Distributional Properties of Inhibited Random Positions of Mobile Radio Terminals

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Abstract — **Randomly generated points representing the deployment of N radio terminals (nodes) in an area can be unrealistically close together (clustered) and the resulting connectivity measure for the wireless network (such as average number of neighbors per node) can have a relatively large variance. When a minimum internode distance is imposed according to an "inhibition" random process, the simulated node positions are more regularly spaced and the connectivity measure has a smaller variance. In this paper, analytical methods are presented that characterize the spatial distribution ("area coverage") of the nodes under random and inhibited point generation processes, and expressions are found for the mean and variance of the number of neighbors.**

I. INTRODUCTION

The analysis of signaling protocols for mobile radio networks usually requires making assumptions concerning the physical connectivity between the radio terminals (nodes). Such assumptions may be perfectly appropriate for the physical scenario even if they ignore or use a simplified model of the effects of propagation on the strength of a radio transmission from one node's position that is received at another node's position. One level of modeling is simply to assume that each node can "hear" each other node and that the effect of the medium is adequately modeled by a propagation delay (e.g., [1]). Another level of modeling assumes that mobile nodes within a certain distance from a transmitting node can hear it, and that the probable number of terminals in a particular area is proportional to the area (e.g., [2]). A slightly more detailed level of modeling assumes that the propagation loss experienced by the signals is proportional to some power of the distance, thus ignoring position-dependent variations in the loss, and assumes a random placement of nodes (e.g., [3]) or, in modeling link power control in a cellular system, that the nodes are distributed uniformly in either area or distance from a base station ([4]).

When the placement of nodes is modeled, it is often convenient to assume that the nodes are uniformly distributed in some area. Figure 1 shows the random placement of 100 points on a square grid, assuming uniform distributions for the x and y components of the node positions, a common simulation model for the relative positions of radio terminals (nodes) in a mobile communications network. Note that the value of the distance between any pair of nodes in Figure 1 can vary from 0 to $\sqrt{2}$, the diagonal dimension of the grid. Depending on the scale of the grid relative to the communications range of the nodes, the example random node deployment modeled in Figure 1 may or may not provide connectivity between every pair of nodes, even using relays, because some groups of nodes are close together ("clustered") while others are isolated.

Noting this possible clustering property of node locations that are selected randomly using uniform distributions for the x and y components of the the locations, Cheng and Robertazzi [5] investigated an "inhibition point process" due to Matern [6] as an alternative procedure for selecting random node locations that is more realistic in terms of minimum separations between nodes and reflects a deployment strategy that provides better area coverage by spreading the given number of nodes out more evenly. According to the inhibition process, when selecting the location of the nth node, candidate locations are generated randomly until one is found that is greater than some minimum distance from each of the $n-1$ previously selected node locations. The result of such a process is illustrated in Figure 2 for $N = 100$ nodes with a minimum relative distance between nodes of $d_{min} = 0.075$. As noted in [5], there is a limit to the number of nodes that can be packed into an area A_0 with a minimum separation of d_{min}, estimated in [5] as $N_{max} \approx A_0/\pi\left(\frac{1}{2}d_{min}\right)^2$, the number of nonoverlapping circular areas in A_0 with radius $\frac{1}{2}d_{min}$. For $N_{max} = 100$, this expression corresponds to $d_{min} \approx 0.1128$.[1] However, since node positions are assigned randomly, the regular spacing required to pack the area with nodes efficiently is not preserved, and some random, generally smaller minimum distance is required to find positions for 100 nodes. For example, in generating Figure 2, it was found that d_{min} must be no greater than 0.083 in order to find 100 random positions in a reasonable amount of time.

Manipulation of the minimum distance to other nodes affects the network connectivity. In [5], the probability distribution of the number of neighbor nodes per node, as a function of the number of nodes, was calculated for a two-dimensional Poisson point process model of node locations and compared to empirical results for the same distribution when the inhibition process is used to determine node

[1] Actually, if an $M \times M$ mesh of nodes "just fits" into a $D \times D$ area, the distance between nodes equals $D/(M-1)$. Thus the maximum of the minimum distance between nodes for N nodes equals $D/\left(\sqrt{N}-1\right) = D/9$ for $N = 100$.

