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IMPLEMENTATION OF A BLOCK LANCZOS ALGORITHM FOR EIGENPROBLEM SOLUTION OF GYROSCOPIC SYSTEMS

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Abstract

This paper describes the details of implementation of a general numerical procedure developed for the accurate and economical computation of natural frequencies and associated modes of any elastic structure rotating along an arbitrary axis. A block version of the Lanczos algorithm is derived for the solution that fully exploits associated matrix sparsity and employs only real numbers in all relevant computations. It is also capable of determining multiple roots and proves to be most efficient when compared to other, similar, existing techniques.

Introduction

Gyroscopic structural systems are often encountered in practice. Thus, some structures such as satellites are usually spin-stabilized whereas others, like helicopters and turbines, have rotating parts. An accurate evaluation of their frequencies and mode shapes is of utmost importance in predicting their stability and also in implementing effective closed-loop control of the gyroscopic systems. The usual solution process starts with a finite-element discretization of the structure yielding appropriate stiffness and inertia properties, which in turn are utilized to yield the natural frequencies and associated modes. Such data are next utilized to compute unsteady aerodynamic forces enabling computation of flutter and divergence characteristics. An extension of the analysis yields the state-space matrices enabling open- and closed-loop control analysis of the structure. The accuracy of such an analysis is, however, entirely dependent on appropriate computation of vibrational characteristics of the system.

The equation of free vibration of any structure discretized by the finite-element method and spinning along an arbitrary axis with a uniform spin rate Ω is given by

$$\mathbf{M}\ddot{q} + \mathbf{C}\dot{q} + \mathbf{K}q = 0 \tag{1}$$

in which

$$K = K_E + K_G + K'$$

and

$\mathbf{M}$ inertia matrix

$\mathbf{C}$ skew-symmetric Coriolis matrix, function of Ω

K_E elastic stiffness matrix

K_G geometric stiffness matrix, function of Ω^2

K' centrifugal force matrix, also a function of Ω^2

q deflection vector

For small vibrations, the **K** and **M** matrices are real, symmetric, and positive definite. The solution of Eq. (1) may be achieved by first rearranging the same as

$$\mathbf{A}\mathbf{y} + \mathbf{B}\dot{\mathbf{y}} = 0 \tag{2}$$

in which

$$\mathbf{A} = \begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & -M \\ M & C \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} q \\ \dot{q} \end{bmatrix} \tag{2a}$$

where **A** is symmetric and **B** is skew-symmetric. A solution of Eq. (2) is obtained by substituting $\mathbf{y} = e^{\omega t}$ yielding

$$(\mathbf{A} + \omega\mathbf{B})\mathbf{y} = 0 \tag{3}$$

in which the natural frequencies ω are pure imaginary, the vectors being complex and both occurring as complex conjugate pairs.

The conventional solution process for Eq. (3) involves implicit inversion of **A** to reduce the eigenvalue problem in terms of a single matrix of order twice the original size, which however is rather inefficient due to its nonsparse character and increased order.

A combined Sturm sequence and inverse iteration (SS/II) procedure was presented earlier[1] for the eigenproblem solution of gyroscopic systems that exploits inherent sparsity of constituent matrices of Eq. (2). Reference 2 provides a sur-

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vey of solution methods for free-vibration analysis of structures including spinning ones. An improved version of the SS/II technique has further been presented in a recent paper[3] that also gives details of numerical techniques for computation of in- and out-of-plane forces in a shell and also line elements spinning along an arbitrary axis. The Lanczos method has been applied earlier[4,5] for the eigenproblem solution of real symmetric matrices. Reference 4 also presents the relative merit of the block Lanczos algorithm over the conventional nonblock procedure. In a recent paper,[6] a nonblock version of the Lanczos algorithm was presented that is suitable for the economical solution of the eigenproblem of gyroscopic systems.

The purpose of this paper is to provide details of a related block Lanczos algorithm and its implementation in a general-purpose finite-element computer program, STARS[7] (STructural Analysis RoutineS). Numerical results are also presented that prove the efficacy of the current solution technique.

