The 28th International Conference

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CLASSROOM BUSINESS AS USUAL? (WHAT) DO POLICYMAKERS AND RESEARCHERS LEARN FROM CLASSROOM RESEARCH?

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I have called this contribution:

"Classroom Business same as usual? (What) Do Policymakers and Researchers Learn from Classroom Research?"

I would like to use this opportunity to address a recurring problem in educational research:

> *The problem of change within educational change – or more precisely – the denial of change within educational change.*

This is often framed as a problem for the practitioners. The problem of change – the lack of change – is a problem that belongs to the professionals and the practitioners: to schools and teachers, to the pupils and their parents. In this contribution I will discuss this as a problem – and a challenge – for researchers and policy makers. How come researchers (and policy makers) continue to reproduce schools, teaching and learning in terms of status quo? The research literature tells us that despite a huge amount of reform efforts teachers, students and parents continue to reproduce a rather stable and familiar pattern of interaction and repertoires in schools and classrooms which could be summed up by the following phrase: Classroom business as usual.

In Norwegian a saying goes:

> *"Reformer kommer og går – klasserommet består"*

This might of course be an empirical fact – in the sense that established patterns of activities, communication and interactions in schools – the "grammar of schooling" – are so strong that they continue to set their regime through – despite all sorts of reform efforts.

But it might also reflect an embedded problem in how educational research practices grasp, analyse, document and envision dimensions of change within the same practices.

The epistemologist I.Wallerstein has been occupied with the denial of change within social sciences, which he links to the absence of a critical examinations and analyses of concepts, theories and methodological practices within the social sciences.

Wallerstein states for example that concepts, theories and analytical framework developed throughout the 19th century no longer are adequate for defining and describing political and social changes, movements and activities in today's rapidly changing society. As a consequence social sciences are locked up with "…the denial of change in theories of change" (Wallerstein 1991).

The American educationalist Tom Popkewitz claims that policy studies in education (and he actually uses Norway as an example) tend to reproduce their own common sense understanding because analytical concepts, categories and practices are not critically examined and analysed. This has as one of its consequences the "… denial of change within educational change ".. and where the "… knowledge system of policy and research denies change in the process of change." (Popkewitz 2000: 25)

In this paper I will address the denial of change within educational change by focusing on three factors relevant for how the educational research community frames and approaches the process of change within educational practices.

- Theoretical perspectives underlying the different studies
- Types of data and methodological practices that establish the bases for analyses and conclusions
- Conceptual and analytical framework for analysing the situation.

I will use later empirical research from Norway, Sweden, UK and US to discuss these issues. Especially I will lean on later classrooms studies from elementary and lower secondary schools in Norway. These studies were conducted during a period of large reform efforts in Norway. In the 90's Norway – as a lot of other Western countries – experienced educational restructuring in education implying new ways of funding and steering the educational sector as well as new professional roles for educational stakeholders. A new national curriculum was introduced in 1997 putting new professional demands on the teachers as well as requiring new forms of classroom practices. The comprehensive school system was extended from 9 to 10 years of schooling (meaning that children start at school at six instead of seven).

Along with the reform efforts in Norway a large research program was initiated on the basis of the reform trying to grasp some of the effects and impact the reform had on the daily practices of teachers and schools and on their forms of interaction. This research and evaluation program, Reform 97 (implemented by the Research Council of Norway), had a twofold ambition. Firstly, the program wanted to focus on how the reform functioned and developed and what measures might be taken to make improvements. Secondly, the evaluation program also intended to provide general knowledge and information about the compulsory school. The program would combine the evaluation ambitions with research ambitions.

The program funded 25 different research groups or projects varying from subject specific investigations in school subjects such as written Norwegian, maths, science and the use of drama to the role of textbook as curriculum facilitators, assessing different types of curriculum policy instruments, new challenges for the municipalities a.o. The program implied the most extensive support for school based research in Norway with a price tag of 6 mill. Euro, extending for a period of 4 years (Haug 2003).

The problem of status quo in education

How come educational research tends to arrive at status quo as a way of describing how reform efforts interplay with educational practices?

A vast research literature seems to sum up the relation between policy (such as educational reforms) and practice (in terms of school practices) as the following research titles suggest:

- The persistence of recitation (Hoetker and Ahlbrand 1969)
- The more you change the more it will remain the same (Sarason 1982)
- Teaching Practice: Plus que ça change (Cohen 1988)
- Reforming Again, Again and Again (Cuban 1990)
- The Grammar of Schooling (Tyack and Hanson 1990)
- The predictable failure of educational change (Sarason 1991)
- No news on the reform front (Monsen 1998)

Decades of reforming the curriculum (and school practices) again and again had obviously not brought about the changes that the reform authorities had hoped for. The research on the impact of the new curricula supports this impression even further:

- Most teachers reported that the curriculum guidelines had no or little impact on their lesson planning, teaching, their students' involvement, student achievement, etc.
- The format, size, level of detail, etc. of the guidelines had no or very little impact on how students and teachers cope.
- Higher stakes, added content, etc. led to almost nothing, or rather the opposite.
- The main effect of the external process evaluation tools seemed to be legitimation and the distribution of new argument around the curriculum, neither innovation nor quality enhancement. (Hopmann 2003; 127)

The impact of educational reforms such as curricular reforms on educational practices points to a complicated and complex discussion which I will not go deep into here. David Cohen, Deborah Ball and their colleagues have for example underpinned how:

> "… Schools and teachers simply cannot meet the expectations of the center (reforms), because they do not have the fiscal and human resources that are required, teachers do not have the skills that are asked of them, and/or they are not given the training and education required to develop those skills." (Cohen, Raudenbusch, & Ball 2002)

In this presentation I will take a slightly different perspective on how educational and curricular reforms have an impact on educational practices in schools and classrooms and discuss the lack of change – or the denial of change to quote Tom Popkewitz – as an interior or embedded part of research design and research methodology.

This I will do by getting more deeply into three different – but slightly interrelated – arguments:

i) Theoretical perspectives underlying the different studies (reform perspectives /reform fidelity vs reform hybrids/looking for large scale change)
ii) Methodological tools and types of data that establish the bases of analyses and conclusions
iii) Analytical framework and established concepts for analyses.

But first I will give a brief description of how educational literature describes educational practices in classrooms.

Classroom business as usual? An overview

What defines/constitutes educational practices in the classrooms? According to a vast research literature there are some inhibited patterns of schooling and teaching that seem to continue to define interaction, roles and repertoires in classrooms – the so called "grammar of schooling" (Tyack and Hanson 1990).

The persistence of plenary teaching - Plenary teaching dominates

Teachers dominate, regulate, define and evaluate communication and activities. This communication can be described by the rule of the 2/3 which means that for approximately
75 % of the time teachers talk and/or regulate all official classroom conversation.

The dominant pattern of interaction follows a predefined IRF (E) pattern of communication.
The pupils are left with small possibilities for participation and influence.

If we examine the impact of reform and curriculum on schools and classrooms the picture becomes even more grimy, or, as stated earlier from different studies, teachers report that the curriculum guidelines had little or only limited impact on their lesson planning, teaching, their students' involvement etc. The bottom line could be summed up by one the titles quoted earlier: "Reforming Again, Again and Again" or "The Predictable Failure of Educational Change"

The different studies identify different mechanisms for explaining this situation such as:

- School structure and school organisation
- Epistemological traditions of schooling and teaching
- Teachers' and students' competences and repertoires
- Power relations
- Schools as certificates for social reproduction.

I will not go deep into the different explanations here. My point is that despite reform efforts during different periods researchers continue to report that principal modes of instruction (lecturing, recitation, demonstration, seat work) continue to dominate despite the increased range of possibilities.

In my further argumentation I will penetrate these findings and conclusions by carefully examining how our theoretical, conceptual and methodological framework might lead us to scrutiny of conservatism and status quo.

i) Theoretical perspectives underlying the different studies

The way analytical and theoretical perspectives inform and shape your analyses and conclusions is not a controversial issue and argument in research today. To some degree we all find what we look for in the sense that our theoretical perspectives inform and impregnate our interpretation of the world. (A certain degree of curiosity or astonishment should however guide our research practices – taking the Bourdieu argument on epistemological ruptures seriously.)

For the case of educational reforms we can at least distinguish between two analytical traditions in evaluation approaches. The first tradition, a structural – instrumental – tradition, focuses on structures, implementation tools, legitimacy, etc. Who were involved in the process, central means of the reforms, types of implementation processes etc. A structural/instrumental approach focuses on rational and cognitive structures, tools and implementation processes.

A cultural – institutional – tradition takes a slightly different stand. Instead of focusing on intentions and implementation mechanisms and tools the focus will be on how institutions and their agents meet and interact with the different reform policies. In this approach the focus is neither on the programmatic or the intentional part of the reform nor on how the institutions neglect and counteract towards the reform efforts but rather how institutions and agents selectively negotiate, ignore and adapt to the reform.

In this last perspective rather than seeing how reforms change the schools one is interested in how schools change the reforms.

Larry Cuban is among those who speaks for the value of such a perspective if we want to know more about how reforms impact on schools and teaching and learning. Rather than looking for what is being described as a fidelity or efficiency approach to how reforms impact on schools and teaching and learning he speaks for the value of perspectives that enable us to grasp how schools change reforms such as a popularity perspective or a diffusion perspective. Such a perspective enables us to locate how educational practitioners adapt to innovations to the ongoing lives of their schools and seek coherence where it counts the most – in classroom instruction. Cuban finds it useful viewing reform plans "… not as clearly mandated policies but as concepts to be evaluated on their practical effects, positive or negative, and then reframed accordingly" (Cuban 2004). In his work together with historian David Tyack (1995;64), Cuban argues how reforms should be deliberately designed to be hybridized, to be able to fit local circumstances.

In his overview on how reforms impact on teachers, instruction and learning (based on American experience) Cuban states that over time teachers ignore, combine and adapt different reform strategies. Educational reforms do affect educational practices if they

- i) are built on and reflect teachers' expertise
- ii) acknowledge the realities of the school as a workplace
- iii) accept the wisdom of those teacher adaptations that improve the intended policy

Let me take an example from the Reform 97 evaluation program. One of the projects identifying a fairly high degree of reform success in relation to the new curriculum reform is within written Norwegian in lower secondary schooling. The scholars Evensen et al. underpin a robust and vital picture of Norwegian writing skills based on in depth analyses of National tests in written Norwegian. In their study Evensen et al. highlight two central findings. First of all it has become more difficult to achieve good marks as well as bad marks after the new grading system was introduced. Despite the intention of the new grading system, one is now more likely to achieve an average learning result (and get a mark in the middle) than with the earlier grading system. This is what the scholars call an unintended consequence of the reform. But the second and more important finding is as follows. The writing culture in lower secondary schools in Norway can be described in terms of vitality and pluralism. This vitality can be identified in the way the students write their texts (use of textual tools, approaches, etc.) as well as within established norms for good writing among the evaluators (sensorer). Textual pluralism, trust and confidence impregnate both the students' way of writing and the established norms for good writing within the evaluators' corpus. Evensen et al. underpin how this situation reflects a sensus communis in first language writing skills between literacy teachers' established norms for good writing in upper secondary classes and the way the national curriculum defines textual competence. Process writing has become a national standard for good writing, recognised by both teachers, students, evaluators and curriculum designers. Process writing has been spread and made popular through a systematic and deliberate use of developmental teachers' pioneer work in this respect and is today recognised as the good way of writing among professionals, students and national evaluators and in curriculum texts.

ii) Methodological tools

How methodological tools interplay with conclusions arrived at.
Another way to understand the denial of change within educational change is linked to methods of measurements used in the different studies.

If we look at later studies – and especially the studies identifying some aspects or traces of change – they are all relying on some sort of *in depth studies* and *how data*. If we use the Reform 97 evaluation as an example, the studies identifying new forms of practices are all based on some sort of qualitative data or a combination of survey data and qualitative data. To put it another way: Studies leaning solely on survey information tend to be good at grasping established forms of educational practices in terms of the *what* aspect, but seem to be less able to identify ongoing changes and especially changes related to the *how* aspect. Survey studies enable us to see patterns of distribution and variation across groups, individuals and contexts on a large scale. Survey studies are however less fitted for identifying substantial and detailed variances. Maybe ongoing changes in educational practices are related to substantial rather than structural elements and are better envisaged by *in depth* and *how* related data.

Misunderstand me right. I do not mean to speak for a methodological program – in terms of observation data/discourse analysis data or the like. What I want to address is how our methodological tools interplay with, and define, the conclusion we arrive at. Once again although frontal teaching and teacher centered instructions – and especially the IRF pattern – still define central aspects of classroom organisation in Norwegian classrooms they are differently played out today than those defined by Bellack, Mehan and other well recommended studies. One of the big differences compared to earlier studies is related to the role of the students and their possibility for participation and contribution. In that sense the IRF patterns of today are much more “student centered” in terms of students’ possibilities for initiation, negotiation and involvement.

What data might bring you to the wrong conclusions. The persistence of an activity over time does not mean that we are describing the same activity and phenomenon. If we use *how data* we see that teacher centered questions – recitation patterns of today to paraphrase Hoetker and Ahlbrand – give much more room for student participation and student latitude. Let me give you an example from a recitation sequence in a math classroom at the 9th grade:

Pursuing an interest in details
In English there is a saying: The devil is in the details. In a sense, educational research should play along with the devil and endeavour to go beyond everyday language and search for the epistemological ruptures (the Bourdieu argument). For those of us interested in educational practices and how to cope with change there might be strong arguments for detailed in depth studies (alongside with more comprehensive studies) in education. Carefully designed and clearly focused in depth studies enable us to see how classroom activities interact with ongoing societal changes. The changes in classroom activities and interaction themselves (from plenary activities to seatwork in pairs or groups) ask for in depth studies as well as detail studies, simply because the most common practices in Norwegian classrooms today are desk interaction **and not** plenary teaching.

Context vs Content
So far I have been arguing for qualitative studies – or to be precise the need of both comprehensive data and in depth data – as a way of grasping ongoing changes in educational practices. But in depth data or contextual data could be grasped in different ways – or more precisely context means different things during different periods and from different perspectives. The shift from studying teaching to studying interaction can illustrate one such shift in perspective. Another aspect of what defines context can be recognised in how a mathematician versus an educationalist interprets and explains classroom interaction.

iii) Analytical and conceptual language

A third road to understand “the denial of change within educational change” can be linked to the established analytical and conceptual language offered for analysing teaching and learning in educational practices. Within the field of education we have a lot of concepts established for analysing educational practices such as:

– teacher centered vs student centered
– traditional vs progressive
– mimetic vs transformative

– monological vs dialogical
– process vs content
– control vs autonomy.

Based on the data played out throughout the qualitative and quantitative descriptions of Norwegian classrooms after the new Curriculum Reform our teachers and students cut through these dualistic and polarised concepts. If we use teacher style as an example our teachers combine and merge aspects of teacher centered methods with student centered methods in a rich, nuanced and flavoured fashion.

Dualistic concepts such as teacher centered vs. student centered or traditional vs. progressive do not offer an empirical, sensitive and synthesizing way of describing the observed classroom practices. In most classrooms the teachers combined aspects of teacher centered organised activities with more student centered and activity organised pattern of organisations. For a lot of classrooms (and especially at the higher levels (grade 6 and grade 9)) the work plan (arbeidsplan) or work schedule seems to be the driving force for the activities during the school day. Rather than describing the classrooms as teacher vs student centered they seem to be activity and work schedule centered. This implies an indirect and written ruling of the classrooms and where the teachers use a lot of the plenary activities to secure, direct and metacommunicate around the predescribed activities. In their comparison of Swedish classrooms from the 70's and the 90's, Lindblad and Sahlström state that although plenary sessions are less frequent in the classrooms of the 90's (where seat work at desks dominates), the teacher as a master and conductor of the activities seems to be more central in the classrooms of the 90's. They state for example:

> "What we also find when comparing the materials (1970 classrooms and 1990 classrooms – speaker's comment) is that there are substantially longer sequences of instruction of *how to perform* in the 90's material, often with a high level of detail."
> And they continue:
> "The introduction of desk work thus seems not only to have introduced a new way of working, but it also affects the organisations of the seemingly plenary teaching." (Lindblad & Sahlström 2004)

Available established concepts and analytical framework might contribute to a prolongation of established practices and an inscription of status quo also during periods impregnated with changes.

Concluding remarks

In this essay I have discussed how educational research relates to, frames and identifies educational change. As the scientific epistemologist Wallerstein has underpinned, concepts and analytical framework (and we could add methodological tools and theoretical perspectives) need critical examination and analyses so they can fulfil their potential as tools for describing social changes, movements, and activities. Without examining the common sense of its own analytical understanding, research can preserve the very systems that are to be interpreted and engaged in critical conversations.

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COMMENTS TO KIRSTI KLETTE:
CLASSROOM BUSINESS SAME AS USUAL? WHAT DO POLICYMAKERS AND RESEARCHERS LEARN FROM CLASSROOM RESEARCH?

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Kirsti Klette offers an interesting shift of perspectives on the problem of a 'denial of change' i.e. the problem that classroom practices seem to stay more or less the same despite decades of reform efforts. Her suggestion is that this problem, often attributed to teachers' reluctance to implement new modes of teaching, may instead be due to inadequacies in the researchers' analytical frameworks, which she urges us to re-examine. I propose that in our scrutiny of current research practices we take into account not only how theories and methods frame aspects of the implementation process but also how we, as educational researchers, relate to reform ideas.

INTRODUCTION

In her plenary talk Kirsti Klette invites us to reflect upon a seemingly obvious fact: despite decades of curriculum reform in Norway and elsewhere there is little evidence of real change in teaching practices. This 'denial of change' is often viewed as a problem that rests upon the practitioners. Klette cites David Cohen and his colleagues who state that schools and teachers often lack the "fiscal and human resources" needed to meet the demands of the policymakers. Teachers may not have the knowledge and skills necessary to implement the changes that the policymakers and agencies hoped for or are not offered the appropriate in-service training required to improve their skills.

In her talk Klette contests this way of framing the problem of a 'denial of change'. Instead she invites us, as educational researchers, to re-examine critically how we frame and identify educational change. She argues that the problem of a 'denial of change' may well be an artefact of our own research practices; scrutinising conservatism and the status quo may be an interior or embedded part of the theoretical and methodological perspectives used to analyse how institutions and agents adapt to the reforms. If our analyses are based on superficial or incomplete accounts of what is going on in the classrooms we may not be able to identify reform success, or worse, we may ourselves be instrumental in reproducing a traditional 'grammar of schooling' (Tyack & Hansot 1990)

Klette argues that in-depth studies are needed to evaluate the impact of educational reforms on classroom practice. She emphasizes that we need to look more closely at the lives and work of teachers and students in order to understand how the policymakers' guiding principles are transformed into classroom practice. I agree with her. An activity such as 'recitation' may easily be identified as such if we describe it solely in terms of *what* is going on in a classroom, but may turn out to be a varied and nuanced activity, maybe not even 'recitation' at all, if we view it in terms of *how* it is done and how it is interpreted by the participants.

I deeply sympathise with Klette's call for self-scrutiny amongst researchers engaged in studies of social change. I would even like to bring her argument a bit further by addressing a question that is not elaborated in her talk: How do we as researchers relate to educational reform, in particular to the reform ideas of today? Is there not a need for greater self-reflection in regard to our own roles and responsibilities when it comes to the relation between policy setting and classroom practice?

TEACHERS' RESPONSES TO CURRICULUM REFORM.

Let us assume, for the sake of argument, that there exists such a phenomenon as a 'denial of change' in teachers' responses to curriculum reform. Following Klette we need to ask ourselves how we should interpret such responses. In a recent article Klette (2002) points out that there are two ways of viewing current educational reforms in the Nordic countries: we may either regard them as efforts of empowerment and professionalisation for schools and teachers, or as tools for trivialising the teachers' work and subjecting education to economic regulations (p. 266). Under the former interpretation we can view teachers' reluctance to implement the required changes in their teaching practice as a manifestation of inertia or even conservatism (or as Klette suggests even as an artefact of the researchers' analytical frameworks). Under the latter interpretation we may view professional resistance to change as both rational and well-founded.

Are there reasons to believe that current school reforms may be detrimental to the quality of teaching and learning? Thematic approaches to curriculum delivery, active, meaningful, cooperative learning, and pupil autonomy are guiding concepts in the official rhetoric behind Nordic efforts to restructure compulsory education (Broadhead 2001). How could such seemingly positive efforts possibly cause concern among practitioners?

WHAT CAN POLICYMAKERS AND RESEARCHERS LEARN FROM CLASSROOM STUDIES?

Klette would like to see more in-depth studies of the interplay between reform efforts and educational practices. Such studies already exist, studies that address issues highly relevant to the debate over current reforms and their practical meaning (e.g. Bergqvist & Säljö in press; Siegler & Hiebert 1999, Siegler 2004). I will refer to some of these studies below, since they shed light on the reasons why teachers may be reluctant to unreservedly implement the policymakers' ideas, and why there is cause to discuss critically the researcher's role in relation to these ideas.

“Pedagogy is never innocent. It is a medium that carries its own message” (Bruner 1996)

Current curricular reforms in the Nordic countries focus on certain qualities in student learning. In doing so other aspects of the learning process may shift out of focus and appear to be less important. A clear message of the current reforms is that meta-cognitive and social skills are of primary importance to schooling, whereas content knowledge plays a secondary or auxiliary role in fostering active, independent and cooperative learners (Bergqvist & Säljö in press). For instance, the concept of the ‘autonomous learner’ seems to have paved the way for patterns of social interaction that “encourage, and require, self-observation, self-control, and meta-awareness on the part of the individual” (*ibid* p. 3). Bergqvist and Säljö draw this conclusion from an extensive in-depth study carried out among children seven and twelve years in six primary schools in Sweden. Their results show that planning one’s work and monitoring the time spent on various tasks have become more important to the teachers and students than engaging in the content of these tasks.

> “It is the demonstration of being able to perform the planning that is the decisive element. In what sense the planning actually supports children’s work remains far from clear.” (*ibid* p. 9)

Since the theme of this conference is *Inclusion and Diversity* it is worth pointing out that this new focus, or rather this new content of learning, seems to benefit students who are responsive to the demands that they self-govern their activities, which in turn may favour students from certain social strata (cf. Bernstein 1971-75).

Be Prepared to Scrutinize the Reform Ideas

The TIMSS study provides a rich offering of 231 video-taped eighth-grade mathematics lessons from three countries, Germany, Japan and the U.S. documented from 1994-1995. In their book *The Teaching Gap,* Stigler and Hiebert (1999) comment on the differences in teaching practices in these three countries. The Japanese and the U.S. lessons stand in sharp contrast to each other. While the Japanese teachers gave the students subtle hints, encouraging them to think for themselves and guiding them towards correct and effective problem-solving methods, the U.S. teachers’ discovery-learning practice left the students more or less to themselves to discover mathematical principles and techniques by ‘grappling and telling’. Stigler and Hiebert conclude that:

> “Japanese teachers, in certain respects, come closer to implementing the spirit of current ideas advanced by U.S. reformers than do U.S. teachers.” (*ibid*, p. vii)

However, the empirical studies give little weight to such a notion. In an independent study of excerpts from the TIMSS video-recordings Alan Siegel (2004) shows that the Japanese lessons include “…more lecturing and demonstration than even the more traditional U.S. lessons” (*ibid,* p. 28) and, perhaps more striking:

> “The video excerpts show Japanese lessons with a far richer content than the corresponding offerings from the U.S. and Germany.” (*ibid*, p. 20).

Even if the videotapes as well as the statistical data gathered within the TIMSS project show that Japanese styles of teaching differ significantly from those in the U.S. (*ibid*, p. 17), Stigler and Hiebert do not find any cause for questioning the reform ideas. Instead, based on the results of the TIMSS study, they conclude that something has gone wrong in the implementation of the reforms. My suggestion is that we, as researchers, prepare to scrutinise not only the key ideas emanating from our own sphere that underpin reform initiatives but also precisely how these ideas may transform classroom practice. The in-depth studies that Klette calls for in her talk can be used for such a purpose as well; in fact, the studies cited above show that such data, in combination with more comprehensive studies, is needed if we want to know how idealised reform goals are met when realised in classroom practice.

CONCLUSIONS

Educational inquiry often develops in close contact and cooperation with policymakers. Not only do we offer our services as advisors or evaluators, we are often active partners in the shaping of educational policy. This may make us reluctant to question reform ideas since in many case they begin with us. We concentrate on the problems of implementing the ideas or of reflecting on the theories and methods we use to make sense of the implementation process. My main comment to Klette is that we should also and maybe first and foremost, concentrate on scrutinising the very ideas that form the basis of these reforms.

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GRAPPLING WITH COMPLEXITY: CO-LEARNING IN INQUIRY COMMUNITIES IN MATHEMATICS TEACHING DEVELOPMENT

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The complexity of mathematics teaching involving both cognitive and sociosystemic factors makes development problematic. Examples from research into teaching reveal factors in complexity and ways in which inquiry between teachers and didacticians can foster deeper ways of knowing in the developmental community. Research is seen as the basis of an inquiry process which develops inquiry in mathematics teaching as both a tool for teaching and a way of being for all learners. Theoretical discussion on the concepts of inquiry and community and their relation to development leads to consideration of approaches in which mathematics teaching development is the object of a design process. Tensions between learning theory and proposals for developmental practice are revealed and addressed. Research that is designed to create and study inquiry communities is introduced.

INTRODUCTION

The question of how mathematics teaching does or can develop and how that development is sustainable is at the centre of this paper. Research is revealing increasingly valuable insights to mathematical learning, both in terms of mathematical content and of processes in constructing mathematical ideas. We see also more sophisticated curriculum construction, some of it taking into account research findings. Yet, we find there are still serious issues in students' achievement in mathematics. Teaching is not achieving the widespread mathematical know-how that society would like to see. So, we question teaching and the education of teachers. Research here has revealed many factors that seem important to teacher and teaching development. Yet, despite a theoretical knowledge of such factors and a sincere desire to foster students' mathematical competence, we experience activity in classrooms that does not seem to foster learning. There are many sources that address issues involved here, particularly issues I might call 'sociosystemic'. I will start with just three references.

In 1987, Desforges and Cockburn, writing about sincere teachers who were aware of the importance of developing students' higher order skills and able to do so in subjects other than mathematics, suggested that "classrooms as presently conceived and resourced are simply not good places in which to expect the development of higher order skills currently desired from a mathematics curriculum" (p. 139). They wrote further, "the teacher's job is more complex than that assumed by those who advise them on how to teach mathematics … [indeed] the job is more difficult than even the teachers realize"; "constraining classroom forces operate in concert and …

teachers' necessary management strategies exacerbate the problems of developing children's thinking" (p. 155).

In 2002, Despina Potari and I discussed episodes from the work of a teacher concerned to offer *mathematical challenge* to students in her class. While some episodes provided clear evidence of challenge, there were others in which challenge was lacking, in which the teacher answered her own questions and offered her own explanations in response to students' apparent inabilities to do so. Discussion with the teacher revealed a complexity of factors militating in this and other situations against teaching that would provide appropriate challenge (Potari & Jaworki, 2001). It was not a case of saying what *should* or *could* have been done, but of addressing the sociosystemic demands on the teacher. I shall return to this research.

In 2002, Razia Mohammad reported research with mathematics teachers who had followed an eight week university course addressing mathematics and the learning and teaching of mathematics from a perspective of developing teaching. She found that the teaching she observed conformed largely to traditional school practices with little evidence of the course having made a difference. Sociosystemic factors were evident – physical conditions, authority structures, attitudes, teacher-pupil relationships, text books, examinations, and time. One teacher challenged the researcher as follows:

> Is it all applicable in this situation? If you were allowed to work here would you be able to maintain the quality of thinking and work you all do at the [university]. (p. 112)

This challenge from a teacher to a researcher/educator captured a gulf between the thinking and conditions at the university and those in the school. The course had developed a rapport between teachers and educators. The closeness of relationship was still evident when the researcher worked with teachers in the school context. But the teacher knew that the researcher's knowledge did not encompass the same understandings of school conditions as the teacher's knowledge – what it was like to work as a teacher, deeply embedded in the social milieu of the school system.

I used here the word "knowledge", but an alternative term which captures multiple forms of knowledge relating to situation and context is "ways of knowing" (Belenky, Clinchy, Goldberger, & Tarule, 1986). The educator in the example above had been for many years a teacher in the same kind of school system, but now her ways of knowing, developed within the university context, differed from those of teachers trying to implement university-knowing in their school context. However close this educator came to understanding the teaching context – *knowing* it from her previous experience and knowing *about* it now – she could not experience it as a teacher now.

In addressing teaching development it seems essential to address the ways of knowing of those who contribute to development which includes teachers and educators. The word "educator" can be seen as divisive: teachers are also educators. Mathematics educators in a university setting are *didacticians* of mathematics – they have a responsibility to conceptualise and theorise learning and teaching of mathematics, to develop knowledge in these areas, which is different from teaching mathematics per se (although they might do this as well). In most cases, they do not teach mathematics in schools and their ways of knowing school culture are different from those of teachers. They might also be teacher-educators with responsibility to teach teachers. These distinctions are important to the discussion that follows.

In my introduction above I used the pronoun "we". For example, I said "we experience activity in classrooms that does not seem to foster learning". For inclusivity of teachers in conceptualizing teaching development it needs to encompass both teachers and didacticians. It is a challenge for both groups to achieve ways of working together that draws on all their ways of knowing in mutually fruitful ways. Sandy Dawson has written about an "inservice culture" which assumes that "there is something wrong with mathematics teaching world-wide, and that *we*, as mathematics educators, must fix it" (1999, p. 148, my emphasis). It becomes more and more obvious to me that didacticians can theorise and suggest, and work to understand how theories and suggestions can be realised the school culture, but they cannot "fix it". How much more powerful might it be if theories and suggestions were to come also from *within* the school culture – from teachers? I have been working on this question for many years (e.g., Jaworski, 1998).

It seems important here to recognise that

a) there are many issues relating to mathematics learning and teaching in schools that need to be addressed and that didacticians bring ideas and concepts that can be explored in such contexts;

b) teachers' ways of knowing mathematics learning and teaching are largely school bound, and often school cultures militate against theories and suggestions from outside the school context;

c) didacticians' ways of knowing mathematics learning and teaching are largely theory based and, although many have been teachers formerly, it is rare for such theoretical knowing to be embedded in a school context.

My focus in this paper lies in how to draw fruitfully on both kinds of knowing for developing practice in the learning and teaching of mathematics. I first offer some examples to illustrate complexity in teaching and teaching development, with teachers and didacticians engaging together in inquiry to improve mathematical learning. I then discuss theory relating to inquiry communities in mathematics teaching development, leading to some discussion of developmental theory and practice. Finally, I introduce a current research project that is rooted in these ideas.

EXAMPLES OF COMPLEXITY IN MATHEMATICS TEACHING

Management of Learning in a Vectors Lesson

Ben's Year 10 class (ages 14-15) was working on vectors. They had considered the idea of a vector AB ($\underline{v}$) as a journey from A to B, and had moved on to considering $2\underline{v}$ and $3\underline{v}$. I had observed the lesson and was talking with Ben about it afterwards.

1 BJ Oh, another thing I recall now, do you remember when you'd got three-AB up there, six, six? (Ben said 'yeah') And you turned round and you asked Luke. And my understanding of that was, Luke's not paying attention. You're checking that he knows what's going on. And you asked him to explain that. And he clearly hadn't listened at all, but he comes up with an alternative correct representation.

2. Ben But that's Luke. That's the sort of person he is, isn't it? I think.

3. BJ I mean, I was quite surprised not to / for you not *then* to make the link, but you decided to go on and …

4. Ben I felt there was so much around, that I had to sort of / it's these judgments again isn't it? You make these judgments all the time. (Jaworski, 1994, p. 191)

Discussion was about the vector (6, 6) which had emerged from three-times the vector $\underline{v}$ ($3\underline{v}$) where $\underline{v} = (2, 2)$. The class had many questions which were being asked and discussed. For example, if the vector *AB* is a journey from A to B, what is the journey related to three times this vector – where is B in 3AB? Students asked their questions vociferously and were answered, equally vociferously, by others. Ben, the teacher, was one voice among many as he responded to and managed the discussion. Luke looked as if he was not attending and Ben addressed him directly. I had expected Luke not to know what was being discussed, but quick as a flash he suggested (6, 6) was 2*AB* plus *AB* ($2\underline{v} + \underline{v}$). This seemed to me like a new way of seeing (6, 6) and I was surprised that Ben did not emphasise it to the class.

In whole class mode, Ben was managing a complex interchange of questions and answers. In the middle of it all, he checked up on Luke. He then returned his attention to other students. Contrary to my expectations he did not take Luke's contribution further. When we discussed this later, he referred to "judgments" which had been a topic of discussion between us many times. What are the factors contributing to each *judgment* a teacher makes? How can the teacher manage his attention to such factors? Can he be aware enough of factors to have the option to deal with explicit choices at the moment they arise? John Mason (2001) talks about "noticing in the moment" and Donald Schön (1987) about "reflecting-in-action". The theory is that the teacher is sufficiently aware of the choices to be made, and possibly the issues involved, that he can act knowledgeably at the moment of choice.

Michael Eraut (1995), in a critique of Schön, suggests that teaching is too complex for reflecting-in-action to be a serious option for most teachers. He suggests that "reflection-in-action involves thinking at a meta level about the process in which one is engaged"; involving "a ladder of reflection, where people move up a rung to reflect at a meta level on what they have been doing then down again to take consequent action". He emphasises "the effect on the mode of cognition of the time available for thinking", recognising that "a teacher has to be constantly assessing the situation, responding to incidents, deciding whether to change the activity, alert for opportunities to tackle difficult issues". This suggests that time in teaching decisions is too short to support the metacognitive activity required; that teaching, is too demanding to allow for noticing, and acting knowledgeably, in the moment (c.f., Desforges & Cockburn, above). Yet, I know from my own experience and reports from teachers (Jaworski, 1994) that reflection-in-action *does* happen with consequences for immediate teaching action.

Ben was engaged in a complex management of the learning environment in the situation described, captured by the term 'management of learning', a theoretical construct I have used as part of a triad, *the teaching triad*. This also includes constructs of *sensitivity to students* and *mathematical challenge*. The three constructs, ML, SS and MC, are deeply linked to account for teaching situations. According to Ben, he saw SS and MC to be subsumed within ML (Jaworski, 1994, p. 144). In the episode, Ben's management of learning included handling the whole class discussion, deciding how best to respond to students' questions, injecting new ideas, and checking up on Luke's involvement. He noticed that Luke did not seem to be attending, so he acted to check on Luke. He said afterwards that he had not followed up Luke's input because there was "so much around": that is, so many factors of which he was aware. Further discussion revealed Ben's attention to students who were struggling with vector concepts, gender issues in trying to include some quieter girls in the dominating questions from other students, and the noise levels in the room which were sometimes unacceptable for thinking and interaction. The choice, a) to value Luke's input overtly and b) to offer the alternative way of seeing the vector $3\underline{v}$ to others in the class, was something *I* saw, as an observer without teaching responsibilities. Ben was focusing on multiple concerns. As we grappled together with such issues and inquired into teaching processes and tensions, our knowing developed relative to our particular contexts: Ben's in terms of his making of judgments, mine in a developing awareness of factors of complexity.

Some further examples

In Potari and Jaworski (2001), we described research which explored the use of the teaching triad as a developmental and analytical tool. We wrote about the teacher Jeanette who wanted her Year 9 class to appreciate relationships between volume and surface area of cuboids and wanted to *challenge* pupils fruitfully (i.e., mathematical challenge directed at conceptual learning outcomes). We point to an episode in which two boys seemed to be developing strong concepts (they used the term 'compact' to describe minimum surface area for a given volume), and were able therefore to react well to the teacher's challenges. There seemed to be *harmony* between sensitivity and challenge: Jeanette's in-the-moment decisions there seemed appropriate in her management of the learning situation. However, later, under the stress of a Friday afternoon lesson, students' unwillingness or inability to offer explanations, and time factors in finishing an activity, this same teacher entered a funnelling process in which she herself explained the concepts she wanted students to address. She was aware of the conflict between her aims and actions, but she needed a closure to current activity and, in the moment, no other actions were obvious. In reflecting *on* the activity later, she explained that what she would have done, ideally, did not fit with time factors and the mood and behaviour of students. This discussion in our research team led to an elucidation of socio-systemic factors that have to be considered in the teachers' design of teaching (p. 372/3).

In an episode we are writing about in another paper currently, two girls had not done their homework. Their teacher, Sam, had asked them to look up the meanings of "mode", "median" and "mean" in a dictionary for homework. They said they thought they needed a *French* dictionary, and did not have one. They had been unable to make sense of his task, and had avoided the necessity to do so. Many students had not engaged with the homework task. Sam remonstrated, students grumbled and the atmosphere became unpleasant. Sam was unable to engage with his planned activity for the lesson. He was irritable; changing his plans on the spot (finding a way to deal with students' difficulties, avoidance of work and current disruptive attitudes) challenged his teaching. At the same time, he experienced a growing awareness of the inappropriateness of the challenge in his task for the students. In our research, he had become aware of his tendency to offer mathematical challenge without attending to the sensitivities involved. He had set himself the task of paying greater attention to his sensitivity to students. In this case, as he worked with the students to overcome the unpleasantness in the classroom, analysis shows how successive interactions addressed students' cognitive and emotional needs and that learning outcomes were more fruitful than might have been expected. Again, as we talked about this together we grappled with complexities in teaching and how design of teaching, both before and in a lesson, could account for all that was 'around'. The teaching triad played an important role in these analyses.

Key factors in complexity and development

These examples just start to sketch the kinds of complexity I see in trying to develop teaching. They include dealing with in-the-moment decisions involving cognitive and sociosystemic factors relating to the diverse needs of pupils in class and beyond: time factors, syllabus demand, mathematical or didactical beliefs, emotions of teachers and pupils and more. Teachers tried to balance challenge and sensitivity within a management of learning that was both inclusive of students (sensitive to their thinking and needs) and focused on deep consideration and development of mathematical concepts. Line by line analyses of classroom dialogue provided a fine-grained insight to a complexity of relationships between challenge and sensitivity.

As we talked about who the teacher attended to at certain times in the classroom, how he or she steered the mathematical discussion, what sociosystemic factors influenced decision-making and so on, we explored issues and recognised complexity for teachers. Our awarenesses of the relatedness of theory and practice, and the corresponding tensions in dealing simultaneously with theory and practice, led to a powerful form of *co-learning* in which *inquiry* was a central element. Seth Chaiklin has written, of social situations where research contributes to development,

> Social science research has the potential to illuminate and clarify the practices we are studying as well as the possibility to be *incorporated into the very practices being investigated*. (Chaiklin, 1996, p. 394. My emphasis.)

My focus on teaching development, considering ways of knowing of both teachers and didacticians in developmental practice, looks into how research itself is a major factor in enabling growth. I offered a framework for analysing the qualities of such research (in Jaworski, 2003) and have taken "inquiry" as a unifying factor between research and the learning and teaching development on which research has focused.

RESEARCH AND DEVELOPMENT – INQUIRY AND CO-LEARNING

So far, I have emphasised aspects of complexity in developing mathematics teaching, the differential ways of knowing of teachers and didacticians, and the centrality of research in teaching development. I want now to explore further the relationship between research and development, linking to notions of inquiry and co-learning.

With reference to her work on "reciprocal teaching", Ann Brown (1992) recalls the significance of her work being dismissed as "only the Hawthorn effect", which claims "… the mere presence of a research team will lead to enhanced performance because of the motivational effects of the attention received by the "subjects""(p. 163). She suggests that the Hawthorne Effect, far from being a factor to be wary of in educational research, is actually one to be valued and built on to enhance knowledge and promote improved practices. These days, we might talk of "participants" rather than subjects: however, I want to go further. In the examples above, teachers are not just participants in empirical research; they are *partners* in *developmental* research. In the research with Ben, in which I set out to do an ethnographic study *of* his teaching, the relationship soon developed a mutuality in which learning was reciprocal. He became far more than a "subject" of this research. However, the learning resulted from there *being* a research project.

What do I mean when I say that Ben was "far more" than the subject of the research? Put simply, I claim that he became a partner in the research because he engaged in inquiry too, for example, into the question of "judgments". His inquiry was different from mine. He was much less interested than I was in generalised research knowledge, and had no wish to write research papers. However, he was very interested in thinking about teaching and exploring ways of enhancing learning. Thus his design of teaching, my analytic observations of his teaching and our subsequent (lengthy) discussions served both our purposes, and moreover our learning was mutually dependent – we learned from each other's activity and expression. This has been true in subsequent projects in which I have worked with didacticians and teachers. The act of engaging together in research has meant that we are all inquiring into the learning and teaching processes in which we have differing roles and goals. The mutuality of inquiring together leads to clearer understandings - *co-learning* - for both partners.

Thus *inquiry* provides a theoretical basis for seeing research as a developmental tool. Chambers' English Dictionary (McDonald, 1977) suggests that *to inquire* means: to ask a question; to make an investigation; to acquire information; to search for knowledge. Wells (1999) sees dialogic inquiry as

> a willingness to wonder, to ask questions, and to seek to understand by collaborating with others in the attempt to make answers to them (p. 122).

Wells' "to ask questions …and … attempt to make answers to them" is one way of interpreting Chambers' "search for knowledge". This search for knowledge and its relation to learning, 'coming to know', forms the essence of the inquiry process.

Of course, inquiry has a long history in mathematics education. I think of inquiry as being at the roots of the problem-solving movement, deriving from Dewey and Polya, and promoted by John Mason and Alan Schoenfeld and others in mathematics education (see references for key sources). Inquiry in (school) mathematics can be seen to follow the activity of research mathematicians, and lead to recognition of the value of processes such as specializing and generalizing, conjecturing, convincing and proving (e.g., Mason, Burton & Stacey, 1982). Involvement in questioning and investigating focuses minds on aspects of mathematics and generates further questions and lines of inquiry, seeking answers and supporting learners in coming to know. For example, in their further work on vectors, Ben asked pupils to draw their own vectors and find their lengths. In addressing *what vectors can we draw*, students recognized what seemed like negative or zero vectors and had to resolve these apparent inconsistencies with the idea of a vector being a journey (Jaworski, 1994). Cognition could be seen to develop through tackling such inconsistencies and arguing them out in class. *Viability* (Glasersfeld, 1995) of constructed knowledge suggested that inconsistency was inappropriate and some resolution had to be found. As students argued and explored, results (like the length of a vector being positive even if the vector seemed to be negative) emerged and were seen to make sense. There was evidence of pupils' growth in mathematical knowledge.

It seems to me that inquiry in mathematics, as a mode of activity for pupils learning mathematics, has processes in common with both inquiry in developing mathematics teaching and inquiry in the research process. Indeed, the research with Ben and other teachers began as a study of investigative mathematics teaching: exploring the practices and issues arising from working in an investigative way with pupils in mathematics classrooms. Investigation was a mode of learning, a way of designing activity for pupils and a way of developing teaching. Thus I see inquiry in three mutually embedded forms or layers:

- *Inquiry in mathematics*: Pupils in schools learning mathematics through exploration in tasks and problems in classrooms;
- *Inquiry in teaching mathematics*: Teachers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms;

- *Inquiry in research which results in developing the teaching of mathematics*: Teachers and didacticians researching the processes of using inquiry in mathematics and in the teaching of mathematics.

In each of these layers we have people as individuals and people as groups inquiring into mathematics, mathematics teaching or into the contribution of research to teaching development. The individual-and-social nature of the processes involved is central to what I see as being the way ahead for teaching development. Jon Wagner talks of "co-learning" in research partnerships, writing

> In a co-learning agreement, researchers and practitioners are both participants in processes of education and systems of schooling. Both are engaged in action and reflection. By working together, each might learn something about the world of the other. Of equal importance, however, each may learn something more about his or her own world and its connections to institutions and schooling. (Wagner, 1997, p 16)

We are all deeply embedded in social and cultural worlds (including political, economic, religious and systemic factors). Knowing can be seen both as situated in the context, community and practices in which we engage and as distributed within a community of practice (Cole & Engeström, 1993). Individual construction of understanding occurs within a 'community of practice' and is rooted in the norms of activity within that practice. Learning is in dialogue in the social plane before being internalized to the mental plane through inner speech (Vygotsky, 1978). Wenger (1998) has emphasised the production of *identity* through participation in a community of practice. Learning is presented as a "process of becoming". Wenger states, "It is in that formation of identity that learning can become a source of meaningfulness and of personal and social energy" (p. 215). He speaks of "modes of belonging", including *engagement*, *imagination* and *alignment*. We engage with ideas through communicative practice, develop those ideas through exercising imagination and align ourselves, critically, "with respect to a broad and rich picture of the world" (p. 218). I believe we can conceptualise inquiry learning in such terms.

I have struggled with the individual/social tension in a shift over the years from a constructivist position on knowing and learning to a more overt recognition of the social embeddedness of learning as expressed briefly above. The commensurability of these positions has been both a source of contention and an inspiration to seek some resolution between them, since both are essential (e.g., Bruner, 1997). Two factors, however, were always clear to me: (1) the power of inquiry in processes of learning; (2) the importance of dialogue in coming to know. Theoretically, I believe that a shift from 'community of practice' to 'community of inquiry' provides a perspective in which reflective development of teaching by individual teachers results in a developing community (Wells, 1999). In a community of inquiry, inquiry is more than the practice of a community of practice: teachers, develop inquiry approaches to their practice and together use inquiry approaches to develop their practice. This indicates a reflexive relationship between inquiry and development (where development implies learning and deeper knowing). Wells describes teachers as “attempting to develop such communities of inquiry and simultaneously making their attempts the objects of their own inquiries” (1999, p. 124). A feature of a community of inquiry that distinguishes it from a community of practice, according to Wells (fitting well with references to Mason and Schön on *reflection* earlier) is

> the importance attached to *meta-knowing* through reflecting on what is being or has been constructed and on the tools and practices involved in the process’ (page 124, my emphasis).

He adds, ‘the construction of understanding is a collaborative enterprise’ (p. 125).

Such a model is an individual process and a community process: as part of a community of inquiry, individuals are encouraged to look critically at their own practices and to modify these through their own learning-in-practice. Developments within the community result from rationalisations, implicit and overt, between ongoing practices. Participants grow into and contribute to continual reconstitution of the community through critical reflection; inquiry develops as one of the norms of practice and individual identity develops through reflective inquiry.

In my view, inquiry is both a *tool* and *a way of being*. In constructivist terms, it can be seen to stimulate accommodation of meanings central to individual growth. In sociocultural terms it is a way of acting together that is inclusive of the distributed ways of knowing in a community. The notion of “way of being” reflects Wenger’s concepts of becoming and belonging. When different communities interact in a mode of inquiry, meta-knowing that results through inquiry processes allows understandings that cross community barriers (c.f., Wagner, above). It is within this theoretical frame that teachers and didacticians collaborate for mutual learning. This view accords with the idea of ‘inquiry as stance’ introduced by Marylin Cochran Smith and Susan Lytle (1999). Teachers taking an inquiry stance “[raise] questions about what counts as teaching and learning in classrooms” and “critique and seek to alter” systemic norms and relationships; further, they suggest, “the work of inquiry communities is both social and political”, aiming to bring about change in traditional ideas of knowledge and develop richer conceptions of practice (p. 289).

However, there is a fundamental tension in addressing teaching complexity through inquiry communities that I will try to capture before going further. The theory expressed above articulates a concept of *learning* through inquiry in communities in which teachers and didacticians are *learners*. The communities both support the inquiry and grow through the inquiry. However, so far, the role of a teacher or teacher educator in these learning processes is hidden. Consider again my three levels: at Level 1 we might expect a teacher to contribute fruitfully to students’ learning of mathematics and at Level 2, a teacher educator might contribute similarly to a teacher’s learning of teaching. Indeed systemic requirements and social expectations demand that teachers and teacher educators have goals for the learning of their students. Certain complexities of teaching arise from trying to reconcile developing teaching through a community of inquiry with expecting that teachers will have clear goals for their students' learning – *the inquiry/goals tension*. My next section will start to address these issues.

DEVELOPING TEACHING: DESIGN, INNOVATION AND INQUIRY

Learning Study (LS)

Inquiry as *a way of being* is fruitful for development, as experience and research show (e.g., Schoenfeld, 1996; Wells, 1999). I see that inquiry as a tool is valuable to induce inquiry as a way of being. The tool needs to be used purposefully. Ference Marton and colleagues in Sweden and Hong Kong (Marton, Tsui, Chik, Ko, Lo, Mok, Ng, Pang, Pong, & Runesson, 2004) have used inquiry as a tool in a developmental process they call "Learning Study". Developing from Japanese *Lesson Study* (e.g., Stigler & Hiebert, 1999), Learning Study encompasses elements of inquiry, design and innovation. Marton et al. write, "Students' learning should not be accidental …" (p. 331). They add, "Teachers' opportunities to learn are a key factor affecting classroom practice ..." (p. 332); and "Intervention studies must change what teachers do … in order to affect student learning." (p. 333). It seems to me that, in developing teaching through inquiry, teachers' learning is not accidental; and in good research, researchers' learning is not accidental. This does not mean that we cannot learn what we did not set out to learn, but rather that, in purposeful activity, we have goals for learning; and moreover, it is problematic if learning does not accord with declared goals. But, how do we achieve our goals? These statements about goals speak directly to the tension outlined at the end of the last section, especially if our goals are for the learning of some person other than ourselves. Is a student (or teacher) in a position of deficit with respect to a teacher's (or teacher educator's) goals? Is this tension instrumental in complexities observed?

In learning study (LS) a group of teachers designs innovative classroom activity, based on agreed theoretical principles, and explores the consequent teaching. *Design* and *innovation* offer purposeful directions. Teachers use inquiry as a tool to explore teaching, alongside didacticians who offer theoretical ideas and practical support and who research the processes of teaching development. Teachers develop their thinking and practice through successive cycles of inquiry. They each work in their own classroom, interpreting a design they have produced jointly. Observation of each other's teaching and group reflections lead to building of group and individual awareness through which *inquiry as a way of being* develops.

LS goes beyond *lesson study* in two major respects. The first is its theoretical basis. Design is based on *variation* theory (Marton et al, 2004). Didacticians and teachers work together to establish a theoretical basis for joint inquiry. The second is its purposeful nature in terms of pupil learning. LS conducts research into pupils' attitudes and understandings throughout the developmental process. Thus teachers use variation theory to design activity related to curricular topics such as *subtraction* or *fractions*, and tests are applied before and after classroom activity to find out what students have learned. Marton et al acknowledge their use of "design" as being in accord with a paradigm becoming known as “design research”: this, I believe, both comes up against *and* offers ways to address the inquiry/goals tension.

Design Research

The design research paradigm in education, developing from the work of Ann Brown and colleagues, uses *design* as a developmental tool. According to Anthony Kelly (2003), design research

> attempts to support arguments constructed around the results of active innovation and intervention in classrooms. The operative grammar, which draws upon models from design and engineering, is generative and transformative. It is directed primarily at understanding learning and teaching processes when the researcher is active as an educator. (p. 3)

If we see *educator* here to refer to teachers and didacticians, both of whom are also researchers, this definition applies well to LS. However, we need clearer distinction on the activity of these partners since is likely that neither their roles nor their goals in research are the same. I will come back to this.

According to Paul Cobb and colleagues, *design experiments* offer a means of addressing complexity. They result in an understanding of a *learning ecology* in which “designed contexts are conceptualized as interacting systems rather than as either a collection of activities or a list of separate factors influencing learning” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9). A learning ecology typically includes

> the tasks or problems that students are asked to solve, the kinds of discourse that are encouraged, the norms of participation that are established, the tools and related material means provided, and the practical means by which classroom teachers can orchestrate relations among these elements (Cobb, et al., 2003, p. 9).

Taken from a special issue of *Educational Researcher* which focused on design research at an abstract level, these papers say little about the roles and involvements of teachers. In most cases, although talking of collaboration with teachers, they seem to suggest that design is the province of didacticians, and that teachers in some way that is not explicit implement such design. Predating this writing, Erich Wittmann (1998), writing about the importance of design in teaching development is more explicit: “Teachers need to be trained and regarded as partners in research and development and not as mere recipients of results” (p. 95). Despite the word *partners*, these words suggest that agency in such partnership rests with the designers who are not teachers. Indeed Wittmann says that design “cannot be left to teachers” (p. 96). “The teacher can be compared more to a conductor than to a composer, or perhaps better to a director … than to a writer of a play” (p.96).

So, an issue for design research, as I see it, is how it conceptualizes the activity of teachers with respect to design and implementation. What kind of teacher agency is evident in the design process? In LS, it is teachers who design classroom activity based on theoretical understandings nurtured by their didactician colleagues and supported by these colleagues. The learning of pupils is a clear goal, but this is expressed directly in terms of the required curriculum, not in terms of learning through inquiry. Thus, the inquiry/goals tension is not so evident in LS.

I will conclude this paper with reference to a research project that aims to build on ideas from learning study and from design research while keeping inquiry as its central theoretical focus. Research is designed to look carefully at the building of communities in which inquiry is used and developed and at learning and teaching goals that are addressed. The inquiry/goals tension is an explicit focus of research.

A RESEARCH APPROACH BASED ON INQUIRY IN LEARNING COMMUNITIES

This project, *Learning Communities in Mathematic*s (LCM)[1], is designed to build communities of inquiry involving teachers and didacticians to develop teaching and enhance learning of mathematics. The theoretical basis of the project is inquiry as an approach to learning mathematics, to teaching mathematics and to researching the processes and practices of building inquiry communities to develop teaching. The project aims to use inquiry as a tool to develop inquiry as a way of being in developing teaching and studying related classroom activity and learning of pupils.

We are establishing agreements with 7 schools, from early years to upper secondary, each with a teacher group of at least 3 teachers committed to the project. Teacher groups in schools will focus on design of classroom activity that both builds in ideas of inquiry and addresses systemic requirements, including the goals of the school and educational system. It is the teachers who will design classroom activity based on inquiry as a tool for learning mathematics.

At the beginning, the role of didacticians is to draw teachers into inquiry in a variety of ways: firstly through workshops (at the college) in which we work together on what inquiry means for us all with respect to mathematics learning. Didacticians design workshops to create opportunities to do mathematics together in inquiry mode. Teachers and didacticians together will inquire into what inquiry looks like in mathematics learning. The role of teachers is to work on developing ideas of inquiry in relation to their own knowledge of mathematics, pupils and schooling, and take ideas back for further development in the school context.

In school, during the same time period as the workshop activity, teachers in each school form an inquiry group to think about what their teaching might look like from an inquiry perspective and to plan classroom activity. Within their own social setting – of curricula, programmes of study and school milieu – teacher groups will design innovative classroom activity that encourages pupils to get involved in inquiry in mathematics. Didacticians will support teachers in thinking about the nature of

[1] We are supported by the Research Council of Norway (Norges Forskningsråd): Project number 157949/S20

inquiry, drawing on experience and literature, getting involved in discussion of mathematical topics and examination questions, providing readings, software, advice on using software, access to mathematics and so on: responding to needs rather than imposing directions.

Didacticians study the design activity and the processes that emerge from implementing design; this includes both the design of our project and teachers' design of classroom activity. Here we expect to address the inquiry/goals tension at a number of levels, and to study how the use of inquiry as a tool in design, and in the tasks designed, promotes inquiry as a way of being. What kinds of interactions take place between didacticians and teachers? How do we address issues and concerns? What is needed at practical levels of ideas and resources? How does the thinking of all of us develop through our joint activity?

Data in the above will be collected through audio recordings and hand written notes from meetings and from personal reflections of the people involved. We shall video-record workshops and classrooms. We are recognising complex decisions in choice of methods and use of technology in data capture and analysis, aware that sophistication introduces its own problems. We expect to have a lot of data, so we have to think carefully about data reduction processes, how we shall recognize and validate significance; how our grain of analysis can be judged to capture elements of the delicate "process of becoming", of "formation of identity [in which] learning can become a source of meaningfulness and of personal and social energy", of "modes of belonging", including *engagement*, *imagination* and *alignment* (Wenger, 1998, p. 215). These theoretical issues are central to our inquiry process.

Although our study of interactions within the project will be ongoing (over a 4-year period), we expect to have two phases of data collection in which we video-record classroom interactions, and audio-record conversations with teachers and pupils individually or in groups. Here we shall be looking at the outcomes of the design process, gaining insight to the thinking of pupils, teachers and didacticians, and teasing out key issues in our developmental process. Classroom data will be related to data from the design process, to explore relationships between design and activity. Between these two phases we shall focus on learning in the project so far, ways of being that we can see developing and issues for dissemination and substainability.

Ultimately we are looking for inquiry models that have a practical foundation in terms of the reality of schools, classrooms and teachers' lives. The communities that develop should be sustainable beyond the life of the project because the people involved have developed ways of being. As we talk with teachers and negotiate delicately the early stages of our relationship, the inquiry-goals tension is already evident. Teachers, enthusiastic to take part in the project, are wary that it may take time from necessary curriculum planning, or require classroom activity that does not address curriculum goals. While excited by the possibilities the project offers, some overtly air their concern that project activity will demand different kinds of planning space and different goals. Shifts in planning and goals are a focus of our study.

The current challenge for didacticians in these early talks with teachers, is how we get jointly to seeing this as a project in which the concerns are shared; in which teachers are not just responding to the ideas and desires of didacticians, but themselves taking on the mantle of the project – with ownership of its goals for learning and teaching within their own sociosystemic setting – and grappling with the tensions and issues that arise. We shall be reporting further on our progress in this and other aspects of the project in the coming years. We welcome interest from, and cooperation with colleagues in other parts of Norway and around the world.

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COMPEX DILEMMAS CONCERNING INCLUSION AND DIVERSITY IN MATHEMATICS EDUCATION RESEARCH WITH TEACHERS.

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Those invited to respond to the plenary papers at PME28 have been given a brief that asks them to bring a new perspective to the theme offered by the plenary speaker, or oppose the focus given. The invitation suggests that this might be done by presenting alternative perspectives or by suggesting dilemmas arising from the perspective put by the speaker. The main aim is to fit in with the theme of Inclusion and Diversity by stimulating subsequent debate on the ideas presented.

Barbara's plenary paper gave me a great deal of food for thought. She has a track record of really trying to work with practicing teachers for the improvement of classroom practice and we have seen and attended the contributions of many of her students here at PME over the years. So her ideas are well developed and have a wealth of thought and experience behind them. I applaud the work that she has been involved with and trust that this new project will be rewarding.

You will gather from this that I do not intend to take up the option of opposing the focus that she has given. Instead I plan to respond by using Barbara's paper and the issues that she has raised as a springboard for raising some of the current unresolved dilemmas that I am having to face in my own work with teachers and their research into their own practice. In sharing these personal dilemmas I hope to stimulate debate on Barbara's paper.

I have been working on teaching for the past 30 years from my various positions as teacher, didactitian and Director of an in-service provider. In addition, for the past five years I have offered a taught Masters module at my university which draws on the work of Davis (1996), Maturana and Varela (1986) for its enactivist approach to understanding learning, and on Depraz, Varela and Vermersch (2003) and Mason (2001) for its techniques on approaches to becoming more aware of one's own practice. The first students using this module as a foundation for their dissertations are in the process of graduating with what I consider to be exciting work. I have also used the above course as a foundation for another set of courses on Complexity and Diversity that I have been running for the past three years at UCT's Graduate School of Business, where my starting point again comes from an enactivist position but also draws on the work of Capra (1997, 2002) and business theorists such as Stacey (1996), and Lissack and Roos (1999).

Using this background I am going to draw on three different sources as a backdrop to my response. The first of these flows from my understanding of Complexity Theory and enactivism.

> Complexity theorists draw a distinction between the descriptors complicated and complex. This new interdisciplinary field begins by rejecting the modernist tendency to use machine-based metaphors in characterising and analysing most phenomena. Machines, however complicated, are always reducible to the sum of their respective parts, whereas complex systems - such as human beings or human communities - in contrast, are more dynamic, more unpredictable, more alive. (Davis and Sumara 1997, 117)
>
> Boundaries that currently define schools and universities should be blurred … so that the relations between that which we call teacher education needs to move away from a model that focuses on mastery of classroom procedures and toward a more deliberate study of culture making. (Davis and Sumara 1997, 123)
>
> In such a (diverse) community information and ideas flow freely through the entire network, and the diversity of interpretations and learning styles – even the diversity of the mistakes – will enrich the entire community. (Capra 1996, 295)

I also want to locate myself within the themes of the conference of Diversity and Inclusion and in addressing this I have been influenced by the following comments which were posted on the conference web page.

> While celebrating diversity … it is vital to develop criteria for centrality.
> (John Mason, Oct 22 2003)
>
> I would like to reverse the phrase "inclusion and diversity" to "diversity and inclusion" (in order to) bring our focus towards enquiring structures of power inherited…
> Sikunder Baber (Nov. 3 2003)
>
> The term "inclusion" in the title "inclusion and diversity" is a recognition of (the) presence of dominant structure, which has the power to "include", and therefore "exclude". Therefore the retention of the term "inclusion" in the theme title is an implicit celebration of the power of dominant structure, an act, inherently counter-productive in the equation of intercultural relationships, and therefore of "diversity".
> Al-Karim Datoo (Nov. 21 2003)

Finally, I have for a long time been interested in the field of Teachers as Researchers.

> … the essence of the Teacher Research movement came from the dissonance and unease that it caused in its quest to improve the education system… The teacher-research movement can assist by causing dissonance and trouble. Trouble that comes from conviction based on evidence drawn from research by those in the field who know that we haven't got education right and who are prepared to put their energies into getting something changed. The minute teacher research becomes comfortable, someone else needs to take over. (Breen 2003, 541).

Lewin and Regine (1996) maintain that the main entry into a complex view of the world depends on the value we attach to the stories we tell and the way in which they are listened to. My response at the conference in July will largely take the form of a collection of personal stories. The problem with these stories is that they are inevitably situated within a specific time, context and current interpretation, and as I sit at this keyboard in May, I cannot know what particular form of story I will want to tell at Bergen in July. However there are three main stories which fill the menu at present.

Story One occurs when, as a didactitian at the time, I was privileged enough to be asked to allow my teaching to be used as a research site by another didactitian. The ensuing interaction gave me some insights into issues for teachers working with didactitians.

Story Two centres around a request to me as didactitian to work with two teachers to assist them 'work on their own practice'. We have written about this elsewhere (Breen, Agherdien and Lebethe 2003), and the issues raised at the time were complex.

Story Three involves three postgraduate students registered for the Masters in Teaching, who attempted to research aspects of their own practice for their dissertations and the challenges this faced for them (as it took them at times in directions in opposition to the academy in general) and for me as supervisor.

In all three of these stories I can most accurately be scripted as a troubled man faced with problems of identity and uncomfortable choices. The issues contributing to my dis-ease have to do with:

- Who initiates the 'project'?
- Whose questions are privileged?
- Whose theories are foregrounded?
- How do participants cope with different agendas?
- What do we learn from each other?
- Who is in control of the process?

These questions are not exactly the same ones that Barbara has raised but they are the ones that come back to me as I think about the dilemmas of a didactitian as s/he tries to set up a project where teachers are included in a community of inquiry. I hope that those in the PME audience when I respond will find some resonance with her paper. These issues are (obviously) crucial for me and they are the largely unresolved questions that I have to live with as I work with teachers and their work in classrooms. In a sense I am reassured by the understanding that I am working in that complex place that is also known as the 'edge of chaos' or 'border of disorder' and that all that I can do if follow Rilke's exhortation to 'live your questions now' (Rilke 1986, 45)!

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THE DIVERSITY BACKLASH AND THE MATHEMATICAL AGENCY OF STUDENTS OF COLOR[*]

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This paper argues that discussions of diversity often avoid the issue of race. Further, it maintains that diversity and structural backlashes to it in the United States in social and economic life shape and are shaped by crises in mathematics education. Attention is paid to the lack of instructional diversity in mathematical problem types and to the mathematical achievement of African American and Latino middle-school students. The paper further argues for the importance of the category of intellectual agency, an under-theorized and under-researched psychological phenomenon in mathematics education, particularly in the literature on minority-student achievement. The paper concludes with preliminary data to show the promise of this line of inquiry for researching the development of mathematical ideas and forms of reasoning among a diversity of students.

The notion of human diversity evokes a wide range of ideas, including apparent and subtle variance among cultural groups; celebration, or at least tolerance, of differences; enrichment of social, economic, academic, and cultural life through incorporating commensurable elements of the other's ways into one's own, and so forth. The content of recent discourse on diversity as an intellectual category as well as scientific, social, and cultural phenomena are by and large virtuous and affirmative. Since the victories of anti-colonial and various civil-rights struggles, diversity in the social sphere has evolved to tolerate and even celebrate both essences and preferences within, for instance, categories of ethnic, socioeconomic, racial, and gender variety as well as expressions of, to name a few, sexuality and intellectuality.[1]

In the United States of America, for example, the ideas and actions of proponents of diversity have influenced researchers and educators of mathematics education as well as educational policy makers. Many national and local initiatives have focused the attention of the mathematics education community to the needs of an increasingly diverse population of students. A significant case in point is one of the several

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[1] The research program, ethnomathematics has championed inquiry into the history of mathematical cognition, including the development of mathematical ideas and forms of reasoning, among a diversity of identifiable cultural groups from decidedly political perspectives (D'Ambrosio, 2001, 2004; Knijnik, 1996, 1999, 2002; Knijnik, Wanderer, & de Oliveira, 2004; Powell, 2002, 2004; Powell & Frankenstein, in press, 1997).