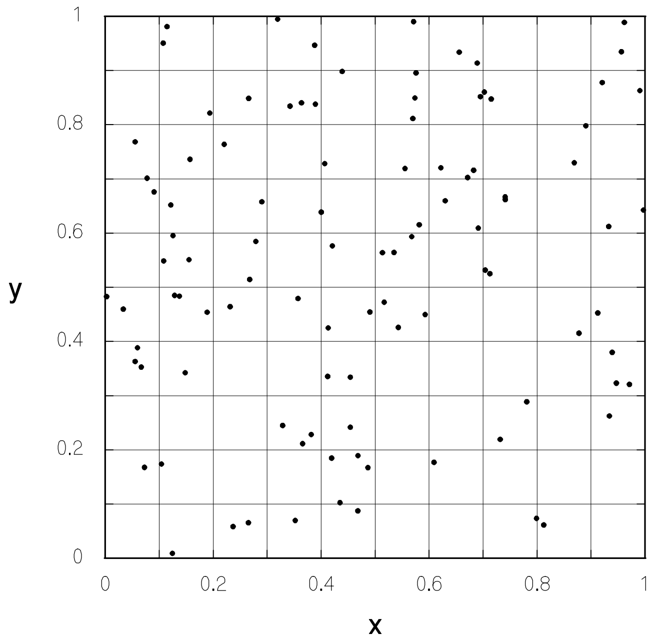

Fig. 1. Relative positions of 100 randomly placed nodes.

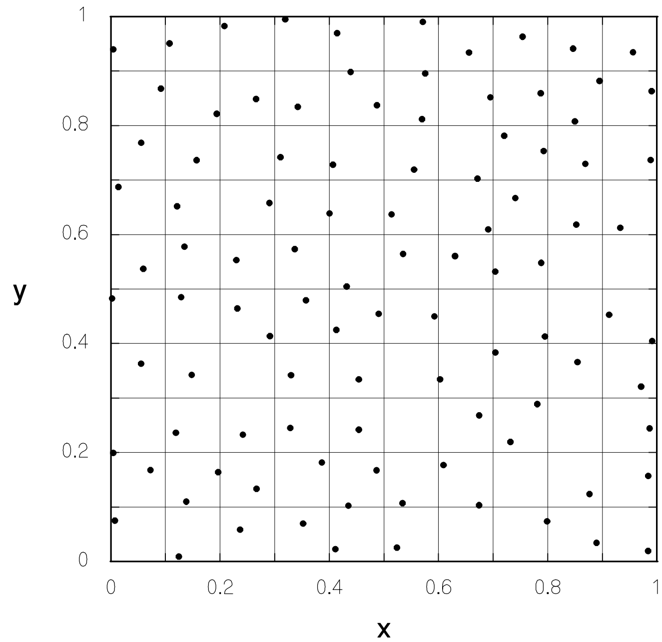

Fig. 2. Random placement of 100 nodes using an inhibition process with $d_{min} = 0.075$.

locations. Any node within the communications range R of a node is considered its neighbor. Suppose that the probability of a node's having exactly K neighbors is desired when the average number of neighbors of a node equals 10; in [5] the authors reasoned that the probability of this event is given by $\mathrm{e}^{-10}(10)^K/K!$ for the Poisson point process. The authors simulated 200 realizations of the inhibition process such that the average number of nodes in the total area equaled 100, and gathered statistics on the number of neighbors for the various nodes for each realization (excluding nodes near the edge of the area)[2]. After averaging the statistics over the realizations, it was found that the distribution for the inhibition process is significantly more peaked and concentrated around the average value, due to the more regular pattern of the node positions as determined by the inhibition process.

In this paper, a simple measure of area coverage is defined that shows the effect of the inhibition process on the evenness of the node location distribution. Then, an analysis of the number of neighboring nodes is performed for uniform and inhibited node location processes, using distributional results for the distances between nodes in a random deployment of nodes [7, 8].