Implementation of a Block Lanczos Solution Procedure

To implement the current procedure, Eq. (3) is first rewritten as

$$(\mathbf{A} - \lambda \mathbf{D})\mathbf{y} = \mathbf{0} \tag{4}$$

in which $\mathbf{D} = i^*\mathbf{B}$ is a pure imaginary Hermitian matrix, i^* is the imaginary number $\sqrt{-1}$, the roots $\lambda = i^*\omega$ are real and occur in pairs $\lambda_1, -\lambda_1, \ldots, \lambda_n, -\lambda_n$ whereas the corresponding eigenvectors occur in complex conjugate pairs. The roots of the original system defined by Eq. (3) may then be simply obtained as $\omega = \lambda/i^*$ while noting that the eigenvectors remain the same for both cases.

To develop the present algorithm, it is necessary to yield a single matrix out of the set of two matrices that define Eq. (4). This is achieved by performing a Choleski decomposition

$$\mathbf{A} = L_A L_A^T \tag{5}$$

in which

$$L_A = \begin{bmatrix} L_M & 0 \\ 0 & L_K \end{bmatrix} \tag{5a}$$

and L_M, L_K are the lower triangular forms of matrices $\mathbf{M}$ and $\mathbf{K}$, respectively. Appropriate transformation of Eq. (4) may then be effected by utilizing Eq. (5), yielding

$$(\mathbf{H} - \gamma \mathbf{I})\mathbf{y} = \mathbf{0} \tag{6}$$

in which $\gamma = 1/\lambda$, $\omega = 1/i^*\gamma$, and the matrix $\mathbf{H}$ is expressed as

$$\mathbf{H} = \begin{bmatrix} 0 & -i^* L_M^T L_K^{-T} \\ i^* L_K^{-1} L_M & i^* L_K^{-1} C L_K^{-T} \end{bmatrix} \tag{6a}$$

It may be noted that $\mathbf{H}$ is a pure imaginary Hermitian matrix and the current transformation retains the banded form of associate matrices. In all subsequent computations, n defines the order of $\mathbf{H}$, whereas m_{11} denotes the half-bandwidth of constituent $\mathbf{M}$, $\mathbf{K}$, and $\mathbf{C}$ matrices.

As the first step toward implementing the procedure using a block size m, a number of relevant matrices are defined as

G_i = n × m complex matrix with orthonormal columns, that is,

$$\bar{G}_i^T G_i = \mathbf{I}$$

E_i = m × m Hermitian matrix

F_i = m × m upper Hessenberg complex matrix

$\mathbf{W}_i$, $\mathbf{X}_i$ = n × m complex matrices

T_i = im × im block tri-diagonal Hermitian matrix with blocks of size m × m

Furthermore, a unitary matrix $\mathbf{J}_p$ of order (p × p) is next utilized to relate complex matrices occurring in the Lanczos method to corresponding matrices that are real in nature. Thus denoting $\mathbf{S}$ as a matrix of columns that are eigenvectors of $\mathbf{H}$ occurring in complex conjugate pairs, the matrix $\hat{S} = \mathbf{S}\bar{\mathbf{J}}_n^T$ is a real, orthogonal matrix of order n. Such a procedure may then be used to recast the block Lanczos algorithm in terms of real numbers effecting considerable saving in solution time. It may be noted in this connection that for an even positive integer p, the matrix $\mathbf{J}_p$ is formed as p/2 replications of $\mathbf{J}_2$ on the diagonal, which in turn is defined as

$$J_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i^* & -i^* \end{bmatrix} \tag{7}$$

Numerical Scheme

Let $\hat{\mathbf{G}}_1$ be an arbitrary, real n × m matrix with orthonormal columns. Then for i = 1, 2, ..., the computational procedure is developed by the following steps.