Centers for Learning and Teaching funded by the National Science Foundation. It awarded a five-year grant of 11.3 million dollars in the fall of 2001 to a consortium project based at a large Midwestern university, whose project is known as "Diversity in Mathematics Education." The project is founded on the recognition that the United States needs to attend to certain immediate, urgent challenges based on parallel changes occurring in the instructional workforce and the student population. The specific changes are stated in its press release of October 2001:

> Over the next decade, the nation's schools will have to replace more than two thirds of the teachers currently teaching. More than half of the university faculty in mathematics education will be eligible for retirement in the next two years and almost 80% will be eligible for retirement in the next 10 years. Over the same period, America's K-12 student population and the next generation of leaders and teachers in mathematics education will become more ethnically and linguistically diverse. (Diversity in Mathematics Education/Center for Learning and Teaching, 2001, 3 October)

This consortium project seeks to address the increasing ethnic and linguistic diversity of students, teachers, and mathematics education leaders. It is interesting to note that this use of diversity to signal ethnic and linguistic variation has gained currency in current discourse on diversity. That is, in the US, at least, the category of race is often disassociated from notions of diversity. We speak of ethnic and linguistic diversity without mentioning the category of race, as in the above quote, even though differential achievement rates in mathematics among different racial groups persists and has worsens. (Evidence for this point will be discussed below.) The discursive tendency to omit race from consideration of diversity in American education signals an apparent desire within the dominant culture to avoid talking about a prickly reality and, in this sense, represents what we view as a diversity backlash. That is, the use of the notion of diversity to circumvent grappling with the social and political realities of race. "Race" as a social concept is real in its consequences, especially within American society's education system where racial and ethnic segregation persist.

Concurrent with the challenges of diversity that the consortium project highlights, the United States suffers from twin interacting crises of the mathematical achievement of its students and of the effectiveness of its mathematical instruction. These crises are especially profound in communities of students of color, especially among African American and Latino students and, as we will argue, supported by a structural backlash against diversity of a certain sort. This diversity backlash presents a significant challenge to social actors—such as, mathematics educators and researchers as well as to students and their families—interested in increasing the mathematical achievement of African American and Latino students. We would like to suggest that research into the mathematical agency of students of color promise to contribute theoretical perspectives, research methodologies, and pedagogical approaches that can address the instructional, racial, and ethnic dimensions of the crises in US mathematics education.

These crises in mathematics education are enmeshed in social and economic realities. While cultural diversity and tolerance are championed, there is a rather strong adverse and, sometimes, violent reaction among some sectors of society toward diversity of race, economic, and social structures. This is part of a backlash to affirmative discourses on diversity such as the debates surrounding affirmative action in the workplace and college admissions. The crises have particularly sharp and pervasive effects on the academic attainment of students of color, particularly African American and Latino students. To discuss the twin interacting crises, we first highlight an aspect of the instructional crisis and then the crisis in mathematical achievement.

CRISIS OF INSTRUCTIONAL DIVERSITY IN MATHEMATICS

Diversity or rather the lack of diversity is an aspect of the instructional crisis in US mathematics education. The most compelling evidence of the underachievement in mathematics of American children comes from the 1995 and 1999 TIMSS studies (Hiebert et al., 2003; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). In these studies, the mathematical knowledge of US students has ranked low among industrialized, “democratic” countries (Hiebert et al., 2003; Stigler et al., 1999). To understand how this might be related to instructional practices, in 1995 and 1999, studies were implemented, using videotape data from a probability sample of eighth-grade classroom in several countries. The 1999 TIMSS Video Study sampled 100 eighth-grade classrooms in each of seven countries: Australia, Czech Republic, Hong Kong SAR of the Peoples Republic of China, Japan, the Netherlands, Switzerland, and the United States. Compared to the six other countries in the TIMSS 1999 Video Study of mathematics teaching, a follow-up and expansion of the 1995 video study, eighth-grade students in the United States scored, on average, significantly lower than their peers (Hiebert et al., 2003). Researchers have analyzed the video data to understand what instructional features might explain differential achievement.

The findings more than anything else underscore the complexity of mathematics teaching. The countries that exhibit high levels of achievement on TIMSS have many similarities and differences in their eighth-grade instructional features. None of the high-performing countries use the same admixture of teaching methods in the same proportions. For example, although both Japan and the Netherlands perform at high levels on TIMSS, the average percentage of problems per eighth-grade mathematics lesson that involved procedural complexity differed radically. Nevertheless, eighth-grade mathematics teaching in all seven countries shared common features of teaching eighth-grade mathematics. Among them we note that “in all the countries at least 80 percent of eighth-graders’ lesson time, on average was spent solving problems” (Hiebert et al., 2003, p. 42).

Besides similarities, discernible variations also exist across the countries in teaching eighth-grade mathematics. In particular, the lack of diversity of implemented problem types stands in poignant contrast in US eighth-grade mathematics classrooms. In most debates about instruction, mathematical problem types are dichotomized: basic computational skills and procedures (or using procedures problems) are placed in opposition to rich mathematical problems that focus on concepts and connections among mathematical ideas (or making connections problems). Classrooms in all of the countries spend time both on problems that call for using procedures and on those that call for working on concepts or making connections. The percentage of problems presented in each category, however, does not appear to predict students' performance on achievement tests. Rather what higher-achieving countries share is the way in which teachers and students work on problems as the lesson unfolds. Expect for the US, the six other nations spend between 8% and 52% of classroom time on making connection problems implemented as such (Hiebert et al., 2003, pp. 103-104). Whereas, in US classrooms, making connections problems as lessons unfold are transformed into procedure problems. That is, only US eighth graders spend nearly all of their time practicing only mathematical procedures (Hiebert et al., 2003, pp. 103-104) and rarely engage in the serious study of mathematical concepts. From the 1999 TIMSS Video Study, it is apparent that diversity of implemented problem types does not exist among the sampled US eighth-grade mathematics classrooms.

CRISIS OF MATHEMATICAL ACHIEVEMENT AMONG DIVERSE RACIAL GROUPS

International assessments, particularly those that innovatively combine quantitative and qualitative data collection and analyses, such as TIMSS, provide rich information and revealing findings but do have limitations. At this stage in development of such research tools, they do not provide a window into the differential attainment among different social, economic, gender, racial, or ethnic groups within a nation. In the United States, The National Assessment of Educational Progress (NAEP), also known as "the Nation's Report Card," is the only nationally representative and continuing assessment of what American students know and can do in various subject areas. Since 1969, assessments have been conducted periodically in reading, mathematics, science, writing, U.S. history, civics, geography, and the arts.

Recently, the National Center for Education Statistics of the US Department of Education (National Center for Education Statistics, 2000) published a report titled *NAEP 1999 Trends in Academic Progress: Three Decades of Student Performance*. The NAEP data reveals trends in educational achievement among White, Black, and Latino students. Interesting patterns can be discerned when the data is viewed from the perspective of the wake of the civil rights movement in the United States and the post-civil rights movement. If we define wake of the civil rights movement as occurring between the years 1970 and 1990, and the post-civil rights movement as occurring after the 1980s, then the NAEP data on educational achievement reveal an important manifestation of the structural backlash to racial diversity. For instance, between 1970 and 1988, the educational achievement of Black and White students narrowed by one half or more (NCES, 2000, p, 108). However, since 1988, the gap has been flat, or in some subjects, is wider (NCES, 2000, p, 108). Comparing Latino and White students between 1970 and 1990, the differential in educational achievement narrowed by one half or more, but sadly since 1990, the gap has been flat, or in some subjects, is wider (NCES, 2000, p, 108).

What does the NAEP data indicate about the differential achievement in mathematics among US students concerning racial diversity? Nationally, in 2003, eighth grade African American and Latino students lagged behind their White peers in mathematics. Mastery of school mathematics up through the end of eighth grade was measured on a three-part scale: below basic, basic, and proficient to advanced. African American and Latino children scored at a level of proficient to advanced 12% and 14% of the time respectively, while white students scored at this level 39% of the time. Similarly, 39% and 43% of African American and Latino children scored at a level of basic or better. White children achieved this level 74% of the time. Only 26% of white children scored below basic on this test, as opposed to 61% and 57% of African American and Latino children respectively. Unless genetic causes are assumed, these differential achievements can perhaps be explained by a structural analysis of the political economy of the US society. Whatever non-biological accounts one accepts as explanatory of the paucity of high achievement in school mathematics by African American and Latino students, the continuance of present achievement trends points to an eventual narrowing of diverse participation in the intellectual life of a nation.

LOCAL DAMPENING OF RACIAL DIVERSITY

Data that compare academic achievement of African American and Latino students, on one the one hand, and White students, on the other hand, exist within the economic and social nexus of life in the United States. Mathematical achievement simultaneously shapes and is shaped by interactions between social and economic forces. Estimates are that 40% of all African American children live in poverty, are the least likely to have access to high-quality education (Patterson, 1997), and have a rather small possibility of enjoying mathematics instruction that reaches beyond the procedural.

During the economic recession of 2000-2003, the unemployment rate in urban centers in the US rose sharply. In New York City, for instance, the increase in unemployment was worse for men than for women, and particularly acute for black men. This reality is revealed in a study by the Community Service Society (Levitan, 2004), a non-governmental organization that fights poverty in New York City and struggles to strengthen community life for all. Based on data from the federal Bureau of Labor Statistics, the study reports on the employment–population ratio—the fraction of the working-age population with a paid job. It found that in 2003 only 51.8% of African American men between the ages 16 to 64 held jobs in New York City. The rate for white men was 75.7%; for Latino men, 66.7%; and for black women, 57.1%. The employment-population ratios for African American and Latino men were the lowest since 1979. According to Scott (2004), economists admit that these findings are consistent with trends in the racial gap in male employment of other Northern, Midwestern, and Central cities where manufacturing jobs have disappeared in recent decades. Such a reality underscores a structural backlash to diversity within the US society. That is, if unemployment trends continue in their current direction, the economic and ultimately the biological viability of certain racial and ethnic groups will become precarious, at best, and most certainly, decrease significantly racial diversity in high levels of schooling as well as other facets of social and economic life.

MATHEMATICAL AGENCY OF STUDENTS OF COLOR

In the US, despite the academic underachievement of many non-White students and the relative economic poverty of their communities, in their early scholastic career, students of color express pleasure with mathematics. Martin (2000) notes that studies have found that "African American children consistently express the most positive attitudes towards mathematics among all student groups and identify mathematics as one of their favorite and most important subjects" (p. 12). Martin's research suggests that African-American parents and community members express also beliefs consistent with dominant societal folk theories of mathematics learning. However, their life experiences are such that at the same time they express beliefs that reflect perceptions of their limited opportunity to participate in mathematical contexts as a result of differential treatment based on their African-American status. Notwithstanding positive attitudes toward mathematics, when researchers examine the course-taking and persistence patterns in predominately African-American high schools, 80% of the students take no more mathematics than what is minimally required to graduate (Martin, 2000, p. 15).

Explanation and insight are required into ways to ameliorate this striking discrepancy between early positive attitudes and identification with mathematics and subsequent failure and avoidance of it. Martin observes that

> Because few studies have focused on academic success among African-American students and fewer have focused on students who do well in mathematics, issues of individual agency, success, and persistence remain largely underconceptualized. Success, for example, has been defined only in terms of external measures such as grades and test scores, and persistence has been defined only in terms of course-taking patterns. (p. 28).

We consider critical Martin's point about individual agency and view agency as potentially pivotal to the involvement of African Americans and other students whose subject position is not identified with the dominant culture and to overcoming societal-engendered failure and avoidance of the discipline. Understanding agency is particularly important since both failure and success can be located within the same set of social, economic, and school conditions that usually is described as only producing failure. Avoiding deterministic theories of educational anthropology, urban education, and sociology of education that tend to focus on discussions of culture, ethnicity, stratification, opportunity structure, and African-American status, Martin's conception of agency is informed by Bandura's (Bandura, 1982; 1997) notion that human agency and individual motivation can manifest and prevail in opposition to larger, countervailing forces.

From a similar theoretical position, we have initiated a new, three-year research project, currently in its initial year. A salient question that we propose to investigate concerns individual agency in mathematical problem solving. Under a grant from the National Science Foundation (REC-0309062), we are gathering data and observing students' initiative and ownership of ideas. Our analysis develops from examining student-to-student discursive practices as individual students in collaboration with peers build mathematical ideas and forms of reasoning (Powell, 2003; Powell & Maher, 2002, 2003). We are conceptualizing agency in terms of the mathematical ideas and reasoning evidenced from learners' individual initiative to define or redefine as well as build on or go beyond the specificities of mathematical situations on which they have been invited to work. Learners' use of their agency also manifests itself as they create heuristics to resolve mathematical tasks or aspects of them. This conceptualization recognizes learners' independent and autonomous mathematical performances through student-to-student discourse. It also corresponds to the work of other investigators (Delpit & Dowdy, 2002; Perry & Delpit, 1998) who suggest the need for further research into relations between the discourse of urban, African American students and their academic achievement.

Our study is set in a particular social context. The setting is an informal after-school program at Hubbard Middle School in Plainfield, New Jersey, an economically depressed, urban school district with 98% African American and Latino students. Sixty-four percent of the students of the school are eligible for free or reduced-cost lunch compared to the statewide average of 28% (Education Law Center, 2002). In the Plainfield School District, the high school graduation rate is 52% compared to a rate of 67% in districts of comparable levels of poverty (Education Law Center, 2002).

In our study, we are investigating how African American and Latino students from a low-income, urban community build mathematical ideas and engage in mathematical reasoning in an after school, informal setting. According to Friedman (2002), resources for after school programs throughout the United States merely replicate and extend the curriculum of the school day with "skill and drill" education. In middle schools, this content and instructional approach contributes mightily to the failure and disenchantment of students with mathematics (Stigler et al., 1999; Stigler & Hiebert, 1999). Hence, it would be more than insidious to replicate and extend this approach into the informal settings of after-school programs. The problem of "skill and drill" education is especially acute for low-income students who, as noted by the United States Department of Education (1997, October), finish high school without the rigorous mathematics courses needed for college entrance. Recently, educators and educational policy makers have identified the critical need for opportunities for academic and social development based on student initiative or agency in contexts outside of traditional school hours (National Research Council, 2002; Urban Seminar Series on Children's Health and Safety, 2001). Our research is designed to document student discourse and to promote the exercise of agency by inviting students to engage in meaningful mathematical tasks and to study over time how they change their participation role in mathematics from what Larson (2002) describes as, “overhearers” to “authors” of mathematical ideas and texts.

The content and pedagogy in our research, while consistent with the vision and philosophical perspective of the Plainfield Public Schools, differ substantially from the mathematics curriculum used in the Plainfield schools. The mathematical content of the project focuses on strands in combinatorics, algebraic thinking, and probability, incorporating the use of technology as a tool. A critical difference between the curriculum of the school district and our study is an inevitable result of school realities. Unlike mathematics instruction in public school districts, constrained by administrative, political, temporal, and other limitations, mathematical activities of our study will not be directly affected by the pressures of grading, standardized tests, and curriculum coverage. These non-mathematical constraints affect instruction in ways that can cause even reform-intended mathematics curricula to fall far short of idealized scenarios. Instead, different pedagogical processes guide our work (Maher & Martino, 2000). It is important to note that also unlike mathematics instruction in US middle schools our tasks on which we invite students to work involve making connections and are implemented as such rather than transformed in the unfolding of the session into problems focusing on basic computational skills and procedures (For problem task examples, see Harvard-Smithsonian Institution Astrophysical Observatory, 2000).

The previous longitudinal work of the Robert B. Davis Institute for Learning (Graduate School of Education, Rutgers University) has shown that students use their agency in the direction of greater and successful participation in mathematics as authors of mathematical ideas and texts when the contexts in which students explore mathematical ideas provide challenging problem tasks and when students are given opportunities to think deeply about mathematical situations over time (Harvard-Smithsonian Institution Astrophysical Observatory, 2000; Maher, 2002; Maher & Martino, 2000; Powell, 2003; Speiser, Walter, & Maher, 2003).

INSTANCES OF STUDENTS EXHIBITING MATHEMATICAL AGENCY

Twenty-four sixth graders volunteered to be participants in our study in the context of an after-school mathematics program. The main sources of data are as follows: (1) discourse patterns and other activity of students as they work on mathematical investigations recorded on videotape; (2) students’ inscriptions, collected and digitized; (3) researcher and observer notes and reflective diaries, collected and digitized, and (4) research team’s planning notes and debriefing session recorded on videotape. Our framework for analysis, developed from earlier work, is discussed in Powell, Francisco, and Maher (2003).

In the first three cycles of our study, there are respectively eight, eight, and six research sessions, each lasting one and a half hours. Here we report on instances of student mathematical agency during the first cycle of our study. In this cycle, we invited students to build physical models using Cuisenaire rods to explore relations among them that evoke certain kinds of reasoning: organizing and ordering by categories, hypothetical reasoning about number relationships (whole number and fractions), proportional reasoning, reasoning by contradiction, recognizing and predicting patterns, and generalizing.

Students engaged in building mathematical models with Cuisenaire rods, a tool with which they had not previously worked. We invited them to work on problems in which a rod of certain length was given a number name and for which they were to find a rod that had a comparative number name of the given rod. For instance, in the first session, after the students were invited to explore the Cuisenaire rods, Lorrin stated that even though her partner suggested that the white rod could be called 2 that she was thinking the it could be called 5. A researcher then asked, “What if you called the white rod 5 instead of 2?” Lorrin replied that the orange rod would be called 50.

In our theorization, an aspect of intellectual agency applied to mathematical learning is taking risks to venture beyond a stipulated situation to explore and further develop a set of ideas. Agency is also manifest when learners develop problem-solving heuristics to address tasks. Such an act requires that learners author their own procedures or strategies. In all instances, we attend particularly to the mathematical ideas and forms of reasoning evidence in learners’ discourse and inscriptions as they exercise agency in mathematical situations. Our initial data, collected in the first sessions of our project, provide a preliminary glimpse into the frame of agency and development. The following are three instances:

> Instance I: A researcher invites students to find which rod would be called one-half if the blue rod were called one and further inquiries what they say or do to convince someone of their result. Herman, Malika, and Lorrin each place two light green rods end-to-end alongside a dark green rod. Later, Lorrin places end-to-end two yellows rods and lays them alongside an orange rod. She says, “I’m going to do all of them.” She proceeds to find rods whose length is the same as a train of two rods of the same color. As Malika helps, Lorrin tells her, “I’m talking about half and half.” In this she seems to mean that her goal is to find all rods whose length can be constructed with two other rods of the same color. Later she separates the rods that can be so expressed from the others Lorrin points to the blue, black, light green, and yellow rods and says that “they don’t have halves.”

> Instance II: Two sessions later, students continue to consider which rod could be called half of a blue rod. Some reason that the light green rod has a length that is one-third the length of the blue rod. Some students exhibit novel ways to show this, using

multiplication or addition. Jeffrey reasons that the red rod would have the number name two-ninths if the blue rod is one. He later shows the class his model of a blue rod alongside a train of rods in the following sequence: red, light green, red, and red. He then challenges the class to find the number name for the red rod when the blue rod is called one.

Instance III: During the fourth session of the cycle, students were invited to work on the question, "If the blue rod is 1, what is yellow?" Many students manipulated the rods to observe how many white rods they needed to place end-to-end to construct a length equivalent to the blue rod. Malika lists how many white rods make up each of the other rods. She calls the yellow rod 5, and later she and Lorrin say that yellow is five-ninths. Building a model of a blue rod alongside a train of one yellow and four white rods, with a purple rod beneath the white rods, Lorrin and Malika show that the purple rod is four-ninths. The students at their table determine number names for all the rods, except that they are uncertain about what to call the orange rod.

Eventually, this group of students resolves what number name to give to the orange rod. One student remarks that ten-ninths is an improper fraction. A male colleague [off camera] says assertively, "It's still ten-ninths. That ain't gonna change it because it's an improper fraction. That makes it even more right."

In each of the three instances discussed above, students play with a variation on a theme introduced by the researcher and improvise in the sense that they act the given materials and compose ideas without following a prescribed script. Students often posed problems for themselves and for others to solve. In one instance, students initiated an investigation to find which rods have a rod that can be called one-half. Their reasoning indicated that they connected meaning to the symbols they used in their problem solving with rods. Through their actions, observations and reasoning, they progressed in building a foundational understanding of ideas about fractions and their operations, fraction as number, comparing fractions, upper and lower bound, equivalent fractions, proper and improper fractions. Certain earlier "beliefs", such as "the numerator cannot be larger than the denominator " were examined individually and by the whole class, eventually resolved by reasoning from the patterns they observed in the models they built.

DISCUSSION

Our study is in the first year of its project three-year tenure and we are just beginning to analyze our initial data. From our investigation, two of our intended outcomes are the following: fundamental knowledge of the mathematical ideas and forms of reasoning built by African American and Latino youngsters of middle-school age engaged in working on deep, open-ended mathematics tasks in technology-rich, informal settings in a high-needs public school district; and evidence of the mathematical achievement of students of color as a byproduct of their engagement of their agency.

These goals are significant since first and foremost, the notion of being biologically ill-equipped for high cognitive functioning has influenced attitudes and actions toward students of color throughout history (Gould, 1981) and some lay people and scientists still promote it (Herrnstein & Murray, 1994). Moreover, some researchers in mathematics education (Orr, 1997) conclude that the linguist structure of African American speech is at fault. Racism has not ended with the successes of the Civil Rights Movement. Rather, as the African American novelist Alice Walker writes, "racism is like that local creeping kudzu vine. It swallows whole forests and abandoned houses; if you don't keep pulling up the roots it will grow back faster than you can destroy it" (Walker, 1983, p. 165). It might be that racism roots itself in our theoretical assumptions, our methodological approaches, our observational lenses, as well as our interpretation of data. Not assuming that students of color have intellectual agency that can be used in the learning and teaching of mathematics may unwittingly derive from certain assumptions about their intellectual capabilities. Whereas, research methodologies that incorporate a focus on the intellectual agency of African American and Latino students in mathematical situations and the mathematical ideas and forms of reasoning develop through the exercise of agency promise to inform the mathematics education community not only about cognitive diversity but also to engender respect for students of color based on evidence of their mathematical intellectuality.

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MATHEMATICS EDUCATION RESEARCH, DIVERSITY AND INCLUSION

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This paper comments on Arthur Powell's plenary paper "The Diversity Backlash and the Mathematical Agency of Students of Color". A highlight of some of the main arguments in Arthur's paper is offered, and questions are raised concerning elements of importance in setting a research agenda committed to equity in mathematics education.

In the international community of research in mathematics education Arthur Powell's work has provided insight into the multiple predicaments of African American students' mathematical learning, from an ethnomathematical perspective where issues of power are connected to school mathematical knowledge and its learning. His work has challenged not only research with an embedded racist assumption about the mathematical learning of these students in the USA, but also even progressive research concerned with issues of equity in the access to participation in mathematics education practices. His paper "The Diversity Backlash and the Mathematical Agency of Students of Color" summarizes the concerns that motivate his and his colleagues research work, as well as the selected approach. A discussion of "inclusion and diversity" in mathematics education –with advances and backlashes– without a consideration of Arthur's work would be incomplete.

Arthur's sentence "*It might be that racism roots itself in our theoretical assumptions, our methodological approaches, our observational lenses, as well as our interpretation of data*" caught my attention. It touches one of the points that I consider to be central in a discussion of inclusion and diversity in mathematics education. Mathematics education researchers have constructed a discourse about the practices of the teaching and learning of mathematics. Such a discourse is not neutral since it provides frames of action for researchers (but also for teachers and policy makers) to address the multiple problems of mathematical instruction (Valero, 2002, 2004b). As Arthur indicates, it is possible to conjecture that mathematics education research and the discourse it produces are implicated in the "diversity backlash".

THE THESIS OF THE DIVERSITY BACKLASH

The thesis of the *diversity backlash* contends that the current diversity discourse, with an emphasis on linguistic and ethnic diversity, omits a direct mention of race, while racial segregation is still a crucial problem. Despite the relatively high public attention to the multi-ethnic, -cultural and -linguistic composition of the population in the USA, little advancement is really being made in the provision of equality of access to a variety of resources to different racial and ethnic groups. The gap between these two is actually a mechanism of the dominant culture to maintain the statu quo.

The thesis invites to discussions of the relationship between structural inequalities and access to participation of different groups in (mathematics) education. It is clear in Arthur's work (see Powell, 2002 in his reference list) that such a connection is indispensable in research concerned with equity issues. For research in mathematics education this means that considerations of the social, political and economic context in which mathematics education practices take place need to be incorporated. This poses many challenges for researchers because, it not only opens the focus of attention of research from the details of learning processes in mathematics to broader social spaces of action where mathematics education practices get constituted, but also because it demands the use of theoretical and methodological tools that have not been widespread in mathematics education research (see Valero & Zevenbergen, 2004; Vithal & Valero, 2003). The challenge becomes finding significant ways of connecting the macro-contexts in which structural inequalities happen with the micro-contexts of mathematical learning.

CRISES (OF ACHIEVEMENT) IN MATHEMATICS EDUCATION

The diversity backlash is associated with *mathematics education instruction and achievement crises*. Arthur argues that USA students' low achievement in international tests can be associated with the dominance of a procedural instruction – while students from countries with a balanced conceptual and procedural instruction achieve higher. This is what he refers to as the instruction crisis. At the same time, the achievement crisis refers to the fact that students from particular racial (ethnic and linguistic) groups continue to have a significantly lower achievement than white students in the USA. The systematic lower achievement of particular groups of students is an alarming sign for politicians about the crises of educational systems, and it is an important justification behind investments in reforms and research in mathematics education. It has directed the attention of researchers towards particular ethnic groups, as well as towards students with learning difficulties, girls and working class students.

But what is behind the focus on issues of achievement? Research has shown that measures of achievement are measures of the ability of students to cope with the social framing of tests rather than a measure of students' mathematical competence (see Wiliam, Bartholomew & Reay, 2004). Mathematics tests fulfill a double function of providing a categorization of students according to criteria of ability determined by the test makers, as well as that of exercising a normalization of students, that is, a classification of each person according to what is considered to be normal (and therefore outstanding and deficient). The average (and related concepts of superior or inferior) is defined in terms of the characteristics of the dominant cultural group, in this case middle-class, white, male population. Measures of mathematical achievement operate as important classification and normalization tools in society in relation to dominant groups. If we adopt this thesis, then underachievement says something about the position of those groups in society, but does not necessarily say something about their actual mathematical ability.

Furthermore, if tests are analyzed from this socio-political perspective, high achievement of different groups may be interpreted as a success in an assimilation of different groups to the dominant cultural discourse. I doubt that the aim of diversity (with or without consideration of race) is that we all become “White, Middle-class Americans”. That would also represent a disaster for diversity (and may not necessarily secure equality of access to participation in social, economic, cultural and political resources). A challenge for mathematics education research with a concern for equity and diversity is unpacking the discourse of (under)achievement and finding other tools to talk about what different groups of students actually can mathematically (instead of starting from a deficit perspective).

THE THESIS OF THE INTRINSIC RESONANCE

It is of paramount importance that African Americans and Latinos do well in mathematics since “mathematical achievement is simultaneously shaped by and shapes the economic and social well being of communities as well as of nations” (see Powell, this volume). Arthur argues that the recent crisis of unemployment in male African American population will result in more poverty in that group and, consequently, in lower school participation, lower mathematical achievement, lower participation in the work market and so on. This cycle compromises the “biological viability of certain racial and ethnic groups”.

Mathematics has been associated (in the Western culture) with economic wealth. The more mathematical (technological and scientific) production a society has, the wealthier the society becomes. Since the time of the “Sputnik shock” this argument has been at the roots of justifications for expanding mathematical research and improving mathematical instruction. Part of the concern for achieving equity in access to the participation in mathematics education is precisely that of giving access to excluded people to wealth. In other words, good mathematics education in itself empowers people.

Behind these formulations there seems to be a belief in the intrinsic goodness of mathematics (education). Mathematics and mathematics education are given positive characteristics such as being “empowering” or “wealth-provider”. Such assumption of goodness diverts attention from the operation of mathematics (education) in larger social and political spaces where both mathematics and school mathematics are power-knowledge used as resources for the creation of “wonders and horrors” (Skovsmose & Valero, 2001). Therefore, it is necessary that researchers examine critically the ways in which mathematics (education) forms part of larger systems of reason and is used in the construction of unjust as well as just social, economic and political structures.

INDIVIDUAL, INTELLECTUAL AGENCY AND POLITICAL AGENCY

A key notion in the study of African American and Latino students’ participation in mathematical instruction is *individual intellectual agency*. Such agency is defined as the learner's individual initiative and ownership of ideas to define, redefine, build, take risks and go beyond the specificities of a mathematical problem. The concept of agency is bounded to the particularities of the context defined by the mathematical problems through which the research will invite students to display and build their intellectual activity. This notion of agency is focusing on the characteristics of those students as learning, cognitive subjects engaged in mathematical activity.

Much of mathematics education research has concentrated on describing and analyzing the individual, intellectual agency of students in diverse mathematical contexts. I have argued (Valero, 2004a) that such research has constructed a view of the learner as a "schizomathematicslearner". Such a discursive object portrays students as mathematical cognitive agents, decontextualized from the social, historical, political and cultural arenas where they exist. The focus and interest in understanding one aspect of students' thinking has almost eliminated the other components of students as fully real, living, and acting human beings. The notion of cognitive, intellectual agency has to be encompassed with a notion of political agency understood as the students' action in complex social situations where mathematical initiative is one of the multiple possible ways of influencing their life conditions. An interesting challenge for research is finding ways to enlarge the notion of agency in order to connect the micro-context of the mathematics classroom with larger context of action in which students participate (and where exclusion/inclusion is also in operation). In other words, the challenge is link the individual learner (and his/her intellectual agency in mathematics) with his/her larger social setting, within which disadvantage on the grounds of race and ethnicity has been historically constituted.

ELEMENTS OF A RESEARCH AGENDA FOR DIVERSITY AND INCLUSION

That research in mathematics education is implicated in the maintenance of exclusion is a contention that has been examined in different ways (see Skovmose and Valero, 2002; Popkewitz, 2002). Theoretical frames, problems and methodologies contribute to the creation of a discourse (and of a practice) that leaves unattended fundamental issues of access of different groups of students to various resources of power. When thinking of a research agenda committed with diversity and inclusion there are some necessary issues to consider: (1) The connection between macro- and micro-spaces of action in search of explanations for and interpretations of exclusion of certain groups of students. (2) The deconstruction of the discourse of achievement as a measurement of mathematical capacity, and analysis of the social processes operating through the measurement of achievement. (3) The critical examination of the discourse around mathematics (education), power and equity. (4) The expansion of notions of agency to encompass both intellectual and political dimensions of students' actions.

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FROM DIVERSITY TO INCLUSION AND BACK: LENSES ON LEARNING *

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The ideas presented in this lecture are based on the observation of processes of construction and consolidation of knowledge by individual students learning in groups within classrooms along a sequence of activities. Whereas the uniformity of the basic elements used to describe the knowledge construction processes may be seen as inclusive, there is a lot of diversity in the different ways in which individual students combine these basic elements into their personal learning trajectories.

INTRODUCTION

There is a thread, which links this plenary talk to the one I gave at PME 23 (Hershkowitz, 1999). In the previous one I asked, "Where in shared knowledge, is the individual knowledge hidden?". My point was that research should focus more extensively on the investigation of the development of individuals when they evolve in different social settings and construct of knowledge about different topics through successive activities.

At that time many researchers in mathematics education were attracted by the investigation of the construction of the "shared knowledge" of a community of students (e.g. Cobb, 1998; Hershkowitz, and Schwarz, 1999). Most researchers' lenses were focused on the ensemble. Individuals were observed as "members" and the knowledge of the individual was seen as a contribution that transformed the knowledge of the ensemble, where the ensemble designates "the smallest group of individuals who directly interact with one another during developmental processes related to a specific activity context" (Granot, 1998). Research on shared knowledge was mostly based on the interpretation of various episodes in different social settings. The episodes were mostly taken from one lesson, and even when the sample of episodes where taken from a sequence of activities the data were accumulated by observing different ensembles within the classroom, populated by different students with no possibility to trace the learning trajectory of specific students along a sequence of activities.

Less research effort has been invested in the opposite direction, namely in investigating the shared knowledge, constructed by a group of students or by a dyad, with the aim to better understand the development of the participating students' individual knowledge. The work of Kieran and Dreyfus (1998) is an example in this opposite direction. Kieran and Dreyfus observed student dyads solving problems, and

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then right away interviewed each student individually using an analogous problem in order to check the effect of the dyad work on each individual.

Recently I have invested efforts with colleagues and students in this opposite direction, to describe and understand how individuals construct new (to them) structures of knowledge in peer interaction and consolidate it (or not) in subsequent activities. We did not observe students or small groups when isolated in laboratory conditions; rather, we traced the participation of individuals and groups by studying talk in the classroom. This was done along a sequence of activities (Tabach, 2001; Shtein, 2003).

Our empirical approach led us to focus primarily on process aspects of construction of a new structure of knowledge rather than on outcomes. We focused on a particular kind of construction of knowledge, the process of abstraction, which we defined as a process in which students <u>vertically</u> reorganize previously constructed mathematics into a new mathematical structure. In order to empirically study abstraction, we looked for observable actions relevant to the construction of knowledge. Following Pontecorvo and Girardet (1993), we called these actions epistemic actions. We identified three epistemic actions relevant for processes of abstraction: <u>Recognizing</u>, <u>Building-With</u>, and <u>Constructing</u>, or short RBC. Two case studies in which we observed students evolving in laboratory settings led us to initiate the elaboration of a model of abstraction: we started with an interview with a single student (Hershkowitz, Schwarz & Dreyfus, 2001), and then turned to the observation of dyads working in collaboration. In the second case study, the shared knowledge of the dyad and the construction of a new structure of knowledge of each individual in the dyad were investigated by analyzing pair interactions between the two students. Interaction was investigated in detail as a main contextual factor determining the process of abstraction (Dreyfus, Hershkowitz & Schwarz, 2001a). A crucial feature of the model is that the epistemic actions are nested within each other. We therefore called it the nested epistemic actions model of abstraction in context, but usually refer to it simply as the "RBC-model". The model is described in detail in these references. Shorter descriptions may be found in PME proceedings (e.g., Dreyfus, Hershkowitz & Schwarz, 2001b).

We were aware that the contexts in which the model was elaborated were quite limited. Social interactions and other contextual factors in school classrooms are often much more complex than in research interview situations. Therefore, we began about two years ago to expand our program of research in two directions. The first one is concerned with the construction of knowledge in teacher-led whole-class discussions. We initially focused on the role of the teacher (Schwarz, Dreyfus, Hadas & Hershkowitz, 2004). The second direction is at the heart of learning and development: We decided to develop theoretical and experimental tools to follow individuals participating in <u>successive</u> school activities such as collaborative problem solving sessions or individual problem reporting, in order to possibly identify construction or abstraction of the individual in a wider time-scale. One of the main questions we asked was whether it is possible to speak about consolidation (or its opposite: fragmentation) of knowledge along a sequence of activities.

DIVERSITY AND INCLUSION: A SOCIO-CULTURAL PERSPECTIVE ON THE PSYCHOLOGY OF MATHEMATICS EDUCATION

The dialectical approach I adopted (with many other math educators in our community) is exemplified in this plenary: we investigate how shared knowledge is constructed and, to do this, we need to go back to research on knowledge construction by the individual. However, this individual is not isolated like in a laboratory; she or he learns in a context, and the researcher constantly faces the "problématique" of isolating and investigating the development of individual knowledge within the shared knowledge of a changing/developing community.

The link of my personal interest to the interest of my changing community, the PME community in this conference, is then, I think, quite obvious. I would like in this plenary session to focus on *diversity and inclusion* of learning processes within a group of individuals, and to express it via the RBC model of abstraction. I will present data from two girls who participate (actively or passively) in the same class dialogues, and collaborate in the same small group (a group of three). The different combinations of constructions of knowledge, whose trajectories vary from one girl to the other, show *diversity* within a group of individuals. On the other hand the expression of this diversity and its analysis for each girl are based on the use in the same three epistemic actions, as they are reciprocally nested among them. We may relate to these basic ingredients, which characterize abstraction processes, as to the *inclusion* of these processes. Individuals will have also different ways of consolidating what they abstracted earlier -- we are again facing *diversity*.

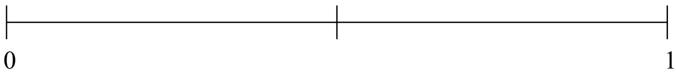
Let me provide an analogy to clarify this idea of *diversity and inclusion* in learning theories. Let's think about the relevance of a "good" micro-world to learning. It provides well-defined primitives that are easy to use. These primitives afford "*inclusion*" because they are the same and provide the same learning opportunities for each learner. However, this *inclusion* has the potential to produce, within a community of learners, a *diversity* of ways to solve a given problem and this diversity is due to the *inclusion* the tool affords. Because the primitives are easy to manipulate, the learners can use them to produce many different combinations, each of which expresses a different way to solve a given problem. Like any analogy, this one has its limits: while the primitives of the technological tools are designed beforehand by designers and are made visible, the observability of RBC actions as primitives of abstraction depends on our judgment.

In the following sections I will use data from the two girls mentioned above to reflect on inclusion and diversity in the above sense by analyzing the intertwining of RBC combinations nested in each other along a sequence of tasks.

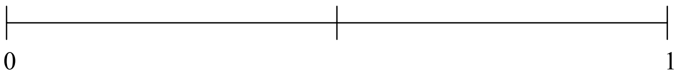
THE RESEARCH SETTING

The research took place in grade 8 classrooms during an 8-lesson unit on probability, organized in five activities including tasks for small group collaborative work and for whole-class discussions. The activities were designed so as to create opportunities for construction of knowledge. One set of tasks in the second activity was designed to introduce students to issues related to repeated events, by asking them to locate the probability of various repeated events on a chance bar (tasks 5, 6, 7 and 11). Students come back to this issue in a written (individual) final quiz of the unit (task Q2) as well as in an individual interview (task T3). These tasks are presented next.

5. You spin a Chanuka dreidel 100 times (the letters that appear are **N**, **G**, **H**, **P**). Mark approximately, on the chance bar, the letter that designates each event, and explain:
   A: The outcome will be **N** all 100 times.
   B: The outcome will never be **N**
   C: The outcome will be **N** between 80 and 90 times.
   D: The outcome will be **N** between 20 and 30 times.
   E: The outcome will be **N** exactly 25 times.
   F: The outcome will be **N** exactly 26 times.

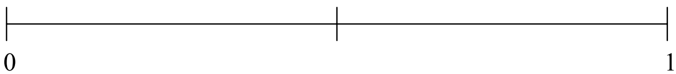


6. You flip a coin 1000 times. Mark, approximately, on the chance bar, the letter that designates each event, and explain:
   A: The outcome will be heads all 1000 times.
   B: The outcome will be heads between 450 and 550 times.
   C: The outcome will be heads between 850 and 950 times.
   D: The outcome will never be heads.



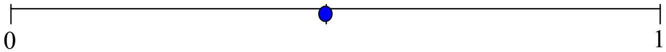
7. Which of the following events has a bigger chance to occur? Mark, approximately, on the chance bar, and explain:

   A: The outcome will be heads between 450 and 550 times.
   B: The outcome will be heads exactly 500 times.



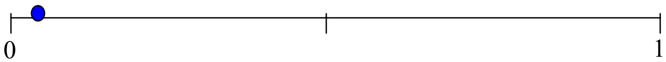
11. A regular die was thrown 100 times and the students were asked to mark, approximately, on the chance bar the probability to obtain 50 times an even number.

Amir marked the middle of the chance bar:



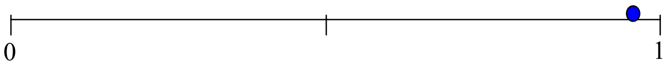
Amir explained: *I marked it this way because the chance to get an even number is **one half**.*

Shira marked a position close to zero on the chance bar:



Shira explained: *The chance to get an even number is one half; but **no way** will there be **exactly** 50 times even. Maybe there will be only 46 times even, or 52.*

Nir marked a position close to one on the chance bar.

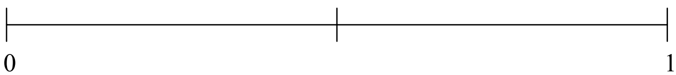


Nir explained: *I marked close to 1 because the chance to get an even number is one half. Therefore in half of the throws, the outcome will be even and thus it is **almost certain** that there will be 50 even numbers.*

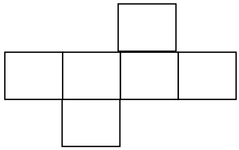
Who, do you think, is right? Explain!

Q2. You throw a die 1200 times. Mark approximately, on the chance bar, the letter that designates each event:

A: The outcome will be 6 exactly 200 times.
B: The outcome will be 6 exactly 202 times.
C: The outcome will be 6 between 100 and 300 times.

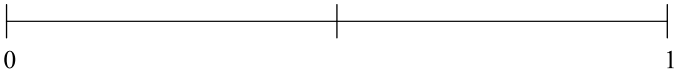


T3. In the eighth grade booth at the school fair, they use various dice for chance games. The flat view of one of the dice is. This die is thrown 900 times. Mark, approximately, on the chance bar the letters that designate events A and B:



A: The outcome will be 1 exactly 450 times.

B: The outcome will be 1 between 400 and 500 times.



Answering these questions requires a number of knowledge structures; however, in order to keep the discussion focused, I will focus on the construction of the following principles: *The probability that the frequency of an outcome is in a specific range of given length is large if the range includes the expected value, and small if the range is far from the expected value. The probability that the frequency of an outcome is equal to the expected value itself is very low.* Most items listed above require these principles at least partially. The remaining ones (5A, 5B, 5F, 6A, and Q2B) have been added for completeness and coherence. For brevity, I will refer to these principles as “the focus knowledge structure”.

At the beginning of tasks 5 and 6, students are given opportunities to learn that *the probability of the frequency of a repeated event is smaller than the probability of the corresponding simple event, and that the probability of the frequency of a repeated event decreases as the number of repetitions increases*. Schwarz et al. (2004) analyze the role of the teacher in a detailed discussion on items 5A and 5B. They also show how the difference between the probability of a simple event and the probability of the corresponding repeated event is constructed as shared knowledge about relative frequency in the classroom. In the sequel, I will refer to this as “the preliminary knowledge structure”. This construction of the preliminary knowledge structure is an epistemic action in its own right, and we will see that it is nested in the construction of the focus knowledge structure.

In the following subsections, I will present a classroom discussion of tasks 5C, 5D and 5E. Then I shall focus on two girls, Yael and Rachel as they participate in subsequent activities.

5C: The outcome will be N **between 80 and 90 times**

Guy marks C at 1/4 on the chance bar.

Yael 83: *It’s much less than he marked. It’s close to B. It can’t be a chance of 1/4 that it happens…, it’s not…*

Ayelet 84: *That N comes up between 80 and 90 times means that the other three letters come up between 10 and 20 times; that’s much less than what Guy marked.*

Yael 85: *It's much closer to B. A little larger than the B but very close to it, like the distance between A and B.*

We can see here that the students who participate in this discussion agree that the probability that the frequency of an outcome in a given range, that is far from the expected value, is small. The shared construction of this part of the focus knowledge structure appears to be unproblematic for the students. We presume that this is so because its low probability conforms to the low probability in the preliminary knowledge structure.

5D: The outcome will be N **between 20 and 30 times**

Eliana goes to the board and marks a point close to 1/4

Adi 93: *I think … 30%*

Adi 95: *There is a greater chance …*

Adi 97: *It's closer to the middle.*

Teacher98: *Does somebody have a different impression, wants to support or object? What do you think, Guy?*

Guy 99: *I think it is much higher.* [Teacher asks how much.] *80%, because there are 4 sides, right? And the chance it falls on one of them is 25%, and you said it falls between 20 and 30, so …*

Yael 100: *Thus it is 25%. It's not 80%.*

Guy 101: *No, that it falls on this 25 times, on this … out of 100 … 80, about 90%.*

Guy 103: *Just a second, can I continue this? It's not how many times the outcome …*

Omri 104: *What I'm trying to see, if I understood Guy: that there is one chance in four … thus that there is a very high percentage that it will be between 20 and 30.*

Omri 108: *What he says is that every time you spin, there is a chance of one in four that it will fall on N. In other words, now 25% out of 100 that's about the number of times it will fall on N. That's a very high chance.*

Teacher 109: [To the class:] *What do you think?* [To Rachel:] *You nod your head – with whom do you agree?*

Rachel 110: *With Guy.*

Michael 111: *Guy is right. As Omri says, it's not sure that if you spin once, it will come out 1/4. More times you spin, there is a greater chance.*

Itamar 112: *I agree with Guy, it is 75%. If I say that's 25 it's once, then maybe because I think it's high I deduct 25% from the certain.*

Yael 115: *I am still not sure. Guy succeeded in convincing me, but in the beginning I thought it was half but still …*

Teacher 122: *That is you expect an answer between 20 and 30; that's something we expect will happen. Thus, if that's what we expect to happen, then the chance is large, close to 1.*

In 5D, the students face two challenges: The first concerns the fact that for the first time they see a case in which the probability of a repeated event is close to 1. The second challenge is that the range includes the expected value. The second challenge naturally invites students to mark the probability of the simple event at 1/4. The class as a community seems to construct a new structure of knowledge, another part of the focus knowledge structure. That such a construction has indeed occurred can be inferred, for example, if the structure is being used in later tasks. This is exactly what I will show. I will focus now on the two girls Yael and Rachel, who always collaborated when the class was asked to work in small groups. I first reflect on their participation in the class discussion. I will then trace their behavior in subsequent activities.

Yael

We first follow Yael in the class discussion. Yael marks the probability for events 5A and 5B very close to 0 on the chance bar. She is not very active during the discussion on these questions. However, later on, during group work on task 6, she uses explanations raised during the class discussion, in order to convince Rachel. This suggests that she tacitly participated in the shared preliminary knowledge structure that was publicly agreed upon. She also capitalizes on this construct in 5C.

In 5D, Yael is trapped by the challenge of a range including the expected value – a crucial part of the focus knowledge structure, and this pushes her to estimate the probability of 5D according to the probability of the simple event 0.25 (Yael 100). Later on, while the discussion continues in the class, she becomes convinced that the probability is high and marks D close to 1. However, we will see later that she did not *consolidate* this part of the focus structure, suggesting that she perhaps never even *constructed* it (Yael 115).

Yael marks 5E (N exactly 25 times) close to 1. It seems that she *recognized* in this task a relationship to a non-relevant part of the focus knowledge rather than to the relevant one. Specifically, 5E following just after 5D, she may have been led by the answer to 5D (which also refers to the expected value) that was still quite fragile for her, rather than by the fact that in repeated experiments the probability to obtain the same outcome exactly k times is very small (the preliminary knowledge structure), even in the case where k corresponds the expected value (the final part of the focus knowledge structure).

Rachel

At the end of the discussion on 5A, B and C, Rachel marks all events close to zero (A the closest, then B, and then C). So we can conclude that like Yael she agreed upon the shared knowledge concerning the preliminary knowledge structure as well as the first part of the focus knowledge structure.

During the discussion on 5D Rachel agrees with Guy (Rachel 110) after he and Omri co-explain why the probability of D is high (Guy 99, 101; Omri 104, 108); accordingly she marks D close to 1.

Rachel marks E lower than 0.5, in contrast to Yael, and the third girl in the group, Noam, who both mark it close to 1. We may assume that Rachel not only *constructed* the preliminary knowledge structure, but also *consolidates* it when using it in a difficult case (for the expected value itself) to answer 5E. I infer this from the fact that she *recognized* it, and *constructed* with it the knowledge that in repeated experiments the probability to obtain the same outcome **exactly** k times, where k corresponds the expected value (5E), must be smaller than the probability that the frequency of an outcome is in a specific range that includes the expected value. Nevertheless, she did not draw the correct conclusion that it must be close to zero.

I suggest that Rachel *constructs* her knowledge gradually but certainly: After having *recognized* in the discussion on 5C and 5D the problem of the (non-)inclusion of the expected value within the range, she undertakes all subsequent tasks dealing with this issue correctly (see, Rachel 161, and later her answers in the final quiz and interview). The issue that the probability for obtaining the same outcome **exactly** k times, is very low, is still fragile as we can see in 5E and later on (Rachel 138, 140).

Yael and Rachel in subsequent peer interaction on Task 6

Task 6 has been carried out in small groups. The three girls have a long discussion on the probability for the same outcome to repeat 1000 times. The preliminary knowledge structure is relevant in 6A (the outcome will be heads all 1000 times), and later in 6D (the outcome will never be heads). Following are some utterances, in which one can see how Yael convinced Rachel that such a probability is close to zero.

Yael marks A at 1/4 on the chance bar, but then immediately corrects herself:

Yael 132: *It's like when you throw a coin 10 times and you get 5 times heads and 5 times tails, you can't say that in 1000 that's 500 times heads and 500 times tails.*

She moves her mark close to zero, and later explains why the number of times counts:

Yael 137: *Yes it does! As you add more throws, your chances drop.*

Rachel understands that the event in 6A can hardly happen:

Rachel 138: *But if there are two sides, and you say yourself that there is not much chance it will come out 1000 times heads, then there are many times it will come out on tails. That's really what you are saying because the coin has only two sides.*

And later:

Rachel 140: *If you say that there were few heads, then many times, 1000, there was tails.*

But she still does not conclude that the probability is close to 0, and marks it at 1/4. Yael reacts:

Yael 143: *No, all the 1000 times you got heads, all the 1000 times?*

And later:

Yael 148: *1000 throws – the chance is low. It's not 1/4, it's much less. It is almost illogical that it should fall 1000 times on heads.*

It is worth noticing that these two utterances evidence Yael's construction of the preliminary knowledge structure.

Rachel seems to be convinced but still does not change her marking. Only after co-solving 6B and 6C, Yael returns in 6D to her explanation:

Yael 165: *Listen, could there be a case where all 1000 throws it came out only heads?*

She passes the eraser to Rachel. Rachel erases and corrects and puts a mark close to 0. She is not very active but seems willing, quite convinced, as can be seen from her answer in 6D and in the following tasks.

In 6D (The outcome will never be heads) all three girls declare together: *It's exactly like A.* They mark it at the same place as A, close to zero.

6B: The outcome will be heads between 450 and 550 times.

Noam 159: *That's at the half!*

Yael 160: *No, there is a much greater chance, it's what Guy explained.*

Rachel 161: *Right, she [Yael] is right.*

The three of them mark B close to 1.

Again it seems clear that Yael constructed that the part of the focus knowledge structure that concerns the probability that the frequency will be in a range that includes the expected value, presumably when they worked on question 5D.

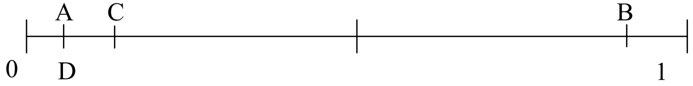
6C: The outcome will be heads between 850 and 950 times

Noam 162: *A little before A!*

Yael 163: *A little before? A "little" after!*

Noam 164: *Yes, about here. !*

The three of them mark the events on the chance bar at the same places. They mark C after A and D at about 0.15.



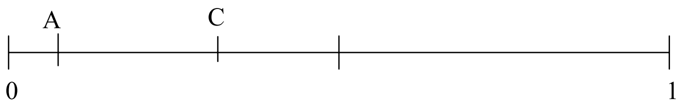
Further tasks

The girls did tasks 7 and 11 as homework. Thus their work can be evaluated according to their worksheets only. In 7 they all did approximately the same: They marked event A (heads between 450 and 550 times) close to 1 and event B (heads exactly 500 times) close to zero without any explanation.

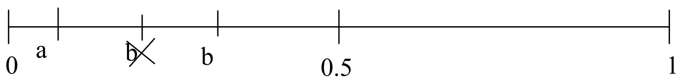
In task 11, Yael and Rachel wrote that Shira is right. Yael's explanation is: *Shira is right, as at average we will get 50 times an even number, but there is very little chance that it will be exactly 50.* Rachel's explanation is: *I think that Shira is the closest to be right as there are not many chances that the die will fall exactly half of the times on an even number but I personally would have marked a little bit closer to half.*

Yael and Rachel's answers were quite similar. It seems that both of them *constructed* the focus knowledge structure. But our conclusions may be somewhat different if we also look at their final quiz and interview.

In Q3, in the final quiz, Yael correctly marked A (6 exactly 200 times out of 1200) close to zero, but marked C (6 between 100 and 300 times) around 1/3.



She acted similarly in T3 in the interview: She marked A (1 exactly 450 times) close to zero but B (1 between 400 and 500 times) around 1/3. From her worksheet it can be seen that she hesitated as she marked B first closer to A and then erased it, and moved her mark to the right.



Rachel answered the final quiz question correctly. She also acted in the interview correctly and similarly to the way she acted at the end of the activity in tasks 6B and 11. She marked A in the final quiz close to zero and C close to 1. She marked A in the interview close to zero and B close to 1.

Summary of Yael's and Rachel's actions

In Task 5 Yael is not very active, but from her marks on the chance bar and from her explanations to Rachel in Task 6, we may conclude that she had *constructed* the preliminary knowledge structure. This construction was not fully *recognized* in 5D and 5E, where the expected value is involved (probability that the frequency will be in a range that includes this value, or that it will correspond the expected value exactly). In these cases her actions are not systematic. This knowledge seems to be constructed by Yael in the four tasks of the activity (5, 6, 7 and 11) but was not *consolidated* at all as can be concluded from her responses in the final quiz and the interview.

Rachel, in contrast to Yael, did not appear to *construct* the knowledge in the class discussion in task 5, but her discussion with Yael during the group work on task 6, may have acted as a catalyst for this construction. It is very typical for Rachel that constructing the structure of knowledge went together with an immediate consolidation, which can be seen in her responses in the final quiz and the interview.

DISCUSSION

Researchers and theoreticians of learning (including learning in mathematics) have traditionally tried to find general features characterizing large populations (age groups, high level performers, experts, etc.). These attempts delineate an inclusion. Such an *inclusion* approach always conflicts with thorough and fine-grained analyses of empirical data concerning specific learning features of the various individuals in the community. It opposes *diversity*.

During the past decade, theories and research methodologies concerning the ways in which learning characteristics of various individuals should be observed and analyzed, have undergone deep changes: “Subjects” interviewed in laboratory conditions have been replaced by observations of (groups of) people in natural contexts, in various social settings (ensembles). Clearly, as the number of students in the ensemble increases, the difficulty to follow a single student becomes bigger and the information that a researcher is able to retrieve about the learning processes of the single student decreases. Noam, the third girl in the group with Yael and Rachel is a case in point – we have very little information on her.

The complexity of data collected on ensembles of students in “natural” settings can be enormous. The data presented in this paper are especially compound, because we started to follow the students in a whole class discussion (Task 5), moved to group work (Task 6), then to home work (Tasks 7 and 11), where they work separately, and eventually to the final quiz and the interview, which were also taken individually but in situations with very different risks for the students.

An second difficulty we face is the fact that we chose to investigate the *constructing* of knowledge of rather high complexity: First the difference between two connected probabilities: the probability of a specific outcome of a single event and the probability that the frequency of this same outcome in a repeated event has a specific value or is in a specific range. And then the idea that the probability that the frequency of an outcome (in a repeated event) is in a specific range of given length is large if the range includes the expected value, and small if the range is far from the expected value.

Moreover, a relatively short and interrupted time was allotted for this purpose: The tasks 5, 6, 7 and 11 that appeared in the second of five activities and the final quiz and interview that were carried out after the fifth activity.

In the first task (Task 5), which was discussed by the class as a whole, we could only assume whether and what the two girls had *constructed* while the constructing process of the class's shared knowledge took place. The second task (Task 6) was carried out by a group of three of which the two girls were the more active ones. From observing the group work, we obtained more detailed information on the girls' constructions, and were also able to make our previous assumptions concerning their *construction* (or not) in Task 5 more reliable. The homework tasks (7 and 11), which, was done individually and differently provided some information on the girls' knowledge structures immediately after the learning episode; we thus had some more information on what was constructed by each of the two girls (or not). Fortunately, we also had the final quiz and the interview and were able to see not only what was constructed but also what was *consolidated*. We note that knowledge may be constructed but remain available only for a short while; in a later stage the student may not recognize it as an already existing structure and thus not build-with it, and possibly not even be able to reconstruct it. This means that no consolidation of this short-term construction has occurred.

From an epistemological point of view, two constructions were involved in the short flow along the four tasks of Activity 2: the preliminary construction and the focus construction.

We were able to see that Yael, in spite of her ability to explain the preliminary construction to Rachel and in spite of her correct responses to some of the questions relating to the focus construction, did not appear to have constructed the focus construction. Thus her preliminary construction was not nested in any additional construction.

Rachel, in contrast, was able to consolidate the preliminary construction, to recognize the resulting knowledge structures during her focus construction, and thus her preliminary knowledge structure became nested in her focus knowledge structure.

In conclusion, I want to emphasize that even in such a short flow of constructing and consolidating actions, during which social and other contexts kept changing, it was possible to use the *inclusion* of the RBC model in order to obtain significant insight into two individual students' constructions of knowledge, enough insight to observe the *diversity* inherent in the differences between the two students' processes of abstraction.

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LIVES, LEARNING AND LIBERTY
THE IMPACT AND RESPONSIBILITIES OF MATHEMATICS EDUCATION

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ABSTRACT

In this paper I draw out themes that run through the three plenary panel papers for PME28 (Johnsen Høines, 2004; Santos, 2004; Vithal, 2004). The linking themes for me are children's lives, their learning of mathematics and their right to liberty.

INTRODUCTION

I should perhaps explain my choice of title for this plenary panel - "*Suffer the little children*". Some will know, but I am not presumptuous enough to assume everyone does, that is it a translation of a quote for the Bible. The story is: Jesus was preaching and became an attraction not only for the general public, but understandably for large groups of little children who had been bought by their parents to see the great man. The disciples pushed them out of the way because the great man would not want to be bothered with children; his message was too important. The story goes:

> And they brought young children to him, that he should touch them: and his disciples rebuked those that brought them. But when Jesus saw it, he was much displeased, and said unto them, suffer the little children to come unto me, and forbid them not: for of such is the kingdom of God. Verily I say unto you, whosoever shall not receive the kingdom of God as a little child, he shall not enter therein. And he took them up in his arms, put his hands upon them, and blessed them. (Mark 10:13-16; Luke 18:16-17)

Now believe it or not, that is the first time I have *ever* quoted THAT source! It resonates with a post card I have on my office wall that I bought in Mozambique in 1979. In it a Mozambican girl is smiling and holds in her hand a literacy book. The slogan goes "*Forge simple words that even children can understand*". So the message stretches to the Marxist revolution in Mozambique in the early 80s of which I am proud to have played a small part as a mathematics teacher.

Of course, I am playing games with the English language here, (well it is my language and control of language gives one power!) and in particular the word "suffer" but the message is one that I think can be metaphorical. Let us consider Jesus as a metaphor for mathematics - and I apologise to anyone who finds that offensive. But it does suggest that there is the view that the power is too great for children to appreciate. Yet, if we can't make it understandable and more challengingly meaningful to children, which is surely why we are all here, then we are lost. The kingdom of heaven will not be ours – whatever your heaven is to you.

To find exactly where that quote came from, I did a Google search, and I found the following photograph, which made me stop in my tracks and want to cry. Some of you as old as me may be able to remember the day (9.15 am on Friday, October 21, 1966) when a coal waste heap slid onto a primary school in Aberfan, Wales, and killed 116 very small children. I think I still have the newspaper of the day. You will all have similar catastrophes that stand out for you. This one is pertinent to us because it happened while they were in school. Was it a natural disaster? Was it just one of those things, an "act of god" as they say? Well, we were shown in the opening plenary to PME25 in Utrecht, where the manned space rocket exploded just after take-off. In Aberfan, someone somewhere did not use mathematics enough to work out the dynamics of coal dust and water. But it probably was a question that was not even asked. In such communities, the coal is king. The communities are secondary; coal is after all what the houses and the school are there for. Michael Apple has suggested however, that many "natural disasters" may be "natural" but are far from "neutral". He asks why they usually seem to befall people on the margins of society. There are clear answers to this as he points out.

THE CULPABILITY OF MATHEMATICS

The argument then is that if you cannot understand mathematics as simply as children on the margins, then you do not understand it well enough. The challenge for us is how do we ensure that pupils from disadvantaged backgrounds achieve highly when the mathematics we present is intended, organised and structured to advantage the more prosperous student? That is of course a controversial claim which I hope many of you will engage with at this conference, whose theme is "*Inclusion and Diversity*". I go further and ask how we satisfy the needs of pupils from diverse cultural backgrounds when the mathematics we present is fundamental white and Euro-centric. I offer here a quote from Claudia Zaslavsky:

> It is the content and methodology of the mathematics curriculum that provides one of the most effective means for the rulers of our society to maintain class divisions. (Zaslavsky, 1981, p. 15)

If that does not get you going, little else I can say will! Notice here she maintains that not only is there a problem for those of us concerned about equity in mathematics education, but that the culpability lies both with what we teach as well as how we teach it. Consequently we all bear some of the responsibility for the failings of mathematics education and therefore need to consider what we can do to change things. Before I go onto consider what we might do, I need to consider in some detail just what I see as the problem. I argued this in PME21 and PME25. The Australian mathematics educator, Sue Willis, forcefully argues:

> Mathematics is not used as a selection device simply because it is useful, but rather the reverse. (Willis, 1989, p 35)

In other words, mathematics education plays its part in keeping the powerless in their place and the strong in positions of power. It doesn't only do this through the cultural capital a qualification in mathematics endows on an individual. It does this through the authoritarian and divisive character of mathematics teaching. Mathematics thus performs a social function, and by engaging in mathematics teaching, teachers are consequently involved in a social function. Hence in order to understand better the nature and functioning of mathematics teaching we need to look for foundations, predilections and structuring frameworks that would support a social model for understanding the discipline (Gates, 2000).

Yet unfairness, injustice and prejudice are not abstract concepts of some macro-social analysis of an internecine class war. They are felt through the disappointment, hopelessness and frustrations of ordinary people as they get though their everyday lives. They exist in the knots in the pit of the stomach and the tears in the eyes. Injustice exists in the disappointments many children face when they are not endowed with financial resources to have what other children have and take for granted. Injustice exists in the frustration, anger and self-depreciation when a pupil is placed in a low set for mathematics based on some assessment procedure over which they have no control and which they feel is unfair. Injustice is a process that goes on all around us, even when - and arguably especially when - we do not look for it or recognise it (Gates, 2001).

There is a rather nice mathematical problem doing the rounds at the moment, thanks to Michael Moore (Moore, 2001).

1. Who won the 2000 presidential election in the USA?
2. Why then isn't he the President of the USA?

Why is this a mathematical question? Well because it demonstrates the fallibility of numbers. God may have created the integers, but we do the counting, and of course, it's unfair. But look what damage a disagreement over a few numbers has done to the world. (I hope that is not too controversial) But it does demonstrate that mathematics is often not far from issues of power, whether it is being used to take control, or to construct a reality that permits the continuation of control.

When I was writing this paper, a UK magazine for teachers published an article titled "*Stolen Lives*" (Monahan, 2004) which describes how millions of children around the world are forced into work that robs them of their basic human rights. According to the International Labour Organisation (ILO, 2004) there are 246 million children between the ages of 5 and 17 who are deemed to be involved in child labour (Monahan, 2004, p. 9). According to the World Bank, 1.2 billion people subsist on incomes of less than one dollar a day. Now THAT is an awful lot of people.

Jerome Monahan offers teachers some lesson ideas on child labour, offering activities in religious education, citizenship, geography, history, English - all of which are really helpful. But, hold on. Something's missing here isn't it? Isn't one of the purposes of mathematics to help us understand and operate on our world? So why is it so common for mathematics not to appear for purposes such as this? And when it is, it is used in a perfunctory way?

It does not have to be like this of course and there are examples of how mathematics may be used to challenge the ills of society – so called critical mathematics education (Ernest, 2001; Gates, 2002; Powell & Frankenstein, 1997; Shan & Bailey, 1991). The issue here – and this is reflected in each of the panel papers here today - is, how is mathematics culpable in the social exclusion of children on the margins. The questions for us are, exactly how does it happen and what can be done about it? This panel and all the research associated with it, is a part of that response. What is particularly illuminating in all three of the papers, are the insights into children's daily lives, for it is here that we will find many of the answers to the two questions.

THE CULPABILITY OF PSYCHOLOGY

And what has it all got to do with PME anyway? Now I want to get controversial – yes, quite unusual for me I know. I want to ask, how many of these plenary panel papers would have been accepted as research reports to this conference? In my view it is not at all clear any of them would and as a member of PME since PME10 I make no apologies for having a view on this. Michael Apple throws some criticism at psychology for the damage it does to certain people and to the discipline and this resonates greatly with me and I am sure with many who have had papers rejected:

> In the process of individualising its view of students, it has lost any serious sense of the social structures and the race, gender and class relations that form those individuals. Furthermore, it is then unable to situate areas such as mathematics education in a wider, social context that includes larger programs for democratic education and a more democratic society. (Apple, 1995, p. 331)

This clearly makes some sense when one looks at the examples that are used in many school mathematics textbooks and resources. School mathematics has the effect of alienating certain social classes but also of pathologising them. Valerie Walkerdine (Walkerdine 1988), has written about the process by which school mathematics alienates women and racial groups for example. Barry Cooper has shown how the national Standard Assessment Tasks in the UK can result in discrimination between pupils of different social classes (Cooper 1996). Renuka Vithal draws our attention to this in her contribution (Vithal, 2004).

Two other quotes seem pertinent here, one from one of our own past presidents.

> Traditional psychology, for all that its field of study is human behaviour, has offered little that can help to improve society. (Lerman, 2001)

> Modern psychology has been incapable of making serious contributions to Third World development…it is important to point out that mainstream psychology has also failed to make significant contributions to national development and the lives of the poorest sectors of Western societies. (Harré, 1995)

Of course, this begs the question of whether it ought to be focussed on contributing to the lives of the poor. But we are at a conference whose theme is "*Inclusion and Diversity*" so I am taking that as read.

