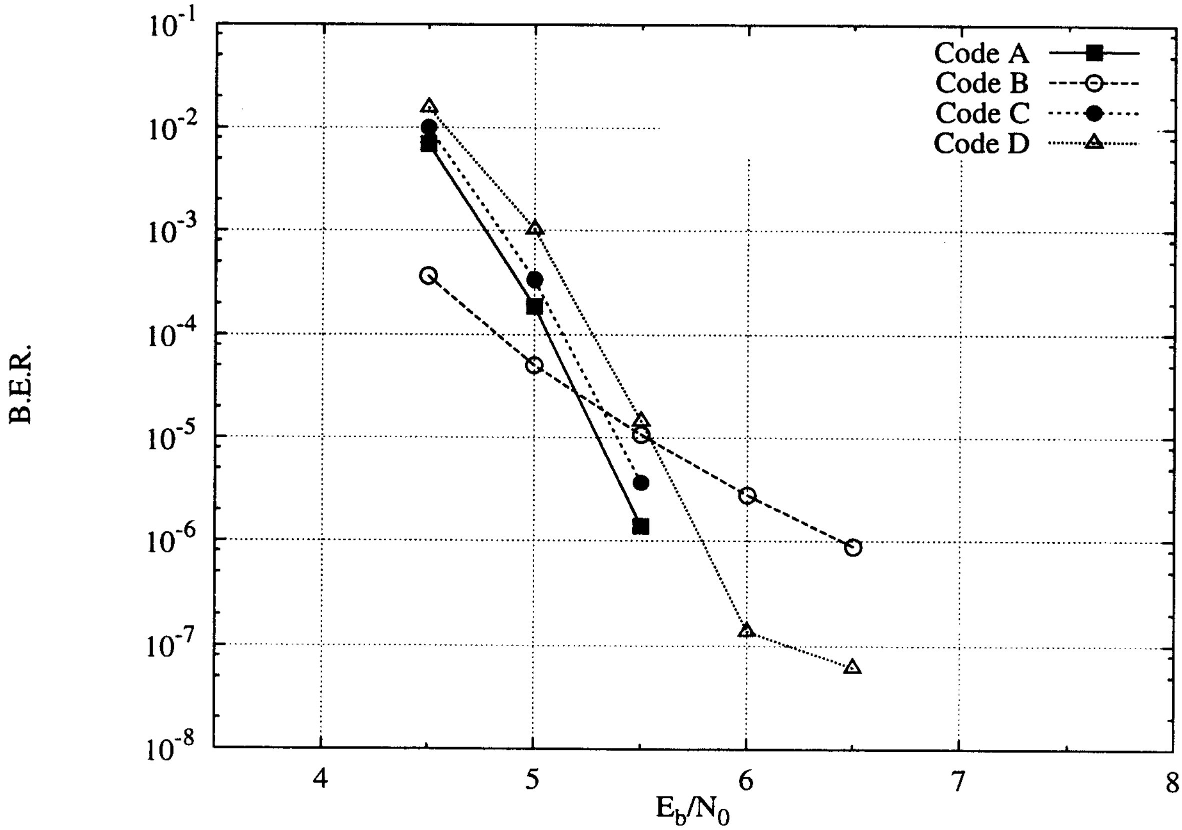

Fig. 6. Simulation results for four SCTCM schemes of spectral efficiency 2 bps/Hz employing an interleaver yielding an input delay $N = 1000$ and 8 iterations.