Step 1. Perform matrix operation

$$\hat{\mathbf{W}}_i = \hat{\mathbf{H}}\hat{\mathbf{G}}_i \qquad \text{(for } i = 1\text{)}$$

$$= \hat{\mathbf{H}}\hat{\mathbf{G}}_i + \hat{\mathbf{G}}_{i-1}\hat{\mathbf{F}}_{i-1}^T \qquad \text{(for } i > 1\text{)}$$

in which the matrix substitutions are

$\mathbf{H} = i^*\hat{\mathbf{H}}$ — $\hat{\mathbf{H}}$, n × n real skew-symmetric matrix

$G_i = \hat{\mathbf{G}}_i J_m$ — $\hat{\mathbf{G}}_i$, n × m real matrix with orthonormal columns; that is, $\hat{\mathbf{G}}_i^T \hat{\mathbf{G}}_i = \mathbf{I}$

$F_i = i^* \bar{J}_m^T \hat{F}_i J_m$ $\qquad \hat{F}_i$, m × m real upper triangular matrix

Step 2. Compute the reduced order matrix

$$\hat{E}_i = \hat{G}_i^T \hat{W}_i$$

where the following substitutions are made

$E_i = i^* \bar{J}_m^T \hat{E}_i J_m$ $\qquad \hat{E}_i$, m × m real skew-symmetric matrix

$\mathbf{W}_i = i^* \hat{W}_i J_m$ $\qquad \hat{W}_i$, n × m real matrix

Step 3. Produce the matrix

$$\hat{X}_i = \hat{W}_i - \hat{G}_i \hat{E}_i$$

the substitution being

$X_i = i^* \hat{X}_i J_m$ $\qquad \hat{X}_i$, n × m real matrix

Step 4. Use a standard procedure such as the Givens, Householder, or Gramm Schmidt method to obtain the "QR" factors of $\hat{X}_i$ namely $\hat{G}_{i+1}$ and $\hat{F}_i$ satisfying $\hat{X}_i = \hat{G}_{i+1} \hat{F}_i$ and $\hat{G}_{i+1}^T \hat{G}_{i+1} = \mathbf{I}$.

Step 5. Form the block tri-diagonal matrix $\hat{T}_i$ of order im

$$\hat{T}_i = \begin{bmatrix} \hat{E}_i & -\hat{F}_i^T & & \\ \hat{F}_i & \hat{E}_2 & \ddots & \\ & \ddots & \ddots & -\hat{F}_{i-1}^T \\ & & \hat{F}_{i-1} & \hat{E}_i \end{bmatrix}$$

a solution of which yields the eigenvalues and vectors of the system as the ith stage approximation. Furthermore, it may be noted that $\mathbf{T}_i$ and $i^* \hat{T}_i$ have the same eigenvalues that are real, occur in pairs, and have opposite signs. Also if the eigenvectors of $i^* \hat{T}_i$ occurring as complex conjugate pairs are denoted by, say, $\hat{\mathbf{v}}$ and $\bar{\hat{\mathbf{v}}}$, the corresponding vectors for $\mathbf{T}_i$ may then be obtained as $\mathbf{v} = \bar{\mathbf{J}}^T \hat{\mathbf{v}}$ and $\mathbf{z} = \bar{\mathbf{J}}^T \bar{\hat{\mathbf{v}}}$, which are not mutually conjugate. The corresponding pair of real roots (p, -p) are Ritz values of $\mathbf{H}$, the corresponding Ritz vectors being

$$\alpha = \mathbf{u}_i \mathbf{v} = \hat{\mathbf{u}}_i \hat{\mathbf{v}}$$

$$\beta = \mathbf{u}_i \mathbf{z} = \hat{\mathbf{u}}_i \bar{\hat{\mathbf{v}}}$$

where

$$\hat{\mathbf{u}}_i = \mathbf{u}_i \bar{J}^T = [\mathbf{G}_1, \ldots, \mathbf{G}_i]\, \bar{J}^T$$

$$= [\hat{\mathbf{G}}_1, \ldots, \hat{\mathbf{G}}_i]$$

Since $\hat{\mathbf{u}}_i$ is real, α and β are mutually conjugate so that $\beta = \bar{\alpha}$.