CHILDREN'S SOCIAL WORLD

There is much research in our field on children's differential ability in mathematics. It is often supposed that one can do maths or one can't, but an accusation or admission that you 'can't do maths' is more than just plain fact of capability; it is a positioning strategy – something that locates one in particular relations with others. It locates you as unsuccessful, and lacking in intellectual capability; it locates you on the edge of the employment and labour market, as virtually unemployable. Mathematics education thus serves as a "*badge of eligibility for the privileges of society*" (Atweh, Bleicher, & Cooper, 1998, p. 63). How do these badges get given out - or more importantly, what hurdles are there in the race to collect the badges (Gates, 2002)? These badges of eligibility, of which success at mathematics is one is tightly regulated by their place in society and by their consciousness – which, as Bernstein argues

> … is differentially and invidiously regulated according to their social class origin and their families' official pedagogic practice. (Bernstein, 1990, p 77)

Of course, this is all very well and good, but it so easily (and so often) remains at the level of theory. Here is another offering from Pierre Bourdieu

> The attitudes of the members of the various social classes, both parents and children, and in particular their attitudes towards school, the culture of the school and the type of future the various types of studies lead to, are largely an expression of the system of explicit or implied values which they have as a result of belonging to a given social class...the same objective conditions as those which determine parental attitudes and dominate the major choices in the school career of the child also govern the children's attitude to the same choices and, consequently their whole attitude towards school. (Bourdieu, 1974, p. 33)

What we need, if we are to improve pupils' lives and their attainment in mathematics, are more studies of the detailed mechanisms and interrelations that bring about the global processes of exclusion. One such has been provided by Andrew Noyes, who has illustrated how teachers of mathematics contribute, sometimes unwittingly, but very definitely, to the gradual process of social reproduction through the way they interpret, process and respond to historical, cultural and attitudinal evidence they take from children who suddenly appear in their classrooms at age 11 (Noyes, 2004).

And this differentiation extends to reducing the opportunities to non-white ethnic groups through the assessment structures of the mathematics curriculum.

> Black pupils were significantly less likely to be placed in the higher tier, but more likely to be entered in the lowest tier. This situation was most pronounced in mathematics where a majority of Black pupils were entered for the Foundation Tier, where a higher grade pass (of C or above) is not available to candidates regardless of how well they perform in the exam. (Gilborne & Mirza, 2000, p. 17)

Jan Winter, who has been engaged for some while now in a study of mathematics and children's home context, puts it quite forcefully:

> I believe that we cannot teach children to be numerate if we do not pay attention to the broader experience of their learning. The mathematical skills that are so highly prized are meaningless if a pupil does not have the personal, social and moral education to make sense of the world and thus know when to use them. So, at all levels, mathematics and real life are all part of the whole experience of children and it is up to us to find ways of making our teaching of mathematics reflect that. (Winter, 2001, p. 211)

MATHEMATICS AS AUTHORITY

In "*Do We Welcome Children's Mathematics?*" Marit Johnsen Høines raises the issue of authority and reminds us that one does not have to be at the margins of society to experience the "*formatting power of mathematics*" (Skovsmose, 1994). For as Ole Skovsmose writes

> Mathematics not only creates ways of describing and handling problems, it also becomes a main source for reconstructing of reality. (Skovsmose, 1994, p. 52)

This is nowhere more true that in the old South Africa, where as Herbert Khuzwayo indicates, mathematics was constructed to bring about an "*occupation of our minds*" (Khuzwayo, 1998). Yet, things can change with changing social circumstances. Renuka Vithal (Vithal, 2000) has looked at establishing a social, cultural and political approach in South Africa, where she integrated, project work, critical mathematics education, and ethnomathematics (Powell & Frankenstein, 1997). This created a reflective atmosphere where democracy and authority were seen as complimentary because they were made explicit. In her contribution here "*Researching, and learning mathematics at the margin: from "shelter" to school*" Renuka reminds us of the ways in which the social conditions of some children in South Africa impinge upon and restrict their opportunities for learning mathematics.

> Many mathematics classrooms are permeated by communication forms that assume the existence of an omniscient authority, represented, if not by the teacher, by the textbook or by technological tools. Communication, then, gets structured around a bureaucratic absolutism, according to which no particular justification for the different learning activities presented for the students is needed. (Skovsmose & Valero, 2001, p. 50)

Mathematics colonizes part of our reality and reorders it (Skovsmose, 1994) contradicting the purist view of mathematics that it is a neutral sublime purity. Marit tells us of her involvement with another Norwegian – Stieg Mellin-Olsen whose premature death left a great hole for many of us. Yet when discussing his words and ideas for mathematics education, can we ignore who or what he was and in what he believed? Of course the same is true for all teachers.

In "*Learning (and researching) as participation in communities of practice*" Madelena Santos introduces us to the ways in which mathematics is being used outside of what many of us would see as normal everyday activity. But while this activity might be outside most children's activity, it is exactly the activity these children are engaged in.

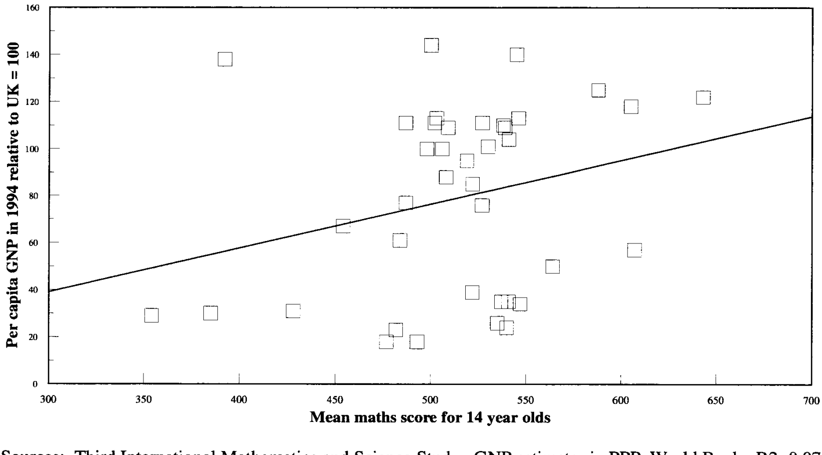
IS MATHEMATICS IMPORTANT? FOR WHAT?

I am sure, we all would support the claim that mathematics is important for all children to learn. So why is it important? I do not actually think the answer to this is as clear cut as we would like to hope. All the papers in this panel have pointed to difficulties between children's lives, their liberty and their learning of mathematics. Yet we go on teaching it to all children. One key answer to this question is, yes of course mathematics is vitally important, because it is one way in which both people and countries can develop and improve. It is important to raise living standards; it is important to improver the GDP of a country.

So let me give you some data from the TIMMS study, and taken from Peter Robinson's pamphlet on *Literacy, Numeracy and Economic Performance* for the Centre for Economic Performance (Robinson, 1997). **Figure 1** shows the correlation between attainment in mathematics and per capita GNP for 39 of the 40 participating countries. The correlation is so weak as to be meaningless. "*There is effectively no correlation between doing well in international tests of attainment in mathematics in 1996 and overall economic performance as measured by per capita GNP*" (Robinson, 1997).

Figure 1

The correlation between mathematics attainment in 1995 and average living standards in 1994

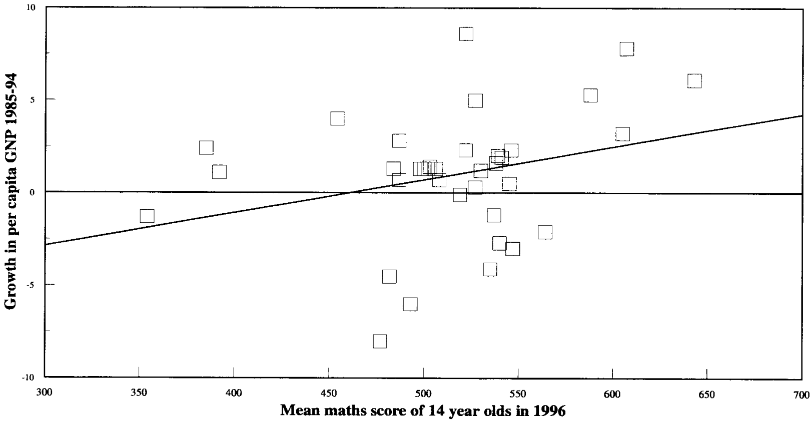


Sources: Third International Mathematics and Science Study. GNP estimates in PPP, World Bank. R2=0.07.

Figure 2 shows the correlation between mathematics attainment in 1996 and economic growth over the previous decade for 36 countries. "*the relationship is so weak as to be meaningless*" (Robinson, 1997).

Figure 2

The correlation between mathematics attainment in 1995 and economic growth 1985-94



Sources: Third International Mathematics and Science Study. GNP growth data, World Bank. R2 = 0.09.

Robinson's argument is backed up and further substantiated by Alison Wolf in her book "*Does Education Matter. Myths about education and economic growth*" (Wolf, 2002). What she does point out however, is the good-news story; the only UK post-16 A-level qualification that has any bearing on the labour market, is mathematics.

> Even after allowing for every other factor imaginable, people who took A-level mathematics earn substantially more – around 10 per cent more – than those who did not.
>
> (Wolf, 2002, p. 35)

Of course, you can guess where this is going – which social group is most represented in those children who go on to study mathematics A-level? Surely you do not need me to tell you they tend to be the already advantaged. Peter Robinson goes on to conclude, from analyses of longitudinal studies in the UK that the single most important factor in children's attainment in numeracy and literacy was their measure of social and economic disadvantage. All other factors were relatively insignificant (Robinson, 1997).

FINALLY…

One clear message for me in all these papers, is that for many people, many children, life and learning mathematics is a dally struggle. We think of problems for them to solve and strategies for them to learn. But for many children, our problems pale when compared to theirs. I ought to apologise for taking up so much time of the conference but like the three panel presenters today, I feel it is so vitally important for us to understand the lives of the children we teach, and how it impinges upon their learning. For too long, mathematics education has tried to remain neutral to the daily struggles of the children we teach and the politics behind it. I'll finish with the words of Ole Skovsmose and Paolo Valero

> Breaking political neutrality demands deliberate action to commit mathematics education to democracy.
>
> (Skovsmose & Valero, 2001, p. 53)

The struggle for me, and I know for many of you, is to use mathematics as a tool for liberty and liberation of the soul, the spirit and the poor; hence my title.

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DO *WE* WELCOME CHILDREN'S MATHEMATICS? WHO ARE *WE*?

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ABSTRACT

I have different approaches to this contribution. My concerns as a teacher and a teacher educator are to discuss how we manage to organise for inclusion of the variety of children's mathematics. How do we organise for the mathematics to be included into their mathematics? A message from children about what they expect mathematics to be has impact on this complexity. Their voice is a voice of the culture and affects the position of "we". What impact has the authoritarian nature of the mathematics for our discussions about diversity and inclusion?

THE "INFORMAL MATHEMATICAL KNOWLEDGE"

I begin by referring some aspects from teaching in Norwegian primary schools, and move back to around 1980. One could say that the situation was quite simple: at that time we did not have children from diverse ethnic backgrounds in the average Norwegian school. At that time children started school seven years old. It is only since 1997 that children in Norway have started school at the age of 6 years. All children were included in mainstream classes; we did to a minor degree have schools or classes for special needs. This was our reality. The challenges seemed demanding and complex: to educate *all children*, taking different background related to differences in gender, culture, class, and cognitive aspects into account.

We argued for mathematics to be taught on the bases of established knowledge; for the importance of making *it* concrete; making *it* understandable and actual; making *it* simple enough. Move slowly enough "up the stairs". The curriculum was used as point of departure when we worked to develop the best teaching methods as possible. We put the mathematics nicely into the communication. We aimed to help *them* understand the mathematics; to help them build a mathematical language. All the time, we explained that we used the children's knowledge, their experiences as the bases for our approach. I can still repeat the words we used, I still can memorise those voices. However, we realised we did not do what we claimed to do. The basis was the tradition of school mathematics. "Somebody" had decided what was important, what the children should learn. This somebody was an authoritarian somebody, represented by the textbooks (authors?), the curriculum (– makers?) or perhaps the very nature of school mathematics itself. We worked on making the mathematics concrete.

We tried to help the pupils understand; to think and express themselves in the way we expected them to. *We,* as teachers, used what *we* saw as relevant from their knowledge, to illustrate, to make it easier for them to enter our world. We did not take the knowledge of the children into account if it was broader than we needed it to be. As we were narrow-minded, we used the children's expressions *if* they helped in making links to the language we wanted them to learn. Observing our interaction with the children, we realised that our point of departure was the curriculum.

The children interacted with questions like: *Is this correct? Is this how to do? Show me once more, and then I'll remember how to do it. Am I to divide or multiply?* They visualised a conflict between our ideal thoughts and what really happened in the classrooms. We argued that children who enter school have competencies in mathematics, they have knowledge, they use that knowledge, and they communicate it. They have developed different ways of expressing themselves that function socially. The children's use of language should be characterised by the way it changes due to the contexts and due to whom they are communicating. We realised that the children show a wider range of mathematical knowledge when they argue themselves; but that they are more limited or narrow when they answer "the teachers' questions" or "the tasks of the textbooks". It became challenging to make the children's competencies actual in the school setting. It was not enough to "make them understand", it was not enough to show that the knowledge was actual. How could we inspire the processes of children's own argumentations; their mathematising; their investigative activities? At that item I worked with Stieg Mellin-Olsen (1987) who took part in the discussions, and the actuality of his theory on rationale for learning became obvious.

The contradictions mentioned above generated a project. The aim of the study was to get insight into children's ability in symbolisation. It focused on getting to know the children's mathematical reasoning, their developing of and use of mathematical language. The focus of the project was on the school starters. "The formal language of mathematics" was not introduced. The children did not even write numbers as digits in the first school term. They elaborated, investigated, and developed signs and drawings as written language. They explained their reasoning, listened to one another in a more interested way than we had experienced from first graders before. Lots of them moved between low and high numbers in a competent way. The problem of differentiation was easier to cope with than we were used to. Some children worked on numbers below 5 and others worked on higher numbers and even experimented with numbers above 1000. This work showed evidence of diversity, concerning the children's reasoning and argumentation, their way of representing, and the contexts they (we) made relevant for mathematising. The teacher's voice was important; to stimulate, actualise, and develop the classroom discourse. We found it of great importance that the teachers were concerned about learning to communicate *on the children's terms* - to learn about their way of reasoning and of expressing themselves. We found the mathematical interests of the adults to be important.

The insight developed through this study showed how the children's use of language could be characterised as flexible, investigative, argumentative, actual and descriptive. The diversity became evident.

When reflecting on this now, 15 years later, one methodological tool becomes important: The formal school-mathematics that usually is part of the curriculum was not introduced. The mathematics that was defined by the curriculum was, of course, reflected in the work – but more important, the work implied a wider range of mathematics - and other ways of symbolising. *Excluding the formal mathematical language became a tool to get in touch with the diversity of children's knowledge. In the study this was seen as a tool for releasing the children's use of mathematical tools. It was a tool for learning about the mathematics of the children, to get in contact with and learn about the diversity.*

Work related to this study became basis for Teacher Education in the Nordic Countries. The documentation of children's use of knowledge and their linguistic and communicative abilities motivates teachers and teacher students to investigate, make use of and stimulate children's mathematical competencies. To an increasing extent it serves as an aim for teachers to actualise the formal mathematics as ways of reasoning, ways of expressing – in context of the diversity of children's mathematics (one way among other ways). (Johnsen Høines, 1998). When we started focussing on these aspects, we were met by ignorance and arrogance. Children have mathematical knowledge without being formally taught? Which mathematics can they do? What do you mean by mathematics when you claim this? We had to present examples of children's mathematics and their language. The responses often sounded like: "Oh yes… but this in not *real* mathematics…." (How did we dare to *touch the mathematics*!)

However, something has happened. When communicating with students, teachers, parents or "the average person" today, telling that children develop mathematical competencies outside school and before they start school; most people understand what it is about. They want some stories, and bring new stories. Do we see some movements concerning the attitude to mathematics in the society? How far we have moved?

If (or when) we are to teach pupils formal algorithms today, it is quite often organised in an investigative atmosphere. The pupils develop their own methods individually and by cooperating. They investigate the quality of their different methods. *To make the formal method to be investigated in the same way, and used accordingly can be seen as an aim.*

However, the processes moving between the informal and formal algorithms often seem to be difficult. The formal algorithms are not easily seen as one method among others. Should they?

DIFFERENT LANGUAGES STRUCTURE THE CONTENT DIFFERENTLY.

When discussing the different algorithms I remember saying: *There are different ways to express the same content*. At a distance I remember the voice of the child that I see supports another approach: *It is the same but it is not the same!* Different texts imply different content. They order the content differently. The content becomes different (Bakhtin, 1998; Johnsen Høines, 2002).

I see a comment from the Conference on Environment in Johannesburg 2002 addressing aspects related to this when it is said that one needs to acknowledge and support the language of indigenous people. The point was being made that their knowledge about the environment is important for the work on protecting the environment. It was argued that knowledge is implied within the language. Through language people structure their observations; they make their categories and their hypotheses. To protect languages is about protecting knowledge. It is important to the people that own the languages, and it is also important to the world (and the scientific field).

This can be seen in the context of the children's language: Their knowledge is implied in their language. This supports an approach to empower the children's mathematical language. It is important to them and it is important to us. It also tells us that the formal mathematical language is characterised by certain ways of ordering. The content is implied in the language. This is supported by the child's voice: *It is the same, but it is not the same!*

THE FORMAL MATHEMATICS – AN AUTHORITARIAN FIELD

When the authorised or formal mathematical language is positioned in this area it is not positioned as equal to the others (even if we try to introduce it that way). The formal mathematics is not easily seen as one alternative amongst others. Mathematical texts are authoritarian texts. We cannot deal with them "the way we want". Mathematics in school has an authoritarian tradition. The tradition is not easily changed, and is implied in the texts. (Text here refers to a text theoretical approach related to Bakhtin and Lotman. However I do not pretend to elaborate this perspective here). I describe the texts as authoritarian in the sense that a kind of loyalty and obedience is expected. The continuation of the text is expected to be in the line of how it is (Wertsch, 1991, p.78). This is embedded in the genre itself. It is underlined by the voices connected to it. These voices are the traces of the tradition. We can hear the "teacher's voice" making "explanations", talking about how to do it. We can identify parents' voices or voices from politicians. Those voices are implied in the text. The following section confirms that children interpret such voices.

CHILDREN HAVE EXPECTATIONS. THEY KNOW WHAT MATHEMATICS SHOULD BE.

Trude Fosse taught first grade pupils. She organised for situations where the children worked in investigative and interactive ways. The teacher and the pupils enjoyed themselves. Fosse saw lots of qualitative mathematical learning. However, the pupils commented: *This is fun, but when are we going to do mathematics?*

In her masters study Fosse (2004) questions: "*Do children have expectations about what school mathematics is to be without having been thought? If so, what do they expect mathematics to be?*" She videotaped children who had not yet started school when they "play school". Through the play they showed how they organise the classroom, how they take different roles as teacher and pupils, how they communicated and what kind of activity they focused on.

The videotape shows learning sessions dominated by correct and wrong answers, by focusing on paper and pencil, by pupils working individually, by focusing on discipline, on certain ways things have to be done and on the teacher as an authoritarian teacher, a teacher that decides which answers that *are* correct. When they played a learning session in Norwegian, the climate, the attitude and the activities showed to be different. They were supportive, polite and working friendly together. This masters study underlines the authoritarian nature of mathematics in school – it tells about what mathematics was expected to be by those children as part of the society. It tells that the teachers are not free to position the mathematics – the mathematics is positioned - even by the children.

COMMENTS

A focus on including children's mathematics into the mathematical classroom discourse is seen as a perspective on *inclusion and diversity*. This focus also implies a focus on how children have the possibilities of including formal mathematics as part of their mathematics. The authoritarian nature of mathematics affects the complexity in this field. It does not seem trivial to touch *the mathematics*. This invites questions like: Is it about avoiding mathematics as authoritarian texts or is it about what it does imply to educate people to touch, handle, and struggle with and investigate authoritarian texts?

This for me is one of the fundamental questions that underpin the theme of this panel.

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LEARNING (AND RESEARCHING) AS PARTICIPATION IN COMMUNITIES OF PRACTICE

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ABSTRACT

In my contribution to this panel I will bring elements from recent research I conducted (Santos, 2004) in Cape Verde aiming to clarify the meaning of learning as participation in social practices – "learning as participation in the social world" (Lave and Wenger, 1991, p. 42). But as my main interest is learning in compulsory education (in Portugal until 9th grade) I looked for an empirical field that allowed me to describe the practice developed by one group of young people involved in activities that are not seen(by the youngsters and by the social world around them as a suitable profession for adults. The object of study was the participation of youngsters in an activity that they approached as something that allows them to fulfill immediate needs and not seen as a way of getting 'a job'. In this sense, the activity was not connected to any sense of 'becoming' a certain kind of person. I identified a group of youngsters in Praia (the capital of Cape Verde) within a practice – selling newspapers in the street. The boys involved in this practice are called ardinas. For me – a mathematics teacher looking for a deeper understanding of the learning of mathematics in compulsory education – the mathematics-in-use was the 'natural' entry point to make sense of the practice and to identify the learning emerging from ardinas' participation.

INTRODUCTION

The theme of the panel, and specially the sub-title given, was something that pushed me to look for a focus for my contribution. Therefore, some moments of my research were flashing back, and in re-viewing them some questions were brought to the fore:

1. Are we taking of tensions and conflicts between what or who? Lived, experienced by whom? What is being learned?
2. Inclusion and diversity of what, of whom, in what? Who decides about it?
3. Opportunities to whom? And what for?
4. How and what for is mathematics present in all this problematic?

Although I will not address all these questions, I feel that they were always present throughout the reflection I share here.

A. MAIN LINES OF MY RESEARCH

In this section I present, very briefly, the elements of the theoretical background used in order to situate the ideas to be discussed. According to Vann and Bowker (2001), "practice is an emergent relation between the 'real work' and the 'designed organization'" (p. 16). As I see such relation constructed (established) by the people-in-action, one fundamental step in the direction of understanding how the *ardinas* participate in social practices, was to characterize what was going on (and emerging) among them as *people-in-action*. This took me to the concept of *community of practice.*

I collected ethnographic data in two periods of the newspaper selling activity that one large group of *ardinas* developed in the streets of Praia. This was a group of boys between 12 and 16 years old, with a variety of schooling backgrounds ranging from 8^{th} grade to none. The *ardinas* sell the newspapers in the street - the only way they are sold in that city. Being with the *ardinas* all day in two different periods of time, I was able to identify similarities and differences in various moments of their selling activity - changes in the group and in the group dynamic as well as in the institutionalised organizational modalities of the integration of newcomers.

To be able to say if the group of *ardinas*-in-action constituted a community of practice demanded the analysis of the social practice the *ardinas* developed together during their everyday participation in selling newspapers – the activity-in-setting – through the observation and description to make sense of it. This orientated my efforts to recognize (or identify) elements in the *ardinas'* social practice in order to describe it as the source of coherence of the community. Wenger (1998) talks of describing the "dimensions of the relation by which practice is the source of coherence of a community relation" (p.72) in terms of mutual engagement, joint enterprise and shared repertoire. The *ardinas'* participation in the selling practice 'put' them in interaction with (in action, in relation with, and within) the *social world* where the newspaper selling was situated or of which it was a part.

I followed very closely some *ardinas'* trajectories, from their first day in the activity until full participation, and I identified changes in their modes of participating, in their calculating procedures, as well as in the various modes of belonging in action and in transformation. Such focus on the *ardinas*-in-action and their practice enabled me to understand and describe how their modes of calculating-in-action took shape and to recognize the situated nature of their mathematical thinking-in-action. Those modes were quite different from the school procedures but were part of their shared repertoire even if they were not made explicit among themselves within the 'ordinary' everyday selling activity nor were they explicitly taught to newcomers. They never speak about the calculation procedures they used in different moments of the selling activity between themselves or the man to whom they pay for the newspapers sold. They spoke about their calculations only with me. The wish of being and acting as good informants was the *ardinas'* 'reason' to describe those procedures or make them visible in several ways, then they were explicit objects of the talking we developed.

I made considerable effort to:

- describe the *ardinas'* practice in terms of relations between the social world, the activity-in-setting and the people-in-action;
- understand the group of *ardinas* in their everyday participation of selling newspapers as a community of practice.

These helped me to clarify the situated nature of the thinking and acting, particular to *ardinas*-in-action, as well as the meaning of the learning of such particular ways as an integral part of the learning of being an *ardina*, which involves a competence, a belonging and an identity.

B. SOME SNAPSHOTS IN *ARDINAS'* LIFE

In this section I will share two small stories in order to bring to the fore the socio-cultural world where the *ardinas* selling activity is taking place. From them I will focus on some tensions and conflicts experienced by some *ardinas* in their selling activity and by myself living with them the research process. With this, I hope to bring to the discussion the relation between *inclusion* (of what and whom) and *diversity*.

The *ardinas* are, in general, boys from poor families but they are not generally considered 'street children'. In fact, selling newspapers, among other available activities that enable poor children to contribute some money to their families, was the considered quite positively in Cape Verde. Traditionally the *ardinas* came from (recruited) a particular borough of the city, although in the first period of data collection (and for the first time in the history of *ardinas*) a group of 12 boys came from a rural area. In their village (a very poor one) it was natural for the children to help the family through engaging in fishing or agriculture, although it was not usual for young boys to go out of the village to gain money for their families. This was seen as an explicit sign of the families not being able to fulfill the needs of their children. So, the social value for the participation of boys in the selling activity was not equally considered among the two groups of children (the rural and the urban).

The two stories will illustrate how learning to be competent in the selling activity relates with belonging to the *ardinas'* community of practice, and how the learning emerged from their participation in such community overlapped and gave shape to their use of mathematics.

Competence may involve tension between various Belongings

Zeze is a boy from the village who sees the participation in the selling activity as a good opportunity (an acceptable 'excuse') for stay one night with his father living in the city. In order to enable this proximity, he needs to be a non-competent seller, that is, he needs to sell very few newspapers in order to justify the need to come back another day. The group of boys coming from the village and the man who delivered the newspapers were aware of Zeze's need, but not the group from the city. To this group, he was seen as an *ardina* that was too slow, that did not learn how to be a competent *ardina* and they complained about it. They frequently argue with the man in charge that Zeze should not receive newspapers to sell, that he should give up the selling. The man in charge, however, accepted the weak engagement of Zeze in the selling.

The *ardinas* from the village had a kind of ritual when they came back to the village. They joined in the small coffee shop (the only one where the men meet together at the end of the day) and they used part of the money they earn in the selling (the part the family allows for their own expenses) to buy candies or drinks for their friends. Those moments were very important to change the way involvement in the selling activity was considered in the village. When Zeze stayed in the city with his father, he was not able to share in this collective moment; he was not contributing visibly as an active partner on such transformation.

Gradually it was possible to see him become more involved in the selling activity, more engaged with others and more accepted as a competent *ardina*; he was now finishing the selling with his colleagues and coming back to the village with them.

What began as useful to exhibit as a non-competence – to refrain from selling and keep newspapers to sell the day after (and stay with his father) – become an obstacle to the sharing of relevant moments with his colleagues to sustain their belonging to the village community. To stay a few hours with his father would not really change his everyday situation in his family, as he lived mainly with his mother, but could put in risk his image as an *ardina*, particularly the part of such identity that involved the regard of the people from his village.

It was useful to participate actively with the others in the re-building of their image – to be seen as boys engaged in an activity outside the tradition but that did not put at risk their belonging to the community. His need to continue to negotiate his belonging to the two communities was visible and explicit within the village and the sub-community of *ardinas* colleagues from the village. But in the *ardinas'* everyday practice, with the urban part of the group, those needs were not usual. In the history of the practice the acceptance of youngsters as *ardinas* was 'natural' and it was a socially valued way of contributing to the family. So, Zeze's condition (the need to organize his participation in a way that allows the conciliation of conflicting belongings) within the global *ardinas'* community did not find a social space for being spoken about, and without the colleagues and the acceptance of the man in charge, it was not been possible for him to pursue his evolution as *ardina*.

There was a tension between the socially defined competence in the community of practice of the *ardinas* and the experience of it by Zeze. To participate in the selling and to be competent in it was not detachable from his life outside the strict time of the selling and he risked being unsuccessful if he was alone. To participate in this activity has attached to it two other dimensions that relate closely to identity dimensions – to be a son and to be a rural boy from a village with particular social and cultural values. I wonder what would happen if it was not possible for him to develop as an *ardina* without being able to negotiate/reconcile his other 'belongings' (family and living community) in a group that supported him.

This brings up the discussion of *inclusion and diversity* (of identities, of values, of knowledge). To be able to develop 'belongings' far away from the ones 'natural' to our socio-cultural heritage can be experienced in a very conflicting way and usually introduces tensions in our lives. The inclusion or exclusion is not totally and completely defined inside the strict temporal and spatial boundaries of a practice. However, the organization designed for that practice and the one that emerges from the everyday participation of the members of a community of practice, may allow (or not) the expression of diversity. Inherent to the visibility of differences it is the valuing of the various modes of belonging and of the various interests in presence. Particularly to the case of young people, the openness for a space and time to explore a new belonging without putting at risk some of their multi-membership (a fundamental characteristic of identity) may provide them with a learned experience of agency. In this way the youngster may find out relevance for other memberships and may see them as empowering, that is, they may experience it as a way of enlarging their possibilities of choice and not as restricting or learning to de-value their own roots and knowledge.

Participation, reification and the meaning of experience

Trying to understand the practice of *ardinas* required me to be aware of the stories they shared and talked about, and to identify the situations in their daily interaction where it was usual for them to speak about facts and moments of their practice. I identified the talking and thinking repertoire developed by the *ardinas*, shared and learned through participation.

For the second story I will bring two boys – Toniko (from the village) and Ntoni (living in the city). Toniko had a very limited experience in school - he left school six years before, during the 2nd grade - and he had some difficulty in understanding the bills. Therefore, sometimes he lost money in the process of giving change to customers. Ntoni was at the 6th grade and he was a newcomer in the selling.

During the selling it was usual to see some *ardinas* checking the number of newspapers against the money they had. This was always a lonely activity, but they accepted well my presence in those situations, video-recording what they were doing and asking them to explain what they were doing and how they were thinking. In those moments their role as informants was clear, and they were aware that they were helping me in the research process. I realized that all the *ardinas* developed common patterns for counting money and for calculate newspaper values. They used multiples of 8 to calculate with newspapers and multiples of 100 for the money.

It was surprising for me to notice that boys like Ntoni with more years of school life described or explained their thinking by giving some sequences of numbers and not nominating the procedures they use. For instance, when they explained me how they found what they earn in selling 43 newspapers they did not say "*I did a multiplication*" but they would say "*cause 8 are 100$, 16 are 200$, 32 are 400$, 40 are 500$ and more 3 does 537$50*". However, boys such as Toniko, with very few years of schooling, tried more frequently to describe it using words such as 'multiplication' or 'adding', usually not corresponding to the procedure they really used. The selling practice did not develop (or use) words for naming the calculation procedures. Those boys could have had access from their schooling to the words of school mathematics but they 'learned' better how much stronger was the social value of schooling compared with being an *ardina.* I belong to that universe they identify with the school (a woman, speaking Portuguese) and so they act as they imagine I would recognize them as 'competent'. Why did the boys like Toniko deny for me their 'natural' way of calculating in the practice? Why did they feel the need to 'translate' their way of thinking in words from another 'world'? And what made the others able to assume a particular way of calculating, the particular and typical way of thinking in the selling activity?

It is relevant to note that the *ardinas* who attend school at the time they were involved in the selling, said to me they felt the need to hide from their teachers the fact that they were selling newspapers. On the other side, Toniko was the boy that the man in charge of *ardinas* trusted more for anything that could involve a great responsibility with money or values. His ability for dealing with numbers and calculations, or for counting money was not as relevant as his trustful behavior, that is, as his respect for authority.

So we have to ask here, what or who is being excluded from what? Who values and what for, the school and the mathematics?

C. RE-ORGANIZING THE QUESTIONING

I will finish this paper by throwing out some fundamental questions that were posed for me in thinking about the theme. The subtitle for this Plenary Panel is "**Working for inclusion and diversity in mathematics education**". 'Inclusion' and 'diversity' are words that push me to think also of their opposites. Is mathematics education, a frame of activity that, for me, includes simultaneously school mathematics teaching and researching mathematics teaching and learning, been assumed (lived, presented) as exclusive and uniform? Who, what and what from, is such mathematics education excluding? What does it mean to be excluded, to be different? Who has the opportunity and the power to include and to exclude?

In what conditions are the inclusion and diversity issues of young people, knowledge and researchers coming to gain relevance to “our” eyes? What are the “communities” we value as the ones to which we think these issues have relevance, and how do we see our role in that discussion? What are the ‘belongings’ that are contributing to the way we are being “people-in-action” in the research and teaching field of mathematics education? What are the tensions and the conflicts that arise when we are taking these issues seriously? Why (and what for) are we valuing to spend time, energy, and imagination to work on these issues? With whom are we sharing stories and what for? There are the fundamental questions this theme raises for me, and which I leave for you to consider.

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RESEARCHING AND LEARNING MATHEMATICS AT THE MARGIN: FROM "SHELTER" TO SCHOOL

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ABSTRACT

I draw attention to the mathematics education of that group of learners who are usually on the margins of society and also on the margins of mathematics education research, theory and practice. These are children who for various reasons have left home, eke out a living on the streets of a city – referred to as "street children", and are often placed in "shelters" and "homes" (Chetty, 1997). I refer to research conducted related to providing mathematics education for such learners, to address firstly, the question of how such children come to engage particular experiences of mathematics education and secondly, the challenges and consequences of doing such research. The story I tell to respond to the theme of this panel – working for inclusion and diversity in mathematics education - is that of Nellie and Wiseman as researched and documented by Sheena Rughubar (2003), but I also reflect on broader issues of doing research and its processes and relations that involves working outside mainstream schooling and on working with research students in such settings.

INTRODUCTION

This opportunity to work with children who are very much at the margins of society and schools began with a project that the Faculty of Education participated in at the then University of Durban Westville. It was a collaborative venture with the Department of Health and Welfare and involved the City Council but the task for the faculty was that of setting up a school at what was a residential shelter for children who were in various ways recruited, rescued, sometimes arrested from the streets and brought here – called the Thuthukani Harm Reduction Centre (Amin, 2001). This is where Sheena began her study closely investigating a small group of learners as they were taught mathematics. It was here that she saw Wiseman, who surprised her with his interest and capacity to do mathematics despite the conditions of his life and the quality of the learning environment.

The intention of the school at Thuthukani was to provide educational support to the learners so that they could be integrated into the mainstream public school system. Sheena intended therefore to follow a group of mathematics learners from the shelter school into a mathematics classroom the following year. But the next year none of the children she had interviewed and observed in the shelter could be found. She searched for several months and almost dropped out of the study. It was then decided to reframe the study and look for another learner from a shelter who had been placed in a public school. After much difficulty Sheena located and met Nellie and the study was rescued. The research focus shifted to investigating how mathematics is taught to and learnt by such learners in two different teaching and learning environments – a "shelter" and a "mainstream" mathematics classroom.

METHODOLOGY AND LEARNING SETTINGS

Both Thuthukani and Sanville Secondary School are characterised by considerable distractions and disturbances for learners. In the case of Sanville the school is close to an airport and both learners and teachers were observed having their interactions interrupted by the level of noise with various consequences such as loss of concentration and disengagement by the children. Thuthukani is in the middle of the city, located in a building in a state of disrepair. There is much traffic noise and intrusions of city life into the workings of schools especially since learners are allowed to leave and return at will. On some days the school does not function because meals are not provided to learners. Any school in its normal run of events copes with events and other disruptions which in various ways coincide and conflict with those faced by a learner within a classroom setting, sometimes significantly reducing actual teaching and learning time. This intensifies discontinuities in learning that learners like Wiseman and Nellie bring into the learning environment.

Mrs. James' classroom at Sanville is overcrowded with forty learners and barely any space to walk between desks that are arranged in traditional rows. Nellie sits in the middle of the class. There is strong discipline and structure that regulates behaviour and attendance in the school, which is also maintained in the classroom by Mrs James who is an experienced and qualified teacher. Nellie sits in the middle of the classroom, is generally quiet and is often observed not participating in the lesson. Thuthukani runs its school in a large hall, with little groups of learners working in classrooms without walls. Although the desks are clustered in traditional rows, learners have to concentrate hard to ensure that they can hear Mr Xulu and also put up with the regular streams of visitors, donors and administrators who walk through the hall throughout the class period. The repeated interruptions require patience and perseverance on the part of learners and teachers. Mr Xulu, a novice, recently graduated and qualified mathematics teacher, who is teaching virtually on a voluntary basis at the school, while hoping to find employment in a public school, is flexible, light-hearted and takes the disturbances in his stride. Wiseman does not miss school, usually sits alone and pays attention during lessons.

Teachers and researchers are seldom prepared for facing the erratic events of classrooms and also the poor material conditions of places like shelters. Nevertheless, a broad range of data in both settings were generated including: video recordings of lessons; photographs; interviews with Nellie and Wiseman; informal discussions with teachers and learners; written reflections and their class documents such as exercises and tests; and a researcher's journal. These were first analysed in terms of categories that examined the learner and his/her mathematics learning and the environment as well as the mathematics teacher, teaching and content. The analysis being focussed on here that emerged from the study is mainly that related to learning and the learners

Wiseman and Nelllie (rather than their teachers and teaching); and this is extended to reflect on the research itself.

DISRUPTIONS, DISTRACTIONS AND DISCONTINUITY

For many learners in shelters or homes, disruption in their schooling in general and mathematics in particular, is marked by moving to several schools and very erratic attendance. Nellie and Wiseman are both fifteen years old, black, and appear to have had some primary schooling. Although it has been difficult to establish Wiseman's primary schooling level, Nellie has had a disruptive primary schooling having attended three different schools. Nellie is quiet in the classroom, and by her own admission scared to speak but volunteers much about her life to Sheena including the abuse that she had suffered at the hands of her mother which caused her to leave home and impacted on her schooling. Wiseman, on the other hand, is articulate in class, even correcting errors in the mathematics that the teacher makes, and volunteering to work out problems on the board, but does not speak about his life.

These discontinuities in learners' mathematical lives are also present in the learning settings. Nellie has been placed in a grade 8 classroom where the teacher is observed leaving the mathematics classroom to attend to other school functions. Mrs James' large class does not leave any time for her to provide any additional support to Nellie to bridge the gaps left by the discontinuities of the classroom or of Nellie's own life. The work Mr Xulu is doing with Wiseman's class is at grade 7 but he often seems unprepared, relying only on a textbook. He has to cope with erratic attendance of learners and constantly changing groups to work with and the distractions of the learning environment.

Establishing background data for learners is not a simple process when school life is linked to painful personal life experiences. For the researcher working in these settings on the margin, disruption and discontinuity in their data production strategies are reflected in the lives learners or indeed even in their (non)availability as well as in the settings in which learning is taking place. Situations of poverty produce uncertainty because acquiring the basic necessities such as food or shelter take precedence over schooling or attending mathematics class. Overlaid with emotional and other injuries, mathematics learning is engaged within and against this whole life experience. Yet learners and teachers continue to do the work of mathematics education as do researchers. The question is how do practices and theories of learning mathematics take account of the whole, often disrupted life of a learner as they interact with specific mathematical tasks- the focus of much mathematics education research.

MATHEMATICAL VERSUS EMOTIONAL AND PHYSICAL NEEDS OF LEARNERS

Despite the hardships endured, both Nellie and Wiseman continue to attend school and mathematics classes regularly. However, Nellie was often observed falling asleep in the mathematics class or not paying attention and is embarrassed by the teacher and other learners when caught. The teacher admits "*I don't know much about her background… but she lives in a home for children in the area*" and states that she treats all the learners the same. Nellie, however, like many of these learners is working through experiences of abuse, neglect and poor health while trying to cope with schooling. She explains to Sheena how she was hospitalised when she fell ill in school. It is not surprising that such learners often lack confidence, have poor self concepts and low self esteem (Booyse, 1991). Yet in many respects Wiseman is different. Not only does the teacher affirm him and regard him as one of his best students who will definitely be placed into one of the public schools, Wiseman is proud and derives confidence from his mathematical ability and assists others in mathematics in the class participating in discussions. Nellie and Wiseman experience the mathematics classroom in quite different ways. Falling asleep and being silent are ways in which to escape or disappear from the classroom when being forced to be there by the rules and rituals of a mathematics class. But a mathematics classroom can also be a place to feel *good* about yourself; hierarchies of needs established in psychological studies do not fully explain why and how learners in poverty and violent situations continue to learn and want to learn mathematics.

Exploring mathematical experiences of these learners forces researchers and teachers to engage much broader needs. When learners disappear from class or are found engaging in illegal or other activities, working with such children also has an emotional impact on the researcher. As the extent of the suffering endured by these children becomes known the researcher's questioning of her own participation in the research or educational endeavour and deeper values and life experiences often surface. Depending on the research paradigm in which the researcher is working, dealing with this could include the generation and analysis of the researcher's biography and engaging issues of the ethics and politics of research more directly and explicitly. This often includes reflections of their own relationships with their students, parents or other life experiences and acting on these in reciprocal relations within the research process or as an outcome of the research. The point here is that mathematical needs cannot be examined or addressed in isolation from emotional, physical and other needs. Sheena, a mathematics teacher herself, repeatedly reflects on how this research experience has made her notice and redirect her gaze in her mathematics classroom; and reshaped her own practices and understandings in teaching mathematics in the mainstream.

ALIENATED AND SHARED IDENTITIES

Street children as a group develop their own identity within a particular sub-culture from having to survive in the harsh street conditions. When they enter the shelter they share those experiences which get played out in the shelter school in the construction of the learning environment. To this extent the notion of "community of practice" (Lave and Wenger, 1991) may be relevant and useful for explaining how Wiseman participates in the shelter math classroom. Even though this community may fragment along other lines of community such as "gang alliances" or identities of age, geographic urban-rural home etc., for the period they are in the mathematics classroom, they are participating in a particular social world that collides and coincides with these different identities. The teacher may not know the full individual histories of his learners but he is aware of their fragility as a group.

In the mainstream school children from shelters are often singled out and face discrimination from both teachers and other learners (Vithal, 2003) and Nellie is no exception. If the classroom is deemed a community of practice then Nellie is clearly located outside this particular community: "*I live at the home and they don't… I feel different… the other children they do not understand… they will laugh at me… tease me*". She is marked as different in this classroom not only by virtue of living in a "home for children", she is a black learner in a school that is predominantly "Coloured" - an apartheid invented racial categorisation that still dominates to refer to people of mixed origins. The equality perspective that the teacher entrenches by claiming to treat all children equally further ensures that Nellie's different personal circumstances are not taken account of in supporting her mathematics learning. So she continues to be "othered" also by her (lack of) competence in mathematics.

No doubt researcher identity is productive of particular data with particular research participants in particular settings. Sheena's relation with Nellie could not be reproduced say by a male researcher. Nevertheless, researchers who come with a particular gender, race and social class identity, have to overcome their own prejudices and experiences of "street children" and develop empathy and understanding. This may be achieved by developing close relationships with individual children over time; and gaining knowledge about the whole life of a child and the severity of their life conditions and experiences. This is necessary to provide a much wider data set within which to place any analysis of their engagement with mathematics teaching and learning.

INTENTIONALITY, INTEREST AND INVOLVEMENT

How much genuine interest, enjoyment and involvement any learner invests in the learning is linked not only to background but also to how they see their present learning connected to a future life scenario – their foreground. Learners come with different dispositions which shape their "intentions-in-learning" both with reference to their backgrounds and their foregrounds (Olro and Skovsmose, 2001). While backgrounds have been overemphasized in explaining mathematics performance and participation, foregrounds have not been adequately factored into studies of learning. One way of understanding Wiseman's interest and investment in learning mathematics may be by noting his hopes and dreams for the future "I want to go to Moment High School. I want to be a scientist. I like science and mathematics". In the shelter schools there was no compulsion to attend school, though non-attendance was questioned. Wiseman came to all classes and paid careful attention, even becoming annoyed when detecting errors made by the teacher.

Nellie's poor performance in mathematics and negative experiences of her interaction with the teacher can be related to her poorer levels of interest and involvement. Despite her low performance, Nellie claims to like mathematics. Nellie's intentions may described as broken or destroyed. As Sheena observes, "*Nellie does not refer to anything in the future but rather continues to reflect on the past*" (Rughubar, 2003, p. 92). The construct of intentionality is useful for locating and linking explanation for learning (or not learning) to aspects both inside and outside mathematics and the mathematics classroom.

Whatever the methodological design, researchers who bring also particular intentions to these settings often get much more deeply involved beyond and outside their research projects. The significantly impoverished situation of the learners and their environment compared to the resources, both physical and intellectual, that any researcher brings means that they are often in a position to contribute to improving the situation. Researchers have the possibility to make a much wider social situation available to learners as possibilities for the future. In they confront in direct ways the objectivity-subjectivity dilemmas of their positioning in the research.

TRANSITIONS, CURRICULUM AND RELEVANCE

In the shelter school, the teacher worked with a small group of between 6 to 10 learners, and this meant that the curriculum could be organised much more tightly around the needs, performance and interests of learners. But the imperative to place these learners back into mainstream often resulted in rather traditional curricula offerings. The notions of "transitions" (Abreu, Bishop and Presmeg, 2002) may be useful for exploring the bridge between the practices engaged in the shelter school and those of mainstream school. The tension that this transition opened is that since not all learners in the shelter school were likely to be placed into public school, a "mathematics for life" versus a "mathematics for school" became visible. For Wiseman this may be described as including elements of a "mediational transition" – where the shelter school learners "interact in an intentionally educational activity designed to change perceptions and meanings before involvement" (Abreu, Bishop and Presmeg, 2002, p. 17) in school mathematics, to facilitate their participation and experience of school mathematics.

Nellie on the other hand may be described as experiencing a "lateral transition" – "moving between two related practices in a single direction" - having much in parallel with that of "immigrant students in mainstream schools" (Abreu, Bishop and Presmeg, 2002, p.17). She moved from a predominantly "African school experience" to a different institutional culture of a "Coloured school" and having to reconstruct her identity as a learner who lives in a "home for children". Unlike Wiseman she is not accommodated or included in this setting, being lost in a large class of over forty learners.

Mainstream research education and training seldom prepares researchers for the trials and turbulences of facing contexts like shelters and learners on the margin in their research. Much of the focus in research has been in what Ole Skovsmose (2004) calls “a prototype mathematics classroom” which are well resourced with well-behaved teachers and learners all interested and engaged in the mathematics. Often well-designed strategies collapse in the face of resistances, lack of trust or the impoverishment of the setting. Learners refused to have photographs taken because of fear of media exposure for criminal involvement, they resist writing a journal because of poor language competence or they fear of having confidences betrayed that could have serious consequences for them.

MARGIN, POWER AND VOICE

The notion of margin is used in this paper in a number of ways. Shelter schools are one kind of margin that exits in relation to mainstream schools. Within classrooms, shifting margins and centres exist. Nellie is excluded and lives on the periphery of the classroom both in terms of mathematics and pedagogy. As a group of children, both Nellie and Wiseman are on the edge of society belonging to what Castells (1998) refers to as the “Fourth World” or regarded as “disposable people” (Skovsmose, 2003). Despite the harsh conditions of life both inside and outside schools and classrooms, these learners still choose in some sense to attend mathematics lessons. How then does mathematics and its mediation participate in their experience of life both inside and outside the classroom? And how is this represented in mathematics education theory, research and practice?

The status of mathematics secures interest and through this power, success in mathematics translates into improved self-concept and self esteem. This is because doing well in mathematics provides not only a gateway to a better life but also bestows prestige on the learner given how it is valued in schools and societies and by the learners themselves. Both Nellie and Wiseman state that they like mathematics and want to succeed in it. Notwithstanding the background each brings into the learning setting, inclusion into or exclusion from mathematics is to a large extent mediated by the teacher. Both teacher attitude and teacher knowledge (in its broad sense) is critical in how empowerment and disempowerment are enacted in a mathematics classroom. Mrs James does not show the caring that Mr Xulu does, while Mr Xulu lacks competence in the content that Mrs James demonstrates. Wiseman is excluded by limited access to mathematics and Nellie by a pedagogy that that does not recognise and account for her difference.

Doing research in such settings draws attention to the plight of these learners to a different audience of mathematics education researchers and practitioners. Issues of voice and who speaks for whom has been extensively debated, especially in gender studies and must always be raised when attempting to speak for those who are powerless and voiceless in society. Yet research and researchers can and must speak for such learners: firstly to address the enormous disadvantage and suffering of such learners but secondly because it redirects the researcher and practitioner’s gaze in the mainstream to other margins by developing a different more empathetic gaze on those learners to fail to learn mathematics. It forces researchers to develop a more caring and creative research approaches with a sharper concern for the ethics and politics of the research setting and research relationships. The focus on the "outliers" in research sites and participants opens for possible refutations and development in the theories of teaching and learning mathematics, a practice well-established in mathematics yet surprisingly lacking in mathematics education research.

As diversity increases, inclusion and exclusion become more acute in mathematics classrooms, requiring teachers and researchers to broaden explanations for failure and success in mathematics learning. Psychological perspectives typically locate such explanations in the learner him/herself; and in mathematics education research tend to keep the focus narrowly on the mathematics and its learning. However broadly the notion of margin is understood, by focussing on the margin, new insights could be gained for the centre; and such insights allow mathematics educators and researchers to account in more authentic ways for the diversity in their classrooms and schools in more equitable way. The failure to learn mathematics lies, perhaps more significantly, outside mathematics, its teaching and learning, than inside. This assertion points to an imperative to bring political, social, cultural, economic and other perspectives into a closer dialogue in mathematics education research, theory and practice with the more dominant psychological perspectives. The papers presented in this theme point to how separating out social, psychological, cultural, political and other perspective in research, theory and practice may have outlived their usefulness for providing understandings and action in mathematics education.

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RF01: AFFECT IN MATHEMATICS EDUCATION - EXPLORING THEORETICAL FRAMEWORKS[1]

Coordinators: **Markku Hannula, Jeff Evans, George Philippou, Rosetta Zan**

This article brings into a dialogue some of the theoretical frameworks used to study affect in mathematics education. We shall present affect as a representational system, affect as one regulator of the dynamic self, affect in a socio-constructivist framework, and affect as embodied. We also evaluate these frameworks from different perspectives: mathematical thinking, students with special needs, and methodology.

INTRODUCTION

Markku S. Hannula

Affect has been a topic of interest in mathematics education research for different reasons (McLeod, 1992). One branch of study has focused on the role of emotions in mathematical thinking generally, and in problem solving in particular. Another branches have focused on the role of affect in learning, and on the role of affect in the social context of the classroom. Affective variables can be seen as indicative of learning outcomes or as predictive of future success. Partly because of this diversity in the research areas, but also partly because of the different epistemological perspectives of researchers, there is considerable diversity in the theoretical frameworks used in the conceptualisation of affect in mathematics education.

McLeod (1992) identified three concepts used in the research on affect in mathematics education: beliefs, attitudes and emotions. He made distinctions among these and described emotions as the most intense and least stable, beliefs as the most stable and least intense, and attitudes as somewhere in between on both dimensions. Beliefs were seen as the most 'cognitive', and emotions as the least so. Later DeBellis and Goldin (1997) added a fourth element, values. Most research on affect in mathematics education has used one or more of these four concepts. However, the theoretical foundation beneath these concepts is not quite clear.

Attitude has perhaps the longest history in mathematics education. Yet several authors (e.g. Di Martino & Zan, 2001; Hannula, 2002a) quite recently point out that attitude is an ambiguous construct, that it is often used without proper definition, and that it needs to be developed theoretically. Regarding beliefs, Furinghetti and Pehkonen (2002) asked a virtual panel to evaluate the definitions given for this concept in the literature. Their main finding was that no definition could be accepted by all experts in the panel. Hence, there is not one concept 'belief' used in the field,

[1] This article has been produced in collaboration and hence the authorship is divided. The first author and the main contributor(s) of each section is mentioned at the beginning of the respective section.

but many closely related ones; some of them are discussed in the recent book edited by Leder, Pehkonen, and Törner (2002).

Emotion is probably the most fundamental concept when we wish to discuss affect. Researchers who have studied the psychology of emotions have used different approaches, and there is no final agreement upon what emotions are. However, there is large agreement on certain aspects. First, emotions are seen in connection with personal goals. Emotions are also seen to involve physiological reactions, as distinct from non-emotional cognition. Third, emotions are also seen to be functional, i.e. they have an important role in human coping and adaptation. (E.g. Goldin, 2000; Lazarus, 1991; Mandler, 1989; Power & Dalgleish, 1997)

However, there is no agreement on how many basic emotions there are, or what they would be - or even if there are any basic emotions. It is well known that emotions are not only consequences of cognitive processing; they also affect cognition in several ways: emotions bias attention and memory and activate action tendencies (e.g. Power & Dalgleish, 1997). However, there is no detailed understanding of this interaction.

Value is the concept that has probably been least used of the four, and thus the relevant research is in its formative stages. However, values education is a dominant theme in various educational systems' goals around the world, and it is important to explore what a research focus on values in mathematics education can offer to our concerns about affect (Bishop, 2001). Of particular interest at the present time is the relationship between beliefs and values, with the focus of the first being on principles and propositions, and with the second being on choices, priorities and actions.

These four concepts do not cover the whole field of affect. Terms such as motivation, feeling, mood, conception, interest, anxiety, and view have also been used in this field. Motivation is an important concept, but surprisingly little research has been done explicitly on motivation in mathematics - at least within PME (Hannula, 2002b). What is its relation to the above-mentioned four concepts?

One important problem in the recent research on affect is the understanding of the interaction between affect and cognition. This problem is addressed in several ways, since here is great variation in the theoretical frameworks that mathematics education researchers have used. Goldin (2000) interprets affect as a representational system - parallel to cognitive systems - that encodes important information regarding problem solving. Some other approaches emphasize the social dimension. Socio-constructivists see affect primarily grounded in and defined by the social context (Op ‘t Eynde, De Corte, Verschaffel, 2001). Discursive practice theory emphasizes positions that are made available by the practices at play in the social context, and that enable and constrain the emotions that can be experienced and expressed (Evans, 2000). In the Vygotskian framework emotions become one dimension of the Zone of Proximal Development (Nelmes, 2003). Quite a different approach is to look at the recent findings of neuroscience and see how that informs our view of affect in mathematical thinking (Schlöglmann, 2002). Embodied view (Drodge & Reid, 2000), self-regulation (Malmivuori, 2001), and psychoanalytic theory (Evans, 2000) provide yet other theoretical frameworks for conceptualising affect.

All these different approaches have their value, and there is space for a multitude of approaches. One approach may be more suitable for certain research questions, while other questions require different theoretical tools. However, there is also a need to increase coherence in this field. Undeniably, there is the need for discussion. We invited Gerald Goldin, Peter Op 't Eynde, Marja-Liisa Malmivuori and David Reid & Laurinda Brown to each present a description of one theoretical framework that they have used to conceptualise affect. We have also invited three persons to evaluate the usefulness of these frameworks from different perspectives: Shlomo Vinner from the perspective of mathematical thinking, Melissa Rodd from the perspective of students with special needs, and Jeff Evans from the methodological perspective.

CHARACTERISTICS OF AFFECT AS A SYSTEM OF REPRESENTATION

Gerald A. Goldin [1st theoretical framework]

The prevailing view of mathematics as a purely intellectual endeavour, where emotion has no place, is perhaps just one reason for the relatively little attention devoted to research on affect in mathematics education. The methodological difficulty of designing and carrying out reliable empirical studies in this domain also poses an obstacle. We do not now have a precise, shared language for describing the affective domain, within a theoretical framework that permits its systematic study. Let us consider some ingredients of a possible theoretical framework for discussing mathematical affect, based partially on joint work with Valerie DeBellis (DeBellis, 1996; DeBellis & Goldin, 1997, 1999; Goldin, 2000, 2002, and references therein).

Affect as a system of representation and communication

The idea that affect has a basic *representational* function seems to be a less-than-usual perspective in psychology. More often emotions are described merely as accompanying cognition, or occurring in parallel with cognitive activity. Usually they are regarded as *consequences of* cognition, and often as having immediate *consequences for* cognition, either facilitating or impeding cognitive activity. Going beyond these evident features, we propose to regard affect much more fundamentally as one of several internal, mutually-interacting *systems of representation* within the individual human being. That is, the affective system functions *symbolically* so as to encode essential information. Loosely speaking, our emotional feelings and the complex structures involving them have *meanings,* even when we may not be consciously aware of those meanings, or able to articulate them.

Among the kinds of information commonly encoded affectively are: (1) information descriptive of the external physical and social environment in relation to the individual [as when *fear* may signify immediate danger or threat, and *relief* signify that a transition from danger to safety has occurred]; (2) information regarding the individual's own cognitive/affective configurations [as when *surprise* may signify the unexpectedness of an event as it occurs, or *frustration* signify lack of perceived progress in achieving a goal]; (3) information about other peoples' cognitive and affective configurations [as when *attraction* to another may encode the other person's favourable interest in one's own personality]; and (4) information about social and cultural expectations in relation to the individual [as when *pride* may signify fulfilment of societal role expectations, or *shame* signify failure to fulfil them]. Each description of such an encoding suggests a sense in which affect is *situated* [see the discussion by Op 't Eynde below].

Of course, to say that information is *represented* does not necessary imply it is *true.* This pertains whether the representational system is ordinary spoken language, visual imagery, or affect. The feeling of fear may occur when danger is only imagined, not actual. One may be unaware that one's emotional feelings are in this sense non-veridical; or one may know it perfectly well, as when feelings occur while reading an engaging novel. Furthermore, in real-life examples, the information encoded by emotional feelings typically cuts across more than one of the above categories, and is highly nuanced. For example, the feeling of *pride* may signify not only the actual fulfilment of sociocultural expectations, but also the proud feelings attributed by the person to others (e.g., the individual's parents or teachers), as well as the high value placed by the individual on the opinions of particular others. The meanings of affect often have to do with *complex, self-referential* information, such as "what I think someone else thinks of me."

In doing mathematics, the affective system likewise encodes information relevant to mathematical problems, and especially relevant to *the person in relation to the mathematical activity.* The feeling of *bewilderment* in approaching a problem in mathematics may simultaneously suggest that certain standard problem interpretations or problem-solving strategies do not work, because the problem is nonroutine [information about the mathematical structure of the problem], and the person's lack of specific *knowledge* or even more specifically, inability to formulate a subgoal [information about the state of the problem solver]. The feeling of *anxiety* may represent certain beliefs the person holds about his or her inability to do mathematics, or about possibly negative opinions others may form if he or she is unsuccessful. Affective states may evoke heuristic strategies; thus frustration [encoding the absence of apparent progress in solving a mathematical problem after repeated efforts] may evoke a major change in strategy [for example, a decision to try a special case, or to solve a simpler, related problem].

Cognitive representational systems function partly but very importantly by evoking affect and the information it encodes. This applies specifically to the internal verbal/ syntactic systems, imagistic systems, formal notational systems, and strategic/ heuristic systems of representation discussed elsewhere in formulating a model for mathematical problem-solving competence. In short, affective representation is not auxiliary to cognition; it is centrally intertwined with it. Affective configurations routinely signify, evoke, enhance or subdue, and otherwise interact with cognitive configurations, in ways that are highly context-dependent and person-specific.

In addition to its internal, representational function, affect also provides a *language for communication* among human beings. Here we highlight those "messages" conveyed through steady or intermittent eye contact and pupil dilation (or the absence thereof), facial expressions, gestures, posture and "body language", interjections, intonation, song, way of breathing, laughter, tears, blushing, and so forth. Much of the communication that thus takes place happens tacitly among individuals; we find it difficult or impossible to say what it is specifically in someone else's expression, half-open lips, intermittent-to-steady gaze, or raised eyebrow, that gives the impression they are curious, or amused, or very serious. Very likely, in everyday life, our judgments are often wildly inaccurate. Yet the system works extraordinarily well, underpinning and motivating virtually all human activity. Each individuals' affect normally interacts with and evokes affect in others, so that information is exchanged and people in pairs or groups can share affect and function effectively together.

While there is considerable evidence of "emotion" in the world of other mammals, affective communication in the complexity that we experience it seems to be (like natural language) an essentially human phenomenon. It seems plausible that our system of affect, and the communication it makes possible, evolved as human beings evolved. Perhaps it enabled humans to function effectively at the tribal level, as well as in family groups, and facilitated the evolution of children who *learn* for many years before becoming biological adults. If this perspective is correct, it should not surprise us if the development of powerful affective structures turns out to be the key to effective mathematics learning and teaching.

Affective pathways: Local and global affect

Local affect refers to the changing states of emotional feeling experienced by individuals as they engage in mathematical (or other forms of) activity. Recurrent sequences of such states, one leading to the next depending on the context, may be termed *affective pathways.*

For instance, in a highly idealized example, a student approaching a problem in mathematics may initially experience *curiosity,* followed by a sense of *puzzlement* or *bewilderment* if the problem is unfamiliar or difficult. Repeated unsuccessful attempts may evoke *frustration.* Perhaps after one or several changes of strategy, the student experiences some *encouragement* as progress seems to occur, *elation* at a new insight or breakthrough, followed by *satisfaction* with having solved a difficult problem or understood a new mathematical concept. Alternatively, the student's frustration may lead to *anxiety, anger, fear,* and/or *despair,* evoking avoidance strategies and defence mechanisms—a very different pathway.

As they recur, such affective pathways lead to the construction of *global affect* within the individual—long-term affective structures that in the first case might facilitate future enthusiasm, engagement, expectations of success, and a positive mathematical self-concept, but in the second case might lead to future distaste, avoidance, expectations of failure, and a negative self-concept.

Subdomains of affective representation: A tetrahedral model

McLeod and his collaborators proposed three components of the affective domain, which DeBellis and I proposed extending to four, creating a sort of tetrahedral model. In order of increasing stability in the individual over time, and degree of cognitive involvement, they are: (1) *emotions* or emotional feelings; the rapidly-changing states of feeling experienced during mathematical (or other) activity; emotions may range from the mild to the very intense, and are seen as local and contextually- embedded; (2) *attitudes,* orientations or predispositions toward having certain sets of feelings in particular contexts (e.g., mathematical contexts); attitudes are seen as moderately stable, involving a balance of interacting affect and cognition; (3) *beliefs,* discussed a bit further below, which involve the attribution of some sort of truth to systems of propositions or other cognitive configurations; beliefs are often highly stable, highly cognitive, and highly structured, but affect is nevertheless interwoven with them; and (4) *values,* including ethics and morals, the deep personal "truths" held by individuals that help to motivate priorities; values are stable, usually highly affective as well as cognitive, and may also be highly structured.

Each vertex of the tetrahedron (emotions, attitudes, beliefs, and values) may be understood as interacting dynamically with the others in an individual. For example, emotions influence attitudes, beliefs, and values; one mechanism for this influence is the construction of global structures as a result of the recurrence of certain affective pathways. In addition, each vertex interacts interestingly with the corresponding component in the affective domain of other individuals.

Affective competencies and affective structures

In the study of mathematical cognition we discuss *competencies*—the capabilities of an individual to perform particular tasks, take particular cognitive steps, or process information in particular ways in particular representations. Related cognitive competencies form complex, cognitive structures. Analogously, *affective competencies* refer to the capabilities of an individual to make effective use of affect during mathematical activity—for example, to act on curiosity, or to take frustration as a signal to alter strategy. Likewise we see the need to characterize and discuss the most important *affective structures* in relation to mathematics—for example mathematical intimacy [structures of emotional feelings, attitudes, beliefs, and values associated with intimate and vulnerable engagement in mathematical activity]; mathematical integrity [affective structures associated with the commitment to truth and understanding in mathematical activity], and mathematical self-identity [affective structures associated with the sense of self, "who I am" in relation to mathematics; see the related comments by Malmivuori below]. It is well-known that many students and adults have global affective structures that impede mathematical learning—in common parlance, "math anxiety"—but we do not have a straightforward way to change this. Thus it is important to study *mechanisms of change in global affect,* in analogy perhaps with how acts of forgiveness or self-forgiveness can permanently transform structures of anger, resentment, or guilt.

Meta-affect

An idea that has assumed a central role in our thinking is *meta-affect,* referring to affect about affect, affect about and within cognition that may again be about affect, the monitoring of affect both through cognition and affect. Our hypothesis is that meta-affect is *the most important aspect of affect.* It is what enables people, in the right circumstances, to experience *fear* as *pleasurable* (e.g., in experiencing a terrifying ride on a roller coaster). Towers of meta-affect occur often, and when they do they are usually very powerful—thus one may feel *guilt* about one's *anger* about the *pain* of perceived rejection by a parent whom one *loves.* At the core, perhaps, may be the love; but the negative meta-affect transforms it into something painful, and the anger and guilt contribute to an enduring if dysfunctional structure.

Consideration of meta-affect suggests that the most important affective goals in mathematics are *not* to eliminate frustration or to make all mathematical activity easy and fun. Rather they are to develop meta-affect where the feelings *about* emotions associated with impasse or difficulty are productive! Beliefs and values also play a role here, as they influence the ecological function of the emotion in the individual's personality. For example, the feeling of frustration with a mathematical problem *could* and *should* indicate that problem is nonroutine and interesting. The feeling should carry with it anticipation of possible elation at understanding something new, so that the *frustration itself* is experienced as interesting, curious, and anticipatory of joy in success. Related "cognitive" beliefs and values in relation to mathematics—belief that success is in fact likely, the value placed on achieving a challenging goal—can contribute to the construction of powerful meta-affect.