II. Measures of Area Coverage

A measure of the area coverage of a random placement of N nodes in a $D \times D$ area can be based on the statistical variation of the number of nodes across regular subdivisions of the area, say "cells" of size D^2/N. On the average, for a random distribution of N node locations, one would expect one node per cell. The variance of the number of nodes per cell then would reflect the uniformity of the distribution of the node locations among the cells and hence the degree to which the node location process produces an even pattern of coverage for the area.

A. Examples of Area Coverage Measures

For example, the difference between the distributions of the nodes in Figures 1 and 2 can be quantified by calculating for each figure the experimental probability P_n that one of the 100 square cells contains exactly n nodes. For either the uniform or the inhibited node location process (provided that the distance between cells is greater than d_{min}), the placement of a node into one cell or not is a binomial trial, with probability of success on each trial of $p = 1/N = 0.01$ and the expected value and variance of the number of nodes in a particular one of the areas equal $m = Np = 1.00$ and $\sigma^2 = Np(1-p) = 0.99$. Another statistic that is not quite binomial because the numbers of nodes in the different cells are slightly correlated (they must add up to N for a given trial of selecting N node locations) is the number of nodes in any cell. By counting the numbers of nodes in the cells in Figures 1 and 2 and in a similar figure for which $d_{min} = 0.05$, the data in Table 1 are obtained in comparison to the binomial distribution. As expected, for each case the sample mean of the number of nodes in any cell is very close to m, but the sample variances for these examples are much different: the variance for the uniform node location process is close to σ^2, while that for the inhibition process is significantly less. Also as expected, the sample variances

[2] The authors of [5] presumably chose d_{min} to correspond to an average of 100 nodes and chose a value for R that corresponds to one-tenth of the area to make the average number of neighbors equal to ten.

Table 1. Statistics for number of nodes in a cell for $d_{min} =$ 0.0 (Fig. 1), 0.05, and 0.075 (Fig. 2) when $D = 1$, $N = 100$

	d_{min}=0		d_{min}=0.05		d_{min}=0.075		Binomial
# nodes, n	# cells	P_n	# cells	P_n	# cells	P_n	P_n
0	41	0.41	28	0.28	12	0.12	0.366
1	30	0.30	47	0.47	75	0.75	0.370
2	18	0.18	22	0.22	13	0.13	0.185
3	10	0.10	3	0.03	0	0.00	0.060
4	1	0.01	0	0.00	0	0.00	0.015
Sample mean	1.00		1.00		1.01		1.00
Sample variance	1.09		0.62		0.25		0.99

decrease in proportion to the degree of inhibition, that is, in proportion to the value of d_{min}.

B. DIRECT ANALYSIS OF SPATIAL DISTRIBUTION

Without loss of generality, we assume a unit area ($D = 1$) and the case of N points generated by the inhibition process and calculate the probabilities of various numbers of points falling into a particular cell with area $1/N$. Let the normalized minimum distance be denoted $\xi_0 = d_{min}/D$ and let A denote the area surrounding a selected point that is excluded in the selection of locations for subsequent points. Then, ignoring the location-dependent effects of the edges of the cells and the total area on A, the first several probabilities are given by

$$\Pr\{0\} = \frac{N-1}{N}\,\frac{\frac{N-1}{N}-A}{1-A}\,\frac{\frac{N-1}{N}-2A}{1-2A}\,\frac{\frac{N-1}{N}-3A}{1-3A}\cdots\frac{\frac{N-1}{N}-(N-1)A}{1-(N-1)A} \tag{1a}$$

$$\Pr\{1\} = \binom{N}{1}\frac{1}{N}\,\frac{\frac{N-1}{N}}{1-A}\,\frac{\frac{N-1}{N}-A}{1-2A}\,\frac{\frac{N-1}{N}-2A}{1-3A}\cdots\frac{\frac{N-1}{N}-(N-2)A}{1-(N-1)A} \tag{1b}$$

$$\Pr\{2\} = \binom{N}{2}\frac{1}{N}\,\frac{\frac{1}{N}-A}{1-A}\,\frac{\frac{N-1}{N}}{1-2A}\,\frac{\frac{N-1}{N}-A}{1-3A}\cdots\frac{\frac{N-1}{N}-(N-3)A}{1-(N-1)A} \tag{1c}$$

$$\Pr\{3\} = \binom{N}{3}\frac{1}{N}\,\frac{\frac{1}{N}-A}{1-A}\,\frac{\frac{1}{N}-2A}{1-2A}\,\frac{\frac{N-1}{N}}{1-3A}\cdots\frac{\frac{N-1}{N}-(N-4)A}{1-(N-1)A} \tag{1d}$$