Step 6. Perform convergence tests using vectors computed in step 5 and matrix $\hat{F}_i$ obtained in step 4. If the analysis needs to be continued, then a selective orthogonalization of matrix G_{i+1} must be carried out so that its columns are orthogonalized relative to some of the current Ritz vectors.[4] Thus denoting $\mathbf{g}$ as a column of $\mathbf{G}_{i+1}$ and expressing a Ritz vector α in terms of two real n vectors as

$$\alpha = \theta + i^* \phi$$

$\hat{\mathbf{g}}$ may be orthogonalized with respect to θ and ϕ and a new real vector $\hat{\mathbf{g}}$ may be obtained as follows:

$$\hat{\mathbf{g}} := \hat{\mathbf{g}} - \frac{(\theta^T \hat{\mathbf{g}})}{(\theta^T \theta)}\, \theta$$

and

$$\hat{\mathbf{g}} := \hat{\mathbf{g}} - \frac{\phi^T \hat{\mathbf{g}}}{(\phi^T \phi)}\, \phi$$

from which the orthogonalized $\mathbf{g}$ is simply obtained as $\mathbf{g} = \hat{\mathbf{g}} \mathbf{J}$.

All computations in the above procedure are performed in real arithmetic that has been implemented in the STARS[7] program and proves to be most efficient in the solution of vibration problems of complex, gyroscopic systems.

Numerical Examples

The newly implemented block Lanczos procedure employing real numbers (BL/R) is used to solve an extensive number of test cases. Such results are compared with solutions obtained from other existing similar techniques such as the block Lanczos technique using complex arithmetic (BL/C) and the SS/II methods. Because all such procedures have been implemented in the STARS program, it was used to perform analyses of a number of test cases presented here, employing a Digital Equipment Corp. VAX 11-750 computer.

Spinning Cantilever Beam

A spinning cantilever beam (Fig. 1), discretized by 12 line elements for the natural frequency analysis, has the following relevant properties

Element length (ℓ)	5.0
Moment of inertia (I_{YY})	1/12
Moment of inertia (I_{ZZ})	1/24
Cross-sectional length (A)	1.0
Young's modulus (E)	30×10^6
Element mass/unit length	1/5
Uniform spin rate (Ω_Y)	0.33 Hz

and results of such analyses employing various procedures are given in Table 1.

Spinning Cantilever Plate

Figure 2 depicts a rectangular cantilever plate spinning along an arbitrary axis with a uniform spin rate Ω_R. A 10 × 15 finite-element mesh employing thin-shell elements is used to model the plate that has the following structural characteristics

X-side length (ℓ_X)	10
Y-side length (ℓ_Y)	15
Thickness (t)	0.1
Young's modulus (E)	10×10^6
Mass density (ρ)	0.259×10^{-3}
Poisson's ratio (ν)	0.3
Number of degrees of freedom	1056

The plate was first analyzed for a spin rate $\Omega_Z = 0.7\omega_1$ and subsequently for a resulting spin vector $\Omega_R = 0.7\omega_1$ having components $\Omega_X = \Omega_Y = \Omega_Z = 0.7\ \omega_1/\sqrt{3}$. Results of such analyses by the three solution techniques are given in Table 2.

A comparison of results presented in the two tables amply demonstrates the significant advantages of the present procedure.

Concluding Remarks

A block version of the Lanczos algorithm has been presented that exploits matrix sparsity and further performs all numerical computations in real numbers for the eigenproblem solution of gyroscopic systems. While each solution step in the block algorithm is costlier than the conventional nonblock Lanczos method,[6] fewer steps are needed. The overall saving in solution time is comparable to that effected by block multivector inverse iteration in place of the single-vector iteration process. Furthermore, this procedure is capable of effective determination of multiple roots in which the usual nonblock procedure is deficient.[4] Also, although some experience may be necessary in choosing an optimum block size, a range between 2 and 4 has been found to be effective. From the results presented in the two tables, it is apparent that the current procedure is considerably more efficient than other similar existing solution techniques.