Belief systems, meta-affect, and sociocultural contexts

Finally, let us comment briefly on beliefs, systems of belief, and meta-affect. We have noted that beliefs establish meta-affective contexts for the experience of emotion. Reciprocally, affect stabilizes beliefs. The beliefs to which people hold fast may or may not be true, but they are *comfortable.* To say this is not to assert that they are necessarily pleasant; a belief may be somewhat painful [e.g., the belief by a child that she is "no good in math"], but it may be helping to shore up defences against greater hurt [e.g., being "no good in math" the child cannot be *expected* to perform well, and so will not disappoint her teachers or her parents].

Systems of beliefs allow for redundancy and mutual support, further stabilizing them. "Math is for boys," "You have to be really smart to do math," or "You have to be sort of a nerd to like math," may fit together well with "I'm no good in math." *Socially or culturally shared* beliefs and affective structures contribute substantially to the way in which meta-affect and belief systems sustain each other. In general, the strongest affect and most stable belief systems are those such as nationalist fervour, or religious reverence, that are shared and culturally embedded.

All of this suggests we give considerable explicit attention to the affective dimension in understanding the persistence of belief systems that are counterproductive to powerful mathematical learning and teaching.

A DYNAMIC VIEWPOINT: AFFECT IN THE FUNCTIONING OF SELF-SYSTEM PROCESSES

Marja-Liisa Malmivuori [2nd theoretical framework]

Newly rediscovered theoretical constructs, such as metacognition, consciousness and self-regulation, afford opportunities to consider cognition as more closely linked to affect and behaviour in learning and education. The role of personal constructive and self-regulatory aspects of affective responses in social, contextual and situational environments is emphasized in the suggested dynamic viewpoint. More generally, the view connects these aspects closely to the functioning, qualities and development of students' self-systems and self-system processes in respect to learning mathematics. The qualities and functioning of significant self-system processes ultimately determine the power and role of affect in students´ personal learning or performance processes in mathematical situations. The perspective applies recent cognitive, socio-cognitive, constructivist, as well as phenomenological views of learning and links affect strongly, naturally and in a dynamical way to cognition. Moreover, the chosen conceptualisations and developed learning model try to overcome the restrictions often caused by the use of traditional and static affective concepts.

Affect in personal learning processes

This view considers affective factors and emotional experiences as essential features of *personal learning processes* and functioning. In addition to affective experiences, we use also such terms as affective arousals, states and responses, each of which relates both to biophysical, mental and expressive human aspects or processes. Students as historical and social individuals or *selves* constitute, evaluate, develop, and regulate themselves and their own affective experiences and learning processes in relation to mathematics. These are essential aspects of personal functioning and development. With respect to powerful affective arousals and experiences especially important are students' self-perceptions in social contexts and situations. The related highly influential affective responses can be called '*self-affects*'. They are connected with students' experiences of self-esteem, self-worth, and/or personal control with respect to mathematics, which can be described as the aspect of 'how one feels about one's worth' (cf., Harter, 1985). The significant relationship between the self and affect is acknowledged in the classical psychological theories. Within education research domain it appears in the close measured relationship between students' self-concept, self-esteem, self-confidence or self-efficacy and their highly intense responses, such as anxiety, and further the qualities of their motivational or learning outcomes (e.g., Covington & Roberts, 1994; Schunk, 1989).

Mathematics education research has found these kind of affective responses often negative and inhibiting in nature, resulting in disturbance of students' mathematics learning, problem solving, or performances. For example, significant and constant gender-related differences are measured in students' perceptions of their mathematical abilities as well as in their self-affect, such as anxiety or pride and shame, and performances or achievement behaviours (e.g., Fennema & Hart, 1994; McLeod, 1992). The arousal and role of students' affective responses are seen here to be closely connected with their personal and situational self-perceptions, efforts, goals, and self-regulation in the social and contextual mathematics learning environment (cf., Pekrun, Goetz, Titz & Perry, 2002; Skaalvik, 1997). More specifically, these central aspects of learning are considered here as the qualities, functioning and development of students' self-systems and self-system processes in learning and doing mathematics. In addition to mathematical knowledge systems, understanding and skills, students' personal *self-systems* involve their self-beliefs and self-knowledge systems, mathematical beliefs and belief systems, related affective responses, and the related behavioural patterns in mathematical situations. By students' *self-system processes* it is referred to the functioning of their mathematical self-systems in unique social mathematical situations, with their active self-regulation and personal agency to varying extent as included. Moreover, the aspects of personal self-systems and self-system processes represent different degrees of abstraction in students' mental processes or cognition. In this way, the varying levels of consciousness or *self-awareness* in these systems and processes are also applied here as an unconstrained path from cognition to affect and vice versa.

Affective arousals in social mathematics learning situations

The arousal and development of students' highly influential affective responses (self-affects, e.g., anxiety, fear) to mathematics are intertwined with their situational or learned habitual beliefs, perceptions, and appraisals of the self in mathematics learning contexts and social environments. These constitute central occasions for the dynamic interplay of students' cognition and affect in learning mathematics. In this, essential arguments are given for such unique situational and constantly ongoing self-system processes as *self-appraisals* and *self-judgments*. Personal, situational and social environmental features and conditions create a context for a significant self-evaluative situation to emerge and, thence, for the evoking of essential personal self-beliefs and self-appraisals with mathematics. The related unique evaluations and judgments of the self in a mathematical situation are accompanied by affective arousals and constructive or directive processes with affect and behaviours, implying important *affective self-states* for doing and learning mathematics (cf., Lewis, 1999). Especially important are students' perceptions and appraisals of their personal capability, agency and control with respect to mathematics and mathematics learning.

Students' appraisals or judgments are influenced not only by personal but also by unique contextual and socio-cultural features of mathematics and its learning. Influential self-appraisals mediate not only the effects of students' past personal mathematical history (e.g., personal beliefs), but also those of the fundamental socio-cultural and contextual features of mathematics learning on their affective responses to mathematics (cf. Malmivuori, 1996). In this *individual-environmental interaction*, the characteristics of an actual learning context, or unexpected, new, or rapidly changing occurrences in this context, represent more direct environmental influences on students' self-appraisals and on such self-affects as anxiety or test anxiety. Less direct environmental influences on students' self-appraisals and affect are again linked to particular kinds of socio-cultural beliefs about mathematics and mathematics learning or about performance situations that are reflected by students as well as by the larger social environment (e.g., perceptions of the difficulty of mathematics, attributions for mathematical successes or failures).

In referring to the *constantly operating mind* and general flow of affective mental processes and states, we indicate that different appraisals and processing activities can coexist at different levels of consciousness or self-awareness, and cause several (continuously flowing or changing and perhaps conflicting) affective experiences or self-states that are more or less influential in students' mathematics learning processes (Malmivuori, 2001). The scene of the related mental activity can be called a *student's contextual consciousness* that is conditioned by internal personality aspects as well as by various external features of mathematics learning situations and contexts. That is the primary personal and unique situational scene for the individual-environmental interaction to occur and develop in doing and learning. Within this scene, affective responses do not only arouse, tone or disturb students' learning or performing processes but also serve them as a significant source of information about their own mental content and ongoing processing activities, of their action conditions, and of their self-states with respect to mathematics learning.

Self-regulatory features of affect

Self-regulation processes represent the central combining feature of self-system processes with affect. In addition to self-appraisals and self-judgments, these metalevel mental processes involve students' self-directive constructions, self-control and self-regulatory actions. They represent the other significant aspect of the dynamic affective-cognitive interplay that are then accompanied by and/or directed towards affective responses and states. The most common approach to this interplay can be referred to as *affective regulation* that illustrates the property of affective experiences to form a kind of affective feedback system that dominates the cognitive evaluation system or behaviours at a relatively low level of control without clear notions of self-regulatory mental activity (cf., Leventhal, 1982; Taylor et al., 1997). It includes preventive effects of affect such as mental blockages, simplification of mental processings and hindering of the maintenance of higher order metalevel processes, or again, intensification of mental processes and change of content of thoughts caused by promotive positive affective responses (McLeod, 1988).

Affective responses also give rise to, accelerate, or sustain additional interpretations, personal meanings, and beliefs with several evaluation processes going on at the same time at different levels of consciousness. They further establish a set of *additional behavioural goals* related to or independent of students' specific goals or objectives with ongoing original learning intentions or behaviours, and cause differing and possibly conflicting action tendencies (Evans, 2000; Leventhal, 1982). In this way affective responses have important organizing, motivating, and adaptive functions, and directly induce or regulate also other affective responses (e.g., interest attenuating fear and sadness or shame attenuating joy; cf., DeBellis & Goldin, 1997; Goldin, 2000; Taylor et al., 1997). Integration of this kind of whole level affective-cognitive dynamics or self-system processes has a major impact on the organization of personality with important individual differences in affective development, as well as in the development of self-system or self-regulatory personal processes in general. The dynamic view connects this integration to students' self-conscious monitoring, assessment and judgments of their own affective arousals, responses and self-states, to their self-conscious decisions and choices directed toward these responses or states and the causes or effects of these, and to their conscious control over their own affective responses. Students' affective arousals and responses thus become objects of their conscious evaluations and regulation and their unique situational mental processes have significant power in affecting the arousals, experiences and effects of their affective responses in learning or doing mathematics. The dynamic view refers to these kinds of self-system processes as *active regulation of affective responses*.

The essential difference between these two forms of the interplay of affect and cognition is linked here with the varying degrees of students' consciousness or *states of self-awareness* in the functioning of their self-system processes. Thereby, affective regulation represents lower level or more automatic self-regulatory processes with weak self-control beliefs or personal agency and lower states of self-awareness, while active regulation of affective responses relates to enhanced self-control beliefs and high personal agency with efficiently integrated self-regulatory processes and promoted self-awareness. This variation in the qualities of students' self-system processes determine the role that affective responses play in their personal learning processes and performances. It is the key feature of students' contextual consciousness in any mathematics learning situation. With respect to the individual-environment interaction we may characterize active regulation of affective responses as individually and situationally directed personal processes with affect. The interaction between environmental features and students' mathematical affective responses is then less direct and more flexible or independent of the instant environmental conditions and specific social features of school mathematics learning, but also of their own stable or habitual self-systems (i.e., self-beliefs, mathematical belief systems, affective responses, behavioural patterns). Instead, affective regulation can be considered as basically retaining functioning, in which the interaction between environmental features and students' affective responses is rather direct. Arousal, repetition and effects of similar strong and often hindrance affective responses (cf., global affect; DeBellis & Goldin, 1997) depend then mainly on the qualities of students' stable self-systems and/or on the particular contextual and socio-cultural features of mathematics learning.

Theoretical applications

The offered dynamic viewpoint is designed to deal with the complexity of affect-cognition interplay in social learning situations. It supports the idea of personally and situationally unique affective constructions and also considers these constructions in interaction with social environment (cf. socio-constructive views by Op 't Eynde, local affect by Goldin). Examination of the functioning of powerful processes of personal learning and affect (i.e., significant self-system processes) in situations offers better opportunities for understanding not only the importance of students´ self-identity or self-referential information (Goldin, Op 't Eynde) but also their personal involvement and self-regulatory features with their affect. The emphasis on self-reflection and self-regulatory processes in the model also relate to the important ideas of meta-affect presented by Goldin. Furthermore, linking affective aspects to mental, behavioural, and control or regulatory processes at different levels of abstraction and personal functioning will connect affect more closely to cognition, and also such concepts as embodied cognition and affect (Brown & Reid) can be fitted to the model. On the other hand, the role and impact of important affective responses are seen here to vary along with the qualities and functioning of personal self-systems in mathematical situations. In this, a basic qualitative distinction is made between students' *fully functioning self-system processes* and personally powerful learning or doing of mathematics and, in turn, their *defectively operating self-system processes* and learning with self-defending, habitual or retaining, and externally directed performance behaviours, often filled with negative affect.

A SOCIO-CONSTRUCTIVIST PERSPECTIVE ON THE STUDY OF AFFECT IN MATHEMATICS EDUCATION

Peter Op 't Eynde [3rd theoretical framework]

The study of the role of affect in mathematics education typically is not only determined by the way affect is defined but, more generally, also by the researcher's view on learning and instruction. One's view on mathematics learning determines the key aptitudes and processes to be investigated and how this is done. More specifically, it clarifies which role affective aptitudes and processes might have in learning and sets the stage for the affective processes looked for, how they are conceptualised and how they should be studied. Therefore, in introducing our perspective on the study of the role of affect in mathematics education, more specifically on the study of students' beliefs and emotions, we first need to explicate our view on mathematics learning in general.

Learning, engagement, and identity

From a socio-constructivist perspective learning is conceived as a fundamentally social activity. Learning is getting acquainted with the language, rules and practices that govern the activities in a certain community, in our case the mathematics education community. By engaging in the practices of this community people discover meaning, come to know. Meaning, then, becomes jointly constructed in the sense that it is neither handed down ready-made nor constructed by individuals on their own. Well established meanings might be implied in practices characterizing a specific community for many years, but it is through engaging in such a practice anew that the individual experiences meaning and renegotiates the currently accepted meanings. Greeno, Collins, and Resnick (1996, p. 26) clarify that

> "The view of learning as becoming more adept at participating in distributed cognitive systems focuses on engagement that maintains the person's interpersonal relations and identity in communities in which the person participates"

In this way, students' learning in the classroom is characterized by an actualisation of their identity through the interactions with the teacher, the books, the peers, they engage in. On the one hand, these interactions are determined by the class and school context they are situated in and as such the *social context* is constitutive for students' identities. But, on the other hand, students bring with them to the classroom the experiences of numerous other practices in other communities they have participated or are participating in. Continuously challenged to integrate them in one self, this wide spectrum of past experiences determines the specific way students find themselves in the class context and its practices, discover meaning, and renegotiate or *construct* new meanings through *their* way of engaging in the class activities.

The way students engage in classroom activities is function of the interplay between their identity and the specific classroom context. Their motivation to participate in a specific way in certain classroom activities is grounded in the way they find them"selves" in that context. However, their self, their identity, is only partially transparent to them. Who they are, what they value in this context, what they find worthwhile acting upon, is seldom known a priori, it emerges in the situation. It is through their experienced motivations and emotions that subjects recognize the value a situation bears for them. More specifically, students' emotional reactions toward mathematics are the outcome of consciously or subconsciously activated personal e*valua*tive cognitions or appraisals of mathematics, the self, and mathematics learning situations (Malmivuori, 2001). Students' beliefs about mathematics and the mathematics classroom, and especially their self-beliefs related to math (e.g., their expectancy and value beliefs) have been shown to be influential factors determining the interpretation and appraisal processes constituting their affective responses and emotions (see Mandler, 1989; McLeod, 1992). Students' mathematics-related belief system as well as students' mathematical knowledge can be identified as the central mental structures underlying students' understanding of and functioning in the mathematics classroom (see De Corte, Op 't Eynde, & Verschaffel, 2002). An *understanding* that never is only cognitive in nature but always function of cognitive-affective linkages due to the value-loaded character of some of the underlying cognitions.

Students' emotions: A situated and integrated approach

Taking into account the embeddedness of students' knowledge as well as beliefs in the social context (see e.g., de Abreu, Bishop, & Pompeu, 1997) the interpretation and appraisal processes that ground students' emotions in the classroom (e.g., anger, fear, etc.) are fundamentally constituted by the social-historical context in which they are situated. Harré (1986) points out that emotions can differ depending on the social context they are embedded in and this as well in terms of the different kinds of emotions that are experienced, as in the specific characteristics of what at first sight appear to be the same emotions. In line with Paris and Turner's (1994) characterization of situated motivation, one can claim then that every emotion is situated in its instructional context by virtue of four characteristics. First, emotions are based on students' cognitive interpretations and appraisals of specific *situations*. Second, students construct interpretations and appraisals based on the knowledge they have and the beliefs they hold, and thus they vary by factors such as age, personal history and home culture. Third, emotions are contextualised because individuals create unique appraisals of events in different situations. Fourth, emotions are unstable because situations and also the person-in-the-situation continuously develop.

There is, however, much more to emotions than the appraisal processes that determine them and their cultural situatedness. Taking seriously the accumulated findings from emotion research, what is needed is an emotion theory that explains (see Scherer, 2000):

- *both* the phenomenological distinctiveness and the intricate interweaving of cognition and emotion
- *both* the dynamic nature of emotional processes and the existence of steady states that can be labelled with discrete terms (e.g., anger, happy, proud)
- *both* the psychobiological nature of emotion and its cultural constitution

A component systems approach (Mascolo, Harkins, & Harakal, 2000) presents a promising and integrative conceptualisation of emotions that reconciles these dichotomies by bringing them to a synthesis. Three main principles are at the basis of this approach. Firstly, it emphasizes the emotion *process* characterizing emotions as an emotional episode within which appraisal-affect-action systems coact. Emotional experiences are perceived as emerging on-line in a specific context through the interactions between 5 distinct systems: (1) the cognitive system (appraisal); (2) the autonomic nervous system (affect); (3) the monitor system (affect); (4) the motor system (action); (5) the motivational system (action). The mutual feedback processes between these systems in a specific context constitute the experienced emotions explaining their dynamic nature. Secondly, a component system approach points to the non-chaotic nature of these feedback processes clarifying that emotions self-organize in real time as well as in ontogenesis. Framed by the specific socio-historical context emotional experiences tend to self-organize into a finite number of stable patterns, i.e. basic emotions. However, different patterns of component systems interactions will be produced even within each 'basic' emotional category. The sensitive dependence of emotional experiences on initial conditions account for numerous variations found within each basic category. Variations that are not trivial and, although many times labelled with the same emotional term, can refer to large differences in the organization of component systems and thus in the nature of the emotion. A final principal characterizing a component systems approach deals with the social nature of emotions. Emotional experiences are always situated in the immediate and broader social-historical context. This does not imply, however, a denial of the relevant biogenetic and organismic processes. On the contrary, socio-cultural systems always coact with biogenetic and organismic systems in every emotional experience and they all together influence an individual's emotional development.

Investigating the role of emotions in mathematics learning

Combining a socio-constructivist perspective on learning and a component systems approach of emotions to study the role of emotions in mathematics learning necessarily implies: (1) holding a conception of emotions as consisting of multiple component systems that mutually regulate each other in a specific context, i.e. the mathematics classroom, and (2) holding a conception of learning as an engagement in the practices of a specific community that maintains the person's interpersonal relations and identity in a particular socio-historical context. To our opinion, the integration of both perspectives provides a comprehensive and promising theoretical framework for the study of the role of *emotions* in classroom *learning*, involving a clear shift in the methodologies and instruments used to investigate these phenomena.

Studying the student-in-the-classroom. The situatedness of emotions or emotional experiences, and of classroom learning in general, forces research from this perspective to take place in the classroom. A study of the role of students' emotions in classroom learning has to document how students engage and reorganize their ways of participating in classroom practices and clarify the role of emotions in this process. This approach stresses intentionality and emotionality, next to intellectuality, and takes activity and meaning as its basic currency. Emotions are not treated as objects that can be studied as independent and detachable from the specific individual and context. On the contrary, emotions are perceived as an act of participating in certain practices and contexts. To study, for example, joy then implies an analysis of joyful acts as they occur in the concrete world of contexts and activities, in our case, in the context of the mathematics classroom.

Taking an actor's perspective. This focus on the meaning structure of emotional activities and of learning activities in general, implies a shift for researchers from an observer's perspective to an actor's perspective (Cobb & Bowers, 1999). What matters is not so much students' activities and the classroom environment and practices as observed by the researcher, but the meaning students (and teachers) give to it and upon which they act.

Measuring the different component systems. To grasp this dynamic interplay between the student and the class context that fundamentally determines his emotional experiences and learning behaviour in general, a variety of research methods has to be used. Interviews, observations and discourse analysis seem to be more appropriate methods for revealing the meanings that students give to situations and how they are constituted through interactions in class, than, for example, questionnaires. On-line questionnaires, experience sampling methods, video-based stimulated recall interviews, are examples of appropriate techniques in view of reaching the intended goals as far as the continuous flow of interpretation and appraisal processes is involved (see e.g., Prawatt & Anderson, 1994). However, an emotional experience is constituted by the mutual interactions between different component systems of which the appraisal system is but one. The use of facial coding systems and registration systems of physiological parameters (see e.g., DeBellis, 1996) that grasp the evolutions in respectively the action system (e.g., the motor systems) and the affective system should complement the information about the appraisal process to get a more solid and comprehensive picture of the emotional experience.

From an isolated to a multidimensional approach. Analysing the emotional dimension of students' activities in the classroom can not take place in isolation from the study of cognitive and conative processes. Although different in nature, we have shown above that there are close interactions between these processes. On the one hand, the emotional experience itself consists of multiple interactions between affective, cognitive (appraisal) and conative (motivational) processes. On the other hand, and highly related, within learning activities students' emotional experiences are intricately linked to the learning goals strived for and the cognitive and metacognitive strategies used.

A multilevel approach for a deeper understanding. The analysis of the emotional experiences of an individual student in the classroom can reveal how he continuously interprets and appraises the situation and acts upon it. A meta-level analysis of the appraisal processes and the actual learning activities can disclose some of the beliefs and knowledge underlying these emotions and actions, leading to a deeper understanding. However, to fully understand the nature of these beliefs and the consequences of the actions, an analysis of the norms and practices that characterize the classroom the student is a member of, is also necessary. One might even take it one step further and study the rules and values that are dominant in the school community and the society as a whole. A "multilevel" approach that incorporates these three planes of analysis, corresponding to personal, interpersonal, and community processes will probably result in the most complete understanding of the emotional experiences and learning activities studied (see Op 't Eynde, De Corte, & Verschaffel, 2001; Rogoff, 1995).

EMOTIONAL ORIENTATIONS AND SOMATIC MARKERS: IMPLICATIONS FOR MATHEMATICS EDUCATION

Laurinda Brown and David A Reid[2] [4th theoretical framework]

Our interest in emotional orientations and somatic markers is related to our interest in how teachers and students make decisions in mathematics classrooms.

Experienced teachers deal with situations where there are many different possible responses all the time. How do they decide what to do? In the first years of teaching there is little past experience on which to draw and student teachers report an emotional roller-coaster ride. We are concerned with finding ways of working with students so that they are not taken over by strongly negative or strongly positive emotions - becoming incapable of acting as teachers. How can they learn what to do when they do not know what to do and their actions can conflict with their beliefs? They need to develop complex decision-making strategies where there is not one simple right answer of what to do.

Students learning mathematics face similar challenges. Many come from experiences of mathematics that have led them to expect that mathematics is a safe domain of predictable rules and procedures. At some point they encounter teaching approaches and subject matter where their past experiences are insufficient, and they experience the stress of not knowing what to do. Their teachers hope that they will become capable of dealing with complex mathematics situations without being taken over by emotions that leave them unable to act. How can students of mathematics learn what to do when they do not know what to do and their actions can conflict with their beliefs? They also need to develop complex decision-making strategies where there is not one simple right answer of what to do.

Somatic markers

Damasio (1996) has studied the making of such decisions through the neurological characteristics of people who no longer seem able to make them. He has put forward the somatic marker hypothesis to explain what he has observed. The term "somatic marker" is used for the juxtaposition of image, emotion and bodily feeling we have that informs our decision-making:

> Because the feeling is about the body, I gave the phenomenon the technical term *somatic* state ("soma" is Greek for body); and because it "marks" an image, I called it a *marker*. Note again that I use *somatic* in the most general sense (that which pertains to the body) and I include both visceral and nonvisceral sensation when I refer to somatic markers (Damasio, 1996, p.173).

We would suggest that your somatic markers come into play when you judge some actions to be likely actions of a teacher and others to be unlikely. In their work on teachers' complex decision-making, Brown and Coles (2000) state:

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[2] This paper is a collaborative work. The authorship is equally shared.

> Somatic markers act to simplify the decision as to which behaviour to try. Negative somatic markers mean that the behaviours do not even come to mind as possibilities for action. A positive somatic marker means that the behaviour becomes one of a number available for use (p.168).

Somatic markers can be based on “primary emotions”, in which case we make decisions on the basis of inborn reactions. For example, one might move to avoid a snake on a path before even recognising it as a snake. The fear of snakes is part of the inborn structure of the human brain, and is “primary” in that sense. But we are also capable of feeling an emotion without the stimulus being present. If one opens a laundry hamper and discovers a snake inside, one might learn through this experience to feel fear in similar situations, whenever one encounters a laundry hamper, and so one might change ones behaviours in the future. The fear of laundry hampers is an example of what Damasio calls a “secondary emotion” triggered by the feelings we have associated with an event. A somatic marker has been created: a linkage of thought, emotion and feeling that inclines one to do, or in this case not to do, an action.

> Somatic markers are thus acquired though experience, under the control of an internal preference system and under the influence of an external set of circumstances which include not only entities and events with which the organism must interact, but also social conventions and ethical rules (Damasio 1996 p. 179).

We believe that Damasio’s notion of ‘somatic markers’ helps us to describe the development of teachers and students engaged in mathematical activity in classrooms. You have a constellation of “teacherly” somatic markers that are active in teaching situations and a constellation of “mathematical” somatic markers that are active in mathematical situations. While another teacher or another mathematician would make different decisions than you would, at the same time you can recognise similarities in the choices your somatic markers would guide you towards.

Emotional Orientations

What we have called a constellation of somatic markers can be seen as what Maturana (1988a) would call by the name "emotional orientation". An emotional orientation is what characterises someone's actions as appropriate to a context, like teaching. Maturana would call teaching a "domain of explanation", characterised by a community whose members can recognise in others behaviours appropriate to the community, although they probably could not give specific criteria for doing so. Many communities are like this, and communities can overlap, contain other communities, or subtly blend into other communities according to the behaviours different members accept as legitimate. For example, algebra is a domain of explanation, just as teaching is:

> …if someone claims to know algebra, that is, to be an algebraist, we demand of him or her to perform in the domain of what we consider algebra to be, and if according to us she or he performs adequately in that domain, we accept the claim. (Maturana, 1988b, p. 3)

In responding to a student's question a teacher can act in many ways, but as a teacher you might recognise that not all possible responses are appropriate coming from a teacher. As we noted before, a teacher's choice must be based on something other than conscious reflection as there is no time for conscious reflection here. Instead something like a somatic markers, or a constellation of somatic markers, must be at work, and in the larger context of all her or his teaching the somatic markers that guide a teacher implicitly define a preference for certain behaviour s/he, and we, would see as appropriate for a teacher.

> …whether an observer operates in one domain of explanations or in another depends on his or her preference (emotion of acceptance) for the basic premises that constitute the domain in which he or she operates. Accordingly, games, science, religions, political doctrines, philosophical systems, and ideologies in general are different domains of operational coherences in the praxis of living of the observer that he or she lives as different domains of explanations or as different domains of actions (and therefore of cognition), according to his or her operational preferences. (1988a, pp. 33-34)

Maturana suggests the phrase "emotional orientation" to name the set of criteria (which we read as somatic markers) appropriate to a domain of explanations. He uses "emotional" because it is a bodily predisposition rather than conscious reflection that is operating when we perceive some behaviours as appropriate and others as not.

As different kinds of explanations are appropriate to different domains of explanation we have different emotional orientations appropriate to each domain. We observe the actions of others as being appropriate to a particular domain according to our own emotional orientation for that domain. So a mathematical emotional orientation allows us to recognise the activity of others as being mathematical, and hence to identify those people as mathematicians. A teacherly emotional orientation allows us to recognise the activity of others as being teacherly, and hence to identify those people as teachers. We have many emotional orientations as we belong to many communities, characterised by different (probably overlapping) constellations of somatic markers.

Looking ahead

Our current collaborative research looks at the ways in which somatic markers influence teacher decision-making and students' reasoning, and the degree to which those markers can be observed by us, by colleagues, and perhaps by the teachers and students involved. Because somatic markers are a part of unconscious mental activity they cannot be observed by introspective reflection. In fact, the stories we tell after the fact about our decision making are likely to include inventions to account for the influence of somatic markers of which we are not aware. How then can we research something we cannot observe? The process described above, of examining decision points in a person's actions, seems to hold promise. We can observe changes in behaviour, indicative of unconscious decision making, and consider what markers based on past experience might account for those decisions. Our work with colleagues has indicated that mathematics educators see similar events as suggesting the sort of unconscious decision-making accounted for by Damasio's hypothesis of somatic markers. This leaves us optimistic that it will be possible in our work to observe the effects of somatic markers in a range of contexts, to distinguish positive and negative somatic markers, and to suggest ways in which they form and evolve in mathematics classrooms.

MATHEMATICAL THINKING, VALUES AND THEORETICAL FRAMEWORK

Shlomo Vinner [1st evaluative perspective]

I would like to use my role as a reactor to reflect about certain tendencies in the community of PME. These tendencies are general. Since they are general, they might be relevant also to specific cases. However, I am trying not to relate to specific cases here. I believe that part of a reactor's role in forums like this is to relate to broad and essential aspects of the topic under consideration. So essential that it can be compared with the essential aspects raised by the question "to be or not to be." However, the question is not "to be or not to be." The question is what to be? It is related to the identity and character of the PME community. In fact, Nicolas Balacheff (1996), a former president of PME, already raised this question when he called PME members to question the aims and directions of their activities as PME members.

The way I understand the history of PME (and I am not a historian), it really started with mathematical thinking. We saw our role to describe and to explain mathematical thinking. We did not use the name "Psychology of Mathematical Thinking" because we believed that the name "Psychology of Mathematics Education" has a broader scope. By doing this we opened the door to all kinds of issues related to education. An important issue related to education is the issue of values. In the context of values, questions about the educational goals of learning mathematics, about its merit and about its contribution to the moral development of the students could have been discussed. The overall impression is that it almost did not happen. The reason for it can be the fact that we consider ourselves as experimental researchers and discussions about values are not within the domain of experimental research. We took for granted the current situation in which mathematics is taught to certain extent to everybody. For instance, in this forum we discuss students with special needs. It is really a thoughtful gesture. These people have the rights to have normal and worthy life. However, we do not ask why they should study mathematics and to what extent. So, we did not enter the door that we opened for ourselves by choosing the name "Psychology of Mathematics Education." We remained in the domain of mathematical thinking.

But here we discovered that mathematical thinking is determined not only by purely cognitive factors. It is influenced by emotional factors of all kinds, as well as by social and cultural factors. Therefore, if we really want to describe and explain mathematical thinking we should relate also to these factors. Some of them are included in what we call affect in mathematics education. Thus, for instance, if we have to analyse the mathematical thinking of a student who solves a given problem on given a test, we should consider his or her emotional state. Under emotional pressure people might have difficulties to form certain thought processes. It is important to characterize thought processes which people can perform under pressure and others which usually are not expected to occur under pressure. It is reasonable to assume that algorithmic thought processes can be produced under pressure while heuristic processes, in the majority of people, hardly occur under pressure.

By saying that I have made a claim which has a general character. It explains and it can predict. Is it a theory? A grounded theory? A theoretical framework? It does not look like a theory. I will not discuss here the question what is a theory. However, I would like to ask what makes a set of claims to look like a theory. It seems to me that one of its features is terminology - special terms, technical notion, big words used in ways different from the way they are used in ordinary language. Why do we need such theories? Take physics, for instance. The majority of people have an intuitive theory about stable balance of physical bodies. We know that if we tilt a chair it might fall down. But when we introduce the center of mass and its laws as theoretical constructs we will be able to say much more. We will be able to explain and predict many events that we were not able to handle earlier. However, if a theory in the above sense does not explain or predict more than what we know intuitively, then it is quite superfluous. It serves mainly the researchers who invented it and others who develop it at the theoretical level to the extent of giant dimensions, but fail to tie it again to the real world. Such a development is mentioned already by Vygotsky (1927) when he discussed the crisis in Psychology. It seems to be an ongoing crisis.

The need to explain phenomena and events is perhaps imprinted in us by evolution. It gives us an evolutionary advantage. It helps us to survive dangers and disasters. From here, the distance to theory production is very small. However, theories of the above type are not crucial to the survival of mankind. They might be crucial for the survival of some university professors. It is acceptable, let us say, when we consider physicists. But is it acceptable in a group of people who consider themselves Mathematics educators?

SPECIAL STUDENTS FEELING MATHEMATICS

Melissa Rodd [2nd evaluative perspective]

For our purposes, here, students with ‘special needs’ are children who are developing differently from typical children in any way such that adaptations have to be made so that they are able to access the standard curriculum. Children who have sense impairments (e.g. deafness), medical, mobility and developmental conditions (e.g. cystic fibrosis, cerebral palsy, autism, respectively) and children who have suffered extreme adverse social conditions (e.g. abuse) are examples of students with ‘special needs’. Clearly, these students are close to the heart of the conference theme of *Inclusion and Diversity* and their affective responses to mathematics are central to their participation in mathematical activity.

While outlining distinct perspectives, our four theoretical frameworks are linked in several ways – for example, a somatic marker is a way of representing emotion in the body; self-regulation, as a dynamic process, is very strongly influenced by social norms and practices, indeed it is central to developing identity, which is most relevant in the consideration of children with special needs. The subconscious aspects of affect are noted and the question of managing emotion is particularly relevant for children with slow emotional development. All the theories are in some way trying to grapple with the relationship between learning and feeling, understanding that you cannot separate these two aspects of life; cognition and affect are integral parts of, in Damasio's words, "the feeling brain".

Students with special needs are disproportionately emotionally vulnerable in the rough-edged social world of school. Because they are, by definition, at the margins of the assumed normal distribution for some attribute, other children notice difference and may test out their own position in the social order by teasing or bossing the special needs child. And what starts in the playground seeps into the classroom. The social context of the classroom, as Op 't Eynde's theoretical framework emphasises, is central to a positive feeling about mathematics. The nature of interactions with the teacher develop a self-image of being a mathematical person. In the UK, at least, where most children with special needs are educated within the mainstream, special needs students frequently have a non-mathematician 'teaching assistant' to help them. While this arrangement facilitates access, the presence of the other adult dilutes the relationship between the mathematics teacher and the special needs student and thus the intensity of inspiration from the mathematics teacher is diminished, reducing the possibility of a neophyte relationship and a burgeoning mathematical identity.

Malmivuori's theoretical framework is particularly relevant when teaching students with challenging behaviour, as it starts from the individual, while incorporating essential social or contextual features. These students' self-regulatory systems are impaired relative to the norm of the mainstream mathematics classroom. These students get emotionally flooded very easily, a very small environmental impact, can arouse the student beyond their self-control. The frequent result is that the teacher takes firmer and firmer control, thus preventing the development of the crucial self-regulation that other children develop more easily. Indeed, in the context of mathematics, it is important that these students do experience challenge, both in mathematics and in the social space of the classroom. Pages of sums to do in silence at separate desks may be a teacher's solution to quell the difficult behaviour. Yet, a curriculum involving, for example, small groups playing maths games, should develop their interest, self-esteem and positive attitude to the mathematics classroom, which, in turn may help to improve their delayed emotional self-regulation.

Frustration is commonly experienced by students with special needs and Gerald Goldin's framework can be used to explain why their problem-solving capability may be limited and so how this frustration could have arisen: they may not be motivated by *values* that direct towards doing well in school, nor may they have the *belief* that maths is not for them, their *attitude* towards self-improvement may be wanting, and, indeed, their *emotions* may be intense and difficult to control, making successful mathematical progress less likely. The complement to student frustration is student satisfaction and this framework gives a way of connecting different aspects of affect which impinge on student frustration or satisfaction.

The neurologically-based theory of somatic markers is also helpful in understanding learners' responses and also in recognising that mere telling students not to panic, for example, has very little effect! The framework outlined by Brown and Reid explains how a critical mass of somatic markers can lead to a charged emotional orientation towards mathematics. Furthermore they implicate the unconscious both in learners' attitudes and their competencies. It shows us that teaching involves working to re-position students' somatic markers. It also shows that all fine words about awareness and self-regulation are challenged by learners' sub-conscious embodied orientations.

From the perspective of championing special needs students, these frameworks don't yet incorporate an explanation of how learning styles have an affective dimension. And teaching students with specific needs demands acknowledgement of their specific learning styles otherwise frustration and possibly anger or panic arise. So while there are obvious teaching methods for sense-impaired students (e.g. high use of visuals for the hearing-impaired), other tacks are required for other special needs in order to give them every chance to succeed. Examples: students on the autistic spectrum, who are impaired in their grasp of social situations, may be more comfortable accessing mathematics via pattern and logic rather than via a (social) context; attention-deficit hyper-active students respond to kinaesthetic activities, as their need to move can be channelled into embodying mathematical relationships.

Education in the training of the emotional mind is another issue to consider: techniques for meditation and cultivation of positive attitudes have existed for millennia. The one-pointed thinking experienced in mathematical concentration is a means by which the over-easily aroused emotional mind may be soothed; routinisation of mathematical concepts helps fluency and mathematical intimacy as well as building self-esteem.

Affective issues are clearly central to devising effective teaching methods for children with special needs. And in attending to learners with urgent and distinct needs we may well find that our raised awareness of individuality and of culture will provoke better learning environments for all students.

METHODOLOGICAL QUESTIONS IN RESEARCHING AFFECT

Jeff Evans [3rd evaluative perspective]

Here I consider the four theoretical presentations from the point of view of a set of methodological questions:

1. What is the role of theory in the study of affect in the particular approach to mathematics education research? What are the objects of study in this approach?

2. To what extent is affect understood as a social (rather than simply individual) experience? How do we need to take account in research of the social context of experiencing emotion?
3. What sorts of research questions are generated by a particular approach to affect?
4. What research strategies are preferred in this approach, and what sorts of data are appropriate? What methods of operationalising key concepts are preferred?

The exponents of all four theoretical approaches indicate their aim to establish a basis for *description* for the affective area, and all aim to *explain* the relationship of the affective to mathematical thinking and problem solving. Brown and Reid are the most explicit about wanting to provide a basis for *intervening* to help 'students of mathematics learn what to do when they do not know what to do', though all of the other contributions also mention or imply interests in students' development / change.

Goldin analyses the affective as one of several 'mutually-interacting' *systems of representation* that 'encode essential information' He presents four components of the affective domain – emotion, attitudes, beliefs and values – as a tetrahedron, thereby resisting the tendency (e.g. in McLeod, 1992) to rank them as to ‘intensity’ or ‘stability’ over time – although Goldin's *local / global* dimension resembles the latter.

Malmivuori's basic framework seems highly compatible with Goldin's; here, her ongoing, interacting processes relate to *self-systems*. Her main objects of study are students' *self-perceptions*, and their related *self-affects*, the latter connected with 'experiences of self-esteem, self-worth and /or personal control with respect to mathematics'. Malmivuori's *self-systems* are *self-regulating*, from which flows at least part of their 'dynamic' quality; put another way, *metacognition* is central to self-regulation. Goldin takes this one step further, by emphasising the importance of *meta-affect*, the 'monitoring of affect both through cognition and affect'.

Op ‘t Eynde presents two levels of analysis of affect: a *component-system approach*, which is also explicitly *situated* in the social-historical context. This framework aims to explain a number of key issues: the interweaving of cognition and emotion, along with their apparent distinctiveness; the dynamic nature of emotional processes, and the existence of 'steady states that can be labelled (e.g. anger, happiness, pride)’; and the physiological nature of emotion and its sociocultural constitution.

Brown and Reid bring together the concepts of *somatic markers* (Damasio) and *emotional orientations* (Maturana), that can be considered as constellations of somatic markers: the latter 'characterises someone's actions as appropriate to a context, like teaching [or doing mathematics]'. Teaching can be considered a *domain*, which is characterised by a *community*. This linking of emotional orientations to communities allows bringing the social into their analysis of emotions.

The other exponents also emphasise the individual-social relationship. Malmivuori points to the 'individual - environmental interaction', and its effects on students' self-appraisals and on self-affects. Goldin emphasises the role of affect as a *language for* *communication* among human beings; here, he refers presumably to physiological and behavioural aspects of the emotion component of affect. This idea also has reverberations with the claim that emotional expression functions as 'sign(s) in the network of social relations' (Burkitt, 1997, p.45, quoted in Evans, 2000, p.113). Op ‘t Eynde's bringing together of 'psychobiological' and cultural addresses the same issue.

The sorts of research questions proposed here include:

- How to understand the persistence of belief systems that are counter-productive to the goals of mathematics education? (Goldin)
- How to understand students' personal involvement and self-regulatory processes with their affect? (Malmivuori)
- How can one research the occurrence of somatic markers? Can one distinguish positive and negative somatic markers? How do somatic markers influence teacher decision-making and students' reasoning processes? How are somatic markers themselves formed? (Brown & Reid)
- How to develop a theory of emotion that addresses the three 'key issues' above? How does an individual student in a classroom continuously interpret and appraise the situation, and act upon it? Does this allow the uncovering of the beliefs and knowledge underlying these emotions and actions? And how are the rules and values that are dominant in the school community and society as a whole implicated? (Op ‘t Eynde).

Because of the emphasis on a theoretical focus and the space constraints, there is little discussion on other methodological and 'methods' issues. Op ‘t Eynde's response to the problem of how to take account in research of the social level emphasises studying the student in the classroom, and 'taking the actor's perspective', both ethnographic approaches. His suggestions for 'revealing the meanings that students give to situations and how they are constituted through interactions in class' are interviews, observations, and discourse analysis; for grasping 'the continuous flow of interpretation and appraisal processes', he recommends on-line questionnaires, 'experience sampling methods', and video-based simulated recall interviews.

There may be scope for making more explicit the overlaps and communalities in the approaches described. It may be fruitful to compare in more detail the 'systems' of Goldin, Op ‘t Eynde, and Malmivuori. For example, we could ask how Goldin's 'mathematical self-identity' relates to Malmivuori's self-affect; how his 'mathematical intimacy' relates to Op ‘t Eynde's 'engagement' in a mathematical community; how his 'mathematical integrity' relates to Cobb et al.'s (1989) 'socio-mathematical norms'.

The absence of reference to psychoanalytic perspectives is noticeable in the chosen frameworks. True, most, if not all, of the approaches mention 'unconscious' processes, but this is generally used more in the sense of 'non-conscious'. The test is whether the idea or memory has been repressed into the Unconscious, or just 'forgotten' in the subconscious; in the former case, defence mechanisms may become apparent, when certain topics are raised (Evans, 2000, pp.140-145).

SOME CLOSING (OPENING) REMARKS

George Philippou & Rosetta Zan

The effort to encompass in a single paper multiple theoretical frameworks for affect in mathematical education constitutes in itself a gigantic task. Even so, what appears in the above short presentations seems to cover the domain reasonably well. The four presenters have summarized, each from a different perspective, the most recent developments in the field; the reactions summarize how these theoretical frameworks could serve the needs of special students, and provide an appraisal of the methodological questions involved.

One is impressed by the evolution of research in affect in mathematics education. Looking back to the long history of studies in this area, it can be useful to underline the renewal of interest that in the 80's was stimulated by problem solving research. This new trend is represented by the pioneer book 'Affect and mathematical problem solving' (Adams & McLeod, 1989), in which several papers contain words such as emotions, beliefs, and attitudes in the real context of a mathematical activity. These constructs were used to better *interpret* students' mathematical behaviour that a purely cognitive approach was not capable to *explain*.

This need of interpretation instead of explanation is strictly linked to the shift from a normative (positivist) paradigm to an interpretative one, which seems necessary if we want to take into account the complexity of human behaviour, and the fact that human beings act intentionally. According to the interpretative paradigm researchers search for understanding students' intentional actions in the context of mathematical activities, and not for explaining behaviour with general rules based on a cause-effect approach. But how can we understand and interpret human actions without considering affect?

The most recent research in the field of affect in mathematics education aims at developing theoretical frameworks, also in order to increase the coherence between observing instruments and the theory itself: the presented contributions perfectly reflect this aim. As a way to open the stage for the discussion to follow, we would like to point out some unifying elements in the presentations, out of the many that the reader can easily locate and propose to discuss them.

First, one gets the impression of a more or less a consensus among the contributors about the ingredients (components) of "affect" (emotions, attitudes, beliefs and values) as well as concerning the meaning of each variable. On first sight, accepting the "tetrahedral model" refutes the view that there is "considerable diversity in the theoretical frameworks" (Hannula) and the statement of the absence of "a precise, shared language" (Goldin). Even though the ubiquity is only resolved at rather general level, one could consider this as a departure point toward more precise and operational definitions of the constructs, to analyse specifically where and why variable conceptualisations stem.

Of the other important unifying elements of the presentations that might be amplified, we would specify the following: Goldin redefines the relationship among cognition and affect; he views affect as one of several internal representation systems that "functions symbolically so as to encode essential information" that is often "complex self-referential information". How does this perspective relate to Op 't Eynde's demand to clear up "the distinctiveness and the intricate interweaving of cognition and emotion"?

Goldin has also drawn attention on affect as a means of communication; how does this relate to Op 't Eynde's socio-constructivist model, in which meaning is constructed through one's engagement in a social setting? Further, Goldin elaborates on meta-affect and one might wonder how does this construct relate to metacognition, motivation, and self-regulated learning that are extensively discussed and analysed by Malmivuori.

As Goldin describes the "messages" conveyed by eye contact, facial expressions, gestures, voice intonation, etc., that are mediated tacitly among individuals, as useful in understanding students affective state. We wonder again how all these signals concern direct implication of somatic markers in mathematics education. We are sure that Brown & Reid, the proponents of the somatic framework, would like to elaborate further in July. Some specific examples on their part might probably make the difference.

Though self-system processes have been broadly discussed by Malmivuori, within the individuals-environmental interaction, we are of the opinion that some mention of specific obstacles and limitations in pursuing research on self-esteem self-concept and particularly on self-efficacy construct (see e.g. Bandura, 1997) might be warranted. In future discussions we would expect some elaboration on the meaning and function of motivation, which has been mentioned in passive in preceding pages, and certainly the connection of this construct to other affective variables.

Discussing about theoretical frameworks for affect is necessary if we want to improve the quality of our research. But it is also necessary not to forget the very nature of our interest in affect as researchers in mathematics education. Vinner's reaction points out the risk of having theories that don't help to *explain* phenomena, or, using an interpretative approach, to *understand* individuals' intentional actions. So the question is: How do these frameworks help us in interpreting mathematical behaviour? We will face this question in the discussion, proposing the four presenters the same episode to analyse, in order to compare their theories in practice.

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RF02: ALGEBRAIC EQUATIONS AND INEQUALITIES: ISSUES FOR RESEARCH AND TEACHING

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Algebraic equations and inequalities play an important role in various mathematical topics including algebra, trigonometry, linear programming and calculus (e.g., Hardy, Littlewood & Pólya, 1934/1997). Accordingly, various documents, such as the U.S. NCTM Standards, specify that all students in Grades 9-12 should learn to represent situations that involve equations and inequalities, and that they should understand the meaning of equivalent forms of expressions, equations and inequalities and solve them fluently (NCTM, 1989; 2000). To implement these recommendations it is crucial to analyze students' ways of thinking about equations and inequalities when designing instruction and in teaching.

Indeed, in the last decade there has been growing interest in the learning and teaching of algebraic equations and inequalities. Discussions regarding related issues have been conducted, for instance, in PME 22 Discussion Group meetings (1998) and continued during PME 23 Project Group sessions (1999). There was a consensus among 1999 PG participants that the meetings of the group should be temporarily postponed, while calling on researchers to invest more efforts in various facets of algebraic reasoning related to the solution of equations and inequalities. Among the benefits of the 1998-1999 discussions were the selection of key research questions and the initiation of collaborative research teams that since then have been working in this area. The Research Forum at PME 28 provides an opportunity for presenting some fruits of the research that has been conducted since then, for discussing theoretical frameworks for data analysis, and for examining the different educational implications that were suggested and tried. Indeed, in their presentations, all the invited speakers of this RF make due reference to previous related PME publications (e.g., Boero, Bazzini, & Garuti, 2001; Duval, 2000; Garuti, Bazzini, & Boero, 2001; Linchevski & Sfard, 1991; Nunez, 2000; Radford, 2002; Tsamir, & Bazzini, 2001).

The theme of PME28, *Inclusion and Diversity,* is well reflected in this research forum. *Diversity* may be found in the different lines of research, different theories to account for the findings and the educational implications that have been put forward by the researchers. For example, among the theoretical frameworks mentioned here to analyze students' solutions are the Vygotskian model and Nunez's grounding metaphors, in Boero and Bazzini – [BB], Duval's theory on semiotic registers and Frege's theory of denotation, in Sackur – [S], and Fischbein's model, in Tsamir, Tirosh and Tiano – [TTT]. Kieran [K] offers three categories for analyzing algebraic activities: generational, transformational, and global meta-level.

The presentations address a variety of difficulties occurring in students' solutions of equations and inequalities, and suggest different reasons for these difficulties. When analyzing students' performances, [BB] and [TTT] mention students' tendencies to make irrelevant connections between equations and inequalities as a problematic phenomenon. It should be noted, however, that [K] presents connections made between equations and inequalities as an important step in solving algebraic problems by means of non-algebraic methods. [BB] mention traditional, algorithmic teaching approaches as a main reason for students' errors, Dreyfus and Hoch [DH] mention the need to enhance the internal structure of equations that students hold, while [S] carefully analyzes difficulties with reference to the various solving methods and indicates that even the functional approach and the use of graphic calculators do not automatically lead to errorless solutions.

However, beyond their differences, the presentations share common goals. One such goal is to investigate ways to promote performance on algebraic equations and inequalities by seeking means for analyzing students' reactions to various representations of equations and inequalities in different contexts, while considering the way this topic was taught. Thus, this forum will also shed light on the more general issues concerning the interplay between theory, research and instruction.

The two reactors intend to react on all the papers and make concluding statements, but their review is made from different perspectives.

Further discussion will address a number of key questions, like:

- What are students' conceptions of equations / inequalities? What is typical correct and incorrect reasoning? What are common errors?
- What are possible sources of students' incorrect solutions?
- What theoretical frameworks could be used for analyzing students' reasoning about algebraic equations / inequalities?
- What is the role of the teacher, the context, different modes of representation, and technology in promoting students' understanding?
- What are promising ways to teach the topics of equations / inequalities? What curricular innovations can we suggest?
- Is there a global theory that may encompass the local theory of equations and inequalities?

The discussion of such issues could give further support to research and teaching. During the sessions at the conference, each of the presenters is allotted only ten minutes to present the central points of his / her ideas, and each of the reactors is invited to react on all presentations during a fifteen minutes presentation. Most of the two sessions are dedicated to the participants' work in small groups, and to whole RF discussions.

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INEQUALITIES IN MATHEMATICS EDUCATION: THE NEED FOR COMPLEMENTARY PERSPECTIVES

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1. Introduction

This contribution deals with inequalities: an important subject from the mathematical point of view; a difficult subject for students; a subject scarcely considered till now by researchers in mathematics education. Our working hypothesis is that *different tools belonging to different disciplines (cognitive sciences, didactics of mathematics,* *epistemology of mathematics) related to mathematics education are needed to interpret difficulties met by students and plan and analyse teaching experiments intended to cast new light on this subject*. Data coming from some teaching experiments conceived in this perspective and other experimental investigations will be used to support our working hypothesis. In particular, we will present the guidelines and some results of a research program conceived according to the above perspective and concerning the approach to inequalities in 8^{th} - grade. We will show how a functional approach to inequalities (i. e. an approach based on the comparison of functions and suggested by the *didactical, epistemological* and *cognitive analyses* of the subject), when suitably managed by the teacher, can reveal (from the research point of view) and allow to exploit (from the curriculum design point of view) a students' potential which goes far beyond the mathematics content involved (inequalities). We will use the *Vygotskian perspective* and the *"didactical contract" construct* to frame the teacher's role in the classroom and analyse the teacher-students relationships. Amongst the cognitive tools, in particular we will use the *"grounding metaphor" construct* (Nunez, 2000) to analyse some aspects of the students' behaviour and open the problem of how to enhance students' use of those metaphors in this mathematical domain.

2. Inequalities: a challenge for teaching.

In most countries, inequalities are taught in secondary school as a subordinate subject (in relationship with equations), dealt with in a purely algorithmic manner that avoids, in particular, the difficulties inherent in the concept of function. This approach implies a "trivialisation" of the subject, resulting in a sequence of routine procedures, which are not easy for students to understand, interpret and control. As a consequence of this approach, students are unable to manage inequalities which do not fit the learned schemas. For instance, according to different independent studies (Boero et al, 2000; Malara, 2000), at the entrance of the university mathematics courses in Italy most students fail in solving easy inequalities like $x^2-1/x>0$. In general, graphic heuristics are not exploited and algebraic transformations are performed without taking care of the constraints deriving from the fact that the > sign does not behave like the = sign (Tsamir et al., 1998). Similar phenomena were described in some studies concerning the French situation (Assude, 2000; Sackur and Maurel, 2000).

We may ask ourselves what are the reasons of this situation. In a *didactical-anthropological perspective* (Chevallard, 1987), one reason could be the fact that equations (and inequalities) are considered (in most of European countries, including Italy) as a typical content of school Algebra; this subject matter is distinguished from Analytic Geometry and does not include functions. This might explain why inequalities (and equations) are not dealt with in those countries from a functional point of view. But even in countries where functions (and Analytic Geometry) belong to school Algebra (see NCTM Standards, 1989 and 2000) the procedural, algebraic approach prevails in many curricula and even in innovative proposals (Dobbs and Peterson, 1991). So the *didactical-anthropological analysis* must be refined and integrated with an *epistemological analysis*: we must consider the big distance between the subject as a school subject, and the mathematicians' professional approach to the subject. Indeed the functional aspect of inequalities plays a crucial role when mathematicians solve equations with approximation methods, deal with the concept of limit or treat applied mathematical problems involving asymptotic stability. We can make the hypothesis that an alternative approach to inequalities based on the concept of function could provide an opportunity to promote the learning process of the difficult concepts involved and the development of the inherent skills (see Harel and Dubinsky, 1992 for a survey). It could also ensure an high level of control of the solution processes of equations and inequalities (Sackur and Maurel, 2000; Yerushalmy and Gilead,1997).

3. The teaching experiments

Keeping the previous analysis into account we have planned two teaching experiments at the VIII-grade level with rather limited aims: investigating the feasibility of an early functional approach to inequalities; and revealing students' potential and difficulties in dealing with this subject as a special case of comparison of functions. According to a Vygotskian perspective, we choose to guide our VIII-grade students in a cooperative, gradual enrichment of tools and skills inherent in the functional treatment of inequalities. Then we have analysed how (in relatively complex tasks) they had been able to use their knowledge and increase their experience in an autonomous way.

As concerns the content, the concepts of function and variable have been approached through activities involving tables, graphs and formulas. According to existing cognitive and epistemological analyses, at the beginning the function was presented as a machine transforming x-values into y-values (machine view in Slavit, 1997), then classroom activities focused on the variation of y as depending on the variation of x (covariance view). By this way a dynamic idea of function gradually prevailed on the static consideration of a set of corresponding pairs (correspondence view). As a consequence, a peculiar aspect of the concept of variable was put into evidence (a variable as a "running variable", i.e. a movement on a set of numbers represented on a straight line) (Ursini and Trigueros, 1997). Finally, the approach to inequalities was realised by comparing functions.

The specific didactical contract demanded to compare functions as global, dynamic entities. Students knew that they had to compare functions by making hypotheses based on the analysis of their formulas. The point-by-point construction of graphs was discouraged. As a consequence, the ordinary table of x, y values was sometimes exploited as a tool to analyse how y changed when x changed (column-vertical analysis) and not as a tool to read the line-horizontal point-by-point correspondence between x-values and y-values. The algebraic and the graphical settings were strictly related (formulas were read in terms of shapes in the (x,y) plane, while graphs evoked formulas). The teachers promoted classroom discussions about "what do we loose and what do we earn" when a function is represented through formulas or graphs or tables or common language. Also different ways of describing given functions have been enhanced (see Duval, 1995: coordination of different linguistic registers). They became personal tools exploited and to compare functions. Even the metaphors used by students to describe the role of different pieces of the same formula have been encouraged and discussed.

4. Some remarks and questions.

First of all, some remarks on the role of metaphors are worth noticing. Since the beginning of the eighties metaphors have been reconsidered as crucial components of thinking. Nunez (2000) describes *conceptual metaphors* as follows: *“Conceptual metaphors are in fact fundamental cognitive mechanisms (technically, they are inference-preserving cross-domain mappings) which project the inferential structure of a source domain onto a target domain, allowing the use of effortless species-specific body-based inference to structure abstract inference".* Considering *conceptual metaphors*, Lakoff and Nunez (2000) (see also Nunez, 2000) make a distinction between *grounding metaphors* (i. e. conceptual metaphors which *"ground our understanding of mathematical ideas in terms of everyday experience"*) and other kinds of conceptual metaphors (*Redefinitional metaphors, Linking metaphors*).

Concerning *grounding metaphors*, our research study aims to show how different kinds of *grounding metaphors* can intervene (as crucial tools of thinking) in novices' approach to inequalities and to discuss possible refinements of the idea of a *grounding metaphor,* deriving from the analysis of students' behaviour and related to the cultural variety of everyday life source domains. Finally we aim to investigate how *grounding metaphors* can become a legitimate tool of thinking for students.

In particular, it would be interesting to discuss the following questions:

I) what theories and what tools do offer the best opportunities to interpret students' behaviors when they deal with inequalities?
II) can the study of teaching and learning inequalities be reduced to the study of teaching and learning functions?

Some research findings have been already presented in Boero & al., (2001); and Garuti & al, (2001). Further results related to on-going research will be discussed.

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THE EQUATION / INEQUALITY CONNECTION IN CONSTRUCTING MEANING FOR INEQUALITY SITUATIONS

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Recent algebra learning research has included a focus on students’ understanding of and approaches to inequalities. For example, Bazzini & Tsamir (2001) have researched 16- and 17-year-old students’ ways of thinking when solving various types of algebraic inequalities. Bazzini, Boero, and Garuti (2001) have studied the feasibility of a functional approach in the teaching of inequalities to eighth grade students. Tsamir, Almog, and Tirosh (1998) have observed high school math majors’ methods for solving equations and inequalities and have noted that the most common were algebraic manipulations, drawing an graph, and using the number line. This body of research has advanced the field with respect to our knowledge of students' conceptions of inequalities in several ways. It has pointed out, for example, the positive role that graphical representations can play in helping students to better conceptualize the symbolic form of inequalities, as well as the pitfalls involved in attempting to apply to the solving of inequalities some of the transformational techniques used with equations. Despite its foray into graphical representations, this same body of research has been quite narrow in emphasis with its almost exclusive focus on the manipulative/symbolic aspects of inequalities.

Theoretical Framework

By means of a model recently developed and presented at the ICME-8 conference in Sevilla (Kieran, 1996), algebra can be viewed according to three main categories of activity: generational, transformational, and global meta-level. For the case of inequalities, meaning for the symbolic form is often derived via the global meta-level activity of contextualized problem solving, which activity tends to then be harnessed to generate the symbolic form of inequalities. However, these two types of activity seem absent from the current research on inequalities. Because the students involved in those studies are often older secondary level students, we presume that they have already constructed meaning for the symbolic form of inequalities; nevertheless, the research remains relatively silent on this issue.

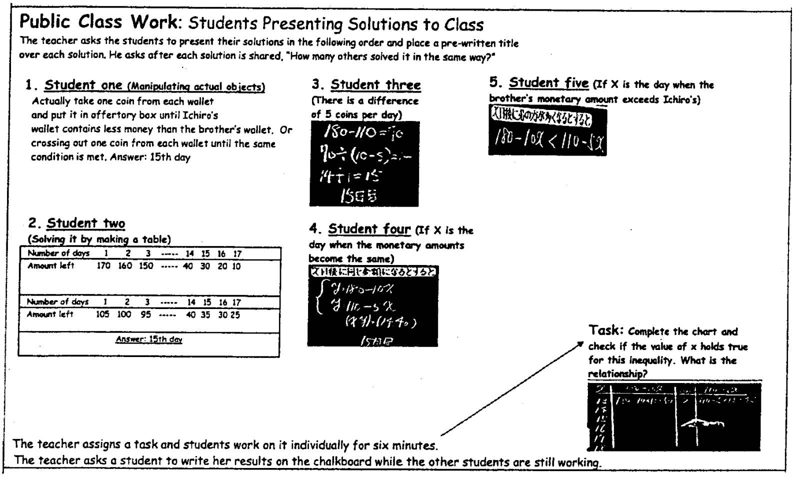
Data Source

The goal of this contribution to the PME Research Forum is to present a brief analysis of a classroom sequence that aimed at introducing inequalities. The data are drawn from the TIMSS-R 1999 video study of 8th grade mathematics teaching in algebra classes around the world (Hiebert et al., 2003). The lesson, which was the first of a set of seven such lessons, involved a Japanese class where the teacher used a specific problem situation to create meaning for mathematical inequalities and for their algebraic form (www.intel.com/education/math). In this analysis, both the global meta-level activity of problem solving and the accompanying activity of generating an inequality are interwoven as we witness the teacher orchestrating both his overall aims for the lesson and particular students' approaches to the solving of the problem situation, which was as follows:

It has been one month since Ichiro's mother entered the hospital. He has decided to give a prayer with his small brother at a local temple every morning so that she will soon be well. There are 18 ten-yen coins in Ichiro's wallet and just 22 five-yen coins in the younger brother's wallet. They decided to place one coin from each of them in the offertory box each morning and continue the payer until either wallet becomes empty. One day after prayer, they looked into their wallets and found the younger brother's amount was bigger than Ichiro's. How many days since they started prayer? (translated version)

Brief Analysis of Student Work

After the teacher spent a few minutes clarifying the problem situation to the class, students began to work individually on the problem. The teacher circulated, taking note of the various solution methods being worked and encouraging students to try more than one method. After some time had passed, the teacher asked certain students to present their solutions at the blackboard in the following order (see figure below):



The above methods that were used to solve the "inequality" problem situation show that the majority of the students who were invited to the board approached this problem as an equality situation, which enabled them--by means of a slight adaptation of the solution to the equality--to provide the solution for the inequality. They used the following language to express this idea:

<u>Student three</u>: "Well, in the beginning, Ichiro had 180 yen, and the smaller brother had 110 yen. And since there is a difference of 70 yen, and since the difference between them becomes smaller by five yen each day, so it's 70 divided by 10 minus 5. And since by the fourteenth day it becomes exactly the same amount of money, so since on the day after that there will be a difference, so 14 plus one is 15 and it's the fifteenth day."

<u>Student four</u>: "On the fourteenth day they become the same amount of money. And the next day since Ichiro puts in 10 yen and the smaller brother puts in 5 yen …. the amount of money put in is bigger for Ichiro. So the next day Ichiro's amount of money left is less so it becomes the fifteenth day."

The student work (Student five) that involved the symbolic form of an inequality right from the start served as a tool for the teacher to introduce to the rest of the class both the inequality symbol and an expression containing this symbol:

S: (wrote on the blackboard: 180 – 10x < 110 – 5x). The one labeled x is that one day after they finished their prayers. On that day since the smaller brother's amount of money was bigger than his brother … and the person who had 110 yen is the amount of money the smaller brother had in the beginning.

T: So these kinds of expressions … the fact is these are the ones we're going to use from now on. Equations that use symbols like this … umm, we'll call them inequalities. … So today I think we would like to find the value for x that holds true for this mathematical expression while actually putting in numbers.

The teacher then asked students to complete a table (see figure above). When the table displayed on the blackboard was filled in, the teacher noted: "x holds true for 15, 16, 17, and 18; these are the '<'. The first value of x [13] was a '>'; the second one [14] was equal -- 'the standard'". The teacher then asked about the 19th day, to which one student responded that Ichiro's wallet was then empty and that the situation was finished.

Concluding Remarks

In this classroom segment, we have witnessed the close relationship between inequality and equality concepts in eighth grade students. In using a problem-solving context involving a situation of inequality, an algebraic activity that we have characterized as being at the global meta-level, the teacher aimed to help students acquire some meaning for the form of algebraic inequalities. The problem provided a backdrop for generating an expression containing an inequality symbol. The solution to this inequality, having already been found by the students by means of non-algebraic methods, was regenerated by substituting values, from the vicinity of the solution, into the two algebraic expressions that formed the algebraic inequality. In this way, the relationship between the solution to the linear equality (180-10x=110-5x) and those of its two related inequalities (180-10x>110-5x and 180-10x<110-5x) could be drawn out – implicitly appealing to a number-line interpretation of these solutions. It is also noted that, among the students' attempted solving approaches to the given problem, no one used a Cartesian graphical representation.

It has been argued from the research carried out with older students (e.g., Tsamir, Almog, & Tirosh, 1998) that there are clear pitfalls involved in attempting to apply to the solving of inequalities some of the transformational techniques used with equations. Yet, if the Japanese students' thinking about inequalities is at all representative of other students of this age range, then the interweaving of inequalities and equalities would seem to be rather deeply rooted. The didactical challenge is to find ways to help students beware of the traps of the equality/inequality connection in their transformational work with symbols, while they still enjoy its benefits in algebraic activity of the generative and global meta-level types.

Some Questions

- The global meta-level activity of contextualized problem solving has successfully been used to provide meaning for inequalities and for their symbolic form. This leads to the question of whether, in a similar way, certain aspects of such contextualized activity can be found to be effective in helping students make sense of some of the exceptional transformation rules used in solving inequalities.
- The properties underlying valid equation-solving transformations are not the same as those underlying valid inequality-solving transformations. For example, multiplying both sides by the same number, which produces equivalent equations, can lead to pitfalls for inequalities. As the differences between the two domains are critical, the following question arises: What is the nature of instructional support that can generate in students the kinds of mental representations that will enable them to think about these critical differences when engaging in symbol manipulation activity involving inequalities?
- In which ways, if any, and for which age-ranges of students, can symbol-manipulation technology be harnessed so as to provide viable approaches for developing students' algebraic theorizing with respect to inequalities and their manipulation?

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PROBLEMS RELATED TO THE USE OF GRAPHS IN SOLVING INEQUALITIES

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I. Introduction

Graphs of functions are used increasingly to solve algebraic inequalities. This phenomenon is most probably in relation with the increasing use of graphic calculators in schools.