Note the requirement that $A < 1/N$, which eventually terminates the series of probabilities for some $\Pr\{n_{max}\}$ for which $(n_{max}-1)A < 1/N$ but $n_{max}A > 1/N$. In a more compact way, we can express the probabilities by

$$\Pr\{n\} = \text{const.} \times \binom{N}{n}\prod_{i=0}^{n-1}\left(\frac{1}{N}-iA\right) \times \prod_{j=N-n_{max}}^{N-n-1}\left(\frac{N-1}{N}-jA\right),\quad n \le n_{max} \tag{2}$$

using the convention that the product equals 1 if the lower limit exceeds the upper limit. For an "uninhibited", uniform process, $A = 0$ and $n_{max} = N$; then this expression reduces to the binomial probability

$$\Pr\{n\} = \binom{N}{n}\left(\frac{1}{N}\right)^n\left(\frac{N-1}{N}\right)^{N-n} \tag{3}$$

For example, let $N = 100$ and let the inhibited area be specified by the minimum relative distance $\xi_0 = 0.05$ such that $A = \pi(.05)^2 = 0.00785$. Then $n_{max} = 2$ and the unnormalized probabilities are

$$\begin{aligned}
\Pr\{0\} &= \text{const.} \times (.99-98A)(.99-99A) \\
&= \text{const.} \times 0.04681 \\
\Pr\{1\} &= \text{const.} \times (100)(.01)(.99-98A) \\
&= \text{const.} \times 0.22031 \\
\Pr\{2\} &= \text{const.} \times (4950)(.01)(.01-A) \\
&= \text{const.} \times 0.10623
\end{aligned} \tag{4}$$

After normalization, the respective probabilities are 0.12538, 0.59009, and 0.28453, giving a mean value of 1.159 and a variance of 0.3846. The corresponding binomial distribution has a mean of 1.00 and a variance of 0.99. Table 2 gives the mean and variance of the number of points in each of N cells as approximated by these procedures, for $N = 100$ and several values of ξ_0. The effect of the approximation is

Table 2. Estimated mean and variance of number of points in a cell, $N = 100$

ξ_0	A	n_{max}	Mean	Variance
0.0	0.0	N	1.000	0.99
0.02	0.00126	8	1.014	0.902
0.03	0.00283	4	0.974	0.669
0.04	0.00503	2	1.001	0.497
0.05	0.00785	2	1.159	0.385

reflected in the variation in the mean, which should be constant at the value of 1.0. The variance decreases, showing qualitatively how the inhibition process affects the spatial distribution of the points.

Note that the variance for $\xi_0 = 0.05$ calculated from (2) is significantly less than the sample value in Table 1. An improved estimate can be obtained by using a value for A that takes into account the fact that the excluded area is dependent on the position of the previously selected point within the cell. It is shown in [7, 8] that the cumulative probability that the normalized distance d_1/D between two nodes that are randomly positioned in a $D \times D$ area is given by

$$\mathrm{F}_1(\xi) = \Pr\left\{\frac{d_1}{D} \le \xi\right\}$$

$$= \begin{cases} 0, & \xi < 0 \\ \xi^2\left(\frac{1}{2}\xi^2 - \frac{8}{3}\xi + \pi\right), & 0 \le \xi < 1 \\ \frac{4}{3}\sqrt{\xi^2 - 1}(2\xi^2 + 1) - \left(\frac{1}{2}\xi^4 + 2\xi^2 - \frac{1}{3}\right) & \\ \quad + 2\xi^2[\sin^{-1}(1/\xi) - \cos^{-1}(1/\xi)], & 1 \le \xi < \sqrt{2} \\ 1, & \xi \ge \sqrt{2} \end{cases} \tag{5}$$

Replacing the area A in (2) by $A' = (1/N)\,\mathrm{F}_1(\sqrt{N}\,\xi_0)$ accounts for the effects within a particular cell but ignores any effects on adjacent cells. Nevertheless, the resulting estimate$s$ of the mean and variance of the number of nodes placed in any cell under the inhibition process is improved (indicated by less variation in the mean as a function of ξ_0), as illustrated by the examples in Table 3. To provide a means for evaluating the accuracy of the estimates, the results of 1000 100-node trials are also given in Table 3.