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Table 1 Results of free-vibration analysis of cantilever beam spinning at rate of 0.33 Hz (2.073 rad/sec)

Mode	Eigenvalue		
	BL/R	BL/C	SS/II
1	3.345	3.345	3.350
2	3.602	3.604	3.603
3	16.403	16.404	16.404
4	22.491	22.490	22.491
5	44.486	44.487	44.487
6	62.258	62.259	62.258
CPU time, sec	20	75	73

BL/R = block Lanczos procedure using real numbers
BL/C = block Lanczos procedure using complex numbers
SS/II = combined Sturm sequence and inverse iteration procedure
Central processing unit time is for 10 modes and frequencies

Table 2 Natural frequencies of a spinning cantilever plate

Mode	Natural frequency (rad/sec)					
	$\Omega_R = \Omega_Z = 0.70\omega_1 =$ 149.50 rad/sec			$\Omega_R = 0.70\omega_1$ $\Omega_X = \Omega_Y = \Omega_Z = 86.32$ rad/sec		
	BL/R	BL/C	SS/II	BL/R	BL/C	SS/II
1	526.69	526.71	526.71	319.46	319.46	319.46
2	780.31	780.30	780.30	522.79	522.79	522.79
3	1375.15	1375.10	1375.14	933.32	933.32	933.32
4	1734.95	1735.03	1735.03	1339.69	1339.71	1339.67
5	1791.21	1791.25	1791.25	1586.76	1586.83	1586.83
6	2613.78	2615.02	2615.02	2049.31	2049.51	2112.78
CPU time, min	15	52	251	14	45	255

BL/R = block Lanczos procedure using real numbers
BL/C = block Lanczos procedure using complex numbers
SS/II = combined Strum sequence and inverse iteration procedure
Central processing unit time is for 10 modes and frequencies

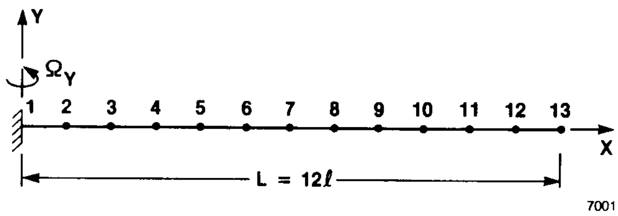

Fig. 1 Spinning cantilever beam.

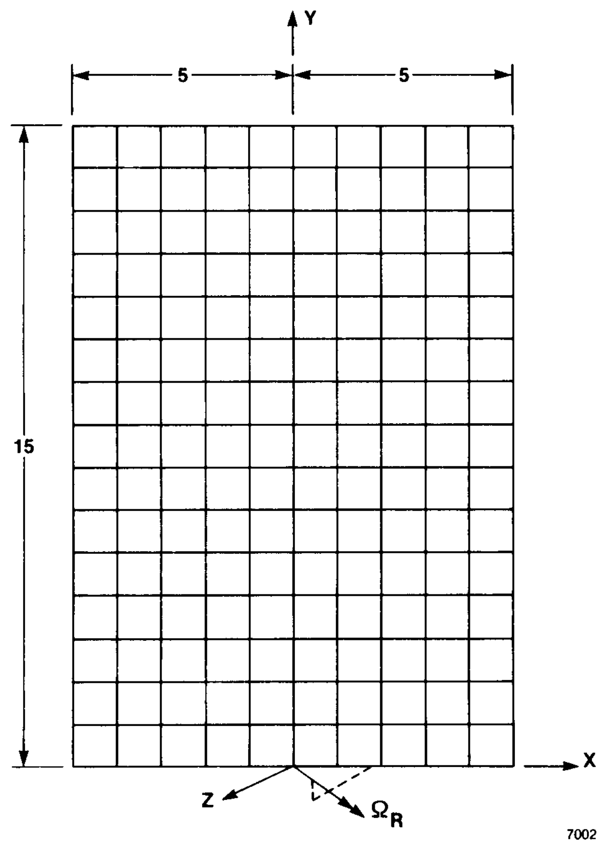

Fig. 2 Rectangular spinning cantilever plate.

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