Most teachers seem to see the use of graphs as something that should help students in their solving of inequalities. In relation to some observations in our classrooms (students aged 15 and 17), we came to consider that this is not always the case and that there is a need to study some of the problems that arise when one changes a problem in algebra into a problem on graphs.

Solving an inequality graphically means, at first look, comparing the position of two curves. Starting from an algebraic inequality, it supposes that the student does the following work:

Inequality → create the two functions→ emergence of the graphs through the emergence of *y* → compare the *y* → come back to *x*.

We will first address Duval's theory on semiotic registers to point out some of the difficulties that can arise. Then, as dealing with graphs means dealing with functions we will question some differences between denotation in algebra and denotation in calculus as they appeared in some recent, and still ongoing, work by Maurel & Sackur.

II. Some Observations

We will first give a quick look to some results coming from the classroom. We asked our students to solve the inequality $3/x>2+x$. As we expected, all the students who used an algebraic method to solve it made the expected error. They multiplied by x whatever the sign of x could be, thus giving an incorrect answer: $x \in]-3;1[$. Quite a few students used a graphical solution, drawing the graphs of the two functions: $y=3/x$ and $y= x+2$. Then we found two types of errors: the first one came from reading the solution of the inequality, the second one from the writing of the solution for x even if the reading on the graph was correct. Older students (age 17) encountered the same type of difficulties on working with graphs. Our purpose is to give some interpretation of these errors and to show that the use of graphs for solving inequalities should be carefully prepared.

III. Duval's theory of semiotic registers

III. 1. The concepts

Mathematics is working with representations of objects. The large variety of semiotics representations for the same mathematical object is stressed as a factor of difficulties for students in learning and understanding mathematics.

A. The registers

Duval considers that there are four different types of semiotic registers in mathematics (Duval, 2000). We will not give any exhaustive description of the registers, those that we will be interested in for this presentation will be described in part III. 2. The most interesting point for us is that two representations in two different registers of the same mathematical object do not have the same content, the same meaning (Frege 1985). Change of register makes explicit different aspects and different properties of the same object.

Duval emphasises the fact that comprehension in mathematics assumes the co-ordination of at least two registers.

B. The two types of transformation of semiotics registers

- Treatment inside one register corresponds to all transformations that can be made on a representation of one type. For instance all algebraic operations on an expression.
- Conversion between two registers is more interesting for us. Conversion is the origin of many difficulties as it is generally not reversible and can be very easy (Duval says *congruent*) in one direction and difficult (*non-congruent*) in the other.

III. 2. Application to our Problem

If we come back to the table in the introduction, we can identify 4 registers involved in the solving of an inequality graphically.

I	II	III	IV
Algebraic	Fonctional	Graphical bi-dimentional	Graphical mono-dimentional
$3/x>x+2$	$f(x)=3/x$ $g(x)=2+x$	$y=3/x$ $y=2+x$	$x\in[\ldots]$

An algebraic resolution of the inequality consists of "treatment" inside register I.

For simplicity, we will consider that students shift directly from register I to register III, and we will now study the two "conversions" I$\rightarrow$III and III$\rightarrow$IV.

To explore the congruence between two registers we have to look at the different ways students act in both of them.

Conversion I→III:

1. First of all students must identify two different graphs in place of one inequality. The emergence of y and its role is a source of difficulty for students (Bazzini & al., 2001).

2. Then the type of transformations that the students have to make in register I have no correspondence in register III. In register I, one writes a sequence of algebraic expressions. The different graphs corresponding to these different expressions do no appear in register III. Another aspect of this conversion is the fact that the transformations in algebra are done for "all" x, whether on a graph one can only visualise the graph for limited values of x.

3. Graphically one has to focus on the y for different values of x, which means looking at the intersection of the curves with straight lines whose equations are x=a. The process of solving depends on the position of the curves, on the number of points where they intersect each other. For simplicity sake we will just observe four different situations as shown in Fig. 1. In the simplest case, "f(x)>0", the conversion is congruent, as this can be translated as "the curve Cf is above the x-axis". Difficulties arise when the slope of the function is steep such as y=1/x when x is close to 0 or y=1000x. The problem is no longer "Cf being above Cg", but Cf belonging to one or other of the parts of plane limited by Cg". One can see, easily, that there is then a difference between solving equations and solving inequalities.

4. Concerning the inequality 3/x>x+2, the situation is interesting in the following way: algebraically, one has to distinguish between x>0 and x<0. To this separation corresponds the fact that one of the graph (y=3/x) has no intersection with the straight line x=0. Thus the algebraic activity has its correspondence in the graphical register and vice versa. See Fig. 2.

Our conclusion is that, most probably, the conversion is not congruent.

Conversion III→IV: The situation here seems simpler. There is a one to one correspondence between the points on the graph of function f where f is greater than g and the abscises of these points. We can then say that the conversion between those two registers is congruent.

The question is to interpret the difficulties that the students encounter to come back to the solution of the inequality with x. As we will see, the situation is different from the one we can observe with an equation and enforces the link between inequalities and functions.

III. The Theory of Denotation

The theory of denotation in algebra is well known (Frege, 1985) and has been used in many situations to interpret the difficulties of the students. We have been lately interested in understanding what could be the concept of denotation in calculus and functional analysis (Maurel & al. 2001). The results we mention here are a very first attempt in this direction.

III. 1. Denotation in Calculus

We studied several situations: primitives, error terms in Taylor formulas. We came to the conclusion that a symbol like $\int f(x)dx$ doesn't correspond to one function (one mathematical object) as it does in algebra but to a class of functions. Ignoring this can lead to some difficulties such as the demonstration that 1=0. Legrand (Legrand, 1993) has emphasised the fact that very often in calculus one has to abandon some information in order to obtain the result. We think that the difficulties of the students in shifting from the graph to the solution in x could come from this aspect.

III. 2. The Case of the Inequality

Very shortly, we can say that, here also, there is not a one to one correspondence between the graph and the set of solutions in x. Different graphs can lead to the same set of solutions as is shown in Fig 3. There is not one point for one x but an infinite number of points. The situation looks very similar to the situation of the primitives. One has to abandon information, the precise graph of the functions, to focus only on the abscises of these points.

IV. Conclusion

Concerning inequalities, the use of graphs induces new difficulties for students, some of them being specific of functions. It should not be taken for granted that when "solving graphically" students learn the same mathematics as when "solving algebraically". Our interest is not so much "how to have students learn to solve inequalities?" but "what do they learn in mathematics when they solve algebraically or graphically?".

Another important question is the apparent similarity between the solving of equations and the solving of inequalities. This issue appears to be a crucial one.

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Figures (these belongs to the preceeding paper)

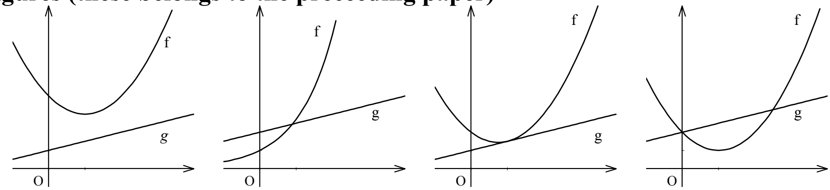

Fig. 1.

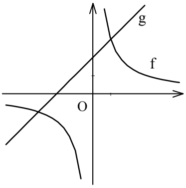

Fig. 2

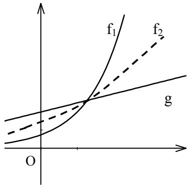

Fig. 3

EQUATIONS – A STRUCTURAL APPROACH

Tommy Dreyfus and **Maureen Hoch** , *Tel Aviv University (Israel)*

Structure

Reading papers on teaching and learning algebra (and other topics in mathematics, including calculus) one frequently meets the term structure. Some examples of papers in which structure plays a substantial role are Sfard & Linchevski (1994), Dreyfus & Eisenberg (1996), Linchevski & Livneh (1999), Zorn (2002). Structure appears to be a convenient term to describe something many of us may have some vague feeling for but cannot grasp in words. In fact, in few papers is there an attempt at defining, or even circumscribing what the authors mean by structure.

According to Sfard & Linchevski algebra is a hierarchical structure. In algebra what may be considered to be an operation at one level can be acted on as an abstract object at a higher level. Dreyfus & Eisenberg variously describe structure as the result of construction; as involving symmetry; as being composed of definitions, theorems and proofs; as being a method of classification; as relationships. Zorn states that "Understanding basic mathematics profoundly means proficiency at detecting, recognizing and exploiting structure, and at drawing useful connections among different structures". While giving no definition of structure he hints that it may be connected to pattern. Linchevski & Livneh discuss students' difficulties with mathematical structures in the number system and in the "algebraic system" but nowhere do they define what these structures are. They also use the term algebraic structure without explanation, and refer to surface structure, hidden structure and structural properties.

Defining or circumscribing what we mean by structure is not an easy undertaking. Many mathematicians, especially algebraists, tend to give the definition of some category that includes algebraic objects such as groups, rings, fields, ideals etc. But this is not helpful for us if we want to deal with high school algebra and with learning it. What does structure mean if we talk about high school algebra, and more specifically about equations with all their technical aspects?

In this contribution, we will remain guilty of the same sin of talking about structure without saying what we mean by it. Further issues of definition of structure and of examining the meaning of the definition in practice are discussed in Hoch & Dreyfus (2004). Here we will concentrate on why one might want to look at structure.

There are two quite different areas where structure is important for equations, recognizing an equation and dealing with its internal structure.

Recognizing an Equation

One would expect that one of the mathematical objects most easily recognized by students is an equation. (In order to avoid the need to distinguish between equation and identity, we include here identities under the general category of equations – more specifically, an equation with the entire substitution set as solution.) We have asked some Israeli high school students who learn mathematics at above average level to say what they think an equation is. Responses included:

1. An exercise where the aim is to find x.
2. An exercise that has a solution, that is, an exercise before you've solved it, and in the end you can do something to it and get to the solution. You need to find the variable.
3. x-s on one side, numbers on the other, an equal sign between them; need to find x.
4. A statement including two sides, an equal sign, and one or more x-s.
5. Two sides connected by an equal sign and certain rules for solving.

We see responses 1 and 2 as being purely procedural, referring to what has to be carried out. The others refer to external form. This might qualify as structure and be useful from the formal language point of view but it remains surface structure. Response 3 also mentions procedure, whereas response 4 focuses on external form only. Response 5 comes closest to indicating that there may be some underlying structure by mentioning "certain rules". The responses do not refer to what we might call the deep structure of equation, the mathematical properties of the object "equation". If these responses are typical, our data indicate that structure is not something that is in the realm of awareness of high school students.

The Internal Structure of Equations

Equations also have internal structure – at a finer level than the one needed to say whether something is an equation or not. Recognizing and using this internal structure may make solving the equation easier and increase success. Internal structure may be the actual or potential structure of the equation. By actual structure we mean the equation as it is given. For example the actual structure of the equation $\frac{1}{x-2} = 3x + 4$ could be described as a rational equation describing the intersection of a rational function with a linear function. The potential refers to what can be reached by transforming the equation. In the case of this example, the potential structure is quadratic, specifically the quadratic equation $3x^2 - 2x - 9 = 0$ whose structure is rather different from the original one. There might be an intermediate case where minor operations such as adding or removing brackets lead to a different structure. The above equation could be written $\frac{1}{x-2} = 3(x-2) + 10$.

Wenger (1987) provides a classic example of where recognizing actual structure is helpful in solving the equation. When solving the equation $v\sqrt{u} = 1 + 2v\sqrt{1+u}$ for v, recognizing the linear structure yields a relatively easy solution process. Many students, of course, focus on the square root sign which is a signal for them to square both sides of the equation. Another classic example is this parametric equation in x: $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-a)(x-c)}{(b-a)(b-c)} = 1$. It appears in Movshovitz-Hadar & Webb (1998). Here "brute force" leads to a solution only at the hands of a very determined and very adept solver while examination of the structure provides a much more efficient solution. Examining a structure that is just below the surface, the structure of the individual terms that make up the equation, reveals that x=a is a solution, and then that x=b and x=c are also solutions. The internal deep structure – the properties of a quadratic equation – provides the information for the final solution, that this equation is true for all values of x.

A typical equation from high school algebra is (x2 - 4x)2 - x2 + 4x = 6. It can be solved by recognising that a simple substitution transforms it into a quadratic equation. Thus a minor operation reveals structure and gives a handle on solving the equation. Another example is $\frac{1}{4} - \frac{x}{x-1} - x = 5 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$. An examination of the structure reveals that this is a linear equation masquerading as a rational equation. For many students however the presence of an algebraic fraction is a signal to multiply by a common denominator leading to a long and error prone solution (see Hoch & Dreyfus, 2004). Here recognizing a hidden structure and transforming the equation (by subtracting the same term from both sides), so as to show this hidden structure, is used to solve the equation. We see that recognizing and using structure is likely to increase success in algebra substantially.

In Conclusion

Our experience is that Israeli students have little difficulty in actually recognizing equations but extreme difficulty in talking about this recognition. They rarely relate to equations in any way apart from the procedural. They usually do not recognize the internal structure of equations. If they do recognize structure they rarely use it (see for example Hoch, 2003) and in fact they have difficulty solving all but the most standard equations. Also, teachers do not seem to be aware of what recognizing and using structure could do for the student. The emphasis in the algebra classroom is on mechanical methods for solving equations. For example, the method of substitution is taught in tenth grade, but usually on a very technical level, and is soon forgotten (see Hoch & Dreyfus, 2004).

We suggest that the forum address the issue of ways of presenting algebra that will focus students' attention on structure.

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"NEW ERRORS" AND "OLD ERRORS": THE CASE OF QUADRATIC INEQUALITIES

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A prominent line of research in mathematics education is the study of errors. The early research on mathematics learning viewed students' errors as flaws that interfere with learning and need to be avoided (e.g., Greeno, Collins & Resnick, 1996). From an instructional perspective, students' errors were traditionally perceived either as signals of the inefficiency of a particular sequence of instruction or as a powerful tool to diagnose learning difficulties and to direct the related remediation (e.g., Ashlock, 1990; Fischbein, 1987). Borasi (1987) argued that errors could and should be used as springboards for problem solving and for motivating inquiry about the nature of mathematics, and Avital (1980) claimed that the best way to address common mistakes is to intentionally introduce them and to encourage a mathematical exploration of the related definitions and theorems.

But what is actually happening in the classrooms with respect to errors? Do teachers intentionally introduce common errors and if so: How? When? To whom? How do teachers address errors made by students? What are the factors that influence their reactions? In this paper we discuss our initial attempts to explore teachers' declarations and practices regarding the role of errors in their classrooms. We describe here the case of Rami, a very experienced mathematics teacher who has a reputation of an excellent teacher. We shall focus here on a variety of ways in which Rami addressed errors when teaching quadratic inequalities. We shall first briefly describe what is known from the literature on students' common errors when solving quadratic inequalities and to their possible sources.

Literature on Quadratic Inequalities: *Errors, Sources and Instruction*

In the last decade there is a growing interest in students' performances when solving various types of algebraic inequalities in general and quadratic inequalities in particular (e.g., Linchevski & Sfard, 1991; Tsamir & Bazzini, 2001). Several *common errors* were identified, including the tendencies to: (1) multiply / divide both sides of an inequality by a factor that is not necessarily positive, (2) deal with products in the following manner: $a \cdot b>0 \Rightarrow a>0$ and $b>0$; $a \cdot b<0 \Rightarrow a<0$ and $b<0$; (3) make inappropriate decisions regarding logical connectives, and (4) reject $\{x | x = a\}$, "R" and "ϕ" as solutions. Several publications mentioned *possible sources* for these errors, mainly relating to possible overgeneralizations from equations to inequalities (e.g., Tsamir, Tirosh & Almog, 1998), and to the grasp of transformable inequalities as being equivalent (Linchevski & Sfard, 1991). Some of these errors are intuitive (Fischbein, 1987), and are, thus, likely to evolve in every class. Consequently, we decided to explore how various teachers address errors in their classrooms. Here we focus on a class that dealt with quadratic inequalities.

The Study

Setting and Methodology

At the time the study was carried out, Rami was the head mathematics teacher in a secondary school. He was a very energetic and highly motivated teacher who invested much effort in his instruction and in establishing open, friendly relationship with his students and colleagues. For the purpose of our study, one of the researchers (ST) observed and videotaped Rami's three lessons on quadratic inequalities in an average, 13th grade class (learning for a certificate of electronic technicians). The videotapes were transcribed and all the "error-episodes" were defined (an error episode consists of an error made in class and the subsequent, related event). Several reflective interviews were then conducted with Rami. In these interviews he was first asked to list students' common errors when solving quadratic inequalities. Then, he was presented with the transcriptions of several "error-episodes" that occurred during his instruction. He was asked to identify the error, to specify its possible sources, to explain the way he addressed it in class, to comment on it and to relate to other, suggested ways of handling this error. Later on, Rami was presented with a list of quadratic inequalities that are known to elicit specific errors. He was asked to list the errors that students are likely to make in each case. Finally, Rami was presented with the typical errors that students commonly make when solving the same quadratic inequalities and asked: "How would you react, in class, to such errors". The interviews lasted about 90 minutes. They were audio-taped and transcribed.

Due to space limitations, we shall focus here only on one main observation regarding Rami's didactical ways of addressing errors in his lessons on quadratic inequalities.

New Errors and Old Errors: A Critical Dichotomy

Our analysis of the "error episodes" revealed that Rami addressed the errors that occurred in his class in two distinguished, clusters of reactions: The *economic cluster* and the *elaborated cluster*. In the *economic cluster* we included his following, typical reactions: 1) ignore the error and go on teaching, 2) state the correct solution, and 3) when having a mix of erroneous and correct suggestions, address only the correct ones. The following, three reactions are representative of the *elaborated cluster* : 1) ask the student to repeat his erroneous solution and to explain his reasoning to the entire class 2) try to find out if other students in the class hold the same opinions, and 3) try to lead the student (e.g., by counter examples) to realize that she erred.

All in all, it was noticeable that the *economic* and the *elaborated* cluster were distinguished in terms of the time allotted for and the effort invested in discussing the errors. An *economic reaction* is a short, local reaction that highlights only the correct solution, with no reference to the incorrect solutions. An *elaborated reaction* unlike the economic ones, is more time consuming and didactically more demanding. Here, Rami explicitly addressed the incorrect ideas, asking the student for further explanations and trying to trace the source of the error.

A question that naturally arose is: What directed Rami's didactical conduct? Under what circumstances was he acting in an economic manner? In what occasions did he prefer the elaborated reaction? Rami's behaviors could be attributed to various factors, some of which are student oriented (e.g., capabilities, gender) while others are timing oriented (in what part of the lesson the error occurred). Our analysis ruled out the "student" option, since Rami reacted to the same student in different ways on different occasions. At first it seemed that the timing was a major factor that guided Rami's reactions: Economic reactions were more frequent at the beginning of the first lesson while elaborated reactions were more evident by the end of this lesson. But this split was not apparent in the other two lessons. A detailed examination of the mathematical content of the episodes that were included in each of the two clusters led us to conclude that the episodes in the economic cluster addressed errors that were embedded in mathematical topics that were studied prior to the lessons on quadratic inequalities (e.g., quadratic equation, parabolas). Elaborated reactions were provided by Rami to issues that were part of the topic at hand (e.g., logical connectives). This observation was confirmed by Rami during the subsequent, reflective interview. Indeed, when Rami was asked to relate to various error episodes that occurred during his lessons, he clearly stated that the nature of the error, in terms of being "new" or "”old" is a main factor that influenced his reaction to the error.

Summing Up and Looking Ahead

Our results indicate a phenomenon that at first glance seems obvious, i.e., allotting more time and didactical energies to errors in the new topic, and less time and efforts to those that relate to mathematical topics that were studied previously. This observation raises many issues for further explorations, three of which are: (1) Is this conduct a general characteristic of Rami's instruction or is it typical only to his teaching of quadratic inequalities? (2) Is the "new errors"/"old errors"' split typical only to Rami or to other expert teachers / novice teachers? (3) What are the pros and cons of this approach? We shall deal with these issues in our presentation.

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REFLECTIONS ON RESEARCH AND TEACHING OF EQUATIONS AND INEQUALITIES

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In reacting to this forum on 'Algebraic Equalities and Inequalities', I take a problem-solving approach, first, asking 'what is the problem?' then looking at the five presentations to see what can be synthesized from their various positions (acknowledging that they are here limited to very short summaries).

The 'problem', as initially formulated, focuses on the *algebraic* manipulation of equations and inequalities. Tsamir et al [TTT] focus mainly to this aspect by considering how a teacher might cope with errors that arise from the inappropriate use of earlier experiences in equations that produce errors with inequalities. This focus is broadened in the list of 'Key Questions', to encourage the consideration of different theoretical frameworks—and the use of technology—to see how research can improve teaching and learning. The other papers take the key questions in different directions. Boero & Bazzini [BB] and Sackur [S] consider broader issues, with a particular focus on the switch from algebraic to visual representations where an inequality $f(x) > g(x)$ is visualised by seeing where the graph of f is above the graph of g. Underlying both approaches are relationships between different representations (or semiotic registers, as described in the subtle theory of Duval).

Kieran [K] presents a different overall framework ('generational', 'transform-ational' and 'global meta-level') that may be described as a 'vertical' theory of development rather than a 'horizontal' theory of relationships between represent-ations. Finally, Dreyfus & Hoch [DH] broaden the context to the increasingly sophisticated structure of equations, from a procedure to undo an arithmetic calculation, to solving equations with xs on both sides, to more subtle cases of equations containing substructures and equations solved using specified rules.)

This brings me back to 'the problem'. What is it that this forum is really attempting to address? There seems to be an implicit understanding that we need to help students to understand and operate with equations and inequalities. *But for what purpose?* If the purpose is to solve a given equation or inequality, then a graphical picture may be appropriate. For instance, to 'see' what happens to the inequality $x^2 > x + c$ as c varies, a powerful visual representation is given by the quadratic $f(x) = x^2$ and a straight line $g(x) = x + c$ that moves up and down as c changes. However, if the problem is to enable the student to become fluent in meaningful manipulation of symbolism, then the activities with the graph may involve no symbolic manipulation whatever (particularly if the graph is drawn by computer). [S] considers the strengths and weaknesses of moving between different registers. These focus on different aspects, highlighting some, neglecting others. If an aspect is absent, then its variation does not figure in the link between representations. An example is the *evaluation* of a function by carrying out a procedure: $2(x+1)$ and $2x+2$ are different procedures in the symbolic register but are represented by precisely the same graph.

The focus of [BB] on graphs of functions as global dynamic entities uses the idea of 'grounding metaphors' of Lakoff & Nunez in a way that 'could also ensure a high level of the control of the solution process'. But *what* solution process? The visual enactive activity can give a powerful embodied sense of global relationships between functions as entities, but how does it relate to the meaningful manipulation of symbols? It emphasizes the strength of grounded metaphors but not the 'incidental properties' of Lakoff's theory, which may be usefully employed in a particular context but have the potential to be the sources of errors in new contexts.

It is my belief that the phenomenon of 'cognitive obstacles' arises precisely because the individual's subconscious links to incidental properties in earlier experiences are no longer appropriate in a new context. Rather than use the high sounding language of 'metaphor' for the recall of earlier experiences, I use the prosaic term 'met-before'. I hypothesise that it is precisely the met-befores in solving linear equations that causes problems in inequalities researched by [TTT]. Students taught to manipulate symbols in equations, will build personal constructions that work in their (possibly procedural) solutions of linear equations but operate as sub-conscious met-befores that cause misconceptions when applied to inequalities.

In a given context there are often several different approaches possible. [K] reveals a spectrum of responses to a problem that may be formulated as an inequality, including a physical representation, the use of tables, equations and inequalities. [DH] presents a compatible spectrum, with different emphases, numerical procedures to 'undo' equations, more subtle manipulation of expressions as mental entities, and seeing sub-structures of equations as mental entities in themselves. Some of these approaches may be more amenable to future development than others; in particular, theories of cognitive compression from process to manipulable mental entities (which are entirely absent from all the presentations) address the possibility that the construction of mentally manipulable entities is likely to be more productive for long-term development.

Later developments in the use of inequalities include the formal notion of limit, where the epsilon-delta method will certainly benefit from meaningful grounding of inequalities, but will also need to focus on the manipulation of symbols and the development of formal proof. Inequalities at a formal level involve axioms for order in a field F, for example, by specifying a subset P of F that has simple properties (if $a \in P$, then one and only one of these holds: $a \in P$, $-a \in P$ or $a = 0$; if $a, b \in P$ then $a + b, ab \in P$.) In this case $a > b$ is *defined* to be true when $a - b \in P$. This use of 'rules' is not a meaningless procedural activity but a meaningful formal approach that has the potential of giving new meanings. For instance, a structure theorem may be proved to show that every ordered field 'contains' the rational numbers and may also contain 'infinitesimals' that are elements in F which are smaller than any rational number. In this way intuitive concepts at one stage (infinitesimals as 'arbitrarily small' variable quantities) can be given a formal mathematical meaning.

An organization such as PME needs to aim not only for local solutions to problems, but also for global views of long-term development. The papers in this forum present essential ingredients to contribute such a wider scheme.

When the 'problem' of equations and inequalities is seen in this way, a wider picture emerges. There are unspoken belief systems that get in the way of our deliberations. For instance, while several of the papers give examples of different individuals using different methods to solve the same problem, no one attempts to say whether one solution is potentially better or worse for long-term development. Differences are apparent in the success and failure in all the examples given. Do we need to look at different solutions for different kinds of needs? Rich embodiments have strengths that may be appropriate in some contexts (perhaps to solve an inequality in a specific problem) and misleading in others (where concepts of constructed that, if unresolved, become met-befores causing obstacles in later learning). Do all students follow through the same kind of Piagetian development or, does their journey through mathematics find them using methods that are more or less suited to long-term development that gives different kinds of possibilities for future development?

In addition to the horizontal framework of registers and the vertical framework of [K], I offer a third that relates to the algebraic spectrum of [DH]. A study of long-term development of symbolism in arithmetic and algebra (Tall *et al.*, 2001) led to a categorization of algebra (Thomas & Tall, 2001) in three levels, which we termed 'evaluation algebra', 'manipulation algebra' and 'axiomatic algebra'. The first encompasses the idea of an expression, say $3+2x$ being used simply for evaluation, say in a spreadsheet or in a graph-drawing program. The second encompasses the idea of an expression as a thinkable entity to be manipulated. The third concentrates on the properties of the manipulation and leads to an axiomatic approach to algebra in terms of groups, rings, fields, ordered fields, vector spaces, etc. In what ways do the papers presented in this forum address problems both at a local level and also in producing a helpful global theory? Much of the discussion could involve evaluation algebra, [TTT] considers manipulation and [DH] looks from manipulation to axiomatic. Do we need one kind of algebra for some students and other kinds for others? Richard Skemp once said to me, 'there is nothing as practical as a good theory'. In our forum it would be practical to look for a global theory encompassing the local theory of equations and inequalities.

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SYNTAX AND MEANING

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A forum is certainly a multi-voiced dialogue, an example of what Bakhtin used to call heteroglossia, or the encounter of multiple perspectives in cultural interaction. With their own intonation and from their own perspective, the papers of the research forum engage in dialogue with each other about pedagogical, psychological and epistemological questions concerning two key concepts of school algebra, namely, equations and inequalities. They offer us valuable reflections on the search for new contexts to introduce students to inequalities (e.g. functional covariance) and a critical understanding of the limits and possibilities of these contexts. They also provide us with fine enquiries about urgent learning problems along the lines of key theoretical constructs that have played a central role since the 1980s in mathematics education (such as structure and the cognitive status of students' errors).

The papers tackle a general problématique against the background of the present context of discussions about cognition. In the past few years, there have indeed been important changes in conceptions of cognition in general, as witnessed by e.g., a recent interest in phenomenology, semiotics, and embodiment. We have become aware of the decisive role of artefacts in the genesis and development of mathematical thinking and we have become sensitive to theoretical claims from sociology and anthropology that emphasize the intrinsic social dimension of the mind. With their own intonation and from their own perspective, the papers of the research forum have engaged each other in a dialogue on the problem of algebraic thinking as set by the general stage of our current understanding of cognition. Since one of the key common themes of the papers is that of syntax and meaning, let me delve into it and comment on what the papers intimate in this respect.

1. Meaning

In the introduction to their paper, Boero and Bazzini find fault with the classical approach to inequalities and claim that the "purely algorithmic manner" that reduces the solving of inequalities to "routine procedures" limits students' understanding. This complaint is not new. In the seminal book edited by Wagner and Kieran (1989) the same reasons led Lesley Booth to object to the considerable attention paid to the syntactic aspects of algebra in the classroom. There is nevertheless a subtle but important difference in how solutions are envisaged one the one hand, by Booth and the structural perspective, and by Boero and Bazzini, on the other.

Booth claimed that difficulties in learning syntax were the result of a poor understanding of the mathematical structures underpinning algebraic representations: "our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations", she argued, and added that "[t]hese structural properties constitute the *semantic* aspects of algebra." (Booth, 1989, pp. 57-58). I do not think that Boero and Bazzini disagree with the important role played by structural properties in the constitution of the semantics of algebra. Nevertheless, they seem to disagree with the idea that, ontogenetically speaking, the understanding of structural properties comes *first* as well as with the claim that these structural properties alone constitute the semantics of algebra. Indeed, in their approach (see also Garuti et al., 2001), the study of the production of meaning is located in an activity that transcends mathematical structures. Their analysis traces elements of students' linguistic activity and body language in an attempt to detect metaphors, gestures and bodily actions that can prove crucial in students' understanding and use of algebraic symbolism. In their analysis of the way in which students make sense of a quadratic inequality, they emphasize the students' allusion to artefacts and to their understanding of symbols in terms of cultural linguistic embodied categories such as "going up" and "going down". As I see it, the covariational functional context that they propose is conceived of as a means for students to produce meaning and understand signs.

The idea that the production of meaning goes beyond mathematical structures and the claim that meaning is produced in the crossroad of diverse semiotic (mathematical and non-mathematical) systems is certainly one of the cornerstones of non-structural approaches to mathematical thinking. And yet, many difficult problems remain. Algebraic symbolism is undoubtedly a powerful tool. Even if some calculators and computer software are able to perform symbolic manipulations, algebraic symbolism is not likely to be abandoned in schools –at least not in the short term. Kieran's reflections on what happens to meaning when students translate a word-problem into symbolism, Sackur's interest in understanding the outcome of meaning in conversion between, and treatments within, registers and Dreyfus and Hoch's concerns about students recognizing the underpinning structures in equations thus appear to be more than justified. Certainly, one of the crucial problems in the development of algebraic thinking is to move from an understanding of signs having been endowed with a contextual and embodied meaning, to an understanding of signs that can be subjected to formal transformations. The meaning that results from noticing that a graph "goes up" or "goes down" supposes an *origo*, that is, an observer's viewpoint. This *origo* (Radford 2002a) is the reference point of students' spatial-temporal mathematical experience, the spatial-temporal point from where an embodied meaning is bestowed on signs. Algebraic transformations, such as those mentioned by Dreyfus and Hoch, require the *evanescence* of the *origo*. Does this amount to saying that symbolic manipulations of signs are performed in the absence of meaning? To comment on this question, let us now turn to the idea of syntax.

2. Syntax

One of the tenets of structuralism is the clear-cut distinction between syntax and semantics. From a structural perspective, the real nature of things is seen not in the world of appearances, but in their true meanings –something governed by the intangible but objective laws that Freud placed in the unconscious, and that structural anthropology, psychology and linguistics, after Saussure and Lévi-Strauss, thematized as "deep structures". Syntax was conceived of as lying on "surface structures", it was merely dead matter, the shadows of deep, structurally governed, mental activity. It is understandable that, in this context, in 1989 Kaput argued that instead of teaching syntax (which would produce "student alienation") we should be teaching semantics (Kaput, 1989, p. 168). Nevertheless, as I have already stated, we have become more sensitive to the claim that every experience, even the more abstract one found in mathematics, is always accompanied of some particular sensory experience, or –as Kant put it in the *Critique of Pure Reason*– that every cognition always involves a concept and a sensation.

How, then, within this context, can we address e.g. Dreyfus and Hoch's legitimate concerns? Recognizing equivalent equations is one of the fundamental steps in learning algebra. The formal transformation of symbols in fact requires an awareness of a new mode of signification –a mode of signification that is proper to symbolic thinking (Radford, 2002b) and whose emergence only became possible in the Renaissance. As Bochner (1966) noted, despite the originality and reputation of Greek mathematics, symbolization did not advance beyond a first stage of iconic idealization where calculations on signs of signs were not accomplished. It is not surprising then that the problem of explaining the formal manipulation of symbols puzzled logicians and mathematicians such as Frege, Russell, and Husserl. While for Russell (1976, p. 218) formal manipulations of signs are empty descriptions of reality, for Frege and Husserl formal manipulations do not amount to manipulations devoid of meaning. In fact, for Frege, equivalent algebraic expressions correspond to a single mathematical object seen from different perspectives: they have the same referent but they have a different *Sinn* (meaning). Adopting an intentional, phenomenological stance, Husserl contended that manipulations of signs require a shift in attention: the focus should become the signs themselves, but not as signs *per se*. Husserl insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the "rules of a game" (Husserl 1961, p. 79).

These remarks do not solve the crucial problem raised by Dreyfus and Hoch, also present in the other papers of this forum. It would certainly be of little help to tell students that a seemingly rational equation is, after transformations, equivalent to a linear equation because they are both designations of the same mathematical object.

Perhaps Husserl's insight intimates that the change in the way we attend the object of attention (e.g. the modeled situation or the equation itself) leading to an *awareness* of the "rules of the game" rests on a process of *perceptual semioisis*, or a dialectical movement between perceived sign-forms, interpretation, and action. Hence, it may be worthwhile to consider the ontogenesis of new modes of signification required by algebraic symbolism as a back and forth movement between interpreting the symbolic expression in its diagrammatic form (Peirce) and the (mathematically structured) hypothetical generation of new diagram-equations.

It might be very well the case that the greatest difficulty in dealing with equations and inequalities resides in: (1) the understanding of the *apophantic* nature of equations and inequalities and (2) the *apodeictic* nature of their transformations.

Number (1) refers to the fact that, in contrast to a symbolic expression like $x+1$, an equation or an inequality makes an *apophansis* or *predicative judgment* (in Husserl's sense; Husserl, 1973): it *asserts* e.g. that $P(x) = 0$. Number (2) refers to the necessary truth-preserving transformations of equations and inequalities –if, for a certain x, it is true that P(x) = 0, then Q(x) = 0, etc., something that Vieta expressed by saying that algebra is an *analytic* art. What I want to suggest is that the predicative judgments $P(x)=0$ or $P(x)\leq 0$, etc. that rest at the core of solving an equation or an inequality should not be confined to the written register containing an alphanumeric string of signs. We need an ampler concept of predication (and of mathematical text) less committed to the written tradition in which Vieta was writing not many years after the invention of printing. We also need a better concept of predication capable of integrating into itself the plurality of semiotic systems that students and teachers use, such as speech, gestures, graphs, bodily action, etc., as shown clearly in the Grade 8 lesson mentioned by Kieran. Predicative judgments would be made up of a complex string of gestures, written signs, segments of speech and artefact-mediated body actions. Their transformations would not be confined to the realm of logic and formal symbol manipulation, for the passage from one step to the next in a semiotic process is not something predetermined in advance by the logic of deduction alone: what seems to be a formal manipulation is in fact continually open to interpretation. There is, in the end, no opposition between syntax and meaning. Every sign has a meaning. Otherwise, it cannot be a sign. Conversely, every meaning is an abstract entity –"a general" (Otte, 2003)– which finds instantiation in signs only.

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SOME FINAL COMMENTS

One main aim of this research forum is to have a rich discussion and enable the participants to address the issues presented in the five presentations. For this purpose we decided to have the following structure of meetings: *In Session One:* each of the presenters will briefly [10 minutes] present their studies together with educational implications, and conclude his / her presentation by posing a number of questions for further discussion. Then, all the participants will be asked to discuss these questions, raise additional questions, dilemmas, doubts and comments that will be addressed by the presenters, reactors and all the others during the second meeting. *In Session Two:* each of the reactors will present his analysis of the approaches presented in sessions one [15 minutes], referring both to the presentations and to participants' remarks made during the first sessions. There will be ample time for the audience to add their own thoughts and analyses to those of the reactors.

Another aim of this RF is to discuss issues of *inclusion and diversity*. This will be done by refining the questions posed by the participants so as to meet the needs, abilities and beliefs of different students, teachers, and classes. For example, when discussing students' erroneous solutions to inequalities, we will address the following questions: What are the difficulties of low achievers vs. high achievers? Boys vs. girls? Those who studied the topic in different ways (e.g., graphical vs. algebraic approaches)? When discussing the teaching of equations and inequalities, we may for instance address the following questions: How do different teachers make their related didactical decisions? What is the impact of different teaching approaches on different students?

Finally, this RF aims to create a wide international network to investigate the teaching and learning of algebraic equations and inequalities by deepening existent collaborations and encouraging researchers from additional countries to enter this endeavor. A selection of contributions discussed during the Research Forum could also yield specific publications on the theme.

RF03: INTERNATIONAL PERSPECTIVES ON THE NATURE OF MATHEMATICAL KNOWLEDGE FOR SECONDARY TEACHING: PROGRESS AND DILEMMAS

Coordinators: **Helen M. Doerr, USA** and **Terry Wood, USA**

This research forum addresses the question: what is the nature of the mathematical knowledge that is needed for secondary teaching? Six international contributors respond by making two claims (one related to an area where progress in research has been made and the other related to dilemmas facing researchers): preparing teachers, teaching practice, and research designs and methodologies. This structure provides a way of focusing the discussion among forum participants and a means to develop international points of view on the nature of the mathematical knowledge that is needed for secondary teaching.

GENERAL FRAMEWORK

Over the past two decades, international perspectives on research about the teaching of mathematics have received considerable and increasing attention at PME and by the research community in mathematics education (Ellerton, 1998; Jaworski, Wood & Dawson, 1999). Yet, progress towards changes in teaching practices remains slow and large gaps exist between the highest achieving schools and countries and the lowest achieving schools and countries. Substantial progress has been made in many areas of research related to students' learning along with the emergence of curricular materials and standards documents that reflect findings of this research (e.g., the early numeracy projects in the United Kingdom, New Zealand and Australia). Nevertheless, translating research on mathematical learning into forms that are useful for teaching practice continues to be a difficult problem that varies substantially across schools and countries and progress has been elusive. Difficulties in preparing new teachers are compounded by the disconnection that pre-service teachers can experience between their teacher preparation programs and their experiences in practice. Furthermore, the complexity that characterizes teaching and learning seems to have yielded a multiplicity of research designs and methodologies with insufficient coherence across these research designs to support the development of a shared knowledge base for teaching.

KEY QUESTIONS AND THEMES

There is substantial agreement among mathematics educators that the quality of teachers' subject matter knowledge is necessary but not sufficient for effective teaching. Subject matter knowledge is just one category among many that attempt to capture the complexity of the nature of the mathematical knowledge base that is needed for teaching (Hiebert, Gallimore & Stigler, 2002; Shulman, 1986). Hence, the central focus of this research forum is the nature of the mathematical knowledge that is needed for teaching in secondary schools.

A recent National Research Council report in the United States (NRC, 2003) described mathematical proficiency for students as the simultaneous and integrated acquisition of five strands: (1) conceptual understanding, (2) procedural fluency, (3) strategic competence, (4) adaptive reasoning, and (5) productive disposition. These proficiencies provide one possible framework for considering the mathematical knowledge that is needed by secondary teachers. However, in addition to teachers having this kind of mathematical proficiency, they must also understand (1) how such mathematical proficiencies are developed in curricular materials, (2) the ways in which students' thinking might reveal students' mathematical proficiencies, and (3) how students from diverse cultural and ethnic backgrounds develop these mathematical proficiencies.

Another possible framework comes from the KOM project (Niss, 2003) which provides eight competencies for students, such as 'think mathematically and make use of different representations and translate between them,' that describe the main components for mastering mathematics derived from the work of mathematicians. In addition to having these competencies, teachers must also have competencies in curriculum, teaching, student learning and assessment. We will use these proficiencies and competencies as background for considering the nature of teachers' mathematical knowledge, and then we will describe the challenges and difficulties in designing and implementing research in this area.

This research forum will focus on the following central question: *What is the nature of the mathematical knowledge that is needed for secondary teaching?*

Our goal in this forum is to stimulate discussion on this question through a reporting of research findings that identify areas in which significant progress has been made and where difficulties and persistent obstacles to progress continue to exist. To initiate the discussion, contributors from six different countries that represent international differences in contexts and perspectives report their findings. Each contributor addresses the question from three views: (a) preparing teachers; (b) supporting teachers in practice; and (c) research design and methodologies. Each contributor makes two key claims related to each of the three views of the above question. The first claim reflects an area where substantial research progress has been made in the contributor's country with respect to the nature of the mathematical knowledge that is needed for teaching secondary mathematics. These claims reflect findings that are of significance to the field and are based on a substantial body of research. The second claim reflects a significant dilemma in research or an area where progress has remained elusive. This structure provides both a broad view of the field (as it is seen internationally) and a way of focusing the discussion among forum participants. The contributors' claims are presented in the next section. Following those contributions, we provide a tentative synthesis of the claims and pose some cross-cutting questions that will provide a beginning point for work of the participants in this forum.

PREPARING TEACHERS—PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: The corpus of research on students' conceptions, thinking and learning in mathematical content areas provides foundation knowledge for a greatly improved teacher education.

Claim 2 Dilemma: This corpus of knowledge needs to undergo substantial didactic transposition before it is maximally useful.

Claim 1 is about creating the scientific basis of a discipline of mathematics didactics (pedagogy) for teacher education. The established sciences and humanities have an accumulated set of well-tested research results, which have been codified and simplified to create learnable disciplines. In mathematics education, we are now reaching a point where we too have a sufficiently strong scientific foundation to undertake this task. We know enough about students' thinking patterns, conceptions and their development to begin the "didactic transposition" from raw research results to learnable and organised material which could form the basis of a new teacher education. I expect these outcomes to be very much more effective than teacher education based around general theories of mathematical development (as was tried, for example, with Piagetian research in times past).

What is the evidence for Claim 1? The extent of the research knowledge is evident from the accumulated proceedings of PME, the handbooks of reviews of research and so forth. The need for this material to undergo a didactic transposition is evident in the lack of textbooks on student's thinking and learning for secondary mathematics teacher education (indeed no textbook is widely used in Australia) and the consequent practice of referring teacher education students directly to research reports rather than to scholarly accounts written for them.

My claim also requires evidence that this new content of teacher education would "make a difference." Two large scale elementary teacher development projects provide some confirmation. *Count Me In Too* (Bobis, 1999) is a New South Wales government professional development initiative where mathematics education researchers turned international research on children's early number development into support material for professional development. Teachers learned about how children's knowledge progressed, assessed children's learning carefully and selected teaching materials to move them along the framework. The *Early Numeracy Research Project* in Victoria had a similar mission and adopted a similar approach, although differing in detail. Both projects, although focused on elementary schooling, demonstrate improved outcomes for students across large numbers of schools, some of them sustained. A difficulty with using a program evaluation as evidence for my claim is that improved learning outcomes are a result of the whole program, rather than one component, such as improved teacher knowledge.

I envisage the process of didactic transposition running some decades behind the research. There is clear evidence that teacher education students find such material interesting and relevant to their future work. We have found that presenting case studies of children's thinking about decimal numbers using simple multimedia products has engaged our pre-service teachers very deeply (Chambers, Stacey, & Steinle, 2003). It has been a powerful way to expose and remediate their own misunderstandings (e.g., Stacey et al., 2001). Several years after working with this material, some of our pre-service teachers have spontaneously recalled the case studies by name and by misconception.

The didactic transposition is not, however, unproblematic. In a teacher education course with limited time, what is the right "grain size" for knowledge about student's learning so that it can guide teaching actions? Our research catalogue of decimal misconceptions (e.g., Steinle & Stacey, 2003) with 12 major types is probably too large for teachers to act upon in real time in classrooms, without technological assistance (Stacey, et al., 2003). (Even in this paper, we only use the simplest examples.) *Count Me in Too*, for example, presented a considerably finer analysis of early number learning than the Victorian *Early Numeracy Research Project*. There are many other questions to be answered. Are there (or where are there) strong commonalities in learning trajectories that will assist transfer of knowledge between different teaching areas? For reasons such as these, Claim 2 is a call for research on our research.

Brazil (Marcelo Borba)

Claim 1 Progress: The notion that we need to search for the particularities of mathematics that should be taught to teachers with a broad view of mathematical content. Adding more content to teacher preparation programs is not a solution.

There are a significant number of teacher educators who are investigating what specific mathematics should be included in teacher education programs. In Brazil, this discussion has taken on new dimensions, since there is an established tradition of research on ethnomathematics that began over a quarter of a century ago. The idea that different cultural groups produce different mathematics (D'Ambrosio, 2001; Borba, 1987) is very well-accepted. Various researchers seem to have chosen to extend this idea into teacher education, and to consider pre-service teachers as members of the mathematics education community (Lave & Wenger, 1991), even if they do not necessarily address the problem using the specific constructs of ethnomathematics or "community of learners."

Different researchers in Brazil have emphasized that simply adding more content is not the solution to the problem of what should be taught to pre-service mathematics teachers. Instead, such content should be seen as embedded in cultural and social issues regarding the context, philosophical themes related to what should be taught and historical aspects of the change in mathematical knowledge over time. Mathematics teachers, beginning in their pre-service education, should become members of the mathematics education community and not members of the mathematics community.

Claim 2 Dilemma: Although there seems to be a consensus that the education of mathematics teachers should be different from those who will go on to do research in mathematics, there is no consensus regarding whether or not the education of these two types of students should be totally distinct.

On the one hand, there are teacher educators who believe that it is important for future teachers to interact with professional mathematicians and with students who will become mathematicians. On the other hand, there are mathematics educators who believe that it is impossible to do so, and that to keep these two kind of students in the same structure means that pre-service teachers will be educated like mathematicians for two years and only in the final years they will be prepared to become teachers. According to Fiorentini et al (2002) dilemmas such as this, involving the tension between mathematicians and mathematics educators, have been present in Brazilian research about teacher education for a more than a decade.

Israel (Ruhama Even)

Claim 1 Progress: Regular university or college mathematics courses do not support the development of adequate mathematical knowledge for teaching secondary school mathematics.

A traditional approach to equip secondary school mathematics teachers with adequate mathematical knowledge is quantitative in nature: “more is better.” This approach is based on the premise that teachers already learned, and therefore know, school mathematics; and that teachers should know more mathematics than the mathematics their students have to learn, and therefore, advanced mathematics studies are a good indicator of adequate teacher mathematical knowledge. However, several research studies that examined teachers’ mathematical content knowledge (e.g., Even, 1990, 1992, 1998; Knuth, 2002; Lipman, 1994; Shriki & David, 2001) suggest that secondary school mathematics teachers often do not hold a sound understanding of the mathematics they need to use and teach in school. This includes fundamental concepts from the secondary school curriculum, such as functions and proof.

For example, the following problem was presented to 162 American (Even, 1992) and to 45 Israeli (Lipman, 1994) prospective teachers from several universities (U.S.) and teacher colleges (Israel), in the last stage of their formal pre-service preparation.

> A student said that there are 2 different inverse functions for the function $f(x)=10^x$: One is the root function and the other is the log function. Is the student right? Explain.

Many did not answer correctly. Some chose the root function as the inverse function, using a naive conception of “undoing” as their interpretation of inverse function. The xth root of 10 seemed to them to “undoes” what 10^x does in the following manner: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. Accepting the root function as an inverse function because of its "undoing" appeal created for many of the American prospective teachers a cognitive dissonance; they remembered that log was the appropriate inverse function and that the inverse function for any given function is unique. To solve this uncomfortable situation these students decided that the log function and the root function were both inverse functions of the given function *since they were the same function*. For example, "I believe that there is only one function. The root function and the log function are just two different ways of representing the same function."

Such findings indicate that relying on advanced mathematical studies at the college or university level to account for adequate teacher mathematical knowledge of secondary school mathematics is problematic. Apparently, even though teachers have already learned as students the mathematics they need to teach, and then studied even more advanced mathematics, they still need to re-learn the mathematics they have to teach.

Claim 2 Dilemma: What would support the development of adequate mathematical knowledge for teaching secondary school mathematics?

Several programs and courses for in-service secondary teachers in Israel include as one of their components the deepening of the participants' knowledge of the mathematics they need to use and teach at school (e.g., Even & Bar-Zohar, 1997; Zaslavsky & Leikin, 1999). However, this is less common in pre-service teacher education. At any rate, we do not have enough research findings to provide adequate answer to the above question.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: The problematic nature of mathematics teacher education in Norway is at last being recognized and addressed.

Norway is a long and thin country in Northern Europe (Scandinavia) covering 324,000 km^2 of land, approximate in size to Poland. It is covered largely with lakes, fjords, mountains and forests and is only sparsely populated: its population is 4.5 million, of which 0.6 is in the capital, Oslo. Thus, for geographic and demographic reasons, many schools are small and this affects the organisation of education. Multi-grade teaching is common, and teachers need to teach a wide range of subjects.

As a consequence of this geographical spread and an educational philosophy of inclusion, all teachers educated in teacher education colleges in Norway are general teachers. This means that they have formal competence to teach *all* subjects in grades 1 to 10 (age 6 to 15). Mathematics has been a compulsory subject in teacher education only since 1992, which implies that there are many teachers in Norwegian schools (including lower secondary school, grade 7-10) teaching mathematics without any formal competence within the subject.

To start teacher education study, students need what we call a general study competence from upper secondary school (grades 11-13). The first year in upper taught 5 lessons a week out of a total of 30 lessons. To obtain general study competence, students need only this basic course in mathematics from upper secondary school, although it is possible do more. In teacher education before 1992, a short course in mathematical didactics was taught to all pre-service teachers who, at that time, could also choose to study mathematics (per se) for one-fourth, one-half or 1 year of study. From 1992 to 1998 studying mathematics became compulsory in teacher education for all pre-service teachers for one-fourth year of study. Since 1998 it has been compulsory to include one-half year of mathematics, with the option of up to one and one-half years of study. Currently, in upper secondary school, mathematics teachers usually have 1-3 years of education in mathematics from a university, some having a degree in mathematics.

In 1995, a group of experienced mathematics teachers from all levels in the school and college system were asked by the Ministry of Education (KUF) to undertake a survey of the subject of mathematics from primary school to university level. The goal of this 3-year project, MISS–MATEMATIKK I SKOLE OG SAMFUNN (Mathematics in School and Society, Bekken, 1997), was to improve the teaching of mathematics for all students by identifying basic problems and suggesting strategies and initiatives for improvement of teaching competence with reference to teacher education (pre-service and in-service) and to textbooks and teaching material. The background for the project included changes in the need for computational skills in the light of computer technology with increased emphasis on understanding of concepts. Students' and pre-service teachers' attitudes to mathematics were an important focus.

The work of the group relied mainly on three sources: the experience of the members of the group, findings in other (research) documents and some small investigations done by members of the group. Outcomes from the work are to be found in articles in three reports, June 95, 96 and 97 and in a final report from November 97. The articles were written by individuals, discussed and sometimes revised by the whole group before being printed. Thus they vary in reflecting individual or group perspectives.

The final work of MISS concludes with 42 proposals for changes, where 14 are labelled as key proposals. Those significant for mathematics teaching and teacher education include: to give teachers a sabbatical year to study more mathematics; to establish a forum for the didactical development of mathematics teachers; to enable teachers and teacher educators to collaborate in developmental projects in schools; to start research and development projects designed to create an extensive plan for in-service teacher education; to establish a requirement for at least two years of mathematics from upper secondary school in order to start higher education studies involving mathematics (science, economics, and teacher education).

Claim 2 Dilemma: Students entering higher education involving studies in mathematics do not have command of all basic skills in mathematics that one would expect at this level. This is especially dramatic within teacher education. Norway has

Norsk Matematikkråd (NMR), the Norwegian Mathematics Council, has constructed a survey (Halvorsen & Johnsbråten, 2002) which is administered to pre-service teachers at the starting point of teacher education to analyse their performance in mathematics. The test is also given to students entering other studies in mathematics such as engineering or computer science. Ninety percent of the items relate to mathematics that is covered by the syllabus in lower secondary school (grades 8-10). The items test mathematical skills, procedural knowledge and facts rather than students' conceptual knowledge in mathematics. The survey was conducted every other autumn between 1982 and 1991, and every autumn since 1999. Results show that there has been a decrease in performance in recent years. In 2001, 4,737 students participated with an average number of correct answers of 52%. These figures include 732 pre-service teachers. Of all groups pre-service teachers had the lowest average number, 29.5%; 516 of these students had only the basic course in mathematics from upper secondary school. These results reveal that among students starting higher education involving mathematics, pre-service teachers are those who perform lowest with regard to basic skills within the subject. With this background, The Norwegian Mathematics Council has suggested at least two years of mathematics from upper secondary school should be required in order to start higher education studies in teacher education.

Taiwan (Fou-Lai Lin)

Claim 1 Progress: Based on research process and results, several mathematics education courses have been developed in teacher education program.

Both the MUT (Mathematics Understanding of Taiwanese students) program carried out in the eighties (cf., Lin, 1989) and CD–MIT (Concept Development-Mathematics in Taiwan) program conducted recently (cf., Lin & Chen, 2003) studied students' conceptual understanding of most topics in school mathematics. The MUT program has generated a course "Mathematics Learning" for pre-service teachers and the CD–MIT program enhanced its content. The HPM (History and Pedagogy of Mathematics) program in Taiwan has developed more than thirty learning units based on historical text and have published their results in a monthly newsletter, *HPM Forum* (cf., Horng, 2002). Those results shaped the content of "Mathematics History" course towards a pedagogical orientation.

On his website (http://math.ntnu.edu.tw/~cyc/), Chen, Taso and others have demonstrated many learning activities developed with GSP (Geometric Sketch Pad). Those learning activities serve as the foundation for a "Computer and Mathematics" course. Some other mathematics education courses, such as "Mathematics Activity and Thinking", "Mathematics Problem Solving", "Mathematics Teaching and Assessment" have also benefited from the results of varied research projects. By taking those courses, pre-service teachers have experienced multiple didactic views about mathematics, such as mathematics as a model of thinking, school mathematics is about students' thinking and strategies, and mathematics has a cultural and dynamic nature.

Claim 2 Dilemma: Pre-service teachers are still experiencing two contrasting views about learning mathematics, one from university mathematics courses and the other from mathematics education courses they take.

A survey (Huang, 2001) aimed to investigate the learning phenomena of mathematics pre-service teachers has revealed the seriousness of the conflict. To reflect the multiple didactic views about mathematics, instruction in mathematics education courses very often is activity-based, process-oriented and includes multi-media aids. Such a process-oriented view about learning was challenged by pre-service teachers because of their own experiences in learning university mathematics. Within university mathematics classes, how much content should be covered still is the main concern among most mathematicians. To cover the content, their instruction very often keeps a traditional exposition on formal structured content. Pre-service teachers, therefore, have no choice but to experience two contrasting approaches of learning--process-oriented versus content-oriented.

USA (Helen Doerr)

Claim 1 Progress: Pre-service teachers tend to hold beliefs about the nature of mathematics and its teaching and learning that are at odds with views put forth by the Standards documents (NCTM, 1989, 1991, 2000) and by teacher educators.

It is widely accepted in the US that pre-service teachers come to their teacher preparation programs with beliefs about mathematics "as a set of discrete rules best learned through repeated practice. Based on their own experiences as students, prospective teachers think of 'doing math' as a matter of completing a page of forty problems" (Feiman-Nemser & Remillard, 1996, p. 70.) A view of mathematics as doing procedural problems is generally accompanied by an image of teaching as clearly presenting, showing and explaining to students how to follow the rules of mathematics and to do particular problems. In 1992, Thompson provided a detailed review of the beliefs of teachers and later work by Cooney and colleagues (1998) has provided more detailed descriptions of the beliefs structures of pre-service teachers. Work by Frykholm (1996) has documented the difficulties and challenges that pre-service teachers face when attempting to implement standards-based teaching practices that attend to the conceptual development of mathematical ideas through a focus on problem-solving, reasoning, communication, and connections. Frykholm found that the pre-service teachers lacked the tools to implement standards-based lesson and were influenced more by their cooperating teachers who did not make the standards a primary focus of their teaching than by their university-based methods course that did.

The perception of mathematics as centered on the knowledge and application of rules is aptly illustrated by Kinach (2002) who routinely found that pre-service teachers' descriptions of explaining the operations with integers to someone just learning it focused on giving students rules for signs. As Kinach observed, none of the pre-service teachers had any representational notions (other than arrows) to draw on or initially saw the inadequacy of simply telling students the rules rather than providing a good mathematical explanation.

While the beliefs of pre-service teachers, as I have just described them, would resonate well with the experiences of teacher educators and researchers, I have found no systematic, large scale study of the beliefs about mathematics that pre-service teachers bring to their preparation programs. However, I am not suggesting that such studies be conducted, but rather that we shift our focus from the nature and structure of pre-service teachers' beliefs systems—which can often appear to be impermeable and not particularly easy to directly address—to an examination of the issue of the mathematical knowledge that is needed to begin learning to teach.

Claim 2 Dilemma: One of the central dilemmas of learning to teach is found in the struggle of moving past the apprenticeship of observation and the years of experience as a learner of the rules and procedures of mathematics.

Unlike their elementary counterparts, who often found frustration and confusion as they encountered difficulty in trying to make sense of mathematics, pre-service secondary teachers were by and large successful (and often very successful) in their experiences as learners in K-12. Hence, pre-service secondary teachers are less likely to find a practice focused on the mastery of procedures to be problematic. Furthermore, there is no clear evidence as to how or to what extent pre-service teachers' undergraduate experiences in mathematics reinforce notions of mathematics as a fixed body of rules to be mastered. Field experiences at the secondary level (as noted above) may reinforce traditional views of learning mathematics; secondary practice in the US has been especially resistive to change. This, of course, situates pre-service teachers in the gap between the realities of classroom practice and the goals of their preparation programs.

This leaves teacher educators and researchers facing two difficult issues: (1) How and what do pre-service teachers learn about the nature of mathematics as a discipline in their undergraduate experiences with mathematics? And how does this influence their beginning ideas about how others might learn mathematics? (2) How do pre-service teachers negotiate the constraints and limitations of field experiences? To what extent do those experiences impede and support their understanding of the mathematics that is needed for teaching?

PRACTICING TEACHERS—PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: Many examples demonstrate that teachers' deep content knowledge and extensive pedagogical content knowledge improves students'

Claim 2 Dilemma: There is insufficient evidence to convince a skeptic that teachers' deep content knowledge and extensive pedagogical content knowledge improve students' learning.

Evidence for Claim 1 arises in most studies of classroom learning which gather relevant data. For example, for a variety of interesting reasons, about 10-15% of secondary school students are likely to believe that a decimal which looks smaller (e.g., 0.45 looks smaller than 0.4567) is actually larger, and another significantly sized group has a great deal of difficulty with zero, as the number and as a digit in decimal numbers. Teachers who understand these problems can address them in their teaching—others will not and as a result, misconceptions cluster in classes and schools (Steinle & Stacey, 1998). This illustrates an unfortunate but unavoidable feature of research in this area; it is easier to trace the impact of errors and misunderstanding, than of teachers' good understandings. Other research demonstrates that a minimal intervention which demonstrates to teachers how students might be thinking about decimal numbers and provides some targeted teaching tasks can make a long-term difference to children's understanding (Helme & Stacey, 2000). Some of the difficulties that students have, often for years, are not necessarily difficult to fix, but a teacher needs to understand their importance.

I find Claim 1, supported by many small examples, compelling because it gels with my own experiences of teaching mathematics. However, there is little hard data to support it: hence Claim 2. The most internationally influential studies are far from conclusive. Ma (1999) asserts, from only tens of examples, that differences in teachers' ability to make connections among mathematical ideas are largely responsible for the difference in performance between Chinese and U.S. students. The work of Ball (2000) very usefully emphasized how a myriad teaching decisions, such as what questions to ask, what test items to set, what examples to choose, are affected by teachers' knowledge, but again this information is case-based.

There is some large-scale quantitative data to support Claim 1. In considering teachers' characteristics and their association with children's numeracy performance in Britain, Askew et al (1997) identified teachers' recognition of deep connections between mathematical ideas as one of the few predictors of high learning gains by children. On the one hand, effective teachers of numeracy saw mathematics as richly connected and adopted classroom strategies that helped children to make links. On the other hand, the correlations found are surprisingly low. In sum, the data convinces the believers, but not skeptics. Furthermore, the above studies focus on elementary mathematics and elementary mathematics teachers; it is even less clear how to generalize this evidence to the secondary level.

Brazil (Marcelo Borba)

Claim 1 Progress: Online support has been shown to be useful in continuing teacher education projects as means of collaborating with teachers in the implementation information and communication technology in the mathematics classroom.

Online support has been used in continuing teacher education projects, both for research and extension courses. One of these projects (Borba, in press; Gracias, 2003) has been developing 100% on-line courses for teachers in Brazil and in countries like Argentina and Venezuela. This appears to be a solution for continuing education in countries like Brazil—with huge geographical size but with a concentration of research centers in just one small region. For some, the concern with mathematics teacher education is almost synonymous with mathematics education, as pre-service and continuing education can be seen as the "trunk" for all other aspects of mathematics education.

Among the researchers who investigate how teachers deal with the introduction of information and communication technology (ICT), there seems to be a tendency to share one certainty these days: short term courses are positive, but they are far from enough if teachers are to incorporate changes in the classroom. An alternative that goes beyond courses, but without discarding them, is one based on collaborative practices. Penteado and Borba (2000) developed a project that merged short-term courses on basic use of technology and on mathematics education software together with support for teachers to use them in the classroom. Teachers would prepare classes using software with the help of members of the research team who had more experience with given software, and who, at the same time, would help to frame the problems to be investigated (and maybe solved) during this interaction which joined extension courses and research. More recently, researchers such as Penteado (http://ns.rc.unesp.br/igce/matematica/interlk) have been leading a project in which there is collaboration between researchers and teachers in order to provide support for teachers who want to use geometry, function or other types of software in the classroom. Different research projects focusing on the relationship among members of this support network are developed and, at the same time, provide solutions for problems and help to bring teachers into graduate programs.

Studies have shown the transformation of the interaction in these courses, which focus on trends in mathematics education, when we compare it to the regular interaction we had in graduate courses in which teachers and researchers take part. For instance, when we have a synchronous interaction in a chat, multiple dialogues may happen at the same time. Participants may switch from one discussion to another and the teacher may have to deal with several questions and issues at the same time. (See http://www.rc.unesp.br/igce/pgem/gpimem.html for papers on these types of interactions.)

Based on the assessment made at the end of each course (five have been offered so far), this model has had a significant impact in terms of bringing members of different communities into the discussion regarding mathematics education and giving them access to professors from one of the most prestigious mathematics education graduate programs in Brazil with whom they would otherwise not have an opportunity to interact.

Claim 2 Dilemma: Online support also raises problems that are far from being solved

The first problem is related to continuous support, as discussed before. In this sense, we need to have an increasing number of people giving support to teachers who participate in the course if they are to bring change to the mathematics that is taught in the classroom. On the one hand, this can be considered to be more of an extension course problem, but on the other hand, it is a logistical problem for researchers if we want to assess change with teachers who participate in such courses.

The second question is related to the very notion of what mathematics should be taught once a specific function or geometry software is in use. Pre-service teachers should be exposed to changes that software brings to the mathematics in the classroom, as most Brazilian researchers on technology believe. However, there is no such discussion regarding the case of the Internet and distance education. The question, "What kind of change will be brought to mathematical content as Internet use becomes more intense, in face-to-face as well as distance education?" has only recently been posed (Borba, 2004) and as of yet, not even a tentative answer exists. Posing the question in another way, we can think of an example: Does it make sense to spend too much time on techniques of differentiation if we have software that does this rapidly? Is there an equivalent change in content in the case of the Internet?

Another open problem is related to education for teachers who will teach distance education courses. Is it possible to have education for teachers who will teach or participate in distance education courses? In fact, is it possible or desirable to have full distance pre-service education courses? What should be done when participants drop out of courses like this? These are some of the questions which have been addressed in more detail (albeit not answered) by Borba (2004).

Israel (Ruhama Even)

Claim 1 Progress: Conceptual frameworks for mathematical knowledge for teaching are being developed.

A general suggestion for a conceptual framework may be found in Shulman's influential paper (1986) which emphasizes two kinds of understanding of the subject matter that teachers (not necessarily of mathematics) need to have—knowing *that* something is so and knowing *why* it is so. This may seem an almost trivial statement when mathematics knowledge is concerned, although research suggests that quite often teachers know that something in mathematics is so, but not why it is so (Ball, 1990; Even, 1993; Even & Tirosh, 1995.)

The National Council of Teachers of Mathematics (NCTM, 1991) suggests a more detailed mathematical perspective on teacher subject-matter knowledge, stressing that the education of teachers of mathematics should develop their knowledge of the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality; and, in addition, develop their perspectives on the nature of mathematics, the contributions of different cultures toward the development of mathematics, and the role of mathematics in culture and society; the changes in the nature of mathematics and the way we teach, learn, and do mathematics resulting from the availability of technology; school mathematics within the discipline of mathematics; the changing nature of school mathematics, its relationships to other school subjects, and its applications in society (NCTM, 1991, p. 132).

Focusing on the quality of teacher mathematics knowledge, researchers further emphasize the importance of teacher understanding of the 'big ideas' of mathematics, and the connections among and within different ideas, representations and areas of mathematics (Ball, 1991; Even, Tirosh, & Robinson, 1993; Simon, 1993), and of teacher "profound understanding of fundamental mathematics" (Ma, 1999). These qualitative approaches, although some of them are the products of studies that focused on elementary school teachers, are helpful as they acknowledge the complexity of knowing mathematics for teaching and they point at some promising avenues that researchers and teacher educators may explore when designing learning experiences in mathematics for teachers. Still, these approaches do not provide satisfactory answers to questions, such as, what is the meaning of teacher knowledge and understanding about a specific mathematical concept or topic? Is it important that prospective teachers think that the root function is the inverse function of an exponential function, or that they think that the log and the root functions are the same thing? Why is it important, or, Why not? What should a mathematics course for teachers on a specific mathematical topic focus on?