III. ANALYSIS OF THE NUMBER OF NEIGHBORS

Although the spatial distribution measure discussed in the previous section is effective in showing how the inhibition process of selecting node locations results in a more even dispersion of nodes, it does not indicate directly the connectivity of the resulting wireless network. In fact it cannot, not the least because an additional parameter is required to characterize connectivity: the communications range. Here we denote the range by R and its normalized value by $\xi = R/D$.

Frequently the connectivity of a wireless network is indicated by the fraction of the bidirectional links that have distances less than R. The link distance distribution function $\mathrm{F}_1(\xi)$ estimates this measure of connectivity for a uniform, random deployment of nodes. A related measure is the average number of neighbors per node; we denote this quantity by ν and proceed to analyze its distribution for uniform and inhibited random node deployments.

Table 3. Improved estimated mean and variance of number of points in a cell, $N = 100$

	Estimates using (2)				Result, 1000 trials	
ξ_0	A'	n_{max}	Mean	Variance	Mean	Variance
0.0	0.0	N	1.000	0.990	1.000	0.989
0.01	0.00029	100	1.000	0.962	1.000	0.960
0.02	0.00105	9	0.998	0.883	1.000	0.893
0.03	0.00215	5	0.999	0.777	1.000	0.793
0.04	0.00345	3	1.000	0.653	1.000	0.682
0.05	0.00483	3	1.002	0.520	1.000	0.576
0.06	0.00620	2	1.062	0.461	1.000	0.475
0.07	0.00745	2	1.134	0.407	1.000	0.387
0.075	0.00800	2	1.169	0.376	1.000	0.347

A. MEAN, VARIANCE OF NUMBER OF NEIGHBORS

The cumulative probability distribution function (cdf) for the normalized link distance d_2/D, assuming that randomly generated other-node distances d_1 greater than d_{min} are retained under the inhibition process of selecting node locations, is given by

$$\mathrm{F}_2(\xi) = \Pr\left\{\frac{d_2}{D} \le \xi\right\} = \Pr\left\{\frac{d_1}{D} \le \xi \,|\, d_1 > d_{min}\right\}$$

$$= \frac{\Pr\left\{\dfrac{d_{min}}{D} < \dfrac{d_1}{D} \le \xi\right\}}{\Pr\left\{\dfrac{d_1}{D} > \dfrac{d_{min}}{D} = \xi_0\right\}}$$

$$= \begin{cases} 0, & \xi < \xi_0 \\ \dfrac{\mathrm{F}_1(\xi) - \mathrm{F}_1(\xi_0)}{1 - \mathrm{F}_1(\xi_0)}, & \xi_0 \le \xi < \sqrt{2} \\ 1, & \xi \ge \sqrt{2} \end{cases} \tag{6}$$

Examples of this cdf are shown in Figure 3 for different values of ξ_0. Note that $\mathrm{F}_2(\xi) \le \mathrm{F}_1(\xi)$, uniformly, and that $\mathrm{F}_1(\xi)$ in (5) is the special case of $\mathrm{F}_2(\xi)$ when $\xi_0 = 0$.