To answer such questions we need a conceptual framework that the mathematics teacher educators could use for the development of mathematics courses for teachers (and the researcher could use to frame studies on teacher subject-matter knowledge of a specific mathematics topic or concept). For this, we draw on a line of research in mathematics education that Dörfler (2003) terms mathematicology–meta-study of mathematics as a human phenomenon and activity. For example, by analyzing mathematical topics from the secondary school curriculum for teaching, using the framework developed by Even (1990). Illustrations of using the framework to analyze the concept of function for teaching (Even, 1990) and the topic of probability for teaching (Kvatinsky & Even, 2002) suggest *what* (but not *how*) needs to be addressed mathematically (e.g., why understanding inverse function is important for secondary school teachers).

Another way to approach the issue of teacher subject-matter knowledge is to adopt a different starting point, as suggested by Ball, Lubienski and Mewborn (2001), and to start with practice in order to uncover knowledge. Ball et al. point out that often teachers do not use what they know, nor does what teachers know fully accommodate the demands of their practice.

Claim 2 Dilemma: What conceptual frameworks for mathematical knowledge for teaching are appropriate?

As of today, there is not enough research in this area. This contributes to the current situation in Israel, where the mathematical preparation of prospective secondary school mathematics teachers is based on traditional advanced mathematics courses with occasionally idiosyncratic innovative courses–the latter dominate enhancement of mathematical knowledge of in-service mathematics teachers.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: The problematic nature of education in mathematics at a variety of levels in Norway is at last being recognized and addressed.

Low competence in mathematics among teachers has been addressed as a possible explanation for low performance among students. However, there have been some positive indications in recent years for practicing teachers as well as pre-service teachers. Now in 2003, teachers who want to take time out to study mathematics can be supported with NOK 100,000 (£10,000). LAMIS, Landslaget for Matematikk i Skolen (the National Society for Mathematics in School) has grown and it receives official support to arrange a conference every summer. There is a nationwide plan for in-service education of teachers (Brekke, et al., 2000) and there have been several collaborating projects in mathematics between teacher educators and teachers in schools. There is an ongoing collaboration project between six colleges of education funded by SOFF, Sentralorganet for Fleksible Læring i Høgre Utdanning (Central Organ for Fexible Learning in Higher Education) where distance-learning or school-based courses in mathematics are offered. In the latter, in-service education takes place as collaboration between teachers and researchers in the classroom.

The project KIM, “Kvalitet i Matematikkundervisningen,” (Quality in Mathematics Teaching) was initiated by the Norwegian Ministry of Education in 1993. Like international studies such as TIMSS and PISA, KIM gives us broad information about students’ knowledge. Its main focus was to direct teachers’ attention to conceptual development in mathematics through materials and guidelines linked to diagnostic testing of students’ conceptions.

KIM developed sets of diagnostic test items. The different sets were linked to a specific part of the mathematics curriculum, and thus intended to cover most of the concepts of school mathematics. Choice of items used in the tests was based on research literature, curriculum and textbooks and was made in cooperation with a group of teachers who conducted trials in several rounds. A national standardisation was carried out at two or three grade levels (e.g., 6 and 9) in which written responses were gathered from approximately 2000 students from 100 schools. A survey of students’ and teachers’ beliefs and attitudes was also conducted. Sadly very few teachers responded to the survey.

Materials produced drew on analyses of the national data obtained from the test items according to identifications of misconceptions and of conceptual obstacles. The associated guidelines for teachers suggest teaching activities designed to create a cognitive conflict for resolution in the classroom. Teachers are encouraged to give students the opportunity to stop and reflect on their actions and experiences in their process of developing a concept. One aim is that students should become aware of their own learning processes. Through use of materials and guidelines, KIM has provided a background for in-service education programs for teachers of mathematics and has become a central focus of pre-service teacher education (Brekke et al, 2000).

Claim 2 Dilemma: Students achievement is lower on national examinations.

Recent research (Alseth et al, 2003) has shown students' performance on national tests to be lower in relation to the L97 curriculum than a similar evaluation of the previous curriculum, M87 (KUF 1987). How should such results be reconciled with materials for teaching development based on students' conceptions and difficulties?

Taiwan (Fou-Lai Lin)

Claim 1 Progress: Multiple didactical views of mathematics are used as content and learning strategies within various teacher professional development programs.

In Taiwan, a generally accepted responsibility of secondary mathematics teachers is helping their students to pass an entrance examination to go on to senior high school at age 14^+ or to the university at age 17^+. The Entrance Examination Center for college organizes workshops to help mathematics teachers develop four types of entrance examination tasks for their students. The four types of tasks developed (conceptual understanding tasks, contextual tasks, argumentation tasks and heuristic tasks) are implemented during workshops. Conceptual understanding tasks assess students' common-sense and intuition of mathematics. Contextual tasks assess mathematics as connections, situational reasoning and modeling. Argumentation tasks assess mathematics as communication and as a deductive system. Heuristic tasks assess comprehension of reading a mathematics text and analogical ability. In addition to the algorithmic nature of the mathematics examinations, these didactic views of mathematics are embedded in the exam tasks and reflect the key nature of mathematics knowledge needed for Taiwan secondary teaching.

Studies on teacher professional development often have designed certain activities as learning strategies for teachers. For instance, analyzing learning cases from practice in which the cases may reveal students' mathematics cognition or affect (Lin, 2003; Leung, 1999; Lee, 2003), developing generic examples, either historical or phenomenological examples (Horng, 2000; Lin, 2000), and communicating the underlying rationale of ones' own teaching to reveal ones' pedagogical values (Chin & Lin, 2000; Leu, 2001). Such activities reflect researchers' didactic views of mathematics.

Claim 2 Dilemma: Didactic views of mathematics besides those of examination mathematics are hardly implemented nationally.

Teaching in Taiwan secondary schools is examination driven. The algorithmic nature of mathematics found in examination mathematics (Lin & Tsao, 2000) is widely adopted by the majority of secondary mathematics teachers in their classrooms.

Cooperating with the Entrance Examination Center to cover different didactic views of mathematics, such as students' conceptions and inappropriate strategy/reasoning and connections, in exam tasks is an effective approach for implementing such views. However, the limitation of examination mathematics, such as a time limit for doing the task, still contradicts some views necessary for teaching well from a teacher's perspective, (e.g., modeling, mathematics with graphic calculators and mathematics investigations). Teachers accept the views of mathematics relevant to the entrance exam very passively. Teachers are waiting for a systematic textbook related to the particular didactic view of mathematics, such as modeling, generic examples, dynamic geometry and so forth. From teachers' professional autonomy point of view, engaging actively in designing learning activities to expand their didactic views of mathematics seems a necessary process for teachers' development.

USA (Helen Doerr)

Claim 1 Progress: The importance of subject matter knowledge is widely agreed upon (CBMS, 2001), despite some claims (Begle, 1979; Darling-Hammond, 2000) that would suggest a ceiling effect beyond which teachers' additional knowledge of mathematics has no added influence on student learning.

Several important areas of secondary teachers' subject matter knowledge have had important beginnings, notably studies on teachers' knowledge of algebra and functions (see Doerr (in press) for an extensive review of this area), but large areas of teacher subject matter knowledge remain relatively unexplored: e.g. statistics, probability, rational numbers, geometry, measurement, and topics in advanced mathematics. For example, several researchers have documented how a limited understanding of the concept of function can restrict the kinds of tasks that teacher choose for students to engage with, the depth of questions that are posed, and the connections that are made within mathematics (Haimes, 1996; Heid, Blume, Zbiek & Edwards, 1998; Wilson, 1994).

Other researchers (Chazan, 1999; Lloyd & Wilson, 1998) have shown how the well-connected content knowledge of the teacher can be used to shift from a procedural approach to a more conceptual approach in the teaching of algebra. Such a conceptual approach emphasized a co-variation as well as a dependence approach to functions, the use of graphs to understand patterns and families of functions, the flexible use of multiple representations and the use of meaningful discussions to support student learning. This line of work suggests that well-connected subject matter knowledge is a necessary condition for expertise in teaching algebra, but such subject matter knowledge is not sufficient for expertise in teaching. In the case of functions, the teachers had transformed their own understanding of the concept into an understanding of the concept *for teaching*, or what Shulman (1986) would call pedagogical content knowledge.

Claim 2 Dilemma: The dilemma facing teacher educators and mathematics education researchers is in understanding how subject matter knowledge becomes transformed into the understanding of the subject that is needed for teaching.

What does such subject matter knowledge look like in practice and how do teachers acquire it? At the secondary level, we are lacking the fine-grained accounts of such mathematical understanding as have been generated around topics in elementary mathematics teaching. See, for example, Ball, Lubienski and Mewborn (2001) for a detailed account of the knowledge needed for teaching multiplication of decimal numbers. Moreover, in elementary mathematics education, the development of teachers' knowledge seems to be enhanced by focusing on their understandings of how students think about various topics and how students' ideas might develop (Fennema et al., 1996). In other words, using students' conceptions is a guiding principle for driving instruction at the elementary school level. Almost no work has been done investigating this same principle at the secondary level. The central question is how would teachers learn to use student thinking in practice?

RESEARCH DESIGN--PROGRESS AND DILEMMAS

Australia (Kaye Stacey)

Claim 1 Progress: We have mastered the art of in-depth case studies and of the careful quantitative analysis of videotapes of randomly selected lessons.

Claim 2 Dilemma: To provide convincing evidence of the nature of knowledge that really makes a difference to secondary mathematics teaching, we need to bridge the gap.

These claims follow from the discussion above about current practice in Australia. I have been impressed by how fully the large-scale TIMSS video studies have been able to describe classroom teaching. Hollingsworth, Lokan and McCrae (2003), for example, give us an unprecedented look at teaching in a random sample of Year 8 (age 13) classrooms, which can be studied from a cross-cultural perspective or as a description against standards. These studies however cannot reveal much about how teachers' knowledge can impact on students' learning, except as noted above occasionally in the negative. Similar studies that look at how the nature of teachers' knowledge impacted on students' learning would need to be designed differently–a topic for discussion.

Brazil (Marcelo Borba)

Claim 1 Progress: Collaborative investigations are the viewed as an effective means to change in schools.

The main consensus related to a research methodology issue is that collaborative research is the way that investigation in this area can lead to change in schools. No one seems to believe that top down models work or that courses for teachers, that take place during vacations or on weekends, are the only way that researchers and teachers should interact. Instead, teachers should collaborate and become researchers in mathematics education research--and this is already happening in Brazil.

Claim 2 Dilemma: Achieving collaboration is not as simple as it looks.

Although there is a consensus that both research agendas and research practice should be developed in a democratic collaboration, there are issues regarding authorship and ethical issues which can make such collaboration a mere formality. For instance, if the researcher is developing a Ph.D. dissertation, even if there is a genuine collaboration of the teacher in the design and development of the research, the authorship of the report and of the analysis belongs to the researcher. Depending on the school and on the content of the research, the teacher may have to suppress his/her name on papers and reports of the problem under scrutiny. Therefore, collaboration is desirable but hard to achieve within the academic and school culture that exists.

Israel (Ruhama Even)

Claim 1 Progress: There is now more appreciation of, and attention to, the complexity of studying the nature of mathematical knowledge for secondary teaching.

Whereas research on student learning has been part of research in mathematics education for almost three decades, reaching a high level of sophistication by means of focus and research design, this has not been the case with research on teachers and teaching. Early Israeli research on teacher mathematical knowledge was mainly evaluative, aiming to measure teachers' knowledge of mathematics.

Data collection for such studies was based mainly on multiple-choice questionnaires, requiring teachers to solve standard mathematics problems. It took time until the mathematics education community began to employ the same level of complexity and depth used in research on students' mathematical knowledge to research on *teachers'* mathematical knowledge. More recent studies on mathematics knowledge for secondary teaching use varied data sources that provide richer information, mainly, open-ended questionnaires and interviews (e.g., Even, 1990, 1998; Hartman, 1997; Leikin, Chazan, & Yerushalmy, 2001; Lipman, 1994; Tsamir, 1999; Shriki & David, 2001; Zaslavsky & Peled, 1994), aiming at better understanding the nature of teachers' mathematical knowledge instead of measuring it. For example, a study that examined the nature of the cognitive processes involved when prospective secondary school teachers work with different representations of functions (Even, 1998) analyzed data from a questionnaire that included non-standard mathematics problems and from an interview that focused on the prospective teachers' explanations of what they had answered on the questionnaire, and why. The results of this study go beyond the conclusion that the prospective teachers had difficulties when needed to flexibly link different representations of functions. Rather, the study illustrates how prospective secondary teachers' knowledge about different representations of functions is not independent, but rather interconnected with knowledge about different approaches to functions, knowledge about the context of the representation and knowledge of underlying notions.

Claim 2 Dilemma: We do not know much about the nature of the interactions between teachers' mathematical knowledge and the practice of secondary mathematics teaching.

How is teachers' mathematical knowledge enacted in the practice of secondary mathematics teaching? The study of the nature of mathematical knowledge for teaching is still often approached cognitively only, and is usually conducted away from the authentic place where this knowledge is enacted, used and constructed—the actual classroom teaching where socio-cultural aspects interact with cognitive ones and where knowledge interacts with practice.

Mathematics teaching relies on deliberate use of knowledge in context. Similar to the dissatisfaction of the mathematics education community for the limited (although important) information obtained from traditional cognitive studies of students' mathematical knowledge and understanding that are conducted outside the classroom, and the consequent expansion of research on students' mathematical knowledge and understanding to classroom studies that incorporate cognitive and socio-cultural aspects (e.g., Hershkowitz & Schwarz, 1999), there is a need to design research studies that focus on studying the interaction of teachers' mathematics knowledge and the practice of (secondary) mathematics teaching; the enactment of mathematical knowledge for secondary teaching in context. This would mean the use of additional data sources, such as, in-class observations and various artifacts (lesson plans, exams, etc.) to be able to answer these new research questions.

Norway (Bodil Kleve and Barbara Jaworski)

Claim 1 Progress: At governmental level, serious recognition of a need to develop research capacity in Norway is resulting in funding being directed at programmes which simultaneously develop research capacity and include teachers in collaborative developmental practices with a research basis.

Under a general title of "Knowledge, Development and Learning," the Norwegian Research Council has granted substantial funding for a four year project in mathematics education. In this programme, didacticians and teachers will work closely to develop 'communities of inquiry' to design classroom activity involving students in inquiry approaches to learning mathematics. Funding includes provision for doctoral stipends so that new researchers can be trained within this programme. Development of inquiry communities draws teachers into design and research activity through which their thinking and teaching develop. Research will be a fundamental basis for development in three ways: 1) Researching activity in workshops in which teachers and didactical work together to explore mathematics, and processes and practices in the learning and teaching of mathematics; 2) Researching teacher group activity in schools in which teachers, with support from their didactician colleagues, design innovative activity for classrooms; 3) Researching teaching of designed innovative activity in classrooms and the associated learning of students. A parallel longitudinal study will explore the status quo of classroom learning and teaching at the beginning and at two further stages within the project.

A new Doctoral Program in Mathematics Education at Agder University College was started in 2002 and given 4 professorships; 8 doctoral students are now registered in the programme and 5 further stipends are advertised. Most of the research generated within this programme involves studies of mathematics learning, teaching and teaching development. Other current moves to build capacity have also been made. A Quality Committee (Kvalitetsutvalget) set up by The Royal Ministry of Educational Affaires, suggests educating resource-teachers in Norwegian, English and Mathematics, and encourages development of master programmes for teachers in the subjects. Several University Colleges in Norway have already responded by developing masters' programmes in Mathematical Education, and are prepared to offer masters studies beginning in 2005.

Claim 2 Dilemma: Despite gaining knowledge through the KIM study about students' learning, and students' conceptions and misconceptions, a recent study (Alseth et al., 2003) shows that students' performance has not improved. Thus we are more aware of the nature of students' knowledge and understanding, but not yet developing this awareness into practices through which learning can be improved.

Instruments and research approaches for studying students' learning, both instrumentally and conceptually, are now well developed in Norway. Despite progress in research-related understandings of students' learning, and opportunities for teachers to be aware of and to use such findings, it appears that recorded learning outcomes are comparatively poor. Thus, research needs to explore relationships between teachers' learning of teaching (both pre-service and in-service) and students' learning of mathematics.

Taiwan (Fou-Lai Lin)

Claim 1 Progress: (Searching for Simplicity) "Making sense of mathematics" as a fundamental view about mathematics teaching has been tested.

Regarding the complexity of mathematics teaching and learning, a simple slogan "teaching for sense making" has been tested within a teacher education program for six years (Lin, 2002). To enhance student's sense making, teaching is encouraged for: developing students' intuition, both first and second order (Fischbein, 1987); situational connection and analogical connection; and assessing students diagnostically.

Being sensitive to the sense students are making about learning content is addressed as the main focus in the teacher education program. Teaching for sense making has been analyzed as a fundamental view about teaching because the teaching strategies have integrated multiple didactic views of mathematics. A group of 30 pre-service teachers have been educated in this program and eight case studies on their teaching in secondary schools were reported as satisfied (cf., Lee & Lin, 2003; Chang & Lin, 2001; Chen & Lin, 2004; Chiang & Lin, 2002).

Claim 2 Dilemma: It is crucial in mathematics teacher education to design a well-tested research program on the development of teachers' multiple didactic views about mathematics that are necessary for teaching well. Such a research program is expected to be able to develop a learning theory for teachers.

Regarding the domination of examination mathematics in a secondary teacher's mind, a well-tested research design that aims to develop teachers' multiple didactic views about mathematics becomes a great challenge. The challenge is not about teachers' understanding but about teachers' constructing of multiple didactic views about mathematics as their beliefs. Taiwan secondary schools might not provide necessary "doubt and evidence," the key elements that changes one's belief, for teachers to change their view with examination mathematics. The expected learning theory derived from such research program might show a strong societal feature.

USA (Helen Doerr)

Claim 1 Progress: A shift in research on teaching over the past 40 years has been from a process-product paradigm towards more naturalistic inquiry into the complexities of teaching practice.

This shift can be described in Schön's (1983) terms as moving from the high ground of technical rationality to the "swampy lowlands" of practice. This has led to a numerous detailed studies on mathematics teaching, especially at the elementary level. This dominance of investigations at the elementary level is reflected in two recent reviews of teacher knowledge by Ball, Lubienski and Mewborn (2001) and Bransford, Brown and Cocking (2000). We do have some studies that are fine-grained analyses of secondary teachers' learning in practice (e.g. Lloyd & Wilson, 1998; Chazan, 1999). We also have a few medium scale studies that give characteristics of the teaching in effective secondary classrooms (e.g., Henningsen, Smith, 1997; Swafford, Jones & Thornton, 1997). However, the methodologies used at the fine-grained level of analysis do not necessarily scale well to medium- or large-scale studies nor are the results of such research easily aggregated across studies. This presents us with several dilemmas.

Claim 2 Dilemma: Understanding the nature of the mathematical knowledge needed for teaching is important at multiple levels of educational practice. However, the design of research studies is plagued by difficult problems of scale, limitations in the usefulness of the forms of results, and challenges in aggregating results across studies.

At the level of policy making and program funding (whether for research, for professional development or for schools), decision makers are confronted with the need to know what is effective and what works in schools under what conditions. Those who design teacher preparation programs and those who certify teachers for jobs in public schools need to know how to make tradeoffs between mathematical coursework and field experiences and how to design courses and experiences that lead to more effective teaching (particularly as measured by student achievement in the current climate). These needs would seem particularly well-served by studies of larger numbers of teachers over a range of conditions (e.g. Schön, Cebulla, Finn & Fi, 2003).

On the other hand, researchers and teacher educators need empirical work based on close observation and embedded in the complexities of practice, attending to the multiple interactions of students, teachers, tasks, curricula, technologies, local school settings, and state policies and mandates. In other words, learning about how teachers learn to teach must be studied in the context of practice. However, such studies are often of the form in which the number of subjects is N=1 or which involve the self-study of teaching, sometimes using a member of the research team. While such studies do provide us with important insights into teacher learning, it remains difficult to scale the methodologies or the results of such studies to larger numbers of teachers.

Another problem of scale can be seen in the dimension of time. It would appear from current research that studies on the development of teachers' knowledge need to be of the order of several years, rather than the several months (or even weeks) that can be sufficient to investigate the conceptual growth of children. The scope of the data collection and analysis are particular problems for research on teacher learning. The potential data sources for understanding teaching are vast, including volumes of student work, reams of observational notes, and boxes of video and audiotape of teaching episodes. Much of this data is not of the form of artifacts or tools that could be used by teachers in the improvement of practice. Much of the resulting analysis is not of the form where findings can be easily aggregated across studies.

SYNTHESIS

During the forum, three matrices will be presented that summarize the claims made above. In the first matrix on the progress and dilemmas in the preparation of teachers, there is consensus among most of the contributors that there has been considerable progress in our understanding that preparation for teaching mathematics is more than knowing advanced mathematics. Although as pointed out by the situation in Norway, knowing mathematics at some level of competence is necessary, but in addition to teach mathematics there is a need for teachers to acquire a 'different' knowledge of mathematics. However, what this mathematical knowledge for teaching is lacks clear definition. In some cases, this knowledge is defined as school mathematics knowledge with specific 'big ideas' such as function, and in others it is seen as distinct from the mathematics of mathematicians. In addition, progress has been made in gaining knowledge of students' conceptions of mathematics, but transposing these conceptions into teaching knowledge is missing. Finally, in preparing teachers mathematically there is still a disconnection between what students experience as mathematics and teaching mathematics in formal mathematics courses and mathematics education courses.

The second matrix addresses claims about progress and dilemmas in terms of mathematical knowledge for practicing teachers and extends the insights drawn from preparing teachers. Here a wider variety of claims and dilemmas exists that extend from acknowledging progress in providing multiple didactic views of mathematics and teachers' understanding in some areas of mathematics to the use of technology as a tool to support practicing teachers. Questions for discussion might include: What is the 'mathematics' that is needed for secondary teaching? How is this 'mathematics' for secondary teaching fundamentally different from the mathematics of advanced courses or mathematicians? What are the elements that define the critical aspects of the mathematical knowledge needed for teaching? How is the question of this forum, what is the nature of the mathematical knowledge that is needed for secondary teaching, connected to the conference theme of diversity and inclusion?

Other discussion questions might be: How can the knowledge of students' mathematical conceptions be didactically transposed in ways that are most useful for teaching? How is teacher's mathematical knowledge transformed when understanding mathematics for secondary teaching?

The third matrix addresses claims about progress and dilemmas in terms of research design. It is clear that qualitative research design and methodology provides valuable insight into mathematics teaching and collaborative research among practitioners, teacher educators and researchers are a means by which to develop not only teaching but research capacity. Yet there is a need for longitudinal studies of mathematical teaching practices, a need to provide evidence for claims of the impact of teacher mathematical knowledge on student learning and a need to define the mathematical knowledge for teaching that can be understood and influence policy at many levels.

Questions for discussion might include: What kinds of research studies might be conducted collaboratively internationally that would address teacher mathematical knowledge in relation to student learning? How can research on teacher knowledge be designed so as to promote the sharing of results in ways that will lead to the development of a knowledge base for teaching? What research designs directly address how changes in teachers' knowledge are generated and sustained beyond the intervention of the research? In other words, what designs enable us to investigate teachers' learning as it occurs and is sustained over time in practice? What kinds of research studies might also be conducted in the same manner that would influence policy on the mathematical knowledge needed for teaching? What research studies might be conducted to address diversity and inclusion?

Participants in this forum are invited to engage in a discussion of these claims, perhaps providing additional supporting or contradictory evidence and additional insights from their particular perspective.

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RF04: CONTRASTING COMPARATIVE RESEARCH ON TEACHING AND LEARNING IN MATHEMATICS

Coordinators: **Jonas Emanuelsson, Göteborg University** and
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International comparative research in mathematics education is a growing field. Experiences from recent and ongoing studies seem to have huge impact on both the field of research and the field of practise. The very idea of both grasping and making use of diversity lies in the heart of all comparative approaches. However there is an ongoing need for enlightened discussion on how the character of these results relate to the research methods and techniques used and the theoretical and analytical perspectives enacted in the research. The main focus of the forum is how these different comparative approaches, and the consequent and profound differences in project outcomes, can inform our individual and collective ways of understanding learning and teaching in mathematics.

GENERAL FRAMEWORK

The idea is to contrast and discuss different approaches, and to discuss both differences and similarities, especially in the character of what we can learn about the learning and teaching of mathematics in classrooms from these studies. What are the possibilities and limitations associated with different approaches? The different types of comparative research that are represented in this forum are:

OECD-PISA, Organisations doing large scale studies with questionnaires and tests

IEA-TIMSS, Organisations doing large scale video studies

LPS, Researchers doing large studies on their own initiatives

Small scale comparative studies

SIMILARITY AND DIFFERENCE IN INTERNATIONAL COMPARATIVE RESEARCH

Schmidt, McKnight, Valverde, Houang and Wiley (1997) investigated the mathematics curricula of the “almost 50” countries participating in the Third International Mathematics and Science Study (TIMSS). The documented differences in curricular organisation were extensive. Even within a single country differentiated curricular catered to communities perceived as having different needs. Countries differed in the extent of such differentiation, in the complexity or uniformity of their school systems, and in the distribution of educational decision-making responsibility within those school systems. Given such diversity, the identification of any curricular similarity with regard to mathematics should be seen as significant. And there were significant similarities. There were similarities of topic, if not of curricular location; broad correspondences of grade level and content that became differences if you looked more closely; differences in the range of content addressed at a particular grade level, but which repeated particular developmental sequences where common content was addressed over several grade levels. In another international study of mathematics curricula, the OECD study of thirteen countries' innovative programs in mathematics, science and technology found that, "Virtually everywhere, the curriculum is becoming more practical" (Atkin & Black, 1997, p. 24). Yet, despite this common trend, the same study found significant differences in the reasons that prompted the new curricula (Atkin & Black, 1996). These interwoven similarities and differences are the signature of international comparative research in mathematics education (Clarke, 2003).

Schmidt, McKnight, Valverde, Houang, and Wiley (1997) reported that differences in the characterization of mathematical activity were extreme at the Middle School level; from 'representing' situations mathematically, 'generalizing' and 'justifying' to 'recalling mathematical objects and properties' and 'performing routine procedures.' Despite the apparent diversity, it was the latter two expectations that were emphasised in the curricula studied. Given the documented diversity, it is the occurrence of similarity that requires explanation. Some curricular similarities may be the heritage of a colonial past. Others may be the result of more recent cultural imperialism or simply good international marketing.

In attempting to tease out the patterns of institutional structure and policy evident in international comparative research (particularly in the work of LeTendre, Baker, Akiba, Goesling, and Wiseman, 2001), Anderson-Levitt (2002) noted the "significant national differences in teacher gender, degree of specialization in math, amount of planning time, and duties outside class" (p. 19). But these differences co-exist with similarities in school organization, classroom organization, and curriculum content. Anderson-Levitt (2002, p. 20) juxtaposed the statement by LeTendre et al. that "Japanese, German and U.S. teachers all appear to be working from a very similar 'cultural script'" (2001, p. 9) with the conclusions of Stigler and Hiebert (1999) that U.S. and Japanese teachers use different cultural scripts for running lessons. The apparent conflict is usefully (if partially) resolved by noting with Anderson, Ryan and Shapiro (1989) that both U.S. and Japanese teachers draw on the same small repertoire of "whole-class, lecture-recitation and seatwork lessons conducted by one teacher with a group of children isolated in a classroom" (Anderson-Levitt, 2002, p.21), but they utilise their options within this repertoire differently.

LeTendre, Baker, Akiba, Goesling and Wiseman (2001) claim that "Policy debates in the U.S. are increasingly informed by use of internationally generated, comparative data" (p.3). LeTendre and his colleagues go on to argue that criticisms of international comparative research on the basis of "culture clash" ignore international isomorphisms at the level of institutions (particularly schools). LeTendre et al. report yet another interweaving of similarity and difference.

> We find some differences in how teachers' work is organised, but similarities in teachers' belief patterns. We find that core teaching practices and teacher beliefs show little

national variation, but that other aspects of teachers' work (e.g., non-instructional duties) do show variation (LeTendre, Baker, Akiba, Goesling & Wiseman, 2001, p. 3)

These differences and the similarities are interconnected and interdependent and it is likely that policy and practice are best informed by research that examines the nature of the interconnection of specific similarities and differences, rather than simply the frequency of their occurrence. This Forum uses brief presentations relating to five different research projects, each representing a very different approach to international comparative research in mathematics education, as a catalyst for discussion of how such research might best inform theory and practice in mathematics teaching and learning.

KEY QUESTIONS

What can be said about the teaching and learning of mathematics in our own countries and how can results be used to reach better performance within our own educational systems?

We have invited researchers that are responsible for very different studies that draw on different paradigms and use different methodological approaches. Furthermore, in order to give a background to the overarching question above, each contribution will address the following questions in relation to their respective study.

What are the goals of the various international comparative studies?

By studying reports and other documents from the studies above we see different aims in comparing countries. Why do we do it? Is it an effort in trying to find good examples of teaching or organisational aspects such as "Lesson study" and implement them in our own country? Are other countries' practises used as mirrors in the quest of trying to understand the practise of our own country? These two approaches can be related to different ways of interpreting your data. Hence producing results of different character.

What is being studied and how does this relate to teaching and learning?

The object of research varies between studies. The "what" we are trying to understand can be exemplified with: Lesson structure, teacher scripts, , negotiation of meaning, object of learning, patterns of interaction and learner practises.

What are the methods of data collection and analysis employed and with what adequacy do they document teaching and learning and their interrelationship?

The perspectives we adopt in our interpretations of these objects also varies. E.g. some studies take their point of departure in the students' perspective others in the teachers'. Furthermore, the theoretical positions are different. They vary both in type (pragmatic, socio-cultural, constructivist, phenomenographical) and in explicitness. The methods and techniques used in producing data vary considerably. Among the group doing classroom research we find examples of studies using audiotape only and some use video recording. Among those using video recording, the number of cameras used varies between one and up to three. Other studies use interviews both as a principal source of information and as a complement to video recordings. This is also true with regard to the use of test and questionnaires as well. There are studies where test and/or questionnaires are the only way of collecting data, in others they are used to collect supplementary information.

GOALS

The forum is intended to deepen the discussion on international comparative studies in mathematics education and their potential contribution to theorising mathematics teaching and learning. This Forum aims to problematise some of the more superficial readings of international comparative research in mathematics education (e.g. league tables of national performance) and move discussion within the community towards a collective and qualitatively more sophisticated reading and utilisation of the results of current and recent comparative studies. Those of us concerned with advancing theory in regard to mathematics teaching and learning must develop strategies to realise the potential of international comparative research in mathematics education to enhance both theory and practice, both in research and in our educational systems.

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WHAT IS COMPARED IN COMPARATIVE STUDIES OF MATHEMATICS EDUCATION?

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INTRODUCTION

Our aim is to discuss what is compared in international comparisons in Mathematics Education. It goes without saying that what is compared constrains what conclusions that are possible to draw from these comparisons. More precisely, presumptions about the phenomenon in focus govern our theoretical understanding as well as the qualities of facts that are collected. That is trivial from a scientific point of view, but not trivial when dealing with comparative studies in Maths Education. In order to penetrate Maths Education comparisons we need to describe what is compared in well known and significant comparative studies in mathematics education. We have chosen the PISA studies, the TIMSS studies and the TIMSS-R studies.

WHAT IS COMPARED?

In most international comparisons of Mathematics Education (ME) it is achievement in terms of test results that is compared. From such *outcome comparisons* we can conclude that students in some countries are doing better than students in other countries. Why this is the case is impossible to tell without further information. But we might also collect data about the prerequisites for learning mathematics, such as the size of per student investments in education in different countries, class size, number of hours in mathematics teaching etc. If the correlation between achievement and prerequisites variables, like those above, were high, we could possibly come up with conjecture, such as one country could boost achievement in mathematics by increasing its investments in education, reducing class size, increasing the number of class hours etc. But such correlation evidence is extremely scarce. If outcome comparisons have such limitations the next move is in a way self-evident, since we need to know what is happening in the *teaching process* in order to understand the outcomes of this process. And this was exactly what the TIMSS-99 did in the most advanced attempt to produce plausible explanations of differences in Maths achievement between different countries. One hundred year 8 classes were selected by random sampling in seven countries. In each class one lesson was video-recorded. When all the data were collected and analysed the results were published on the internet. We could compare the different countries with regard to, for instance: Length of lesson, Time devoted to mathematical work, Time devoted to problem segments, Percentage of time devoted to independent problems, Time per independent problem, Time devoted to practising new content, Time devoted to public interaction, Number of problems assigned as homework, Number of outside interruption, Number of problems of moderate complexity, Number of problems that included proofs, Number of problems using real life connections, Number of problems requiring the students to make connections, Time devoted to repeating procedures, Number of words said by teacher, Number of words said by students, Number of lessons during which chalkboard was used, Number of lessons during which computational calculators was used. Now, it would not be unreasonable to expect several of such factors be correlated with differences in achievement between countries, given that more or less the same Mathematical content has been covered in different countries. But as should be obvious from table 1 below, this was not the case. This means that not only factors like those presented above, referring to *how* Mathematics is taught varied between the countries but also that the content covered, i.e. *what* was taught in Mathematics varied between the countries as well. This in turn means that the characteristics of ME referred to different things in different ways. Small wonder that basically no correlations with achievement were found!

TABLE 1 (4.1.in original) Average percentage of problems per eighth-grade mathematics lesson within each major category and sub-category topic area, by country: TIMSS-1999 (http://nces.ed.gov/pubs2003/2003013.pdf) p 69.

Topic area	AU	CZ	HK	JP2	NL	SW	US
Number	36	27	18	‡	16	42	30
Whole numbers, fractions, decimals	15	13	5	‡	6	20	17
Ratio, proportion, percent	19	4	10	‡	6	19	6
Integers	2	9	3	‡	4	3	8
Geometry	29	26	24	84	32	33	22
Measurement (perimeter and area)	10	6	3	11	9	12	13
Two-dimensional geometry (polygons, angles, lines)	14	15	17	73	15	17	4
Three-dimensional Geometry	5	6	5	‡	9	4	5
Statistics	9	3	2	‡	10	2	6
Algebra	22	43	40	12	41	22	41
Linear expressions	7	16	11	‡	6	5	6
Solutions and graphs of linear equations and inequalities	15	21	23	12	33	14	27
Higher-order functions	6	6	‡	3	3	8	‡
Trigonometry	‡	‡	14	‡	‡	‡	‡
Other	‡	1	‡	‡	‡	1	1

‡Reporting standards not met. Too few cases to be reported. AU=Australia; CZ=Czech Republic; HK=Hong Kong SAR; JP=Japan; NL=Netherlands; SW=Switzerland; and US=United States.

COMPARING DIFFERENT WAYS OF DEALING WITH THE SAME THING

We believe that these results are extremely important because they show what is not critical. So what is critical?

In order to find pedagogically interesting correlations between what the students learn (outcomes) and what happens in the classroom (teaching process), we would need to keep one of the two aspects of teaching invariant and making statements about e.g. how the same thing was taught (instead of making statements how the different things were taught. As a matter of fact Karen Givven – using data from TIMSS-99 – gives, in this research forum, an excellent example that while differences between countries in the frequency of a certain category (problems requiring students to make connections) was not correlated with differences in achievement (for instance, while the best performing country, Hong Kong, had the lowest frequency of such problems; the next best performing country, Japan had the highest frequency) , differences in the way the same kind of problem were dealt with was correlated with differences between countries in achievement.

Another example of keeping the *what-aspect*, or the object of learning invariant and looking at differences in how the same object of learning is dealt with and thus producing pedagogically interesting characterizations of differences between classrooms, we see in Ulla Runesson's and Ida Mok's contribution to this symposium. They found that in the Hong Kong class in their study more different aspects of fractions were taught at the same time compared to Swedish classrooms, where fewer aspects were brought out in a more sequential (than simultaneous) manner. This means that there were different things possible to learn in the Hong Kong classroom as compared to the Swedish classrooms.

We take a third example from our Learner's Perspective Study. The Swedish team was interested in making comparisons of ME in different countries when the same topic was taught. Emanuelsson & Sahlström (2004) compared, for instance, how the geometrical representation of functions was introduced in a Swedish and in an American class.

In the Swedish class the teacher presents three equations:

$K = 15x$; $K = 10x$, $K = 2x$

and the corresponding graphs. The three lines are drawn in three different colours and the three functions are written in colours corresponding to the graphs.

Here, the contrast is between the lines (notably their steepness) and the corresponding differences between the written expressions (notably the coefficient of inclination). It is basically this correspondence that is possible for the students to learn (and even this was made more difficult by using matching colours of equations and lines).

Transcript 1. Student initiatives in relation to a teacher question.

1. Teacher:	WHAT CAN YOU in other words find out from that term or that- that number that stands in relation to X in a formula like this (3.5)
2. Student:	what
3. Student:	what
4. Student:	what are you on about
5. Student:	what
6. Students:	((laughs))
7. Teacher:	we have (.) three formulas
8. Martina:	yes
9. Teacher:	what can one then say on fifteen (.) ten (.) two. (.) directly when you get a such formula then you can say something about these here (.) anyway on their mutual relation ((points to graph)) (1.0)
10. Student:	eh
11. Martina:	in what order they are or
12. Teacher:	yes- no not what order but how they slope
13. Johan:	high low or eh … in between
14. Teacher:	sure (.)

In this case we are seeing the mathematical content as something that is negotiated in classroom interaction. Thus, in order to understand the meaning of the content we need to understand the meaning(s) of classroom interaction. In the US class the students are expected to "compare and contrast", i.e. to describe similarities and differences within five pairs of equations: $y=3x+2$ and $y=-3x-2$; $0x+3y=6$ and $2x+0y=6$; $y=x^2$ and $y=1/x$; $y=1-2x$ and $y=1-x^2$; $2y=x$ and $y=2x$.

Also in this case the students have the opportunities of learning about the correspondence between slope and coefficient of inclination but this relationship is extended to the case of zero slope (horizontal lines) and no slope (vertical lines). Furthermore the students have the opportunity of forming the concept of linear functions through the contrast with non-linear functions. So to the extent that the students learn different things in the Swedish and in the American class, the simplest explanation would be that their opportunities to learn these things have differed.

CONCLUDING COMMENTS

The close relationship between what the students have the opportunity to learn and what they actually learn is logically necessary and has also been empirically demonstrated (Marton & Morris, 2002; Marton, Tsui et al, 2004) and this relationship has been taken as a point of departure for improving learning in ME. By finding out the necessary conditions for a certain group of students to appropriate a certain object of learning and by bringing those necessary conditions about the likelihood of learning is most considerably enhanced (Lo, Marton, Pang & Pong, 2004). So, if we want to understand differences in achievement in ME between students in different countries we must explore to what extent the objects of learning reflected in the achievement test have been possible at all to appropriate. And in order to do so we have to look at how the same objects of learning have been handled in classes in different countries. Now, to us it seems that the international comparative studies such as TIMSS-R och TIMSS are not designed to be comparative in essence, since they show little interest in e.g. keeping the content invariant. So what are they then? We put forwards two kinds of understanding. First, an important side-effect is making of what is important in Maths Education by means of the items that are used in order to measure knowledge in Mathematics. International comparisons such as the PISA or the TIMSS are not only producing data for comparisons, they also produce conceptions of what is important and of value in Maths Education. They are not only comparing, they are participating in the social construction of curricula in Maths Education. This thought is well developed in the work of Ian Hacking (1999). From this point of view, international comparisons are about homogenisation of Maths Education. Second, going back to the correlations between different variables that are sought in international comparisons we find another thought and that is that given a certain correlation it is predicted that some fact will have an impact on another fact. Given what we know about correlations on one side and explanations on the other side such conclusions are of course problematic. But we think that, on a pragmatic level, even the search for correlations between facts is problematic. What we find is an instrumentalistic system of reason, that construct technical directives (von Wright, 1972) based on abstract numerical relations instead of e.g. didactical arguments. Stated otherwise, what is compared in international comparisons of preconditions and outcomes are educational phenomena shrunk to fit an instrumentalistic system of reason. In a word, international comparisons carried out in this way are examples of intellectual thrift of content as well as of educational reason.

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VIDEO SURVEYS: HOW THE TIMSS STUDIES DREW ON THE MARRIAGE OF TWO RESEARCH TRADITIONS AND HOW THEIR FINDINGS ARE BEING USED TO CHANGE TEACHING PRACTICE

Karen Givvin

LessonLab

If our aim is to improve performance in our educational systems, we must first obtain an accurate picture of those systems as they currently exist. To paint a picture of teaching practices in eighth-grade mathematics classes in the United States (and elsewhere) we sought to document and describe average teaching experiences, not exemplary ones. The approach taken in the 1995 and 1999 TIMSS Video Studies was that of a video survey. The marriage of the two research traditions offers a way to resolve the tension between anecdotes (visual images) and statistics (Stigler et al., 2000). Bringing together the two research approaches allowed us to overcome some of the limitations of each. This, along with cross-national comparison, helped provide a detailed description of "typical" classroom teaching.

This forum presents an exciting opportunity to look closely at different approaches to comparative research on mathematics education. We've been challenged by conference organizers to focus on *what* is to be compared in comparative research on teaching and learning mathematics, and *why*. Only then should we focus on *how* comparisons can be done. The idea is that the nature of learning and teaching mathematics, as the substance of comparative studies, needs to come before a discussion of the means and processes of comparison. Beyond this, the goal is to examine what each approach can teach us about improving students' mathematics performance.

WHAT IS TO BE COMPARED?

I'm here to provide the perspective that guided the TIMSS Video Studies. The question of *what is to be compared* within the TIMSS Video Studies can be addressed at multiple levels. At one level, the goal was to examine "typical" teaching. That is, we weren't interested in documenting a particular approach to teaching nor did we set out to examine high- versus low-quality teaching. Likewise, we were not interested in the differential effects of teaching on different categories of students. What we wanted to capture was simply everyday practice as it is experienced by teachers and students in different countries.

The "what" question can also be asked in terms of the aspects of the classroom lessons we examined. The answer is that we coded for a wide array of variables. The variables were chosen and developed by mathematics educators and cultural insiders, and were guided by both the literature and the desire to adequately capture what was seen in the lessons we collected. The time and manpower we had available allowed us to reliably code more than 60 distinct aspects of the lesson, from codes such as interaction pattern, mathematical content activity, and activity purpose, to myriad codes about each mathematical problem (e.g., evidence of real life connections, graphic representations, procedural complexity, and student choice in solution methods), to judgments of student engagement, lesson coherence, and overall quality.

The "what" question can also be asked in terms of what we intended to describe when we reassembled the discrete classroom elements we examined. What we hoped for was to be able to describe systems of teaching. Our thinking was that the individual codes would come together in a coherent way, with particular codes acting to inform others and with a broad set being used to describe and give meaning to the system.

WHY MAKE COMPARISONS?

Because we regard teaching as a cultural activity we began the study with the assumption that many classroom activities would vary little within each country and would be so familiar to cultural insiders that they would become invisible (Geertz, 1984). To describe teaching fully requires exploring it in relation to that seen in other countries. Examining different cultures helps us see what is commonplace in our own classrooms (Stigler & Heibert, 1999; Stigler, Gallimore, & Heibert, 2000) and being forced to explain classroom events (or the absence of particular features) to cultural outsiders helps draw our attention to details that are otherwise transparent to us. Beyond this, examining practices across cultures can help us discover pedagogical alternatives. One might, for example, see unfamiliar ways to pose problems, to organize how students work on problems, or for teachers and students to interact. Discovering alternatives can in turn lead to a discussion of pedagogical choices. The TIMSS Video Studies were conducted with these goals in mind.

HOW WERE DATA AND RESULTS PRODUCED IN THE TIMSS VIDEO STUDIES?

With some *whats* and *whys* behind us, we may turn to *how* we approached the process of comparing mathematics teaching and learning. The approach we took was that of a video survey. As with traditional survey methods, and in order to arrive at an "average," we began with large, nationally representative samples. Using a national sample provides information about students' common experiences. It is important to know what teaching looks like, on average, so that national discussions of teaching focus on what most students experience. The survey quality of the research speaks to the theme of this year's conference: inclusion and diversity. By conducting a national sample we made a best effort at capturing the full range of teaching, not intentionally limiting what we sampled. By applying to it a wide array of codes, we were poised to capture the diversity in teaching within and across countries.

Unlike traditional survey methods, we didn't use a questionnaire as our primary data source. We instead used video. Videos offer the ability to conduct a detailed examination of complex activities from different points of view. They preserve classroom activity so it can be slowed down and viewed multiple times, by many people with different kinds of expertise, making possible in-depth descriptions of many classroom lessons. The marriage of the two research traditions offers a way to resolve the tension between anecdotes (visual images) and statistics (Stigler et al. 2000).

In the more recent and larger-scale of the two studies, we examined between 50 and 140 lessons (one per participating teacher) from each of seven countries. The TIMSS Video Studies were studies of teaching, so the primary of our two cameras focused on the teacher. The second, stationary camera was fixed on students and was, in the end, used for classroom analysis only rarely. The videos were supplemented by a teacher questionnaire. Items on it were sometimes used to clarify lesson events, but the questionnaire was more generally used to round out the picture of teachers in each country (e.g., years of experiences, education) and their perceptions of the videotaped lesson.

Not surprising based on the data collection design, the TIMSS Video Studies report statistically-based characterizations of the ‘typical lesson.’ For each of the codes examined we can explore the frequency of occurrence across lessons (or across mathematical problems) by country. Examined singly, the codes provide a fine-grained description of classroom practice. Organized by concept, they can paint a nuanced picture of teaching in each country – one that can then be compared across countries.

Although we feel strongly about the affordances of our research approach, we recognize the limitations of it as well. Foremost is the enormous cost of such an undertaking. With respect to the data collection procedure, we can say nothing of how teaching plays out over a series of lessons, how teachers of varying competence teach or the degree of variance within the practices of competent teachers, or of how classroom events are perceived by either the teacher or the students. (Fortunately, for some of these goals one can turn to David Clarke’s Learner’s Perspective Study.) The design of the TIMSS Video Studies also makes it impossible to make a direct link between classroom practice and student achievement.

WHAT CAN BE LEARNED FROM THIS APPROACH AND HOW CAN RESULTS BE USED?

With this approach, we were able to answer questions such as (1) whether teachers in all high-achieving countries teach as those do in Japan, (2) with Japan aside, whether teachers in high-achieving countries share a common pedagogy, and (3) what, if any, features most higher-achieving countries have in common. With regard to the last question, Lindblad and Marten correctly point out that we had difficulty finding lesson features that correlate with differences in achievement. There was at least one feature, however, that appeared to have such a correlation. I’d like to expand on it and on how we've begun to use the finding to make an impact on teaching and learning in the United States.

The code to which I'm referring is called "making connections." In the TIMSS 1999 Video Study, it was found that U.S. students, in typical classrooms, rarely had opportunities to engage in challenging work during their eighth-grade mathematics lessons. Although U.S. teachers posed problems with the potential for rich mathematical learning just as frequently as did teachers in the other, higher-achieving, countries, they almost never maintained the problems at this conceptual level as they were worked on and discussed.

Taking a detailed look, column B of Table 1 shows that the percentage of problems categorized as making connections varied across countries, and even among the high-achieving countries. While Japan exceeded all of the other countries on this dimension (54%), the presentation of making connection problems in the other high achieving countries looked more like that found in the U.S. This suggests it is not necessary to present a high percentage of rich problems in a single lesson in order to produce high levels of mathematics achievement.

Table 1. Types of Problem Presentation and Implementation in the TIMSS 1999 Video Study

	A	**B**	**C**	**D**
Country	Average TIMSS 1995 mathematics score	% of Problems Presented as Making Connections	% of Making-Connections Problems Implemented As Making Connections	% of Making-Connections Problems Converted to Lower-Level Problems
Japan	581	54	48	52
Hong Kong SAR	569	13	46	54
Czech Republic	546	16	52	48
Netherlands	529	24	37	63
Australia	519	15	8	92
United States	492	17	0	100

However, presenting a rich problem is one thing, exploiting it for rich learning opportunities is another. The data in Column B considers only how a problem is presented, not how the problem is implemented as the lesson unfolds. A making connections problem, for example, might be converted into a using procedures problem as a teacher works it through with the class. When we look at how problems were implemented, quite a different story emerged. In Column C, we present the percentage of making connections problems that were actually solved by making connections (i.e., not transformed into using procedures, stating concepts, or giving results only problems).

Comparing Columns B and C reveals a striking difference. The data in Column C reveal something that high-achieving nations have in common. It is not the percentage of rich problems presented but the way they are implemented in the lesson that distinguishes them from the United States (and, to some degree, Australia). Most of the making connections problems in the United States were converted into lower level problems (see Column D). Instead of using these problems as opportunities to explore and reason about mathematical concepts, U.S. teachers typically broke them into procedural elements and took students through the procedures step-by-step (Hiebert, et. al., 2003).

This pattern of results can be interpreted in various ways. First, it is possible that U.S. teachers lack the content knowledge that would be necessary for them to facilitate rich discussions of mathematics (Ma, 1999). Another hypothesis is that U.S. teachers have little experience – either as teachers or when they were students themselves – engaging in conceptually rich discussions of mathematical problems. Again, we argue that teaching is a cultural activity, varying more across cultures than within. If this is true, then it will be difficult for teachers to practice instructional strategies that are rare in their own culture, and thus less likely that they would have observed many examples of others doing so.

Whatever the merit of these interpretations, the making connections results from the TIMSS 1999 Video Study suggest a potentially cost-effective strategy for professional development of mathematics teachers. In brief, these data indicate that:

U.S. mathematics curricula already include rich problems that lend themselves to conceptually rich discussions, and that U.S. teachers typically present as many rich problems to their students as teachers in high achieving countries;

Gains in student mathematics achievement might be obtained if U.S. teachers more often implemented these rich problems by maintaining their complexity, rather than converting them to using procedures problems.

These results suggest a way to improve mathematics achievement using existing curricula and programs, and provide the rationale for our current intervention study. With a grant from the Institute for Education Sciences, we are attempting to teach teachers to identify and effectively implement mathematically rich problems in their pre-algebra lessons. Our plan is to assess the impact of the training on (1) teachers' knowledge of mathematical content for its use in the classroom (i.e., pedagogical content knowledge), (2) teachers' ability to present rich problems in their lessons and maintain high conceptual levels of implementation, and (3) students' mathematical achievement. During the first of two years of implementation, we will compare the teachers and students who receive our professional development training with those who do not. During the second year of implementation, the control group from the first year will receive our PD program as well. The plan will allow us to examine the effects of the program on two groups of teachers, as well as enable us to assess its effectiveness when used over two consecutive years.

SUMMARY

As the pendulum swings to and fro between pedagogical movements and with the comings and goings of popular practice and political policy, what often becomes overlooked is the need to obtain a clear picture of the state of everyday practice. The absence of such an understanding prevents programs from being adequately informed by teacher and student needs.

With the TIMSS Video Studies, our interest was to describe everyday teaching across different countries via a methodology we refer to as video survey. Results indicated possible ways of improving classroom teaching. We're currently pursuing one of those ideas – a focus on implementing problems with a conceptual focus – in our current intervention program.

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LEARNER'S PERSPECTIVE STUDY: DEVELOPING MEANING FROM COMPLEMENTARY ACCOUNTS OF PRACTICE

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By examining mathematics classroom practice over sequences of ten lessons, the Learner's Perspective Study provides data on the teacher's and learners' participation in the co-construction of the possible forms of participation through which classroom practice is constituted. The use of post-lesson video-stimulated interviews offers additional insight into participants' intentions, actions and interpretations. Complementarity of account is possible on at least three levels: Between study participants (teacher and students – through both videotaped classroom actions and post-lesson reconstructive interviews); Between project researchers (through parallel analyses of a common data set); and, Between projects (eg LPS and TIMSS video studies). All three are important.

INTERNATIONAL COMPARISONS OF CLASSROOM PRACTICE

The practices of classrooms are the most evident institutionalized means by which the policies of a nation's educational system are put into effect. Given this, the classroom seems a sensible place to look for explanations and consequences of the differences and similarities identified in international comparative studies of curriculum, teaching practice, and student achievement (see Clarke, 2003).

If we are to engage in international comparative research, there are two quite distinct methodological alternatives:

Alternative 1.

If two groups of objects are to be compared then one approach is to consider these two questions:

Difference – "What is the characteristic about which the *comparison* is to be made?"

Similarity – "How might each group of objects be separately *typified* with respect to that characteristic?" The international comparison of national norms of student achievement could be described as conforming to this approach.

The order in which these two questions are posed is a major methodological signature.

Alternative 2.

If two groups of objects are to be compared, consider these two questions:

Similarity – "Which characteristics appear to *typify* this collection of objects?"

Difference – "What *comparisons* can be made between these two groups of objects using the identified characteristics?"

Posing the questions as in Alternative 2 reduces the danger of constraining the data to a predetermined structure, but may lead to the typification of the two groups by different emergent characteristics, restricting the common bases on which comparison of the two groups might be made. Note: Alternative 2 assumes a domain within which comparison is sought, such as classroom practice or curricular policy.

For example, it might be that for one nation or culture there is no nationally characteristic structure to the lesson as a whole, but that particular types of idiosyncratic lesson events offer the most appropriate typification. For another nation or culture, there could be a high degree of regularity to the composition of lessons, or in the sequencing of particular types of instructional activity in the delivery of a topic. Such differences in the form of typification provide a basis for international comparison that reflects something more essential to each than the identification (imposition) of the same structural level as the basis for the comparison. The methodological choice of Alternative 1 makes the basis for comparison a matter of prescription based on either theory or on the prevailing educational priorities of the country conducting the study. Choice of Alternative 2 makes the identification of possible bases for comparison an empirical result of the research.

DATA IN THE LEARNER'S PERSPECTIVE STUDY

The Learner's Perspective Study documented sequences of ten lessons, using three video cameras, and supplemented by the reconstructive accounts of classroom participants obtained in post-lesson video-stimulated interviews, and by test and questionnaire data, and copies of student written material (Clarke, 1998, 2001, 2003). In each classroom, formal data collection was preceded by a one-week familiarization period in which the research team undertook preliminary classroom videotaping and post-lesson interviewing until such time as the teacher and students were accustomed to the classroom presence of the researchers and familiar with the research process. In each participating country, the focus of data collection was the classrooms of three teachers, identified by the local mathematics education community as competent, and situated in demographically different school communities within the one major city. For each school system (country), this design generates a data set of 30 'well-taught' lessons (three sequences of at least ten lessons), involving 120 video records, 60 student interviews, 12 teacher interviews, plus researcher field notes, test and questionnaire data, and scanned student written material. Data collection is complete in Australia, China, Germany, Hong Kong, Japan, the Philippines, South Africa, Sweden, and the USA; underway in Israel and Korea; and planned for the Czech Republic, England and Singapore. The teacher and student interviews offer insight into both the teacher's and the students' participation in particular lesson events and the significance and meaning that the students associated with their actions and those of the teacher and their classmates.

COMPLEMENTARITY AS ESSENTIAL

Complementarity Between Participant Accounts: Establishing the Co-Construction of Classroom Practice

Like Wenger (1998), Clarke's (2004) analysis of patterns of participation in classroom settings stresses the multiplicity and overlapping character of communities of practice and the role of the individual in contributing to the practice of a community (the class). Clarke (2001) has discussed the acts of interpretive affiliation, whereby the learners align themselves with various communities of practice and construct their participation and ultimately *their* practice through a customizing process in which their inclinations and capabilities are expressed within the constraints and affordances of the social situation and the overlapping communities that compete for the learner's allegiance and participation. By examining sequences of ten lessons, the Learner's Perspective Study provides data on the teacher's and learners' participation in the co-construction of the possible forms of participation through which classroom practice is constituted (cf. Brousseau, 1986). An example of utilizing the complementarity of teacher and student accounts can be found in Clarke (2004), which examines the legitimacy of the characterisation of kikan-shido (Between-Desks-Instruction) as a whole class pattern of participation, and to situate the actions of teacher and learners in relation to this pattern of participation. By drawing on classroom video evidence and juxtaposing teacher and student interview data, it is possible to demonstrate that while engaging in kikan-shido, the teacher and the students participate in actions that are mutually constraining and affording, and that the resultant pattern of participation can only be understood through consideration of the actions of all participants. A key characteristic of kikan-shido, as it is practiced in the Australian classrooms, is the implicit devolution of the responsibility for knowledge generation from the teacher to the student, while still institutionalizing the teacher's obligation to scaffold the process of knowledge generation being enacted by the students. Comparison with the enactment of kikan-shido in other classrooms (Hong Kong, Shanghai, and San Diego, for example) provides significant insight into the pedagogical principles underlying the practices of different classrooms internationally.

Complementarity Between LPS Researcher Accounts: A More Comprehensive Portrayal of Classroom Practice

Classrooms are complex social settings, and research that seeks to understand the learning that occurs in such settings must reflect and accommodate that complexity. This accommodation can occur if your data collection process generates a sufficiently rich data set. Such a data set can be adequately exploited only to the extent that the research design employs analytical techniques sensitive to the multifaceted and multiply-connected nature of the data . . . we need to acknowledge the multiple potential meanings of the situations we are studying by deliberately giving voice to many of these meanings through accounts both from participants and from a variety of "readers" of those situations. The implementation of this approach requires the rejection of consensus and convergence as options for the synthesis of these accounts, and instead accords the accounts "complementary" status, subject to the requirement that they be consistent with the data from which they are derived, but not necessarily consistent with each other, since no object or situation, when viewed from different perspectives, necessarily appears the same (Clarke, 2001, p. 1). In the LPS project, multiple, simultaneous analyses are being undertaken of the accumulated international data set from a variety of analytical perspectives. For example, while Ference Marton and his colleagues in Sweden and Hong Kong analyse the practices of classrooms in Shanghai from the perspective of Marton's Theory of Variation, Clarke is undertaking analysis of the same lessons in relation to the Distribution of the Responsibility for Knowledge Generation. These two analytical approaches do not appeal to the same theoretical premises, but nor are they necessarily in conflict. They represent complementary analyses of a common body of data, aspiring to advance different theoretical perspectives and to inform practice in different ways.

Complementarity Between Project Accounts: Approaches to Studying Lesson Structure

Lesson structure can be interpreted in three senses:

- At the level of the whole lesson - regularity in the presence and sequence of instructional units of which lessons are composed;

- At the level of the topic – regularity in the occurrence of lesson elements at points in the instructional sequence associated with a curriculum topic, typically lasting several lessons;

- At the level of the constituent lesson events – regularity in the form and function of types of lesson events from which lessons are constituted.

A research design predicated on a nationally representative sampling of individual lessons, as in the TIMSS Video Studies (1995 and 1999), inevitably reports a statistically-based characterization of the representative lesson (the first of the alternatives listed above). The analysis of video data collected in the first TIMSS video study (Stigler and Hiebert, 1999) centred on the teacher's adherence to a culturally-based "script." Central to the identification of these cultural scripts for teaching were the "lesson patterns" reported by Stigler and Hiebert for Germany, Japan and the USA, and the contention that teaching in each of the three countries could be described by a "simple, common pattern" (Stigler & Hiebert, 1999, p. 82).

The characterisation of the practices of a nation's or a culture's mathematics classrooms with a single lesson pattern has been problematised by the results of the Learner's Perspective Study (see www.edfac.unimelb.edu.au/DSME/lps). The recent report of the TIMSS 1999 Video Study (Hiebert et al., 2003) employed 'lesson signatures' rather than 'lesson patterns' to characterize differences between the practices of international mathematics classrooms internationally. These lesson signatures characterize national norms of practice in terms of the prevalence of different activity types at different points in the lesson. The resultant 'signatures' remain insensitive to the location of the sampled lesson(s) within a topic sequence. As such, they can give a misleading impression that the structure of any particular lesson is independent of whether it is the introductory lesson at the commencement of a topic, a consolidation or developmental lesson later in the topic sequence, or a summative lesson occurring towards the end of a topic. Nonetheless, the TIMSS data offers the opportunity to estimate the prevalence of a particular activity type identified as significant from LPS data. Similarly, activities identified in the TIMSS project as prevalent within a particular country can be evaluated from within the LPS data in relation to their capacity to stimulate specific responses in students, particularly learning outcomes. The complementarity of these two projects is acknowledged and valued by both research groups.

CONCLUDING REMARKS

This paper has offered complementarity of accounts as an essential methodological and theoretical stance, adopted by the Learner's Perspective Study, for the explication of mathematics teaching and learning in classroom settings, the advancement of theories relating to such settings, and the informing of practice in mathematics classrooms. This paper and the Research Forum of which it is a part embodies the PME conference theme of 'inclusion and diversity' in a very fundamental way.

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LESSONS FROM A SMALL-SCALE OBSERVATIONAL STUDY: AN EXAMPLE OF THE TEACHING OF FRACTIONS

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The aim of this paper is to demonstrate what a small-scale project can tell about features of teaching and learning in two different cultures. We argue that some features, which may not be easily observed within one culture, can become more visible in the contrast in order to get a better understanding of the teaching practice per se, even from a small scale project. We have studied the mathematics teaching in one classroom in Hong Kong and four in Sweden. Based on the assumption, that how the content is taught has an important implication on what students may possibly learn, we compared how the teaching of the same topic (fraction) may differ between the two places. Some profound differences regarding how the same topic was dealt with in the two countries were found. In the Hong Kong data several things were handled in one lesson at the same time whereas in the Swedish data this happened in a sequence of lessons spreading over a substantial period.

INTRODUCTION

Being in an environment constantly, one usually takes things for granted and fails to see the characteristics of the environment as special or different from the others. To bring about a better understanding of mathematics teaching itself is one argument for comparative studies (Stigler & Hiebert, 1999; Lopez-Real & Mok, 2002). However, comparison can be made at different levels and with different focus. Mostly, these studies are to different extent grounded in data from more extensive data sets (e.g. TIMSS Video Study and in the PISA-project). But, are these very expensive and extensive studies the only way to bring about insights about cultural differences? The study we will report here captures a small number of mathematics lessons in Hong Kong and Sweden. In Sweden five consecutive lessons from four different classrooms and in Hong Kong only one lesson were studied. Compared to the extensive studies mentioned, our study can appear to be too thin and insufficient to generalize anything about mathematics teaching in the two countries. However, our aim was to some extent different from these studies the overall aim of which was to compare the teaching practices in different cultures. This was not the goal for our study. Instead we hoped that some features, which may not be easily observed within one culture, would become more visible in the contrast in order to get a better understanding of the teaching practice *per se.* The question whether this is possible even within the frame of a small-scale project will be discussed in this paper.

THEORETICAL FRAMEWORK: A THEORY OF VARIATION

In this study, we approach classroom learning with a specific focus. Assuming learning is always learning of something – it has an object - we study how the object of learning is constituted in the classroom interaction, and with the particular interest in different possibilities to learn in different situations. What is possible to learn, has to do with those aspects of the object of learning that are possible for the learners to be aware of, or to discern. However, only that which is varying can be discerned (Bowden & Marton, 1998). So, the possibility to discern an aspect has to do with whether it is present as a dimension of variation or not (Marton & Booth, 1997; Marton, Runesson, & Tsui, 2004). If the particular aspect is present as a dimension of variation, it is likely discerned by the learner. And further, if the aspects are present as dimensions of variation at the same time, the learners likely discern them at the same time. So, what is studied is the pattern of simultaneous dimensions of variations related to the object of learning that are present to the learners in the situation (Runesson & Marton, 2002). And when studying the differences in possibilities to learn in different classrooms, it is the difference between the patterns of simultaneous dimensions of variations opened in the different classrooms that we describe.

THE STUDY

The current study has its origin in a previous study of Swedish mathematics classrooms, which aimed at finding differences between the teachers as regards how the topic was handled (Runesson, 1999). To shed new light on this data, a similar study in Hong Kong was conducted. The aim was to find differences between how the same topic was taught by contrasting mathematics teaching in two different cultures. However, to be able to see critical features in our own classrooms and one's own culture, we chose the same mathematical topic in order to see how the same topic can be handled in different cultural context. Therefore, the selection of the Hong Kong data set was made on the basis of matching up with the existing Swedish data as much as possible. The Hong Kong lesson was a primary four (age 10, grade 4) lesson on the topic "Comparing fractions". The lesson was carried out in Cantonese and videotaped. The Swedish data is drawn from a larger data set consisting 20 lessons from four different classrooms in grade six and seven. These lessons were audio taped and transcribed verbatim. Our aim was to be as close as possible with regards to the content of teaching. That is, when sections of the Swedish data were selected, this was done at the level of sub-constructs of fractions. The sub-constructs of fractions, which were available in the Hong Kong data, did appear in four of Swedish teachers' teaching. The analysis is grounded on data from all of these classrooms. Due to differences between the Swedish and the Hong Kong curriculum, we could not match the age of the pupils in the two countries. The topic was taught in grade six and seven (age 12 and 13 respectively) in Sweden and in grade four in Hong Kong (age 10). And although, we tried to come as close as possible to study the same content, some differences occurred. In the Hong Kong lesson the students worked with finding the common denominators of two fractions.

In the Swedish lessons the tasks was slightly different; the task was to find another fraction with the same value (e.g. 2/6=1/3). However, in both the Hong Kong and the Swedish lessons, comparison of fractions with different denominators was found. Unlike the Hong Kong data, which is drawn from one single lesson, the Swedish data consist of several lessons.

SUMMARY OF THE RESULTS: TWO DIFFERENT EXAMPLES OF SIMULTANEITY AND VARIATION

The analysis was with a particular focus on those aspects of the topic taught that were opened as dimensions of variation were identified. The Hong Kong lesson appears to have only one objective, i.e. comparing fractions with different denominators. Nevertheless, this objective was visited and revisited via several tasks, which were either in the form of questions in the worksheets or examples on the board. As a result of this, the Hong Kong lesson shows a pattern of variation, which consists many dimensions of variation. For example, some dimensions are: alternative representations of the method of amplification, the denominators, the fractional parts of different wholes and the contrast between the methods of comparison. Moreover, the intertwined relationship between these dimensions of variation forms a special arrangement or simultaneity of variation in a single lesson. Such experience is important because it provides a chance for “fusion” i.e., for the students to consider several aspects of the object of learning simultaneously (Marton, Runesson and Tsui, 2004). The Swedish lessons showed a very different pattern of variation. The most striking difference was perhaps that variation of methods was not opened. The students were presented to one method only, a diagrammatic method. Instead of varying the method, the teacher demonstrated a method on a couple of different examples. The other apparent difference was the sequential character identified in the Swedish lessons. We found that these sub-constructs were commonly never presented simultaneously in the Swedish lessons, but instead they were extended over time and presented as disjoint instances without any connection or reference to previous lessons. So, finding the bigger of two different fractions was taught in one lesson, and "fractions with different denominators but with the same value" was taught in another. The latter was taught with no reference to how this had been presented earlier although the two topics were indeed connected. In other words, since in the Hong Kong lesson several sub-constructs were presented and related to each other at the same time, the Hong Kong lessons were richer in terms of sub-constructs related at the same time.

Comparing to the Hong Kong lesson, the Swedish examples created a narrower space of variation, and in combination with the sequential character, accomplished a quite different space for learning in the Swedish lessons. From the theoretical position taken, we can assume that what was possible to discern of the same thing was different in Hong Kong and in Sweden. In other words, the students’ understanding of the two sub-constructs “comparison of fractions” and “fractions with the same value” are very likely to be different when the students from the two places experience such different space of learning. So, what we can say is, it was possible to discern different things in Hong Kong and Sweden. But, what that means for what the students actually learned, we cannot say, since this has not been studied.

WHAT COULD COME OUT OF A SMALL-SCALE OBSERVATIONAL STUDY?

The study presented here is in many respects a small-scale project, so what could possibly come out of such, as it seems, limited project?

The original purpose of this study was to shed new light on a study conducted earlier in Swedish classrooms. In line with the theoretical framework taken, discovering something new when revisiting the data would be easier if it was contrasted against something different, e.g. by contrasting mathematics classrooms and possibilities to learn in different school systems and educational traditions. The object of research in this study was not possibilities to learn in a general sense, but possibilities to learn *the same thing.* Therefore, it was important to study how the same topic was dealt with, i.e. to keep the content constant. This design has been used in a number of studies (Runesson, 1999; Marton & Morris, 2002; Marton, Tsui et al., 2004) However, it was in many ways a bit problematic to match up with a data set from Hong Kong to the existing Swedish data. From our point of view we wanted to delimit our definition of “the same topic” as much as possible. “The same topic” was defined in terms of how it appears in classroom practice, and on the level of tasks, so we asked the teacher to invite us to study a lesson when fractions with different denominators would be the topic taught. Although, we tried to come as close as possible to study the same content, some differences occurred. Being restrictive to having the same topic, it was not possible to study pupils of the same age, due to different curricula in the two countries. However, from this point of view the result is interesting. In the Hong Kong classroom the pupils were about three years younger than their counterparts, however a space of learning consisting of many simultaneous dimensions of variation was afforded to the learners, whereas for the older Swedish pupils dimensions of variation were brought out in sequence.

It could be argued that this sequential pattern of variation was a result of the longer period of observed lessons in the Swedish data, that the likelihood of such a finding is bigger if several consecutive lessons are observed. It could not be excluded that the sequential character of handling the object of learning, which was found in the Swedish data, would not appear in Hong Kong. This has never been claimed, and it was never the purpose of the study either, i.e. to say anything about the general in the two cultures. What we have described is two different ways of handling the same topic, or two different patterns of variation and simultaneity when teaching the same topic. This was found by comparing two different school cultures.

The way we worked in this study, implied doing a close and narrow analysis, but without the aim of finding more overall patterns or a more general character in the different classrooms. A main difference between, for instance, the TIMSS Video

Study and ours is what we were studying. To us the TIMSS Video Study was a study of teaching, whereas ours is a study of possibilities to learn the same thing. In our study we identified and described how the same object of learning could be dealt with differently by means of examples from different cultures. This was possible to do, even if only one single lesson from one teacher from each country was studied.

Necessarily a small-scale project like this touches the issue of representatives. Our aim was not to come up with something that could tell us something about the possibilities to learn about fractions with different denominators in Swedish and Hong Kong classrooms *in general,* or to explain differences in learning outcomes between the two countries. Instead we wanted this study to open our eyes to that, which is not easily seen within our own culture, so it would become visible, but without saying anything about the typical Swedish or the typical Hong Kong classroom. The most prominent coming out of this study is, that by seeing what could be done differently, what *could be the case*, new light has shed light on what *is* done and *what is the case* in some classrooms our own countries. When the characteristics identified from the two different data sets are used as a mirror, it gives us a better understanding of the practice in our countries. Surprisingly, such understanding could be achieved from a small-scale study like this.

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THE PISA-STUDY: DIFFERENTIATED ASSESSMENT OF "MATHEMATICAL LITERACY"

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More than some international comparative studies before, PISA (Programme for International Student Assessment) referred to conceptions from Mathematics Education. Specific backgrounds of PISA are

- the ideas behind realistic mathematics education as formulated by Freudenthal and de Lange, and

- in the German national option of PISA, the distinction of cognitive activities required in the item, esp. conceptual vs. procedural thinking (Hiebert).

According to that theoretical basis, PISA gives insight into the structure of mathematical achievement. These "profiles" bear messages which come closer to the needs of the development of mathematics education in the countries.

INTENTIONS OF THE INTERNATIONAL PISA-STUDY

PISA, the "Programme for International Student Assessment" is a study initiated by the Organization for Economic Cooperation and Development (OECD). PISA compares the achievement of 15 years old students in the (industrialized) OECD-countries in the domains Reading, Mathematics and Science (OECD 1999). The first test was held in 2000, followed by tests in 2003 and 2006. See OECD (2001) for results of PISA-2000; PISA-2003 will be released in Dec 04.

That PISA tests an age based sample of students, but not a grade based sample as many other studies did, has its origin in the political intentions of PISA. The OECD wanted to gain insight into the outcomes of the educational *systems* in the countries. Therefore, to choose the 15 years olds was decided according to the idea that this is the age of transition to vocational training or to an extended secondary school career. However, this decision once made has the consequence that a "core-curriculum" approach, as e.g. TIMSS has chosen, should not be appropriate. In apparent contrast, PISA focused on "mathematical literacy" which is intended to target the resulting abilities acquired in school up to the age 15, from whatever schooling it may come.

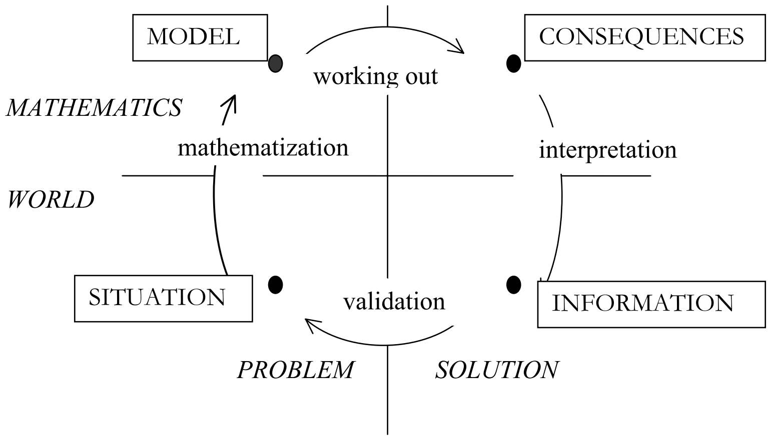
Mathematical literacy focuses on the "functional use of mathematics" (OECD 1999) in various situations, not only realistic ones, or as the framework says: "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments, and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen." (OECD 1999)

MEETING THE FULL RANGE OF MATHEMATICS

Any serious test in mathematics or in other domains has first to clear up its views of the area tested, mathematics in our case. A literacy test as PISA is even stronger urged to do this. The PISA mathematics items are thus constructed as problems which call for using mathematics in context, i.e. mathematical modelling in a very broad sense, including not only contexts from daily life, but also components of the social and cultural embedding of mathematics (OECD 1999). It is not the place here to go into a deeper discussion of what “literacy” should mean: see e.g. Kilpatrick & al. (2001), Jablonka (2003), Neubrand (2003) for an extended discussion. However, one common idea is a kind of red thread through all these discourses: “literacy” is not fully captured if one does not try to meet the full range of mathematics.

A starting point for considerations of what mathematical activity can be is the well known picture of the (idealized) process of modelling (Fig.1):

Figure 1: The Process of Mathematical Modelling



This picture can be applied to the modelling of situations from outside of mathematics, as it was created for, but it may also describe mathematical problem solving when the situation one starts from is a challenging problem inside of mathematics. From the cognitive point of view, both activities have the same structure: Modelling in applications as well as working on intra-mathematical problem bearing situations is essentially characterized by the activity of transformation, translation, seeing in other connections etc., which is called mathematization in the classical sense, but equivalent to the activity of structuring an intra-mathematical situation.

The process of “working out” as in Fig. 1 can follow two cognitive ways: We distinct if the working-out process of an item needs more procedural or more conceptual thinking (Hiebert 1986). This idea was added to the international framework of PISA in the German national option to PISA-2000 (Neubrand et al 2001). Such a distinction essentially defines three classes of items (Knoche & al 2002) which we called *„The Three Types of Mathematical Activity“*:

(a) employing only techniques (abbreviated: „technical items“),

(b) modelling and problem solving activities which lead to use mathematical tools and procedures (abbr.: items with „procedural modelling“)

(c) modelling and problem solving activities which need drawing connections and using mathematical conceptions (abbr.: „conceptual modelling“).

What "technical items" are is very clear. These items consist of just "working out" in the picture of Fig. 1, and the whole lower half-plane in the picture does not even exist in that item. Regretfully, all too often knowing mathematics is considered as just to be able to run an algorithm when the starting point is given. All non-technical items, i.e. those items in which the lower half-plane in Fig.1 is relevant, are called here "modelling problems". Among the "modelling tasks", two kinds of "working out" (in the sense of Fig.1) make the difference from cognitive perspective: "Working out" can consist of solving an equation, doing a calculation, etc., to produce the result, as it is the case in the most "classical" textbook problems. These items are the „procedural modelling items". In contrast, there are items, which can be solved by giving a appropriate argument, or by connecting the situation to a mathematical concept and drawing conclusions from that connection, etc. In this cases we speak of „conceptual modelling“ items.

The "full range of mathematics" is therefore defined by giving items from all three types of mathematical activity. It is this distinction between different aspects of mathematical thinking which rules the German national framework for the national option of PISA in Germany (Neubrand & al. 2001).

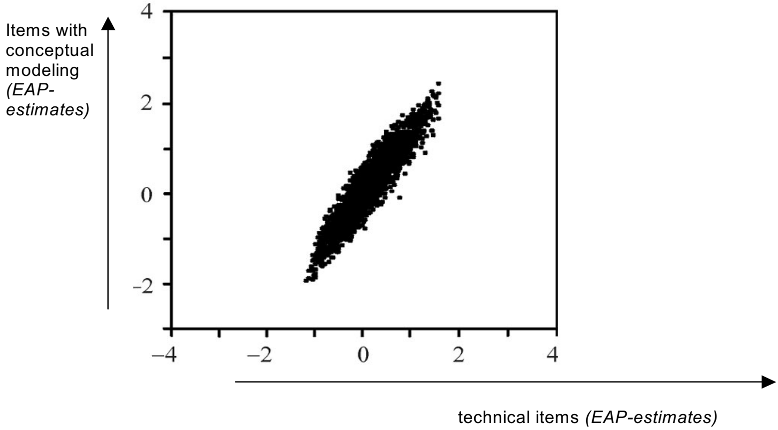
THE IDEA OF "PROFILES" OF MATHEMATICS ACHIEVEMENT

There is empirical evidence, that in some cases a differentiated analysis of mathematical achievement gives insight into the inner structure of mathematical achievement - and is possible on the basis of a full set of items distributed over the three Types of Mathematical Activity. We sketch only two instances of such "profiles" (cf. Neubrand & Neubrand 2004 for details).

(A) Individual differences on the three types of mathematical activities

Knoche & al. (2002) showed this scatter plot (Fig. 2). At least in the German population of PISA, there is not enough reason to conclude that good performance in the technical items is sufficient to show good performances on the conceptual modelling items.

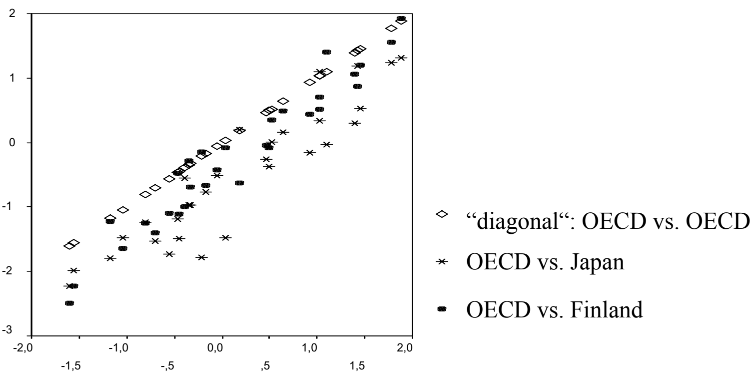
Figure 2: Achievement of German students on “technical” and “conceptual” items



(B) Differences between countries

The following picture (Neubrand & Neubrand 2004) shows the striking different behaviour of Japanese and Finish students in PISA. Against the achievement in the OECD average (the diagonal line) there are plotted the means for every international item in the respective country.

Figure 3: Different achievement of two high scoring countries, Japan and Finland



Apparently, the Japanese population in PISA behaves very different from both, the OECD average, and one other high achieving country, Finland. Furthermore, Finland shows a tendency to score very well at the easier items, but does not show substantial advantages at the harder items.

CONCLUSION: STRUCTURES OF ACHIEVEMENT POINT TO DIFFERENCES IN TEACHING AND LEARNING

Comparisons of mathematical achievement can, or at least should, used to indicate fields of further development of mathematics education. However, this requires a broad and theoretically based picture of the mathematics included in the test. In PISA this reflections were done extensively. The results shown here indicate empirical foundations e.g. that it is not a promising way to restrict mathematics education to a better performance of technical abilities (Fig. 2), nor that "high achievement" can be restricted to the one score point on the "horse race axis" (Fig.3). In both cases, the inner structure of mathematical achievement revealed areas where to act, in Germany on the improvement of conceptual capabilities, in Finland on stronger results also at the more challenging items.

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RF05: RESEARCHING MATHEMATICS EDUCATION IN MULTILINGUAL CONTEXTS: THEORY, METHODOLOGY AND THE TEACHING OF MATHEMATICS

Coordinators: **Richard Barwell, University of Bristol** and
Philip Clarkson, Australian Catholic University

INTRODUCTION

Multilingualism is prevalent in classrooms worldwide. In most mathematics classrooms, however, only dominant regional or national languages are used, often for practical or political reasons. Multilingual contexts may include the presence of:

- migrant communities (e.g. Vietnamese speakers in Australia);
- indigenous communities (e.g. Navajo speakers in the USA;
- historically multilingual communities (e.g. South Africa, Singapore);
- immersion schooling (e.g. English-medium education in Hungary).

Given the prevalence of multilingualism, it is important for researchers in mathematics education to consider the consequences of multilingualism for their research, even if multilingual issues are not the primary focus of their research. In this forum we draw on the increasing amount of research being conducted within the PME community into the teaching and learning of mathematics in different multilingual contexts (most recently, Adler, 2001; Barwell, 2001, 2003a; Khisty, 2001; Moschkovich, 2000; Setati, 2003).

Conducting research in multilingual contexts leads to a number of theoretical and methodological challenges. Classical research methods may be hard to apply, leading to the development of original approaches to research. In particular, issues arise concerning validity, interpretation and the relationship between language, mathematics and mental processes. To tackle these issues, researchers in this field have drawn widely on theories from a range of disciplines, including psychology, linguistics, anthropology and sociology, as well as education. A further challenge for researchers is to draw on their work to inform the practice of teaching mathematics. Our main aim in this Research Forum is to explore the impact of multilingualism on three inter-related issues mentioned above: theory, methodology and teaching mathematics in multilingual contexts.

THEMES

The first theme concerns *the role of theory* in research in this area. Researching multilingualism in mathematics education is by its nature inter-disciplinary. Research into multilingualism within mathematics education has drawn on a variety of theoretical perspectives, including: bilingual education; discursive theories of cognition; discursive approaches to socio-linguistics; Vygotskian approaches to teaching and learning. This inter-disciplinarity leads to a number of questions. What theories are relevant to work in mathematics education? How might these theories be applied in mathematics education? What are the challenges which arise from working with theories from other disciplines? A basis for exploring these issues is provided by Hoffmanova, Novotna and Moschkovich, whose paper also provides the theoretical backdrop for the Forum.

The second theme concerns *implications for mathematics teaching* arising from recent research. Although research has focused on the role of the teacher in supporting mathematics learning in multilingual mathematics classrooms, these classrooms are located within a wide range of different linguistic contexts. Whilst Adler's (2001) research, for example, raises important issues or dilemmas for teachers, these are issues which arise in multilingual South Africa. This context is different from the multilingualism found in Europe, Australasia or Asia. How are such contexts different? And what do any differences imply for the teaching of mathematics? A discussion of these questions is offered by Clarkson and by Halai.

The final theme concerns *methodological issues* thrown up by research in multilingual classrooms, whether or not multilingualism is a focus of the research. Research is necessarily mediated by language. When participants are speakers or learners of several languages, languages which may not be shared with the researcher, many challenges arise for the researcher. Going beyond the basic challenges of collecting and preparing multilingual data are the more complex issues of interpretation. One challenge, for example, concerns the visibility of mathematics in linguistic analyses of mathematics classroom interaction. How can language and mathematics both be kept in focus? Linguistic anthropologists deal with such issues as a central part of their work. How do they deal with these issues? These issues are considered by Staats and by Barwell.

ORGANISATION

Each of the three themes will be introduced by short presentations by the relevant contributors, whose papers follow this introduction. The presenters for each theme will conclude their presentations with a key question, which will form the starting point for focused small-group discussion of that theme. Following consideration of the three themes, the forum will continue with an extended plenary discussion, with the opportunity to raise issues arising from the earlier discussions. The forum will be concluded by Mamokgethi Setati in the role of discussant.

WORKING WITH THEORIES FROM OUTSIDE MATHEMATICS EDUCATION

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We describe why research in mathematics education should consider theoretical views and empirical findings from research on language to provide an accurate picture of the complexity of learning and teaching mathematics in multicultural and multilingual settings. We believe that knowledge of language learning is essential to further progress in understanding the connections between language and the process of learning-teaching mathematics, especially in classrooms where students are bilingual, multilingual, or learning an additional language.

INTRODUCTION

Many of the classrooms where we teach and conduct research include students who speak two or more languages or are learning an additional language. The first part of the paper provides an overview by presenting a brief account of the main theories related to the area of second language learning and acquisition. Special attention is paid to those aspects of the theories and findings relevant to the interaction of mathematics learning and teaching and the teaching of English as a second or foreign language. The second part of the paper describes how a sociocultural and situated framework can be used to frame analyses of mathematical discussions that include more than one language and involve bilingual or multilingual learners. This framework expands "what counts" as the mathematical competence to include the voices of bilingual students and those who are learning English.

THEORIES AND FINDINGS RELATED TO SECOND LANGUAGE LEARNING / ACQUISITION

Although everyone agrees that thought and language are related, the nature of the relationship remains controversial. Traditionally, linguists have studied only the natural languages used by members of human communities to communicate with each other. This, however, leaves out wider senses of communication, e.g. mathematical and logical codes that can be used to transmit messages.

Theories about how we initially acquire language rely on psychological theories of learning in general. They have influenced each other over time. Moreover, different authors bring different models of L2 [1] learning (Ellis, 1994, table 10.2). The very distinction between learning and second language acquisition (SLA) is controversial. We have therefore decided to adopt an eclectic approach to be able to cover the most influential theories.

The Behaviorist Approach

Behaviorists regard language learning as habit formation, as a result of connecting responses to stimuli. Children learn to speak because they are reinforced for doing so.

Correct responses lead to good habits, errors are perceived as bad habits. The negative effect of mother tongue (L1) on students' production of L2, causing errors through analogy with L1, was described as a Contrastive Analysis Hypothesis (Lado, 1964, in Brown, 1993). Critics of the behaviorist position claim that although this view may have an intuitive appeal it provides only a partial explanation of children's early language learning.

The Cognitive Approach

Children do not simply imitate the language they hear, but rather learn to construct grammatically correct sentences they have never heard before by generalizing about language. There appears to be a critical period of language acquisition when SLA can take place naturally and effortlessly (Lenneberg, 1967, in Brown, 1993). From a cognitive perspective, language acquisition occurs in increasingly complex stages as children actively seek ways to express themselves (Brown, 1993). The sequence appears to be universal.

One example of work from this perspective is the psycholinguistic studies comparing monolinguals and bilinguals when doing arithmetic operations (Magiste, 1980; Marsh & Maki, 1976; McLain & Huang, 1982; Tamamaki, 1993). All we can safely conclude from that research at this time is that "retrieval times for arithmetic facts may be slower for bilinguals than monolinguals" (Bialystok, 2001, p. 203). It is not clear whether these reported differences in response to time and accuracy between adult monolinguals and bilinguals during experiments also exist for young learners or would be evident in classrooms.

Such an emphasis on the deficits of bilingual learners or second language learners is described as a cognitive deficit model of learning in L2. As a contrast, other psycholinguistic research has shown that while bilinguals and second language learners may face some disadvantages, they also display some important cognitive advantages over monolinguals. Bialystok (2001) concluded that bilinguals develop an "enhanced ability to selectively attend to information and inhibit misleading cues" (p. 245) [2]. This conclusion is based, in part, on the advantage reported in one study that included a proportional reasoning task (Bialystok & Majunder, 1998) and another using a sorting and classification task (Bialystok, 1999). These results would seem to be closely related to mathematical problem solving.

Linguistic Universals

Universality is one of the most fascinating characteristics of language. Children in all cultures appear predisposed to acquire language through almost the same phases, and may be born with an innate mechanism to learn language – Language Acquisition Device (LAD). Mentalist/nativist theories state that there seems to be one best type of grammatical analysis that all of us are programmed to develop and it is universal to all languages, using the same grammatical forms and relations or linguistic universals, which were later applied to SLA (Chomsky, 1965). We are not completely sure that this so-called universal grammar is accessible to adult learners. After a certain age we are still able to learn a language using such other mental faculties as the logical and the mathematical. The learning of mathematics can be seen as a process parallel to the way children acquire language skills, developing structure in oral ability prior to the more symbolic abilities with writing and reading (Gardella & Tong, 1999).

Social Models

Social models of language acquisition consider that social factors have an indirect effect on all mental processes including SLA. These theories examine linguistic variability rather than universality and claim that children may develop more than one grammar depending on particular situational contexts. A complex view of L2 learning called The Socio-Educational Model explains how individual factors and general features of society interact in L2 learning. The Acculturation Model (Schuman, 1978, in Brown, 1993) suggests that successful learning means "acculturation" – becoming part of the target culture. Learning takes place in society and depends on motivation and aptitude.

The Humanist Approach

The Humanist Approach differs from others in that it focuses on the affective components of learning. For a long time the relationship between cognition and emotion has been a controversial issue. Increasingly, we are becoming aware that cognition, emotion and personality are not entirely independent (Crowl *et al*., 1997). The success of the humanist approach towards teaching depends on the extent to which the teacher caters to learners' affective domain. Critics have a variety of objections, but it would appear that many humanist programs have not been evaluated properly to determine their effectiveness.

Creative Construction Theory

Creative Construction Theory was first developed and described as The Monitor Model (Krashen, 1977, in Ellis, 1994) and later as Creative Construction Theory (Dulay & Burt, 1982, in Ellis, 1994). The theory brings together research findings from different domains. According to Krashen, SLA is subconscious and equals LAD, contrary to Chomsky for whom LAD is but one of various mental organs, a construct that describes the child's initial state. More recently, Chomsky's statements seem more compatible with Krashen's argument that adults and children have access to the same LAD. These ideas have provoked strong criticism. Empirical research studies have shown that the development of L2 is a process in which varying degrees of learning and acquisition can be beneficial. No input is acquired as new language without conscious awareness. Swain emphasizes the role of output in SLA (see Ellis, 1997).

Interlanguage Theory

Interlanguage (IL) is a term introduced to refer to the developing competence of L2 learners, from an initial stage of very limited knowledge about the new language, to a final stage of almost complete fluency. The concept was coined to describe the kind of language that is independent of both the learner's L1 and L2. Recent developments in this area of research try to answer questions concerning the role of L1 (IL is influenced by L1 but the influence is not always predictable), the acquisition of IL (form-function relationship), and the systematicity and variability of IL. The results of experiments provide evidence that mistakes made during bilingual education are both intralingual (within L2) and interlingual (between L1 and L2). Nowadays, IL is considered to be the central concept in SLA (Ellis, 1997). He identifies many external and internal factors that account for why learners acquire an L2 in the way they do.

A SOCIOCULTURAL AND SITUATED THEORETICAL FRAMEWORK INFORMED BY SOCIOLINGUISTICS

Work in sociolinguistics has informed the study of mathematics learning and teaching in multilingual classrooms. This work has contributed theoretical frameworks for studying discourse in general, methodologies (e.g. Gee, 1996), concepts such as register (Halliday, 1978), and perspectives on classroom discourse (e.g. Cazden, 1986; Mehan, 1979). It also provides theories, concepts, and empirical results in second language acquisition, bilingualism, and biliteracy (Bialystok, 2001; Hakuta & Cancino, 1977; Valdés-Fallis, 1976, 1978; Zentella, 1997). This work has provided crucial concepts necessary for studying mathematics learning in multi-language classrooms, such as code switching, as well as important distinctions for example between national and social languages, or among different types of code switching, in different cultural settings such as South Africa (Adler, 2001; Setati, 1998; Setati & Adler, 2001) and in bilingual classrooms in the USA (Moschkovich 1999, 2002).

Psycholinguistics and sociolinguistics differ both in how they explain and explore language practices. While the sociolinguistic perspective stresses the social nature of language and its use in varying contexts, the psycholinguistic perspective has been limited to an individual view of performance in experimental settings. According to the sociolinguistic perspective, psycholinguistics experiments provide only limited knowledge about speakers' competence or how people use language:

> The speaker's competence is multifaceted: How a person uses language will depend on what is understood to be appropriate in a given social setting, and as such, linguistic knowledge is situated not in the individual psyche but in a group's collective linguistic norms. (Hakuta & McLaughlin, 1996)

Code switching has been largely used in sociolinguistics to refer to the use of more than one language in the course of a single communicative episode. In contrast, research that looks at bilingual performance from a psycholinguistic perspective sometimes uses the term 'language switching' to refer to a cognitive phenomenon, the act of switching from a second language to a first language as the language of thinking when a bilingual person is individually engaged in a mathematical task rather than in a conversation. While work from a sociolinguistic perspective also distinguishes between language choice, code switching, and code mixing, sociolinguistics assumes that all of these phenomena are social rather than individual in nature and function.

These two perspectives see bilingualism itself in different ways. From a psycholinguistic perspective we might define a ‘bilingual’ as any individual who is in some way proficient in more than one language. This definition might include a native English speaker who has learned a second language in school with some level of proficiency but does not participate in a bilingual community. In contrast, a sociolinguistic definition of a bilingual would be someone who participates in multiple language communities and is “the product of a specific linguistic community that uses one of its languages for certain functions and the other for other functions or situations” (Valdés-Fallis, 1978). This definition describes bilingualism not as an individual but also a social and cultural phenomenon that involves participation in language practices and communities.

Research in mathematics education should address the relationship between language and mathematics learning from a theoretical perspective that combines current perspectives of mathematics learning and classroom discourse with current perspectives on language, second language acquisition, and bilingual learners. In this section we consider how the situated and sociocultural perspective proposed in Moschkovich (2002) can inform our understanding of the processes underlying learning mathematics when learners speak more than one language.

Moschkovich’s (2002) approach combines a situated perspective of learning mathematics and the notion of Discourses (Gee, 1996). This perspective implies that learning mathematics is viewed as a discursive activity. Learning mathematics is seen as participation in a community of practice (Forman, 1996; Lave & Wenger, 1991; Nasir, 2002), developing classroom socio-mathematical norms (Cobb et al., 1993), and using multiple materials, linguistic, and social resources. This perspective assumes that learning is inherently social and cultural “whether or not it occurs in an overtly social context” (Forman, 1996, p. 117), that participants bring multiple views to a situation, that representations have multiple meanings for participants, and that these multiple meanings for representations and inscriptions are negotiated. Learning mathematics is seen as participation in a community where students learn to mathematize situations, communicate about these situations, and use resources for mathematizing and communicating. From this perspective, learning to communicate mathematically involves using social, linguistic, and material resources to participate in mathematical practices.

This approach also draws on Gee (1996), who defines Discourses as more than sequential speech or writing:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and ‘artefacts,’ of thinking, feeling, believing, valuing and acting that can be used to identify oneself as a member of a socially meaningful group or ‘social network,’ or to signal (that one is playing) a socially meaningful role. (p. 131)

Discourses involve more than the use of technical language, they also involve points of view, communities, and values. Mathematical Discourses (in Gee's sense) include ways of talking, acting, interacting, thinking, believing, reading, and writing, but also mathematical values, beliefs, and points of view of a situation. Gee emphasizes that such interactional and non-language symbol systems, should be included in Discourse analysis. Thus, we should consider the importance of gestures, artifacts, practices, beliefs, values, and communities in communicating mathematically. Participating in classroom mathematical Discourse practices can be understood in general as talking and acting in the ways that mathematically competent people talk and act when discussing mathematics.

A situated/sociocultural perspective focusing on participation in mathematical Discourse practices generates particular questions when analysing mathematical discussions. For example:

1. What are the situated meanings of some of the words and phrases that seem important in the situation?
2. What are the multiple resources students use to communicate mathematically? What sign systems are relevant in the situation (speech, writing, images, and gestures)? In particular, how is "stuff" other than language relevant?
3. What Discourses are involved? What Discourse practices are students participating in that are relevant in mathematical communities or that reflect mathematical competence?

This situated and sociocultural perspective complicates our view of the relationship between language and learning mathematics. A crucial consequence is that it allows us to replace deficit models of bilingual mathematics learners with a focus on describing the resources that students use to communicate mathematically.

We would like to share a word of caution. There are dangers in borrowing isolated concepts while leaving behind the theoretical framework. It is not enough to borrow an isolated concept. If a concept is not connected to the theoretical framework that generated the concept, it can easily become an idea that bears little resemblance to the original idea. For example, we might borrow the concept of "code switching" from its sociolinguistics framework that assumes that language is a social phenomenon but neglect to take the sociocultural view of language along with it. If we do this, we would be reducing code switching to an individual phenomenon. Similarly, if we use "register," a term framed by a sociolinguistic view of language, to mean "lexicon", which unlike register is independent of the social context, we are removing "register" from its sociocultural framework and replacing that framework with an individual view of language.

CONCLUDING REMARKS

Focusing on mathematics is our job as researchers in mathematics education. But focusing on mathematics also has consequences for how we portray students´ mathematical competence. Teaching and research are framed by theories of learning in general, theories of mathematics learning and, in this context, theories of SLA. Whether we are teaching or analyzing a lesson we need to consider the theoretical framework and the assumptions that we bring to our work. We believe that theories and empirical results from linguistics, cognitive psychology, and sociolinguistics have laid the groundwork for the study of mathematics learning as it occurs in the context of learning an additional language.

ENDNOTES

1. L1 – the mother tongue, L2 – the target language.

2. The cognitive advantages of bilingualism seem to depend on some level of proficiency in both languages and "the extent to which an individual is fully bilingual is instrumental in mediating the effect on cognitive performance". (Bialystok, 2001, p. 205).

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MULTILINGUAL CONTEXTS FOR TEACHING MATHEMATICS

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Attention is draw to aspects of teaching that inevitably rely on deep communication and various multilingual contexts for teaching. Examples are given of how different teachers and societies have dealt with these contexts. Little research has been completed in this area. The few studies that are available suggest that informal or exploratory talk in students' first languages is vital before moving to formal mathematical language. In multilingual situations the exploratory talk may be a situation of broken communication, but this may not be recognised by participants.

The great majority of teachers throughout the world now work in classrooms that are multilingual. However it has only been recently recognised that the linguistic backgrounds of students has an impact on students' learning of mathematics, and hence on the teaching of mathematics. For many monolingual teachers the problem of students who have a first language different to theirs has become a reality. Students' language backgrounds have been seen as one factor that means, in the view of some such teachers, that these students are unavailable to learn, or to learn in the way such teachers expect. Other teachers are exploring what tools and strategies they can use to face this growing and complex challenge of changing classroom settings.

Contexts of mathematics classrooms vary enormously, influenced in part by the reactions of individual teachers through to societal judgments having a mixture of impacts. One of the factors in this variation is the different languages of communication that might be present, as well as how both teacher and students view and use them. It is useful to begin by reviewing some of the research that has particularly looked at the teacher's role in situations when there has been more than one language in the mix. In scanning four journals dedicated to research in mathematics education from 2000 to 2003 (*Journal for Research in Mathematics Education, Educational Studies in Mathematics, For the Learning of Mathematics,* and *Mathematics Education Research Journal*), at least in English written contributions (about 300 articles) there are very few articles that focused on the teacher's role in such situations. There are far more, and perhaps understandably so, that have students as the central foci. Hence it may be that more research is needed to clarify the roles that the teacher may play. Three studies seem relevant.

Setati and Adler (2000) discussed the language practices of teachers in some primary schools in South Africa where students' normal out-of-class talk is in a non-English language, but the official teaching language is English. They were interested in the code-switching behaviour of teachers. Although they suggest that it makes a lot of sense for teachers to encourage students to code-switch, and use this as a teaching strategy too, there are challenges in this practice that can not be overlooked. At times it seems that teacher talk is down-played in some curriculum reforms, and yet it is teacher talk they suggest that often illuminates ideas for students. Types of discourse, such as informal talk in students' first language leading to more formal mathematical talk in English, are also critical paths to trace carefully in such complex multilingual situations.

Gorgorio & Planas (2001) were working in classrooms where the teaching language was Catalan. Students were a mix of Catalan students plus immigrant students who spoke a variety of languages at home. The authors suggest it is hard to separate the social, cultural and linguistics aspects of mathematics teaching and learning. Indeed they took the view that it was better to think of broader communication within the classroom than a narrow linguistic one, although language aspects cannot be ignored. In particular they note that in their classrooms, the informal or exploratory talk can often be 'broken communication', particularly for the teacher, since this inevitably occurs in the students' first languages. Therefore helping students to move to the more formal mathematical talking and writing, which often involves a switch to the language of the classroom, can be fraught with unknown linguistic set-backs.

Khisty & Chval (2002) contrast the teaching styles of two teachers who were teaching groups of Latino students in the USA. The two classes were of different levels in English proficiency, and hence there was more frequent use of English in one classroom than in the other by the bilingual teachers. The authors write that a critical issue was the way one teacher used precise and extended mathematical language in her verbal discourse with her class and promoted an expectation that the students would also use such language. The results of the investigation suggested that students did in the end use the formal mathematical language promoted by this teacher. The underlying emphasis is that bilingual students will not learn this type of English, unless they are witnesses to deliberate examples of such discourse.

The above three studies emphasise that the issues of teaching in multilingual contexts are not straightforward. The teachers need to cope in situations where they will not have full management of the discourse, unless they too are proficient in the students' language(s), as well as the teaching language. However the flow from exploratory verbalizing of ideas through to their formalising in a rich mathematical language, both verbal and written, seems to be a given across the contexts. How to manage the flow is an issue that needs further research. What are the teaching strategies that teachers can employ with good effect to this end? In the next section, I consider several examples of the current challenges faced by mathematics teachers in different parts of the world.

CHALLENGES

Various multilingual mathematics classroom contexts can be generated by considering three of the possible interacting sources of language: the students' language or languages, the teacher's language or languages, and the official teaching language (and less often languages). The snippets below discuss various multilingual situations, highlighting the wide range of possible contexts.

In Papua New Guinea, students in a typical classroom will speak a common language, although they may well speak a number of other languages too. The teacher may speak the common student language if s/he comes from the same region, but will also be multilingual. Up to year 3, schools can choose which teaching language they will use, but from year 3 the official teaching language is English, although teachers are encouraged to use a mixture of languages if possible through years 3, 4 and even 5 (Clarkson *et al.*, 2001). Classroom observations suggest, however, that teachers seem to prefer English only when teaching mathematics. It seems that dealing with mathematical concepts is difficult in a vernacular or Melanesian Pidgin. This raises an interesting question. Are crucial nuances lost in translating terms into English. with embedded cultural meanings being marginalized? Should the rule of using English be sidestepped so that the cultural meanings can be explored?

In urban Australian schools, many monolingual teachers teach a mix of multilingual students, many of who are from migrant families, although the migrant community to which they belong may have been in Australia for a number of years. It would seem, however, that few teachers realise the role that a first language plays for these students. This is summed up by the surprise of a primary school teacher, who had recently completed graduate studies in Teaching English as a Second Language, when she discovered how often her year 4 Vietnamese students were switching languages when doing mathematics in her class (Clarkson, 2002). In some European countries too, teachers are faced with teaching many migrants. It would appear that in the main the reaction of teachers has been one of holding a line of orthodoxy. That is, that ‘newcomers’ should learn the ways of the dominant society and integrate with it, including learning the use of the main language in the classroom as soon as possible. But this new context is challenging other teachers to think deeply about their use of language in teaching mathematics.

In Malaysia, at the beginning of the 1970s in Malay schools, the teaching language was changed from English to Bahasa Malay. This was mainly for political purposes to emphasize the unity of the relatively new political amalgamation of historically different kingdoms and states. However from 2003 due to a rather rapid political decision, although the main teaching language remained Bahasa Malay, the teaching of mathematics and science reverted to English. This has interesting ramifications for teachers of mathematics.

In New Zealand, the indigenous Maori peoples have developed a small system of schools were only Maori is used for all communication while present in the school, though both students and teachers live in a dominant English speaking community. Mathematics is taught in Maori. Further, the mathematics curriculum has been translated into Maori, with some changes to include some specific Maori mathematics. In some areas of the Northern Territory in Australia a different strategy has been employed by indigenous communities. Through the 1980s and early 1990s there was political support for the use of the people’s first language to be used as the teaching language. Further, there was insightful curriculum work carried out to devise mathematical curricula that commenced in the early years of schooling with

Aboriginal ideas. Hence in one area in the desert, indigenous spatial ideas became the basis of the early years curriculum, whereas on the north coast the notions of relationships were used as the key framework concept. In these instances not only were the teaching languages changed to that of the students and community, but the mathematics curriculum too was transformed.

CONCLUDING REMARKS

The actions of individual teachers, as well as societies, will be influenced by deep-seated beliefs, which in turn may be manifest in the language(s) of communication in classrooms. A teacher who is a member of the dominant society and who believes that teaching should be in the dominant language will have little inclination to explore any other language options. The perception of mathematics that is held will also have an influence. If mathematics is conceived as a 'language free zone', then the teacher who takes this view will be less inclined to think about the role that the teaching language, or any other, has on the learning of mathematics. If, on the other hand, the teacher accepts that not only does the teaching language impact on the learning of mathematics, but so too may the students' first languages, they may consider which languages can be used in their classrooms and even of what mathematics can be taught.

To address some of the questions raised in this paper, however, a far more detailed meta-analysis of the relevant literature is needed, the state of which is only hinted at in this paper, with the multilingual context of the teaching as a crucial aspect of such an analysis. Such research may allow useful commonalties in teaching mathematics across multilingual contexts to emerge. At the same time, notions and practices that should be seen as context specific may also be identified.

TEACHING MATHEMATICS IN MULTILINGUAL CLASSROOMS

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In this paper, arising from my doctoral research, I highlight issues that emerge for teaching and learning mathematics in multilingual mathematics classrooms. These are classrooms where the language of instruction is not the first, often not even the second language of the learners.

Research about teaching and learning mathematics suggests that the dynamics of teaching and learning mathematics in multilingual contexts take on an added complexity, giving rise to a number of issues and dilemmas. There is need for acknowledging this added complexity and to understand the factors that lead to it. Some of the dilemmas are well recorded in Adler (2001) who highlights three:

- The dilemma of code switching, when learners and/or teachers switch from language of instruction to the first language
- The dilemma of mediation when teachers move towards the learners preferred language
- The dilemma of transparency when the teacher spends time explicitly teaching mathematical language (Adler, 2001, p.)

While Adler's context is multilingual South Africa, in today's increasingly connected world, multilingual classrooms are the norm. Hence, it is important to recognise the centrality of multilingual mathematics classrooms so that reform efforts may take them into account.

My doctoral research (Halai, 2001) involved a study of reform in learning and teaching mathematics. Of particular interest to me was the role of social interactions in students' learning of mathematics. The study was based in two classrooms (henceforth, classroom A and B) in Karachi, Pakistan. These were classrooms where the teachers were using reform-oriented teaching approaches i.e. students were typically set mathematics problems which were open ended in nature and were situated in everyday world contexts. A small group of students (10-12 yr.) doing mathematics was observed in each classroom. Both schools were English medium schools. This meant that the official medium of instruction including the textbooks used and the tasks set in the class were in the English language. In both classrooms, the instruction was almost entirely in English but during group work students spoke in a mixture of Urdu and English. The teachers also reverted to Urdu when they went to the various groups. In reporting the group work to the whole class the students spoke in English with a smattering of Urdu. At times students *took permission* from the teacher to report in Urdu. However, the mathematical terms used in this mixture were invariably in English because these terms came from the textbook, which was in the English language.

The data was qualitative in nature and came from classroom observations and interviews with students and teachers.

ISSUES IN TEACHING MATHEMATICS IN MULTILINGUAL CLASSROOMS

A number of issues pertaining to the processes of teaching and learning arose because of the language of instruction being different from the first language of the learners. These included:

- Understanding the language to make sense of the mathematics;
- Use of everyday language and mathematics learning;
- Using own language to express mathematical thinking;
- Language of the textbook.

Understanding the language to make sense of the mathematics

As students worked at mathematical problems it appeared that their understanding of the problem statement required interpretation at least at two levels. At one level the students appeared to make sense of the language in which the mathematics problem was coded. This involved making sense of the grammar and usage of words. And at another level they appeared to make sense of the mathematics involved. For example, in one problem task students were given the statement "*Sara will be* 28 *year old after* 9 *years. Find her present age"*. Their task was to form an equation for the given statement and to solve it.

Analysis of classroom interactions showed that understanding how the word "will" was used was crucial to the students' successfully doing the task. Knowing that 'will' is future tense had major implications for how the problem statement is converted into a mathematical equation and then a solution is sought. There were other examples which showed that the understanding of the specific structures and usage of words in the English language was important for the students to make sense of the mathematics. For example, in her introductory lesson on ratios the teacher used the task of mixing water and orange juice concentrate to make drinks which were "stronger than", the drink shown as a sample. However, students' interaction showed that one student in the group translated and explained to the others that stronger means 'ziada' which is the Urdu word meaning "more". The result was that the students categorised as stronger the juice that was 'more in quantity' and not as the one, which was more concentrated and hence stronger. This and other similar examples through the research provide vivid evidence of issues that students face in interpreting mathematical tasks that require an understanding of the grammar and usage of words in a second or third language. Questions arise for the teaching and learning processes in the multilingual mathematics classrooms. Was the teacher aware of these language issues arising in the classroom? What could the teacher have done to facilitate students' learning? How else could the teacher have organized his/her classroom so that issues such as those above have been acknowledged and addressed?

Use of everyday language and mathematics learning.

In the classroom that I observed, teachers had prepared tasks that were set in everyday life contexts and made use of everyday usage words. There appeared to be an assumption that using everyday contents and language would facilitate learning. However, these words of everyday usage were in English language and therefore raised questions about assuming that students would be familiar and would be able to make a link with the mathematical concepts embedded in those words. My observation in the classroom showed that the teacher's use of everyday words for mathematical concepts led to difficulties for the students. For example, the teacher used "fair share" for proportional division but students appeared to think that the word "fair' meant "divided equally" or "divided easily". This had implications for how they worked on other related tasks.

A point that I am trying to make is that using discursive strategies to teach mathematics where students are expected to build on their knowledge of the everyday context and language takes on an added complexity in a multilingual context. This complexity arises because of possibly unquestioned assumptions regarding students' familiarity and understanding of the language of the language of instruction.

Students expressing mathematical thinking in their own language

A pattern in the classroom interactions was that the students and the teacher moved back and forth in the use of language. While there is evidence that students change languages, understanding why they do so would be important to making a difference in the way teaching is organised in multilingual mathematics classrooms. Furthermore, changing from one language to the other raises the issue of translation from one language to the other. Now, translation is a nuanced and complex process. In mathematics classrooms translation of key words and phrases would require that the mathematical meaning is also kept intact. Examples quoted in this paper show that students did not always translate in a mathematically appropriate manner. Indeed, on occasions the issue was that there did not exist an appropriate equivalent translation of the key word or translation being used.

Issues pertaining to the status of language also arises (Haque, 1993). Why did the students need to take permission in order to report their work in Urdu? Issues of power and politics of language emerge. Why did students use a mixture of Urdu and English instead of resorting to Urdu only? Is it that they saw English as a more powerful and therefore wanted to belong to the community of English speakers?

Language of the textbook

The teachers and the students in the classrooms where I did my research were all expected to follow prescribed textbooks. The textbook was used as a guide for the subject content to be taught and for providing exercises for practice. Each teacher emphasised to me that if the material provided in the textbook was not covered they would be accountable to the head teacher. This emphasis on the textbook raised issues pertaining to the language being used by the students and that being used in the textbook. Language in the textbook used formal mathematical terminology coded in English. While teachers were using everyday words in English and the students translated these everyday words to Urdu. This rather complex scenario compounded the issue of transfer from the everyday language to mathematical language.

CONCLUDING REFLECTIONS

To conclude, classroom data shared above shows that in the course of teaching and learning mathematics in multilingual classrooms, dilemmas and issues arise, similar to those discussed in Adler (2001). This is because students and teachers in these multilingual classes switch back and forth from one language to the other. This switching requires translation from one language to the other which is complex and not always possible. Furthermore, politics and power of the language of instruction and the students' language also gives rise to issues.

From the discussion so far certain questions arise for the academic and practitioner communities. I will end with some such questions:

- How can teachers organise their teaching to address the issues and dilemmas that arise in a multilingual mathematics classroom?
- In what ways can teacher education prepare teachers for multilingual mathematics classrooms?
- How can research inform practice in mathematics teacher education/ mathematics teaching in multilingual classrooms?

QUESTIONS OF VISIBILITY

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In this paper I explore this issue of the visibility of mathematics when multilingual mathematics classroom interaction is examined from discursive perspectives. The use of such perspectives is related to the linguistic concerns of research in this area. It can lead, however, to a critique that argues that such analyses are insufficiently revealing of the mathematics taking place. I draw on an example from UK data to explore the relationship between mathematics and discursive practice.

The issue of visibility arises from a view that a focus on language issues in research in mathematics classrooms, while interesting, often omits a suitable focus on the mathematics taking place. This view is similar to that encountered in critiques of much socially or politically oriented research in mathematics education: by focusing on the social activity of a mathematics classroom, mathematics 'slips from view' [1]. At the heart of such critiques is an implicit position on what mathematics is. This position is perhaps motivated by the (realist) idea that mathematics exists outside of the social, discursive or political world. In the context of multilingualism (including bilingualism), this position entails mathematics existing somehow outside of language, so that research which explores language practices in multilingual mathematics classrooms is not seen as exploring mathematics. The linguistic, social or discursive practices involved in doing, teaching or learning mathematics are not quite the same as mathematics.

Visibility becomes an issue for me, with the adoption of particular linguistic perspectives on mathematics classroom activity arising from my interest in linguistic phenomena. I conceptualise mathematics classrooms in terms of linguistic practice because my questions concern language. One approach, therefore, is to find ways of keeping my focus on questions of language, and my perspective of linguistic practice, whilst relating what I see to the practices and ideas of the mathematical community. This approach might be termed 'language to mathematics'. An alternative is to focus on mathematical practices and ideas and seek to relate what I see to language-related issues, an approach that might be termed 'mathematics to language'. The danger of the first approach is that mathematics is not seen as sufficiently visible. The danger of the second approach is that issues of language are not seen as sufficiently visible. The challenge is to work with both language and mathematics and keep them both in view. I will use aspects of my own research to exemplify and explore some of these issues.

MY OWN RESEARCH

In research into the participation of 9-10-year-old learners of English as an additional language (EAL) in the UK, I analysed transcribed audio-recordings of students working together. The students are working on the task of writing and solving arithmetic word problems. To analyse the transcripts, I developed ideas from discursive psychology (Edwards, 1997) and conversation analysis (Sacks, 1992). In particular, I drew on the notion of 'participants attention' (Sacks, Schegloff and Jefferson, 1974), seen as a feature of interaction, rather than a form of mental activity. My analysis, a form of the 'language to mathematics' approach, entailed the identification of patterns in students' attention (see Barwell, 2003b). I then explored how attention was used by students as part of the social activity of thinking. The following brief extract and commentary illustrates this approach to analysis [2]. The extract involves Cynthia, who arrived in the UK from Hong Kong, 18 months before this recording. Cynthia speaks Cantonese and some Mandarin. She is working with Helena, who is a monolingual English-speaker. They are starting to write a new word problem, which Helena suggests could involve a character called 'Cynthia' [3]:

H	Cynthia has thirty pounds for/
C	no/ not for her her mum/ if (I bought)/ for my mum
H	for her mum's present
C	if give my mum thirty pound I bought nothing from her/ that not make sense
H	no/ I won't writing for you mother/ I said Cynthia has thirty pounds for her mother's present
C	thirty pound/ I gave thirty pound for my mum present
H	no/ I didn't say give it to her
C	then how why you
H	you **have** thirty pounds [for your mum's present
C	[no
	but/ I think this make sense/ Cynthia has thirty/ pound/ thirty pound/ she bought err something something something/ it's cost something something/ from her mum present/ and how much she left?/ is that make sense little bit

In this extract, my interest is in how the students discursively manage their attention. Different areas or patterns of attention are apparent. The two students attend to the word problem genre, for example, as in Helena's opening suggestion of the opening words of the problem and Cynthia's implicit acceptance of them. This pattern of attention is also apparent at then end of the extract in Cynthia's exposition of a standardised word problem with blanks for the numbers. At other times, attention is on what I will call 'narrative experience', the use of narrative accounts or reasoning (Bruner, 1990) to make sense of the interaction. Cynthia uses attention to narrative experience when she expresses concern that 'if give my mum thirty pound I bought nothing from her'. She is using narrative reasoning to support her claim that the problem does not make sense. Superficially, neither of these two patterns concerns 'mathematics'. They are used, however, by the two students, to make sense of their word problem. Cynthia, for example, gives an interpretation of the opening 'Cynthia has thirty pounds for her mum's present' as meaning that the Cynthia in the problem gave her mum thirty pounds, which for Cynthia, is not a present. A present should be some kind of object, which is bought, at a shop, for example. As the two students trade interpretations, the tension increases a little, with Helena contradicting Cynthia. Cynthia shifts attention to narrative experience to support her claim that the problem does not make sense. Later in the extract, Cynthia shifts attention to genre, by offering her standardised version of the problem. The extract provides a brief snapshot of the two students working together to produce a word problem that makes sense to both of them. It is only by considering how these (and other) different patterns of attention are interwoven, with participants shifting from one to another, that it is possible to understand how they do this (see Barwell, 2003b).

At this point, you may be thinking ‘where is the mathematics here?’ They perform no arithmetic calculation, for example. They are, however, working with the genre of arithmetic word problems, a genre that forms a central part of the discourse of school mathematics. At the very least, therefore, the two students are working within the discourse of school mathematics. The issue, then, is: what of that discourse do we see as ‘mathematics’ and what of it is ‘other practices’. Indeed, can ‘mathematics’ be separated from these ‘other practices’? In the case of the problem Cynthia and Helena are writing, their discussion and preparation of their problem continues for several minutes. When they come to solve the final version of the problem, they use a calculator to find the solution in a few seconds:

C	yeah how much (...) left/ okay/ do it now/ come on/ no no no/ do that/ um/ fifteen and/ one two nine nine and one five oh oh/ okay/ one/ no
H	just like fifteen and twelve
C	no/ I’ve got you’ve got twelve pound ninety nine/ twelve nine nine/ take away/ one five oh oh/ eq-/ no/ not [take away/ it’s add/
H	[no not take away/ add
C	two oh nine nine/ add/ one five oh oh/ two seven nine nine/ two seven nine nine/ and three oh oh oh/ take away/ two/ seven nine nine/ equal/ two pound and one p./ how much she spent
H	she spent
C	yeah/ wait wait
H	twenty seven ninety nine
C	(...) spent/ S PE N/ she spent/ twenty seven pounds and ninety nine p./ left/ and/ she left/ shu left/ she left/ um/ two pound and one p./ done it/mister Barwell

In solving their problem, Cynthia calls out the digits of her calculation. The context of the problem is not explicitly articulated at this stage. The calculation, however, is contextualised by the lengthy discussions which began in the first extract shown above, so that although Cynthia says ‘three oh oh oh’, these digits have accumulated meaning throughout the discussion, starting with Helena’s initial suggestion ‘Cynthia has thirty pounds’ and continuing through a discussion of what the money is for, what having money to by a present means, what is done with the money and so on. The solution ends with attention shifting back to genre, 'how much she spent', the two students thus relating their calculations to their problem. Cynthia's 'three oh oh oh' are not isolated, abstract digits, but a link in a chain (Bakhtin, 1986) of meaning-making. Where, then, is the mathematics? Only at the end of the process outlined above? Or throughout the process? For me, the whole process of developing the word problem is implicated in its solving. It is difficult to draw a line between 'mathematics' and 'other practices' within the discourse of school mathematics.

CONCLUDING REMARKS

My approach to researching multilingualism in mathematics classrooms focuses primarily on social, discursive practices, with the aim of linking this analysis to practices established within a broader mathematical community. The case of Cynthia and Helena shows how such practices are central in mathematical meaning-making. This position, however, is based on a broad notion of what constitutes mathematical practice.

NOTES

1. The comment that mathematics had 'slipped out of view' was made in a review of an earlier version of Barwell (2003b).

2. The interaction between Cynthia and Helena is more thoroughly presented in Barwell (2003b).

3. Transcription conventions: / for a short pause, // for a longer pause, [for overlapping speech, ( ) enclose uncertain transcription, (…) for inaudible speech.

MATHEMATICS DISCOURSE AS PERFORMANCE: PERSPECTIVES FROM LINGUISTIC ANTHROPOLOGY

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This paper draws on perspectives from linguistic anthropology to look at mathematics classroom discourse. In particular, the paper introduces the notion of performance. An illustrative analysis of a mathematics classroom discussion is presented.

While the folk performances that linguistic anthropologists study might seem at first to be markedly different from speech in mathematics classrooms, the two are indeed linked in a fundamental way. In both cases, discourse brings pre-existing knowledge into the social world, often with personal improvisation, presenting it for the evaluation of others who decide whether it was a successful performance or not. This shared process of learning and presenting anew means that in some respects, the analytical tools of linguistic anthropology and folklore are as relevant to education research as the methods of discourse analysis. A great deal of scholarly effort in folklore since the mid 1970s has focused on the concept of performance. In this paper, I outline the dimensions of performance and illustrate how these ideas can be used to examine mathematics classroom interaction.

PERFORMANCE

Richard Bauman writes that performance is a way of speaking that is characterized by "the assumption of responsibility to an audience for a display of communicative competence…highlighting the way in which verbal communication is carried out above and beyond its referential content" (1993, p. 182). Bauman continues "[f]rom the point of view of the audience, the act of expression on the part of the performer is thus laid open to evaluation for the way it is done, for the relative skill and effectiveness of the performer's display" (p. 183). Formal discursive features distinguish performance speech from ordinary, factual, referential speech, for example, opening phrases like "once upon a time" or vocal qualities like the sonorous harangue of a legislator. Still, performance can occur in ordinary, even conversational contexts (Duranti, 1997, p. 16; Silverstein, 1984). Because speakers possess different levels of competence and willingness to perform, performance is an emergent aspect of speech: speakers can achieve varying degrees of performance (Bauman, 1977). Overall, the key attributes of performance are communicative competence, accepting responsibility for a competent expression, highlighting the communicative event as different from ordinary discourse, opening the speech event up for audience evaluation, and the emergent quality of performance.

To what extent do these attributes occur in mathematics classroom discourse? The issue of communicative competence (Hymes' critical response to Chomsky's linguistic competence) and the audience evaluation of competency are clearly typical components of classroom speech. In a traditional classroom that relies primarily on the "Triadic Dialogue" (Lemke, 1990) (the discourse pattern of teacher question, student answer and teacher response), the teacher has the major evaluative role, but in many US reform classrooms, student evaluation of competence is prominent. The question of whether mathematics discourse is distinguished from ordinary speech requires more analysis than is possible in this format, but I can at least note that in folk performance, cross-culturally, one of the most common means of highlighting performance discourse is through the use of specialized vocabulary, as in the mathematics register (Pimm, 1987).

The association of performance and responsibility is most apparent when speakers deny their ability to perform, as in "I don't really know how to tell jokes, but I heard one that went like this" or in a mathematics classroom, when a student addresses the teacher, "I'm not sure, but the book says…" These are ways for a speaker to give a report of previous knowledge rather than take responsibility for a full performance. Still, discourse that opens as a report can nonetheless develop into a more confident portrayal of the speaker's mastery of a topic. Judging from work on the emergent qualities of folkloric performances, several discourse features are likely to indicate that students' speech is a performance of mathematical knowledge rather than simply a report:

- Use of the mathematics register;
- Configuration of speech to control audience critique;
- Use of indexical language to orient the audience to particular aspects of the context;
- Semantic and syntactic parallelism or patterning.

A performance-centered approach to mathematics discourse allows researchers to track the emerging confidence of students' mathematical speech beyond a simple assessment of whether a given statement is factually correct. In the next section, I provide an example of how the above ideas can be used to examine mathematics classroom interaction.

PERFORMANCE IN THE MATHEMATICS CLASSROOM

Performance-centered approaches to discourse have been successful in revealing communicative principles in many folkloric genres and in many languages primarily because they offer formal features that arise across languages. Gee's definition of discourse is relevant:

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artefacts', of thinking, feeling, believing, valuing and acting that can be used to identify oneself as a member of a socially meaningful group or 'social network', or to signal (that one is playing) a socially meaningful 'role' (Gee, 1996, p. 131).

It is not always easy to appreciate the way a "role" is constructed in multiple languages without the sorts of formal features that performance-centered approaches emphasize.

A technique commonly used in folklore and linguistic anthropology for revealing the orderliness and beauty of verbal art is to render discourse in poetic lines. Poetry, rather than a dramatic script, becomes the model for representation of discourse. Parsing sentences into lines can be based on many different features, including breath pauses, intonation curves or parallelism, so that the same text can be divided into lines differently according to the analysis at hand (Tedlock, 1983). Take, for example, a passage of discourse in which Spanish-speaking third graders compare a parallelogram and a trapezoid (reproduced from Moschkovich, 1999, p. 16):

[Julian and Andres have several shapes on their table: a rectangle, a trapezoid and a parallelogram]

Julian: *Porque si. Nomás estas* (Because…Just these) sides get together [runs his fingers along the two non-parallel sides of the trapezoid…] *pero de este* (but on this side only). [runs his fingers along the base and top parallel sides of the trapezoid]

Mario: *Y este lado no* (And not this side)

Andres: *No porque mira, aqui tiene un lado chico* (No because, look, here it has a small side) [points to the two non-parallel sides of the trapezoid] *y un lado grande y tiene cuatro esquinas* (and a large side and it has four corners).

Julian See? They get together, *pero acá no* (but not here). [runs his fingers along the base and top parallel sides of the trapezoid]

Andres: *Acá no*

Recomposing this scene in poetic lines, with stanzas representing different speakers, we have:

Porque si. Nomás estas sides get together pero
de este

Y este lado
no

No
porque mira,
aqui tiene un lado chico
y un lado grande
y tiene cuatro esquinas

See? They get together,

pero acá no

Acá no

Here the lines are indented to display the syntactic parallelism. For example, the line *porque si* is echoed in *porque mira*. Semantic parallelism is present too, in, for example, the phrases *un lado chico…un lado grande.* In this representation of the transcript, the lines are also segmented into stanzas to indicate units of syntactically parallel lines each. This representation of the passage shows that there are more ways to analyze bilingual discourse than simply through code-switching. It demonstrates that performance, along with mathematics understanding, can emerge as a collective achievement of several speakers. A high degree of parallelism develops between the speakers as they repeat each others' sentence structure and word choice. The students developed parallel structures, including shared parallel structures, in both English and Spanish, as Julian's repetition of "They get together" blends into an echo of "pero acá…acá no." The students' language "got together," not just the sides of the figures!

The main advantage of the performance concept for mathematics education research is that it expresses a great deal of what we want our students to achieve. Our pedagogies should foster a student's ability, as Hymes put it, to "breakthrough into performance" (1981), to juggle mathematical ideas even when their level of mastery is tentative and incomplete. When students perform mathematics, we know that they are intellectually engaged.

CONCLUDING REMARKS

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Philip Clarkson, Australian Catholic University, Victoria, Australia

The ideas presented in this research forum have, we hope, served to raise issues and questions concerning the role of multilingualism in research in mathematics education. We have provided an opportunity to explore three aspects of this topic.

On *theory*, the forum has included an introduction to a range of theories of language, of language learning and of language acquisition. Whilst such theories form the basis for an entire field of applied linguistics in their own right, participants may now at least be aware of key reference points and have some indication of where to look further.

On *mathematics teaching*, the forum has highlighted the wide range of multilingual contexts in which mathematics classrooms are situated. Such diversity makes it difficult to make general claims concerning teaching or learning in multilingual contexts. We have, however, seen some general questions which arise more widely, such as the issues of interpretation across languages arising in Pakistan in Halai's research.

On *methodology*, the forum has focused on the issues of relating mathematics to language practices, with the perspective of linguistic anthropology offering an alternative light on the examination of mathematics classroom interaction.

An important motivation for this forum is the awareness that multilingualism is prevalent in mathematics classrooms around the world, yet rarely mentioned in research in mathematics education. It is clear from the exploratory nature of the work presented in this forum, that much remains to be done to take account of multilingualism at substantive, methodological or theoretical levels of our research. This observation leads us to see two clear areas in which PME research needs to take greater interest:

- There is a need for more research specifically focused on the role of multilingualism in mathematics education;
- There is a need for all research to acknowledge and take greater account of the multilingual contexts in which it is frequently situated.

We hope this forum has provided encouragement and a starting point.

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DG01: EXAMINING THESES

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Many members of PME are asked to examine theses submitted by students studying for higher degrees in mathematics education. The rules of individual institutions vary and often concern the aspects of presentation such as length of thesis, binding and time allowed. The content is usually required to be substantial but little guidance is given on the meaning of this expression. Judgement on the suitability of the thesis is made by the examiner according to personal [hidden] criteria. In the Discussion Group we have been sharing these criteria and trying to reach some measure of agreement. In Bergen we will discuss how far an examiner should be allowed to give his/her opinions on suitability of methodology/sample etc. using excerpts from written reports. Students are especially welcome.

DG02: THE ROLE OF MATHEMATICS IN SOCIAL EXCLUSION/INCLUSION: FOREGROUNDING CHILDREN'S BACKGROUNDS

Mike Askew, Peter Gates, Andy Noyes, Jan Winter, Robyn Zevenbergen.

At PME 27 this discussion group considered the theoretical contributions made by psychological and sociological paradigms to understanding inclusion/exclusion in mathematics learning. This discussion group will continue in this direction this year by considering how children's diverse backgrounds, i.e. family, social, ethnic, linguistic, peer group, etc., impact upon classroom learning of mathematics. That there is such an impact is clear but the relationship between school and home is also highly complex. The group will present data from a number of projects that consider the relationship between children's experiences out of school and their learning of school mathematics. In addition a number of theoretical perspectives will be offered as start points for further discussion.

Session I

Introduction of the theme: understanding and developing children's inclusion in classroom mathematics learning by exploring the unseen and diverse background experiences of children

- The middle part of this first session will present case study data from investigations of the relationship between children's experiences inside and out of classrooms. These studies include amongst other things studies aimed at:
    - Developing home-school links; learning from numeracy in the home, accounting for language, etc.;
    - Developing numeracy practices of children from disadvantaged backgrounds;
    - Developing understanding of mathematics learning dispositions as shaped in the family;
- This will lead to some initial discussion around the issues raised from the case material

Session II

- Following a brief recap of session 1 the group leaders will offer a number of theoretical positions that have, or could have, been adopted to make sense of the data presented in the first session. These might include perspectives from socio-cultural, sociological, critical and activity theory.
- This will be followed by a more general discussion of how the mathematics education community might develop these and other frameworks to explore in more depth the impact of children's social milieu upon their learning of mathematics.
- We aim to conclude the discussion with the recommendation of further avenues of inquiry in this area.