Given the location $\mathbf{p}$ of a node in the $D \times D$ deployment area, the number of neighbor nodes out of a total of N nodes is a binomial random variable:

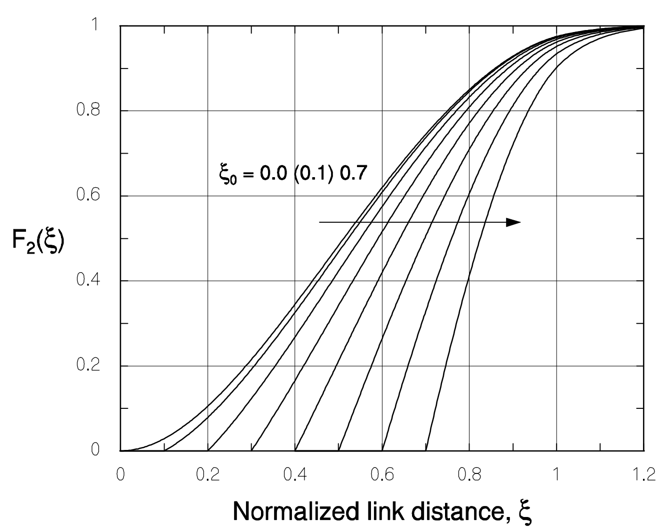

Figure 3. Examples of cdf $\mathrm{F}_2(\xi)$

$$\Pr\{\nu = n|\,\mathbf{p}\} = \binom{N-1}{n}[a(\mathbf{p})]^n[1-a(\mathbf{p})]^{N-1-n}, \quad n = 0,\,1,\,\ldots,\,N-1 \tag{7}$$

where $a(\mathbf{p}) \le \min\{1,\, \pi(\xi^2 - \xi_0^2)\}$ is the portion of the area of deployment that surrounds **p** such that the distance from **p** is greater than d_{min} and less than R. The conditional mean and variance of ν therefore are

$$\mathrm{E}\{\nu|\,\mathbf{p}\} = (N-1)\,a(\mathbf{p}) \tag{8a}$$

and
$$\mathrm{Var}\{\nu|\,\mathbf{p}\} = (N-1)\,a(\mathbf{p})[1-a(\mathbf{p})] \tag{8b}$$

The unconditional mean, mean square, and variance of the number of neighbors are given by

$$\begin{aligned}\mathrm{E}\{\nu\} &= \mathrm{E}_{\mathbf{p}}\{\mathrm{E}\{\nu|\,\mathbf{p}\}\} = (N-1)\,\mathrm{E}\{a(\mathbf{p})\}\\ &= (N-1)\,\mathrm{F}_2(\xi)\end{aligned} \tag{9}$$

$$\begin{aligned}\mathrm{E}\{\nu^2\} &= \mathrm{E}_{\mathbf{p}}\{\mathrm{Var}\{\nu|\,\mathbf{p}\} + [\mathrm{E}\{\nu|\,\mathbf{p}\}]^2\}\\ &= \mathrm{E}_{\mathbf{p}}\{(N-1)\,a(\mathbf{p}) - (N-1)[a(\mathbf{p})]^2 + (N-1)^2[a(\mathbf{p})]^2\}\\ &= (N-1)\,\mathrm{F}_2(\xi) + (N-1)(N-2)\,\mathrm{E}_{\mathbf{p}}\{[a(\mathbf{p})]^2\}\end{aligned} \tag{10}$$

$$\begin{aligned}\mathrm{Var}\{\nu\} &= (N-1)\,\mathrm{F}_2(\xi)[1-\mathrm{F}_2(\xi)]\\ &\quad + (N-1)(N-2)\,\{\mathrm{E}_{\mathbf{p}}\{[a(\mathbf{p})]^2\} - [\mathrm{F}_2(\xi)]^2\}\end{aligned} \tag{11}$$

From (9) and (11) we observe that ν would be a binomial random variable if the area a were not a function of **p**. Because the area does depend on **p**, the unconditional distribution for the number of neighbors has a larger variance than that of a binomial variable with the same mean.

B. Empirical results for mean and variance of the number of neighbors

Simulated random placements of $N = 100$ nodes on a unit square were performed with and without an inhibition based on a minimum distance between nodes. In Figures 4 and 5, respectively, the empirical mean and variance are shown for the number of neighbors as functions of the normalized communications range, ξ, for normalized minimum distances of $\xi_0 = 0.00,\ 0.02,\ 0.04, 0.06,$ and 0.08. For each point, 500 random placements of 100 nodes were simulated. Figure 4 indicates that the mean value of the number of neighbors for no inhibition is well predicted by $\mathrm{F}_2(\xi)$ for small values of ξ_0. In accordance with intuition, Figure 5 shows that the variance of the number of neighbors is close the value for a binomial distribution for small communication ranges (when the area of the "neighborhood" is small and the edges of the deployment region are less significant) but becomes much larger as ξ increases. Also, from Figures 4 and 5 we observe that both the mean and the variance of the number of neighbors are inversely proportional to the minimum distance. A slight increase in ξ for $\xi_0 > 0$ will obtain the same average as for $\xi_0 = 0$, but with a smaller variance.

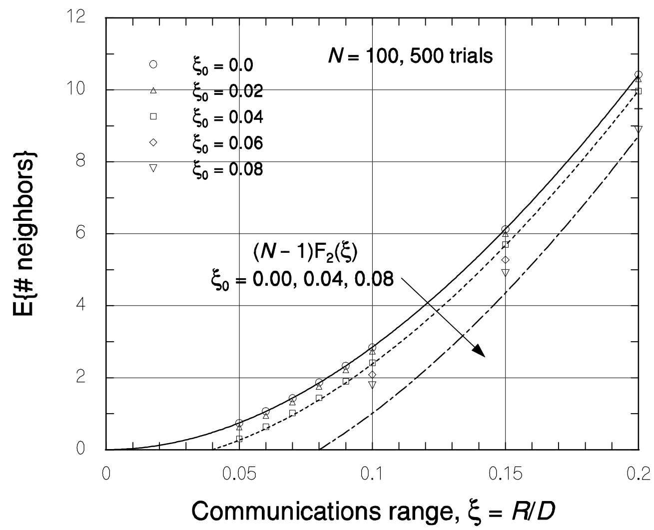

Figure 4. Empirical results for the mean of the number neighbors with different degrees of inhibition in the selection of node positions.

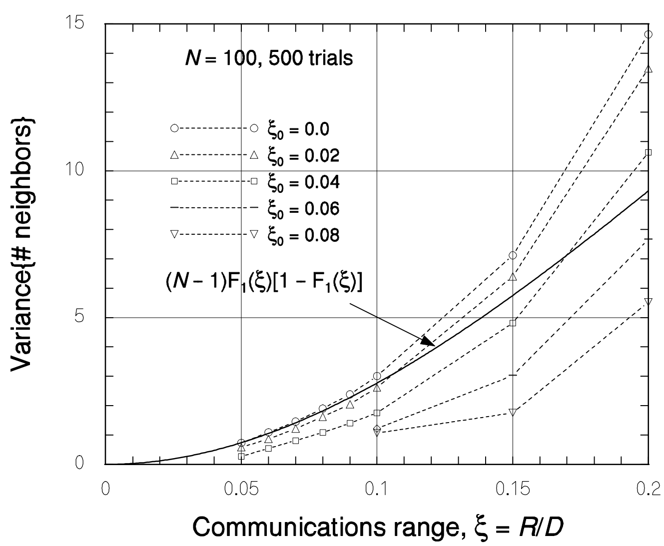

Figure 5. Empirical results for the variance of the number of neighbors for different degrees of inhibition in the selection of node positions.

Regressions were performed on the variance calculations in Figure 5 to find the third-degree polynomials that best fit the variance values. The regressions were of the form

$$\mathrm{Var}\{\nu\} \approx a\,\xi - b\,\xi^2 + c\,\xi^3 \tag{12}$$

and representative values of the coefficients a, b, and c are given in Table 4.

Table 4. Coefficients in the polynomial fit to the variance of the number of neighbors

ξ_0	a	b	c
0.00	7.170	1.0	1666.
0.02	4.564	25.2	1714.
0.04	0.564	87.2	1790.
0.06	17.2	306.	2076.
0.08	23.2	394.	2084.

Additional simulations were performed to determine example values of the mean and variance of ν, the number of neighbors per node, for a fixed area: $\xi_0 = 0.05$ for $N = 100$ and $\xi_0 = 0.025$ for $N = 400$. The scaling down of ξ_0 for $N = 400$ takes into account the increased density of the nodes. Figures 6 and 7 demonstrate that (6) can be used successfully to predict the mean. In Figure 8, the ratio of the standard deviation of ν to its mean is shown for 100 sample networks; the fact that $\sigma(\nu)$ is half as big for $N = 400$ is consistent with there are four times as many nodes being averaged. For these examples, the average connectivity of the sample networks was greater than 90% for $\xi \approx 0.15$ (0.075) for networks generated without inhibition and for $\xi \approx 0.13$ (0.065) for those generated with inhibition, respectively for $N = 100$ and $N = 400$ nodes.

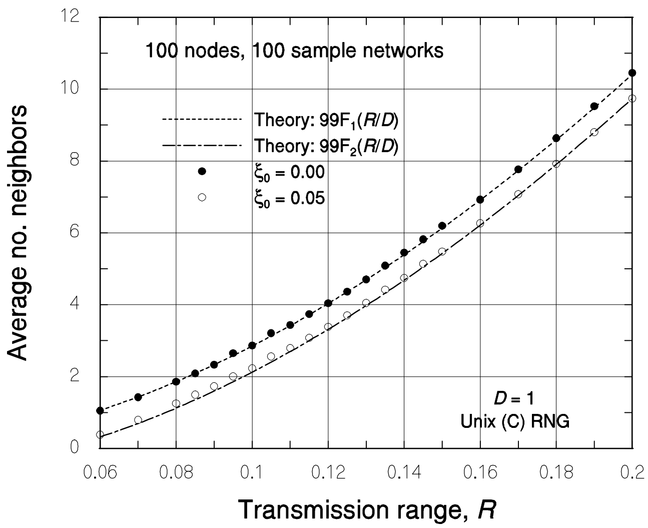

Figure 6. Average number of neighbors per node vs. transmission range for $N = 100$ and $\xi_0 = 0.00,\ 0.05$.

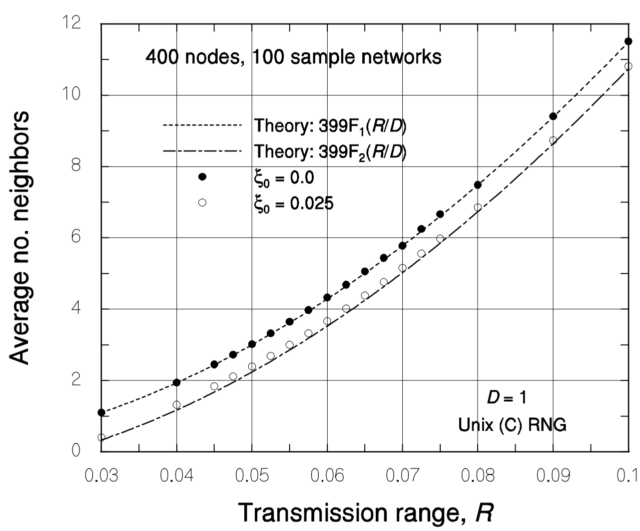

Figure 7. Average number of neighbors per node vs. transmission range for $N = 400$ and $\xi_0 = 0.00,\ 0.025$.

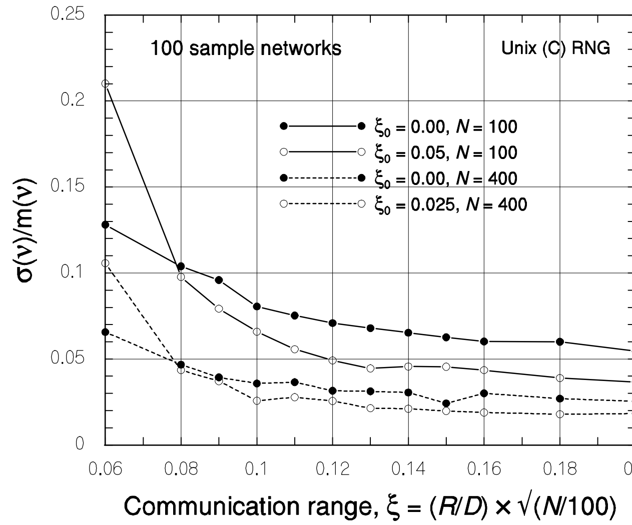

Figure 8. Ratio of standard deviation to mean value vs ξ for number of neighbors per node, $N = 100,\ 400$ with and without inhibition.

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