DG03: KEEPING THE MATHEMATICS IN MATHEMATICS EDUCATION RESEARCH

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This year's sessions refine the focus begun at PME 27 (The Rise and Fall of Mathematics Education Research). Last year, several issues were identified as important to maintaining the vitality of the field of mathematics education research. A key point was that mathematics must be made visible in our research to distinguish our findings from those of general education studies – including a careful consideration of the *mathematics* inherent in assigned tasks, in the analyses of what learners and teachers do, and in the curriculum that is developed and implemented.

This year we will explore this theme of keeping the "mathematics" in mathematics education research using a case study of a Japanese fourth-grade mathematics lesson as a shared resource. The data for the study were obtained as part of the activities of PME 24 in Hiroshima, in which participants visited a local school and observed one of several mathematics classrooms. In one lesson, which covered an introduction to two-digit division, students used personal strategies to solve a given problem and selected solutions were processed in a teacher-led whole-class discussion.

Session 1

Participants, in small groups, will analyze the case study - exploring mathematical perspectives in the classroom context. Aspects of the data that may be addressed include identifying 1) the mathematics taking place among the students and the teacher, 2) the relationship of this mathematics to fundamental concepts, 3) the potential of the activity to lead to deep mathematical learning or connections.

Session 2

We will consider the implications of the work done in the first session. For example, 1) how the analysis of mathematical issues, and how students and teachers grapple with these issues, can strengthen educational research, 2) how the results of such analysis inform the development of curriculum, materials, and classroom practice, and 3) considerations of long-term learning pathways and what kind of research can help us make longitudinal decisions.

A goal of the session will be to identify a specific research task (in which the mathematics is visible) that can be tried out with learners in several places, to trigger next year's discussion.

DG04: RESEARCH BY TEACHERS, RESEARCH WITH TEACHERS

Coordinators: **Jarmila Novotná**[1], Charles University, Prague, Czech Republic
Agatha Lebethe, Mathematics Education Primary Programme, South Africa
Gershon Rosen, Western Galilee Regional Comprehensive School for Science and Arts, Israel
Vicki Zack, St. George's School, Montreal, Quebec, Canada

This new Discussion Group is a follow-up from the Plenary Panel *Teachers who navigate between their research and their practice* held at PME 27/PME-NA 25 in Hawai'i in 2003. We invite all who are interested in practitioner research, especially teachers who are (or wish to be) researchers in schools as well as university people who would like to do collaborative research with teachers in schools.

We are proposing several questions which examine the Panel topic more extensively. The first set of questions are general ones:

- Why do practitioner research? Should all teachers do practitioner research?
- Are there differences in the research results if the direction is teacher ⇒ teacher researcher or researcher ⇒ teacher? If yes, what are the main differences?
- Should faculties of education prepare practitioners to do education research? If yes, how?
- What might teacher researchers working alone, or working in collaboration with university researchers, contribute to educational research?

The second set of ideas deals with practical questions concerning teachers' daily tasks in schools, and their (possible) relationship to teacher research:

- What does it mean when we say that "this approach or activity works"?
- How do we evaluate something to be "better" than something else, as for example a way of teaching a certain topic? For whom is it better? Why is it better?
- Do we have any evidence to show that it works better for certain students? On what basis are we claiming whatever it is we are claiming?
- What was happened in terms of learning and/or teaching?

These questions will provide a general framework for the two Discussion Group sessions.

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[1] The contribution was partly supported by the Research project MSM 114100004 Cultivation of mathematical thinking and education in European culture.

DG05: COMMUNICATION IN MATHEMATICS CLASSES – QUESTIONING AND LISTENING

Coordinators: **Lisser Rye Ejersbo, Learning Lab Denmark**,
Erkki Pehkonen, University of Turku

Using problem solving in mathematics class is quite normal, but its implementation ways vary much. For the quality of pupils' learning, the way it is implemented and the communication used are crucial. One of the main questions for the teacher is how to get his or her pupils involved and motivated in the problem solving process in the first place, and how to keep it going once it has started (cf. Mason 1982, 28). The kind of task or questions the teacher uses will have important implications for the progress of pupils' problem solving. In the discussion group, the following questions will be discussed:

What is a good task or question, i.e. a question which produces meaningful discussion, and why?

What skills are needed for the teacher to make the communication inspiring and effective, and why?

We will try together to develop a classification system for questions, i.e. to define levels of questions.

For the communication part we'd like to focus on the teacher's listening. According to constructivist understanding of learning, it is uttermost important for the teacher to understand his or her pupils' thinking and knowledge base, in order to help them learn meaningfully - the so-called interpretation model (Schoenfeld 1987, 29). Covey (1989, 236) lists five ways of listening: (1) Ignoring the other, (2) Pretending, but not listening, (3) Selective listening, (4) Attentive listening, (5) Empathic listening. According to Covey, a way to achieve pupils' understanding is to try to understand them, before trying to get them to understand you.

Can we use such a classification of listening in mathematics class? In the overall categorization we might distinguish different levels of communication, e.g. the sharing of questions and listening, the quality of questions and the level of pupils' understanding of the content in questions.

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DG07: SEMIOTIC AND SOCIO-CULTURAL EVOLUTION OF MATHEMATICAL CONCEPTS

Coordinators

A. Sáenz-Ludlow, USA Norma Presmeg, USA Carlos Vasco, Colombia

The history of mathematical concepts is a story of the dialectics between thought and symbolization, a story of creation, re-creation, and refinement of mathematical concepts and patterns of symbolization within different socio-cultural frames and throughout different eras. The goals of the discussion group are to continue pondering from the semiotic point of view two related issues that emerged in our last discussion group DG7 in Hawaii. These issues are (a) the usefulness of the history of mathematics as a pedagogical tool; and (b) the use of semiotic tools to refine and critique the synchronic and diachronic interpretations of historical texts. Three presentations will situate the discussions in context. These papers will be posted in the group website (http://www.math.uncc.edu/~sae/) at the end of May for reading prior to the conference.

I. The problem of mathematics education and history of mathematics from a Saussurean point of view

Michael N. Fried — *Israel* — *15 minutes*

Discussion points — *30 minutes*

1. Is mathematics a semiological system, or is it embedded in a semiological system, or is it similar to a semiological system? Are there pedagogical implications for each of the above three positions?
2. How absolute is the division between the synchronic and diachronic views of a sign system? Does this distinction hold for mathematics taken as a sign system?
3. Are there theoretical difficulties in combining history of mathematics and mathematics education?

II. Equalities revisited. A pragmatic analysis

Carlos Vasco — *Colombia* — *15 minutes*

Discussion points — *30 minutes*

1. Are the seven proposed pragmatic intents for equality statements plausible?
2. Can they be reduced to fewer or are there others that might be needed?

III. Beyond the representation given. The parabola and historical metamorphoses of Meaning

Christer Bergsten — *Sweden* — *15 minutes*

Discussion points — *30 minutes*

1. What can be learned from the historical metamorphoses of the parabola, as a mathematical object, from classic and analytic geometry to dynamic geometry?
2. The metamorphoses of meaning in the social process of didactic transposition: from being a mathematical object to being an object of teaching and learning.

IV. Small group discussion and general discussion — *45 minutes*

DG08: TOWARDS NEW PERSPECTIVES AND NEW METHODOLOGIES FOR THE USE OF TECHNOLOGY IN MATHEMATICS EDUCATION

Bibi Lins	Victor Giraldo	Luiz Mariano Carvalho	Laurie Edwards
UNESP	UFRJ	UERJ	Saint Mary's College

Various research studies on the use of technology in Mathematics Education have explored different conceptual frameworks and methodologies for analysing teaching and learning situations in microworlds environment. Early research using more essentialist perspectives (taking technologies as given) often worked within a narrow model. Later research involving more anti-essentialist perspectives (taking technologies as problematic), such as social shaping, cultural studies and actor network models of epistemology, tried to understand the role of technology in educational settings. This discussion group seeks to initiate a dialogue that moves away from current methods and frameworks to new perspectives and new methodologies for considering the use of technology in mathematical education. We are particularly interested in developing international, and possibly alternative, anti-essentialist perspectives that would help us to understand the role of designers, technology and users of technology, such as mathematics teacher educators, teachers and students.

Questions for Discussion

1. What perspectives are used to investigate the use of technology in Mathematics Education in different countries?
2. How would new perspectives allow us to re/think the role of users of technology?
3. What new methodologies would enable us to investigate difficult issues concerning teaching and learning situations in microworlds environment?

Discussion Sessions

We will begin with short presentations about emerging perspectives on the study of the use of technology in Mathematics Education, leading to discussion of the above questions. All participants are encouraged to elicit what problems remain that need to be addressed by research. We intend to form a network of participants to continue discussion via email and possibly develop joint research projects.

WS01: DEVELOPING ALGEBRAIC REASONING IN THE EARLY GRADES (K-8): THE EARLY ALGEBRA WORKING GROUP

Coordinators: **John Olive & Maria Blanton**

University of Georgia & University of Massachusetts Dartmouth

The Early Algebra Working Group's focus is on investigating and describing what we construe as the possible geneses of algebraic reasoning in young children, and in developing and investigating ways to enhance that reasoning through innovative instruction, applications of appropriate technology and professional development for teachers. The EAWG was formed in response to a call from the International Commission on Mathematical Instruction (ICMI) to hold a study conference on "The Future of the Teaching and Learning of Algebra" in December of 2001 in Melbourne, Australia. Following that initial conference, the group has conducted working sessions at PME-NA XXIV in Athens, Georgia, 2002 and at the joint PME 27/PME-NA 25 meeting in Hawaii, 2003.

PLANNED ACTIVITIES FOR PME 28

We plan to hold two 90-minute sessions during which examples of different approaches to fostering algebraic reasoning in young children will be described and discussed. The following researchers have agreed to provide examples: Sergei Abramovich from SUNY at Potsdam (the use of a computer graphics program to enhance second grade students' algebraic thinking), Sybilla Beckmann from the University of Georgia (activities from the Singapore grades 4-6 mathematics text), Barbara Dougherty from Hawaii University (interim results from the "Measure Up" project) and Paul Goldenberg (Education Development Center's materials that are based on an approach envisioned over 40 years ago by WW Sawyer). Participants will engage in prototypical activities from each project followed by discussions of each approach. The following questions will be used to guide these discussions:

1. What are cross-cutting themes (or dissimilarities) of how algebraic thinking is enacted across these activities? What can we learn from this?

2. What are the broader algebraic content issues that need to be addressed? What do we know about what students can do algebraically? What needs further research?

3. What are the policy/implementation/curricular issues that affect the integration of algebraic thinking in the elementary grades?

4. What conversations need to occur and how can they be initiated with secondary mathematics so that algebraic thinking is a connected agenda across K-12? What is currently being done to facilitate this?

5. What do we know about how early algebra impacts student learning in secondary mathematics? What kind of research activities are needed to address this question?

WS02: EMBODIMENT, METAPHOR & GESTURE IN MATHEMATICS

Laurie Edwards

St. Mary's College of California, USA

Ornella Robutti

Università di Torino, Italia

Janete Bolite Frant

Pontificea Universidade Catolica do Rio de Janeiro, Brasil

The purpose of the Working Session is to continue a consideration of the role of embodiment in mathematical learning, thinking, teaching and communication. In specific, the group will focus on the application of work in cognitive linguistics to understanding mathematical thought (Lakoff & Núñez, 2000), and on ways in which gesture is involved in learning, doing and communicating about mathematics (McNeill, 1992; Goldin-Meadow, 2003). Themes and questions to be addressed include:

1. How do gestures relate to speech during social interaction, for example teaching or working in small groups, in terms of content, form, and timing?
2. When are gestures meaningful in introducing a concept, a sign, or an interpretation of a situation?
3. How can gestures be used to condense and manage information during social interaction?
4. What role does gesture play in supporting mathematical problem solving?
5. How are conceptual mechanisms such as metaphors and blends involved in students' cognitive processes while learning and doing mathematical activities?
6. How do gestures and unconscious conceptual mechanisms relate to external representations and technologies used in mathematical activity?
7. What are the relationships between conceptual mechanisms like metaphors and blends, imagery, and gesture?

The Working Session will be structured in two basic parts: during the first part, a common foundation will be established by presenting basic definitions and selected existing research in gesture and cognitive linguistics in mathematics education. The majority of the session will be devoted to contributions from participants, including analysis of data, for example, videotapes of mathematical activity. All interested PME participants are invited to join the Theory of Embodied Mathematics list-serv by sending e-mail to ledwards@stmarys-ca.edu.

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WS03: RESEARCHING THE TEACHING AND LEARNING OF MATHEMATICS IN MULTILINGUAL CLASSROOMS.

Richard Barwell, University of Bristol, UK

Anjum Halai, Aga Khan University, Pakistan

Mamokgethi Setati, University of the Witwatersrand, South Africa

Multilingual classrooms are increasingly the norm in education systems around the world. By multilingual classrooms we mean classrooms in which two or more languages are present. These languages may or may not be heard in classroom talk. They are, however, always available for use by students or teachers during public or private interaction.

The aim of this working group is to raise and discuss methodological issues, which arise in doing mathematics education research in multilingual classrooms. In this year's meeting we will focus on: researching mathematics teaching in multilingual classrooms; working with mathematics teachers in multilingual contexts.

ACTIVITIES

The two sessions of the working group will be devoted to working on video and transcript data from multilingual mathematics classrooms in South Africa and Pakistan, as well as official guidance for teachers from the UK. For each sample of video/ transcript data, we first invite participants to address analytic questions, such as:

- what mathematics is taking place?
- how are different languages used (or not used) in teaching the mathematics?

We then invite participants to reflect on the issues, as well as to consider the UK guidance in their light. Questions for reflection include:

- in what ways can teaching take multilingualism into account?
- how can research support teachers to develop their mathematics teaching in multilingual contexts?

We hope that participants will include researchers who work in multilingual contexts or whose research interests concern the role of language in mathematics classrooms.

WS04: THE COMPLEXITY OF LEARNING TO REASON PROBABILISTICALLY

Carolyn A. Maher — Rutgers University
Robert Speiser — Brigham Young University

NATURE AND TOPIC OF THE WORKING SESSION

This Working Group was formed at PME-NA 20 and has since convened annually at PME-NA. During the joint meeting of PME-NA 25 and PME 27 in 2003, we expanded our working group to include international researchers from 11 different countries. Through shared research, rich and engaging conversations, and analysis of instructional tasks, we continually seek to understand how students learn to reason probabilistically.

AIMS OF THE WORKING SESSION

There are several critical aims that guide our work together. In particular, we are examining: (1) mathematical and psychological underpinnings that foster or hinder students' probabilistic reasoning, (2) the influence of experiments and simulations in the building of ideas by learners, particularly with emerging technology tools, (3) learners' interactions with and reasoning about data-based tasks, representations, models, socially situated arguments and generalizations, (4) the development of reasoning across grades, with learners of different cultures, ages, and social back-grounds, and (5) the interplay of statistical and probabilistic reasoning and the complex role of key concepts such as sample spaces and data distributions. Through our work, we have stimulated collaborations across universities and plan to engage in and support additional research.

PLANNED ACTIVITIES

At PME 27, members of the Working Group decided to create a listserv to promote follow-up contact and subsequent discussion through the use of e-mail. Through this listserv, we suggested several tasks that can be used with a variety of students in a variety of contexts. It was agreed that several participants use a single task with students. Data will be collected in the form of videos and paper-and-pencil work. This data, collected across contexts, cultures, and ages, will serve as a common data set for our continued work at PME 28 (Norway) and PME-NA 26 (Toronto). In particular, during our sessions, we plan to collaboratively analyze videotape data of students' probabilistic reasoning on a task by using several different theoretical perspectives. From this analysis, we seek to generate additional authentic tasks that are appropriate to elicit and extend students' probabilistic reasoning into a broader perspective that includes statistical reasoning.

We are planning to maintain this working session group in both organizations so that international collaborations can continue. Several participants will be attending both PME 28 and PME-NA 26 in order to allow for consistency and communication across groups. It is hoped that our analysis of students' work on this task will lead to a set of papers that describe our work. These papers could be part of a monograph, journal special issue, and joint presentations at future conferences.

WS05: CREATIVE WRITING

Markku S. Hannula & Agatha Lebethe

University of Turku, Finland & University of Cape Town, South Africa

> Writing is also a way of knowing – a method of discovery and analysis. By writing in different ways, we discover new aspects of our topic and our relationship to it. Form and content are inseparable. (Richardson, 2000, p. 923)

Richardson (2000) argues that in qualitative research writing the report is part of the analyses and interpretation of data, not a separate process. The experience is created in, or together with, the text and there is no difference between writing and fieldwork. This blurring of text and experience has led to new forms of experimental writing. Richardson (2000) elaborates on what such 'creative analytical practices' are and lists references to dozens of examples. Hannula (2003) argued that fiction-writing techniques can help both the writer and the reader gain a closer intimacy with the personal experiences that are under investigation.

The aim of this working session is to provide participants a forum where to experiment with and reflect upon different ways of writing.

ACTIVITIES

The session will take a piece of a qualitative research data as a starting point and then the participants will be writing about that data in multiple ways. The session will be structured into periods of brainstorming, writing, and reflective discussions. We encourage participants to use different styles from poetry to tragedy, to amplify different voices (teacher, students, researcher) and to use a variety of metaphors and narrative structures. The participants are encouraged to bring along their own data.

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WS06: SYMBOLIC COGNITION IN ADVANCED MATHEMATICS

Stephen J. Hegedus	David O. Tall	Ted Eisenberg
University of Massachusetts Dartmouth, USA	University of Warwick, UK	Ben-Gurion University, Israel

Symbolic Cognition is the study of the construction of mathematical signs and symbols and the processes involved in manipulating such objects into meaningful concepts, procedures and representations in a variety of mathematical contexts. More practically, it aims to understand the ways in which symbols in many different representational forms, both as static squiggles as well as dynamic interactive objects, help us to do mathematics, build intuitions, develop mathematical concepts and construct powerful mathematical ideas. We investigate this through consideration of the evolution of symbols and their role in the intellectual development of the learner from the early years through to maturity.

Recent work has developed a three-fold mode of inquiry with associated research questions: 1. The use of symbols in human activity and theories of their use, e.g. theories of symbol-systems, semiotics, etc, how they interrelate and their roles, 2. The specific use of symbols in mathematics, with a special focus on advanced mathematical thinking, 3. The role of symbol-use with new technologies. We propose to meet for a fourth year to develop these areas into researchable domains. On-going work has been facilitated by an email discussion group and a constructive body of work (see **www.symcog.org** for details of work to date).

This year, we aim to deepen our inquiry by focusing our study on particular areas of mathematics and in so doing, address the second task type highlighted in the PME Working Session format of “Doing Mathematics”, examining multiple perspectives of teaching and learning particular areas of advanced mathematics and the co-evolution of mathematical notation systems.

We have met for three years and have had a rolling clientele. We will continue to have open meetings for the sake of those attending the conference, but would like now to focus on a publication. For this purpose we will also plan a smaller meeting for those who have something to offer to the work on a multi-author book. Anyone

THE DEVELOPMENT OF MATHEMATICAL CONCEPTS: THE CASE OF FUNCTION AND DISTRIBUTION

Marta Anaya

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The mathematization of physical concepts may require not only the use of mathematical concepts in contexts of different degree of generality and complexity, but also to develop an advanced mathematical thinking. The physical concept of impulse, which is mathematized by means of the δ-Distribution is an example of the result of abstracting and generalizing the concept of (numerical) function, leading to the notion of distribution. A previous exploratory study showed that students' intuitive model of the mathematical impulse as a numerical function, acquired when studying physics, may cause difficulties when formalizing this concept. (Cavallaro &Anaya, 2002).

The historical development of some notions (like number and function), have shown to be a cyclic process which Sfard (1991) refers to, as a long chain of transitions from operational to structural conceptions.

In this work, results will be shown of a research in which the conceptions of functions and distributions were studied following the lines of Sfard (1991) and Dubinsky (1991). This study was carried out with 40 students from the Engineering Faculty of the University of Buenos Aires. Data analysis of two questionnaires on distributions and functions and a modeling activity, show that even if some students could successfully conceive (numerical) functions as mathematical objects, this didn't happen when they were dealing with distributions. Difficulties were found not only in reifying or encapsulating a distribution as a static object on which actions and processes were to be performed, but also in the development of a process conception for distributions (the δ-Distribution case will be analyzed). The cyclic process mentioned above repeats again. Students' conceptions, didactical implications and possible lines of future research will be discussed during the presentation.

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IDENTIFICATION OF MATHEMATICAL MISTAKES BY UNDERGRADUATE STUDENTS

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One of the major objectives of mathematics teaching is the development of students' advanced logical reasoning and critical thinking. To develop this kind of students' ability mathematics teachers have to be able to identify students' mistakes of different types. Checking students' solutions and studying the nature of the errors are critical components of mathematics teachers' everyday work. "Teachers need to know the ideas with which students often have difficulties and ways to help bridge common misunderstandings" (NCTM, 2000, p.17). However, pedagogical experience convinces us that this process of developing logical and critical thinking is one of the major pitfalls of mathematics education. One of the goals of this research is to answer the question: "How do undergraduate students identify mistakes of different types?"
One of the ways to know the level of undergraduate students logical and critical thinking is use of worked-out examples that may include correct or incorrect solutions. "Students [as well as teachers] can use their errors to develop a deeper understanding of a concept as long as the error can be recognized and appropriate, informative feedback can be obtained." (Fisher & Lipson, 1986, p.792)
Sixty-five College's undergraduate students were asked to check four worked-out examples that included incorrect solutions for two mathematical problems (two solutions for each problem). One of the problems was an algebraic one and the second was a geometric problem. Examples for each of the problems included different logical mistakes. The students were asked to examine whether the solutions presented to them were correct and to explain students' mistakes that they found in the solutions.
The worked-out examples were included in the written questionnaire. All the students had the knowledge base to tackle the questions.
In our presentation we will show the worked-out examples presented to the students and will discuss the results of our research.

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SECONDARY SCHOOL TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE

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Current research on mathematics teachers' subject matter knowledge, which includes knowledge of the content of a subject area as well as understanding of the structures of the subject matter (Schulman, 1986, 9), has been investigated in a large number of recent studies (e.g. Ma 1999; Attorps 2003). The research results are essentially the same: teachers lack a conceptual knowledge of many topics in the mathematics curriculum. Current research on the relationship between teacher knowledge and teaching practice has also pointed out the need to carry out more studies involving specific mathematical topics. Furthermore, this research has shown that the way teachers in mathematics instruct in a particular content is determined partly by their pedagogical content knowledge i. e. knowledge that is specific to teaching particular subject matter (Schulman, 1986, 9). In this article I discuss ten beginning and experienced secondary teachers' pedagogical content knowledge concerning the concept equation. Data was gathered by interviews and videotapes and the phenomenographic research approach was applied in the investigation. This approach illustrates in qualitatively different ways how a phenomenon, an object around us is apprehended and experienced by individuals (Marton and Booth 1997). My results indicated that teachers' conceptions about the purposes for teaching equations stress students' procedural knowledge of the concept equation rather than their conceptual understanding of the mathematical notion. Both beginning and experienced teachers presented detailed knowledge needed to identify specific student difficulties of this particular concept. However experienced teachers possessed more rich repertoires, experiments and explanations of the concept than beginning teachers did. They also show a bigger ability than beginning teachers to construct situations and instructional strategies that might assist students to overcome their difficulties with equations.

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EXPLORING THE CHALLENGE OF ONLINE MEDIATION

Jenni Back, Charlie Gilderdale and Jennifer Piggott, University of Cambridge

This paper explores how mediation could be offered on a mathematics enrichment website like NRICH (www.nrich.maths.org) that attempts to engage and challenge students to think mathematically. In classrooms there is often a lot of mediation at the start of lessons in which teachers prepare students for what they are about to do and on-going mediation that offers clarification and reminders. With different children and different tasks, a very varied and rich range of possible options is open to teachers. These features of classroom life are not available to students tackling problems in an online environment like NRICH.

To get some sense of what mediation might help, we asked 70 students aged between ten and eleven to try a non-standard problem from the NRICH site. The students needed to realise that the problem did not have a solution and they were required to provide a convincing argument that showed why. We then administered a questionnaire that asked them about the experience of tackling the problem.

Their responses suggested that some students could be positioned along a continuum. At one extreme of this continuum they appreciated the chance to work on something different and challenging, were motivated to persevere with the problem, saw the potential for learning and understood that working on problems offered the opportunity for gaining new insights. However, at the other extreme they did not recognise that there was anything to be gained from tackling an 'impossible' problem and expressed strong negative feelings about being asked to work on something which did not fit with their preconceptions about what mathematics questions are usually like.

We initially assumed that the mediation that would be necessary would only need to point to the mathematics in the problem and would be in the form of hints. The feedback from some of the students shows that this will not always be enough. Our results have highlighted the need to consider working with learners perceptions of mathematics so students are able to engage with the demanding mathematics that is offered on the NRICH website. How can this be done? We have three suggestions:

- Metacognitive mediation and advice that spells out some of the assumptions about how students are expected to work.
- Looking at solutions: this option would direct students to the solution of the problem and invite them to consider it before tackling another problem which shares some significant features with the original problem.
- Participation in webboard discussions.

Our suggested mediation strategies seek to help pupils manage their learning, offer students a range of choices that attempt to be sensitive to their cognitive and emotional needs whilst encouraging them to engage with challenging mathematics.

DIVERSITY OF GEOMETRIC PRACTICES IN VIRTUAL DISCUSSION GROUPS

Marcelo Bairral — Federal Rural University of Rio de Janeiro, Brazil
Joaquim Giménez — Barcelona University, Spain

Interactive processes turned to professional development usually interest researchers in this area. Nevertheless, the analysis of the various teachers' meanings collaboratively shared in a given virtual community is still incipient in mathematics education. In our longitudinal research we have been analysing the contributions of a virtual environment to the professional development in geometry (Bairral and Giménez, 2003). In this study we will analyse the teachers' discourse and identify hypertextual links (Jonassen, 1986; Gall and Hannafin, 1994) in discussion forum.

Data analysis

We consider that the reflections to the discussion forum are sequences of professional actions and that they must establish different semantic relations. In each teleinteraction it is possible to identify new information on the content of the teachers' knowledge. This information is somehow hypertextually related to the contents of the contribution to which we are referring or to another educational context. The discussion forum was one of the communicative spaces and the interventions to the debate were registered. The procedures followed for the analysis were: **(1)** the creation of a specific file for the texts, by numbering and coding them; **(2)** the transference of contributions to the researcher diary in order to complete them with constant remarks and analysis; **(3)** the characterisation of interactions; **(4)** the summary of ideas and confection of schemata to analyse the dynamic of the debate as hypertextual, and **(5)** the meta-analysis of parts of the debate.

Results

The attention to the personal reflexive processes and their socialisation along the professional development was a notorious fact in the dynamic at the forum. It was a space for the collective immersion in the discussion (with a response action more flexible in time) that presupposes a security and trust in the group. Each teacher participated and contributed in different ways. The analysis permitted us to detect that *argumentative* interventions generate cognitive nodes and often are reported by the teachers to the metacognitive discussion. We also found out that the *informative* interventions generate referential or hierarchical nodes.

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DEVELOPMENT OF INSIGHT OF FUTURE MATH TEACHERS AS A RESULT OF FOLLOW-UP AFTER DEVELOPMENT OF MATHEMATICAL CONCEPTS

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We focus on aspects of interrelations between the stages, levels and modes of percept formation. Mostly we are interested in those related to math teaching and education of math teachers. We assert that the math teachers should be exposed to various aspects of process of development of mathematical knowledge.

In teachers-educating institutions, various sides of this process are being discussed in various courses. The integration of contents of these courses is necessary.
In order to assess the measure of this integration, and to enhance the global aspects of math teaching related to continued structured building-up of mathematical concepts for our students, we planned an experiment in which we induced a number of pre-service students of primary and secondary school programs to follow-up some key concepts in geometry as they develop along the whole school program. In doing this, we meant that a teacher has to survey the perspective of his pupils' former and future learning and to detect correctly his or her place and role in this perspective
The research population included 20 third year in-service and pre-service students who form the primary and the secondary school programs who worked in small heterogeneous groups. The groups were asked to learn a certain geometrical concept; to check how this concept is to be taught according to the school curriculum; to match the Van Hiele level of geometrical thought to the curriculum requirement at each school level; to study 2-3 textbooks in order to appraise all the aspects above as they are (or are not) reflected in them; to draw conclusions.
As we deduced from what the students have written, they come to better understanding of the school curriculum in geometry. As to the contents, the students tended to refer to the textbooks and to compare them, rather than referring to the connections between all the components of the concept teaching. No one mentioned the necessity to teach correctly from the point of view of theoretical mathematical knowledge.

The experiment indicated that the integration of knowledge acquired by the students in different courses is far from granted, and should be worked upon.

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CO-TEACHING BY MATHEMATICS AND SPECIAL EDUCATION PRE-SERVICE TEACHERS IN INCLUSIVE SEVENTH GRADE MATHEMATICS CLASSES

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Special learning strategies have been used to teach mathematics to students with learning disabilities (LD) in resource rooms or self-contained special education classes (Swan, 1998). In recent years there has been a trend to include students with LD in general education classes, providing them with in-class support. Research shows that children who receive in-class support are more accepted by peers and have higher self-perceptions of mathematics competence than those in self-contained classes (Wiener & Tardif, 2004). However, no inclusion program can succeed without changing teachers' attitudes and adapting programs of student-teachers' education (Baker & Zigmond, 1990).

Teacher training in Beit Berl College takes place in P.D.S*, a program involving: children's learning, pre-service teachers' training, in-service teachers' professional development and research. In our project, mathematics and special education student-teachers (ST), cooperated to build and teach a unit in mathematics for seventh grade inclusive classes. The learning unit was originally adapted by the special education ST for children with LD, but was used for the whole class. The planning sessions, the co-teaching experience of the mathematics and special education ST and the reflective process were all videotaped. The seventh graders' performance was assessed as part of a more general test. In our presentation we will show some examples of the study unit and scenes from the videotape showing the collaboration of pre-service teachers, cooperating teachers and college faculty. We will discuss the contribution of the project to all those who were involved, including the project's effect on the achievement of students with LD.

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Professional **D**evelopment **S**chools *

IS IT A MATHEMATICAL PROOF OR NOT ? ELEMENTARY SCHOOL TEACHERS' RESPONSES

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There is a wide consensus in the mathematics education community that teachers should encourage students to make mathematical conjectures. In addition, students should be encouraged to investigate and validate various conjectures (e.g., NCTM, 2000). In this instructional approach, the teacher should relate to the conjectures that arise in his/her class and to the ways that the students use to verify them. It is therefore essential for teachers to be intimately familiar with both formulating conjectures and reacting to arguments that purport to prove or refute mathematical conjectures.

The main aims of this study are to examine elementary school teachers' subject matter knowledge and pedagogical content knowledge concerning proofs and refutations. This paper focuses on elementary school teachers' reactions to common, correct and incorrect, justifications to universal theorems.

Twenty-seven elementary school teachers were asked if they would accept several, given justifications to various universal theorems and to explain their decisions. The given justifications included numerical examples, algebraic proofs and non-formal generalizations. The specific justifications that were used in this study were provided by the same elementary school teachers when they were asked to determine the validly of these statements and to prove their positions. These results were described in a previous PME paper (Barkai, Tsamir, Tirosh & Dreyfus, 2002). Interestingly enough, a substantial number of teachers rejected the justifications that were identical to those that they themselves wrote when they were asked to prove these statements. In the presentation we shall describe this and other results of the study.

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INVESTIGATING USING THE THEORY OF REALISTIC MATHEMATICS EDUCATION TO ELICIT AND ADDRES MISCONCEPTIONS

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This paper reports on some of the results of a case study, consisting of 12 individual cases, carried out in a local urban high school in South Africa. In the study, an intervention for low attaining Grade 8 mathematics learners was implemented in an attempt to improve the conceptual understanding of the participants with regard to place value, fractions and decimals. The intention of the intervention was to revisit familiar topics with an emphasis on eliciting and addressing misconceptions so that these could serve as a source of information (Dockrell & McShane, 1992) that could be used as motivational devices and starting points for mathematical explorations (Borassi, 1987). The literature pertaining specifically to learners with learning disabilities or low attaining learners appears to indicate that these learners on average demonstrate a greater percentage of systematic errors (misconceptions), than higher achieving learners (e.g. Cox, 1975; Woodward & Howard, 1994). Analysis of the error patterns revealed that many of the errors occur due to limited conceptual understanding of the algorithms and strategies taught to learners. For this study, the hypothesis was therefore made that if some of the fundamental misconceptions held by learners could be elicited and addressed during the intervention, their conceptual understanding, relating to the mentioned topics, would improve.

After consulting the literature on low attainers and related terms such as Special Educational Needs (SEN), learning disabilities and difficulties, common aspects that could be included in a working framework for an intervention were identified. The theory of Realistic Mathematics Education (RME) was selected as the vehicle to drive the design and implementation of the intervention as it encompassed all the aspects included in a working framework. The instructional design principle of 'guided reinvention through progressive mathematisation' (Gravemeijer, 1994) also potentially provided a good basis from which to work with the misconceptions. Results from the intervention varied but were overall quite positive and are discussed with specific reference to the hypothesis regarding misconceptions stated above.

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AN INVESTIGATION OF BEGINNING ALGEBRA STUDENTS' ABILITY TO GENERALIZE LINEAR PATTERNS

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This is a qualitative study of 22 9th graders in a public school in California who were asked to perform generalizations on a task involving linear patterns. Our research questions were: What enables/hinders students' abilities to generalize a linear pattern? What strategies do successful students use to develop an explicit generalization? How do students make use of visual and numerical cues in developing a generalization? Do students use different representations equally? Can students connect different representations of a pattern with fluency?

Twenty-three different strategies were identified falling into three types, numerical, figural, and pragmatic, based on students' exhibited strategies, understanding of variables, and representational fluency. Some of the more common numerical strategies include the following: use of finite differences in a table; random or systematic trial and error; or use of finite differences to generalize to a closed formula. Some of the more common visual strategies identified were the following: visual grouping manifesting either a multiplicative or an additive relationship; use of visual symmetry such as seeing concentric or polygonal relationships; visual finite differences; and figural proportioning.

This study is consistent with findings from an earlier study we conducted with preservice elementary teachers (Rivera & Becker, 2003) as well as work done by Küchemann (1981) and Stacey & Macgregor (2000). Overall, students' strategies appeared to be predominantly numerical. In this study we identify three types of generalization based on similarity: numerical; figural; and pragmatic, in accord with findings by Gentner (1989) in which children were shown to exhibit different similarity strategies when making inductions involving everyday objects. Students who use numerical generalization employ trial and error as a similarity strategy with no sense of what the coefficients in the linear pattern represent. The variables are used merely as placeholders with no meaning except as a generator for linear sequences of numbers, with lack of representational fluency. Students who use figural generalization employ perceptual similarity strategies in which the focus is on relationships among numbers in the linear sequence. Variables are seen as not only placeholders but within the context of a functional relationship. Students who use pragmatic generalization employ both numerical and figural strategies and are representationally fluent; that is, they see sequences of numbers as consisting of both properties and relationships. We see that figural generalizers tend to be pragmatic eventually. Finally, students who fail to generalize tend to start out with numerical strategies and lack the flexibility to try other approaches.

SINGAPORE'S ELEMENTARY SCHOOL MATHEMATICS TEXTS AND CURRENT RESEARCH ON WHOLE NUMBER OPERATIONS

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Singapore's grade 1 and 2 mathematics texts were examined using Fuson's 2003 review of research findings on whole number operations as a framework. The texts were found to use a number of strategies that have been demonstrated to be effective by research. Several known effective strategies, notably the use of accessible algorithms, do not appear in the grades 1 and 2 texts used in Singapore.

Fuson's 2003 chapter was used as a basis for analyzing the presentation of whole number arithmetic in the grades 1 and 2 texts used in Singapore. These texts present material in ways that have been shown to be effective by research, notably: 1) The texts introduce addition, subtraction, and multiplication by eliciting stories to go along with pictured situations. 2) The texts support a progression from "counting all" to "counting on" and using thinking strategies for single-digit addition. 3) The base-ten structure of decimal numbers is repeatedly emphasized with drawings of bundled objects. Multi-digit addition and subtraction is strongly supported with these visual aids. 4) The presentation is structured around big ideas, conspicuous strategies are shown clearly, often with the aid of simple diagrams, background knowledge is primed, there are many visual supports with cues for correct methods, material is integrated into complex applications to provide distributed practice, and opportunities for judicious review are provided.

The following items that have been shown by research to be effective are not used in the Singaporean texts: 1) Single-digit subtraction by counting up is not shown explicitly. Instead, subtraction problems are often accompanied by a simple "number bond diagram", showing a number broken into two parts. Counting up could easily be used with these diagrams. 2) Accessible multi-digit addition and subtraction algorithms are not presented in the texts. However, the standard addition and subtraction algorithms are strongly supported with visual aids.

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THE ROLE OF LEARNING COMMUNITIES IN MATHEMATICS IN THE INTRODUCTION OF ALTERNATIVE WAYS OF TEACHING ALGEBRA

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The purpose of this article is to address the main themes of my planned PhD thesis. It focuses on the roles of teachers in the teaching and learning of algebra. A five-levels developmental and analytical model is described with different layers of teachers' reflections emerging from this model. It is suggested that engaging teachers in this developmental model may increase awareness concerning the complexity of the teaching situation.

THE MAIN THEMES OF MY STUDY

The main focus of my thesis is the teaching and learning of algebra. This subject has turned out to be difficult for students and they develop mathematical skills without necessarily exploring the full power of mathematics: reaching a relational understanding of mathematical concepts and seeing the need for the use of symbols. My study involves two or several teachers with development and analysis taking place at five levels. I propose mathematical tasks (related to algebra) for the teachers and study the way they cooperate in solving these (level 1) and the kind of reflections emerging from this process (level 2). The next step is to observe how teachers plan what kind of tasks they can offer to their pupils in their respective classes in order to foster the same kind of reflections that they experienced in level 2 (level 3 and 4). The last level addresses teachers' evaluations of and reflections on the teaching period (level 5). This developmental model evolves as a spiral with teachers' reflections as the main focus. This perspective may offer a powerful conceptual framework for research on the interacting development of teachers engaging themselves and with pupils in problems related to algebraic thinking.

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IS THEORETICAL THINKING NECESSARY IN LINEAR ALGEBRA PROOFS?

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My research took place within the context of a larger study aimed at investigating the potential of weekly quizzes in developing students' theoretical thinking in an undergraduate linear algebra course. We looked at how students prove in linear algebra, as we first believed that this activity would necessarily engage students with theoretical thinking.

Our data consisted of students' solutions to ten quizzes. We have based our analyses of the data on an assumed correspondence between Sierpinska's model of theoretical thinking (2000) and Harel and Sowder's (1996) classification of proof schemes. For example, Sierpinska postulates that theoretical thinking ("TT" in the sequel) is *systemic*. This, in particular, means that, in TT, the meaning of concepts is defined by their relations to other concepts, not by reference to concrete objects and actions. Thus, in TT, proving a general property requires making connections between concepts based on their definitions and on theorems, and not just on reference to concrete examples. Thus systemic approach to proving engages *analytic proof schemes* in Harel and Sowder's sense, while reference to concrete examples engages *empirical proof schemes*. But, our data revealed that the more advanced students in our study were perfectly capable of writing mathematically correct proofs, as if they used analytic proof schemes, while not engaging in theoretical thinking. They seemed to have mastered a discourse or a "genre" on a superficial level, but not the underlying intellectual attitude.

We then started thinking that perhaps the reason for this to be possible is in the nature of the tasks. The quiz questions were designed to foster theoretical thinking, but they did not entirely fulfil this expectation. Therefore we are currently undertaking a research path concerned precisely with the design of tasks in linear algebra that would make better use of students' potential to think theoretically and have them engaged in structural axiomatic and axiomatizing proof schemes (in the sense of Harel and Sowder, 1996).

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DISTANCE EDUCATION IN MATHEMATICS

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In this presentation, we address the theme of Distance Education, which has received little attention in the International Group for the Psychology of Mathematics Education. We have reconsidered the theme in light of the diffusion and availability of new technologies of information and communication.

Over the past five years, the Research Group in Information Technology, other Media, and Mathematics Education (www.rc.unesp.br/igce/pgem/gpimem.html), of the State University of São Paulo (UNESP), Rio Claro, São Paulo, Brazil, has been analyzing non-face-to-face interactions when using information and communication technologies. The analysis is based on the notion that technologies cannot be taken as neutral or transparent; we see them as forming part of a thinking collective, composed of human and non-human actors, that produces knowledge. From this perspective, we analyzed the interactions between human and non-human actors during a 30-hour extension course entitled "Trends in Mathematics Education" offered five times during the last five years.

The educational activities of the course are carried out using "chats", an electronic discussion list, and e-mails, which are used to mediate the educational process, permitting communication among the students and professors. The temporal organization involves synchronous and non-synchronous interactions. The synchronous interactions occur weekly during 3-hour on-line meetings during which the teacher and students discuss, in real time via chat, texts that they have read prior to the meetings. The non-synchronous interactions occur during on-going discussions that take place between meetings via e-mail. A homepage plays the role of bulletin board during the course, where syntheses of the classes, bibliographic references, photos, and other information about the course are available.

The analysis presents the possibilities offered by this configuration of technologies and the pedagogical approach used, indicating that there are modifications in the norms of knowledge production. These modifications are related to the non-linear organization of the dialogues and debates, the need to interpret and attribute meaning, and the extension of imagination and perception, among other aspects. These aspects and others are studied in an effort to contribute to this field, about which there are more questions than answers.

CHANGING PROSPECTIVE MATHEMATICS TEACHERS´ CONCEPTIONS ON ASSESSMENT: A TEACHER TRAINING STRATEGY

Bosch, A.; Romero, I.; Moreno, M. F.; Gil, F.; Frías, A. (U. Almería)

We present a strategy for training prospective Secondary mathematics teachers which is based on a socioconstructivist view of teaching and learning mathematics and of teacher training. The strategy consists of four stages: posing a professional problem to students (contextualization stage); asking them to take a stance on the problem (positioning stage); confronting different positions within the class community (internal confrontation stage); confronting students´ positions with theoretical and curricular approaches (external confrontation stage); and reconsideration of initial stances (reconstruction stage).

Following this strategy, we proposed our students to design assessment tasks for their future pupils by means of which they could value their degree of understanding in two mathematical topics: numbers and area measurement. The development of the strategy allowed us to detect students´ conceptions on mathematics assessment, to characterize their evolution along the implementation of the strategy, and to identify some factors that can influence that evolution.

From the analysis of four students´ cases we can advance some results. The first assessment tasks that students design respond to traditional parameters: mechanical processes, short answers, non-contextualization or connexion to real life, non-reference to attitudinal elements. Our students consider that in order to assess their pupils´ understanding they must check whether they master facts, concepts and arithmetic skills. At the final stage of the strategy, students evolve to consider open tasks that allow pupils to show personal solving processes, are more connected to real contexts and incorporate attitudinal elements. This is more evident in the topic of area measurement. Among the factors that could have influence the students´ evolution we can mention: their attitude towards new approaches in mathematical education, the training strategy (mainly the theoretical instruments provided in the fourth stage), and their previous didactical and mathematical training.

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ELEMENTARY STUDENTS' USE OF CONJECTURES TO DEEPEN UNDERSTANDING

Jonathan Brendefur, Eric Knuth

Developing students' mathematical understanding through reasoning is central to teaching mathematics and is one of the process standards highlighted in the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Furthermore, one element of reasoning is the ability of students to make and test conjectures from observing patterns and to judge the validity of the conjectures through logical arguments and creation of counterexamples (Polya, 1968; Zack, 1999).

This article explores how elementary students reason using conjectures while learning about triangles. The research questions were: What types of conjectures do students in primary grades create when studying triangles? When do students find the need to reason? How do students justify their responses or test their conjectures? And, how do students' conjectures change from primary grades through intermediate grades?

Students in first, third and fifth grade classes, were asked initially to describe the attributes common to all triangles, and to then, observe patterns, write conjectures and test them (Polya, 1968; Reid, 2002). Each of the lessons took place over a two-day period and was videotaped to capture interactions between the teachers and students and among students.

In all cases, students were able to create and test conjectures. The paper highlights the types and sophistication of the conjectures across grade levels and describes the differences among the conjectures created, the need and ability to test conjectures, and the discourse pattern among students.

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MATHEMATICS CONFIDENCE AND APPROACHES TO LEARNING: GENDER AND AGE EFFECTS IN TWO QUITE DIFFERENT UNDERGRADUATE MATHEMATICS COURSES

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I describe early findings from research into mathematics confidence and learning orientations in two very different early undergraduate courses in an Australian university: an introductory statistics service course for students in Science, Business, Commerce and Information Technology (N=179), and a smaller (second) course in calculus and linear algebra mainly for mathematics majors and engineers (N=61). I reflect on age and gender findings in particular. Older students in the bigger class indicated deeper approaches than did younger. Females choosing traditionally very male-dominated courses may be predisposed to the more academic learning approaches, or may assume them in response to the context in which they study.

BACKGROUND FOR THE STUDY, INSTRUMENTS, AND FINDINGS

Wider access to higher education in many countries is increasing the diversity of the student body and changing profiles of gender, age and learning background. Educators seek ways to measure and describe a range of student attributes, and to embed support for the development of desirable approaches to learning. Trends found in recent research support the inclusion of mathematics confidence as a valuable construct in assessing student learning in undergraduate mathematics.

To facilitate meaningful comparisons with the literature, well-researched instruments were used for this study. Mathematics confidence was measured using the USQ scale (developed by the author and others) which demonstrates high internal consistency, test-retest reliability, and validity. Approaches to study were measured using not only the widely used and reported Entwistle-Ramsden Approaches to Study Inventory (ASI), but also scales derived therefrom that are also claimed to measure *meaning*, *achieving* and *reproducing* learning orientations which encompass *deep*, *strategic* and *surface* approaches, respectively.

Findings from the data collected at the end of the second semester, 2002, are described and compared. The reliability of some of the learning scales is questioned, but robust variations illuminate just how different the profiles of students are in these two early undergraduate mathematics classes. As expected, mathematics confidence is much higher in the smaller major/engineering class. The few females (8) indicated significantly higher mathematics confidence and lower levels of surface and achieving approaches on average than the 51 males. However, these gender effects were reversed in the bigger service class (88 females, 71 males). Significant age-group effects were noted in the bigger service class, with the older students declaring on average deeper learning orientations, but variable mathematics confidence levels.

PROJECT MENTOR: MEASURING THE GROWTH OF MENTOR AND NOVICE TEACHER MATHEMATICS CONTENT KNOWLEDGE

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Pacific Resources for Education and Learning (PREL)

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In 2002 PREL received funding for five years from the National Science Foundation (NSF) teacher enhancement program to implement Project MENTOR (Mathematics Education for Novice Teachers: Opportunities for Reflection). Project MENTOR staff work with 4-member teams of mentors drawn from departments and ministries of education and institutions of higher education in the 10 U.S.-affiliated Pacific island communities of American Samoa, the Commonwealth of the Northern Mariana Islands, the Federated States of Micronesia (FSM, which includes Chuuk, Kosrae, Pohnpei, and Yap), Guam, Hawaiʻi, the Republic of the Marshall Islands, and the Republic of Palau. Project MENTOR established a mentoring program for novice teachers aimed at developing in novice teachers the knowledge, skills and dispositions necessary to become effective teachers of mathematics. One goal of the Project is to increase both mentors' and novice teachers' mathematical content knowledge. This report focuses on the tools used for and the results of the initial assessment of the mentors and novice mathematical content knowledge.

A mathematics content test was developed and piloted during early spring of 2003 using a selected group of mathematics and science educators from across the Pacific region who were employed by PREL. After revision, the test was administered to all mentors at regional institutes during May/June 2003. Base line data was gathered from year one and year two novice teachers during the summer and fall of 2003. For comparison purposes, two groups of pre-service teachers from the University of Hawaiʻi voluntarily took the test in fall of 2003. This data has been analyzed and results will be reported during the presentation of the paper.

A second administration of the test will be completed with the mentors and selected first and second year novices in May/June 2004. Preliminary results will be available for the conference so that comparisons with the base-line data collected in 2003 can be made. Future administrations (fall 2004, and summers of 2005, 2006 and 2007) of the test will add significantly to the database of results, and generate research-based evidence of the growth, or lack thereof, of both mentor and novice teacher mathematical content knowledge.

OVERCOMING STUDENTS' ILLUSION OF LINEARITY: THE EFFECT OF PERFORMANCE TASKS

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In a systematic line of research, we have shown that from primary school on, students develop an irresistible tendency to apply the linear model also in situations where it is not applicable. For example, we observed that the large majority of students aged 12-16 believes that if a figure is similarly enlarged with factor k, the area and volume are enlarged with factor k as well. Even with considerable support (e.g., asking to make drawings, providing ready-made drawings or giving metacognitive hints), only very few students made the shift from incorrect linear to correct non-linear reasoning.

In this short oral we will report the first results of a new empirical study, aimed at breaking the illusion of linearity by a new manipulation of the experimental context. Semi-standardized in-depth interviews (registered on videotape) were taken from 21 pairs of 6th grade students, which were assigned (by matching) to one of 3 conditions: (1) *S-condition*: a typical scholastic word problem about the area of an enlarged figure (i.e. about the number of tiles needed to cover the floor of a small and larger doll house) was given and a written, numerical answer was expected, (2) *D-condition*: students received the word problem of the S-condition, together with real-sized drawings, and (3) *P-condition*: students were introduced in the real problem context with the concrete materials and a performance instruction was given (i.e., "Get the exact number of tiles to cover the floor of the doll house."). This means that P-condition students got the same visual information as D-condition students, but additionally, the problem was presented in a performance format. We expected (based on previous research) that students would not profit from the visual information as such, but the tendency to illicitly apply linear strategies might diminish if the word problems were offered with a performance instruction instead of in a scholastic word problem format.

The results showed that all 7 pairs of students in the S-condition committed the linear error. 6 of the 7 pairs in the D-condition and 5 out of 7 pairs in the P-condition found the correct answer (and the others made a calculation error instead of the linear error). The results therefore suggest that the provision of additional visual information (in the D- and P-condition) was already enough to break the tendency to give a linear answer, while the (additional) positive role of the performance task character could not be proven. Further qualitative analyses of the interviews will yield more information about the actual role of the additional visual information in the D- and P-condition on students' problem-solving process, on the one hand, and of the additional value of the performance character of the task in the P-condition, on the other hand.

THE DOTS PROBLEM: THIRD GRADERS WORKING WITH FUNCTIONS[1]

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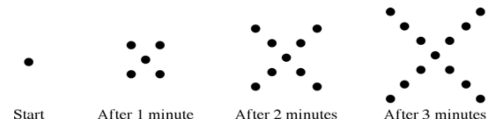
In a lesson popularized in a Japanese professional development demonstration (Bass, Usiskin, & Burrill, 2002), sixth-grade students worked through a problem involving a series of dots increasing in number over time. We gave this problem to third graders (8 to 9 year-olds) in four classrooms in a Greater Boston Public School, midway through a longitudinal Early Algebra study from grades 2 through 4 (see earlyalgebra.terc.edu and Schliemann et al., 2003). The children participated in six to eight 90-minute Early Algebra lessons during each semester. The intervention involved linear functions, tables, graphs, and algebraic notation and focused on algebra as a generalized arithmetic of numbers and quantities. We highlighted the shift from thinking about relations between particular numbers and measures towards thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations between variables, and aimed at the understanding of arithmetic operations as functions. In our presentation we will describe, on the basis of videotape analysis, children's discussions and representations of the dots problem in the penultimate lesson of their third grade year. Students' visualizations and discussions led to meaningful verbal representations and to use of drawings, tables, and algebraic notation that demonstrated their understanding of the function at play. Their work across multiple representational systems provides evidence that, when given the opportunity to work with algebraic concepts and representations, third graders can develop a rich understanding of functions and can represent and solve problems usually taught to be accessible only to older children. Our results support proposals that algebra should become a central part of the elementary school mathematics curriculum.



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COMPARING ASSESSMENT MODES AND QUESTION FORMATS IN UNDERGRADUATE MATHEMATICS

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When presenting an online course within a virtual learning environment such as WebCT, the facility of online assessment becomes readily available. It then makes sense to replace paper assessment with online assessment but before doing so one needs to be aware of the differences in the two modes and the influence it has on student performance.

In order to investigate such differences two broad question types are defined. Distinction is made between *Constructed Response Questions* (CRQs) where students have to construct their own response and *Provided Response Questions* (PRQs) where students have to choose between a selection of given responses. CRQs include open-ended paper questions, essays, projects, short answer questions (paper or online) and paper assignments whereas PRQs include multiple-choice questions (MCQs), multiple-response questions, matching questions and "hot-spot" questions. All the different formats of PRQs are suitable in online courses. In contrast, in mathematics, short answer questions are effectively the only CRQs that can be used online.

Comparisons are made, firstly comparing performance of students in online PRQs to performance in online CRQs and secondly comparing performance in online CRQs to performance in paper CRQs. In the online section of one of the semester tests in a calculus course presented online the same concept was assessed twice in almost identical questions, firstly formulated as a CRQ and later as a PRQ. The experiment was then repeated involving a different concept. In both instances students performed significantly better in the online PRQ.

In the same study the difference in performance between online CRQs, where only a single answer is required, and paper CRQs, where the full exposition of the problem solution is required was investigated. In a test consisting of both a paper and an online section, two similar questions were asked, both CRQs, one in each section. Again the experiment was repeated. In both cases students performed better in the paper CRQs than in the online CRQs, even when discounting partial credit. When taking partial credit into consideration, the difference in performance obviously increased. Although it is not possible to come to any statistically valid conclusion because of the overall nature of the questions the difference is too substantial to ignore.

In a comparison of the performance in paper sections of tests with the online sections, data was collected on performance in both the online and the paper sections over a period of two years from eight semester tests. Students do seem to perform slightly better in the online section in general although this is marginal in most cases and not even always the case.

MATHEMATICS, POPULAR CULTURE AND INCLUSION: SOME FINDINGS AND METHODOLOGICAL ISSUES

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Recent policies on lifelong learning, in the UK and in Scandinavian countries, argue for a substantial return to learning by adults, notably in mathematics and numeracy, to help eliminate inequalities (Parsons & Bynner, 2002). Yet the success of such policies depends on adults' motivation to sign up, and not to resist. Recent studies emphasise the importance of beliefs, attitudes and emotions – for motivations both to learn mathematics, and to use it critically in adult life (Evans, 2000). These affective responses reflect discourses on mathematics, and on people doing mathematics, in popular culture products such as advertisements and films. Initially, using a very small sample, I conjectured that recent films portray the professional mathematician as a 'genius', but who also is susceptible to madness (*Good Will Hunting, Pi, Enigma, A Beautiful Mind*). My sample of adverts portrayed mathematics as something to be disliked, feared and mistrusted; however, it was somewhat dated.

Several methodological questions arise that are relevant to many types of research:

- The findings were based on my *readings* of the data as to the meanings of adverts and films; these are provisional and debatable, and could be interpreted otherwise.
- Initially the samples were 'opportunistic' – but there is scope both for systematic sampling and for 'theoretical sampling'.
- The provisional conclusion explains differences in the portrayal (positive vs. negative) of mathematics and mathematicians by the type of document (advertisements vs. films) – a difference that may, however, be *confounded* with other differences – such as the time period (1985-95 for the adverts vs. 1995-2005 for the films.

The paper will discuss responses to these methodological issues, and consequent developments in the study. Findings from the second phase of the study will be presented.

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MATHEMATICS EDUCATION IN MULTICULTURAL CONTEXTS: A CHALLENGE FOR ITALIAN TEACHING STAFF

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Multiculturalism is one of the most significant changes in many school systems. So far, some European countries, including Italy have paid little attention to mathematics education in multicultural contexts. Mathematics teachers in Italian elementary and lower secondary schools have been asking for refresher courses on mathematics education in multicultural contexts because they realize that minority pupils' culture heavily influences the teaching/learning process. It is not only a matter of language: different cultural variables play a significant role in the appropriation of mathematical concepts. In the presentation we discuss some findings from a survey carried out in the Province of Pisa, Tuscany.

The aim of the survey was mainly to uncover the attitude, vocational education and behaviour of teachers in such contexts and to get a collection of comments and remarks about their experiences which could highlight the different variables to be considered when investigating the teaching of mathematics in multicultural contexts. In this paper we refer to the analysis of answers given in the questionnaires and comments made during the interviews by 108 lower secondary school teachers. The (not so) hidden aim of the questionnaire was to promote in the teachers – step by step, item by item – the awareness of the need for different didactical methodologies, even for a subject such as mathematics (this still sounds strange to most Italian teachers), and, therefore, to ease their search for possible methodological and curricular changes in view of a more effective teaching to minority pupils.

We can point out that, mainly through the questionnaire and the interviews, almost all teachers have had to acknowledge the peculiarity of the new didactical condition, become aware of the partial effectiveness of the activities carried on to tackle that condition; have come to feel it necessary to get a specific in-service training and adequate didactical resources; realized that there is a link, requiring careful investigation, between mathematics education and the culture of foreign pupils.

As for the difficulties met by their minority pupils, teachers mainly refer to the difficulties originated by the foreign pupils' poor knowledge of the discipline and by the language, which are proved both by their poor comprehension of the Italian and an inadequate use of the Italian in mathematics.

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IMPROVING WRITTEN TESTS: WHAT DO STUDENT TEACHERS THINK ABOUT IT?

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Current calls for diversifying assessment instruments in school mathematics have paid little attention to written tests. Yet, written tests do play an important role in the overall assessment process (van den Heuvel-Panhuizen & Gravemeijer, 1993). Thus, teachers must learn how to improve the design of those tests in order to meet school mathematics reform recommendations. As part of a larger project (Tomás Ferreira, 2003), the research reported in this communication was designed to investigate the impact, if any, of the reading and reflective discussion of selected research studies and reform texts on nine Portuguese student teachers' conceptions about classroom assessment in general, and about the role and value of written tests in particular. The participants were enrolled in a 5-year teacher education program characterized by an emphasis on mathematics content courses, and by the absence of mathematics education courses and lack of systematization of the student teaching experience.

This study provided an opportunity for all participants to clarify, broaden, and reflect upon their perspectives about school mathematics assessment and the role and value of written tests. The activities in which they engaged during the study impacted the participants' conceptions differently: their dispositions towards reform-based teaching, and passion and excitement for the teaching profession seemed crucial for embracing a reform-oriented practice of classroom assessment, including the improvement of written tests. School learners' lack of motivation and misbehavior caused some participants to have a sense of helplessness preventing them from even considering alternative written (e.g., de Lange, 1987) tests as practicable. The role of cooperating teachers regarding the improvement of written tests was questioned, and several issues for future research were raised.

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THE ROLE OF ASSESSING COUNTING FLUENCY IN ADDRESSING A MATHEMATICAL LEARNING DIFFICULTY

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In 1992, Fuson argued that automatisation of the number word sequence is essential for the conceptual structures associated with advanced strategies for solving basic addition and subtraction problems to develop (Fuson, 1992, p.76). This presentation will explore the relevance of Fuson's claim for students with learning difficulties in mathematics, who are characterised by an inability to develop either automatised number facts, or fast and effective solution strategies (e.g. Russell & Ginsburg, 1984). Through the case study of a Year 4 boy, I will demonstrate how the assessment of the student's counting fluency was crucial in explaining his puzzling degree of difficulty in learning mathematics, and in the design of a successful intervention to assist him in learning advanced strategies (including memorization) for computing addition facts.

A current Australian assessment framework for young children allows us to directly explore the relationship between counting development and the development of strategies to solve basic addition and subtraction problems (Wright et al., 2000; 2002). In his Learning Framework in Number (LFIN), Wright presents tasks to measure students′ levels of fluency of the forwards and backwards counting sequences, level of identification of written numerals, and stage of strategy development. Assessment data will be presented to show that at the age of 8-7 years the student still had poor mastery of the counting sequences, including a persisting confusion between teen/ty numbers in both oral and written work. These difficulties appeared to be constraining his conceptual development.

Quantitative and qualitative data will be discussed from an intervention designed to assist the student in building an understanding of numbers as a*bstract?* composite units and in seeing the tens/ones structure of 2 digit numbers. A turning point in confidence for the student came as he spontaneously partitioned 10 when asked to make the number ten from other numbers: ‟I want to try it the hard way, not the easy way!″ as he now referred to the ten facts he had previously been unable to remember.

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THE INTERPLAY OF STUDENTS' VIEWS ON MATHEMATICAL LEARNING AND THEIR MATHEMATICAL BEHAVIOR: INSIGHTS FROM A LONGITUDINAL STUDY ON THE DEVELOPMENT OF MATHEMATICAL IDEAS

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The importance of understanding students' views about mathematics, mathematical learning and the educational process, in general, has been extensively documented (Hofer and Pintrich, 1997; Konold et al, 1993,Schommer, 2002).Yet, despite the works of Schoenfeld (1983, 1983), Steiner (1987) and Carey and Smith (1993), more research is still needed that focuses on students' views on their mathematical learning in relation to their mathematical behavior and their educational experiences. For the most part, students' views on the educational process are considered separately from their behavior and the educational experiences in which they take shape. This study addresses these issues by jointly examining the views of a group of five students on their mathematical learning and their mathematical behavior in a longitudinal study on the development of mathematical ideas supporting open-ended student-centered mathematical investigations. The results offer interesting insights regarding such constructs as motivation, mathematical learning, proving and knowing in mathematics supported by compelling quotes from the students and detailed characterizations by the researcher of the students' mathematical behavior.

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A NEWLY-QUALIFIED TEACHER'S RESPONSIBILITY FOR MATHEMATICS TEACHING

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Pehkonen (2001) in her research on teachers' difficulties in changing their teaching practices which were textbook based, found that teachers were committed to "do their job properly" and for this they were respected by their colleagues, principal and parents. They therefore did not see a need to change their practice. Cooney (1999) argued that teachers' knowledge and beliefs are fused with their sense of purpose as teachers and their sense of responsibility given the community in which they teach.

The aim of this paper is to explore the extent to which a teacher's perception of responsibility for his pupils' learning guides his views, actions and decisions in the mathematics classroom through the case study of a newly qualified primary teacher of Y6 (11 to 12 years old) in Greece. The interaction between the school environmental constraints, like parents and the principal, and the development of the teacher's sense of responsibility was also considered.

Data came from the teacher's participation in the research project that one of the authors undertook in the school where the teacher worked. It comprised semi-structured interviews and discussions with the teacher regarding his mathematical background, his teaching planning and its implementation. Observation notes of the teacher working in his classroom, and transcripts of a video-recorded experimental teaching session were also used.

The analysis revealed that the teacher based his teaching mostly on the pupils' textbook, provided free by the Ministry of Education, and which determines the National mathematics curriculum. He used material in a demonstrative way, at times when, and in the way that it was proposed by the textbook, and he organized his lessons from a teacher-centered perspective. His perception of his responsibility for pupils' learning led him to adopt the described approaches in order to find time for revision that he considered valuable for dealing with his pupils' difficulties with certain mathematical concepts and algorithms. These views were challenged when he participated in the research project. He subsequently attempted to change his teaching practice and engaged his pupils in sharing with him the responsibility of learning. However, by basing assessment of pupils' learning on short-term outcomes he considered that his intervention was not successful. He returned to his familiar method of repetition but had become dissatisfied with his practice.

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MATHEMATICS TEACHER UNDERSTANDING AS AN EMERGENT PHENOMENON

Florence Glanfield

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This study examined the research question "In what ways do mathematics teachers grow in their understanding of mathematical processes within the context of professional conversation?" An enactivist view of cognition is used as a frame to consider teacher understanding. Within the professional conversation of four teachers about mathematical processes, individual understanding, collective understanding, and understanding within the body of mathematics was noticed as emerging. Narrative inquiry is used to describe the emergence of mathematics teacher understanding. Two detailed narrative accounts are included to highlight the complexity and complicity of teachers' conversations. From the two narrative accounts, five moments are selected and interpreted further through the frames of individual understanding, collective understanding, and understanding within the body of mathematics. Pirie and Kieren's (1994) theory of dynamical growth of mathematical understanding is used to interpret emergent individual understanding; Davis and Simmt's (2003) work on collective understanding is used to interpret emergent collective understanding; and Davis's (1996) work around understanding within the body of mathematics is used to interpret emergent understanding within the body of mathematics. Some of the patterns that emerged in the interpretations of the selected moments are mathematics teacher understanding is intertwined with teachers' lived histories and student understanding; a teacher may not overtly express their understanding to others, yet changing understanding has occurred; teacher understanding of mathematical processes is affected by the way in which they themselves experienced the processes; changing collective understanding emerges in the collective; developing a shared or distributed understanding within a collective is possible; because conversation itself is an emergent phenomenon, we can see emergent understanding within it; the Pirie-Kieren theory can be used to describe emergent mathematical understanding; and mathematics lives in mathematics teacher conversations.

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GENERIC AND SPECIFIC COMPETENCES AS A FRAMEWORK TO EVALUATE THE RELEVANCE OF PROSPECTIVE MATHEMATICS TEACHERS TRAINING SYLLABUSES

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The work we present is part of a project whose aim is to design instruments to measure the quality of prospective mathematics teachers training syllabuses within the Spanish context. One of the dimensions for evaluating the quality of a syllabus is its *relevance*, that's to say, the degree to which the syllabus provides future mathematics teachers with the preparation and the qualification needed to meet the expectations that society places upon them. In order to characterize this dimension, we have made use of the following referents:

1. The list of generic competences of the Tuning Project (Gónzalez & Wagenaar, 2003) that any university graduate must develop.
2. The list of specific competences that prospective mathematics teachers must develop within the Spanish context, which has been elaborated by the ICMI Spanish Subcommittee (Itermat, 2004).

On the other hand, we consider that the *aims* of a syllabus constitute the main descriptor of its training goals. Therefore, to measure the relevance of a syllabus is to evaluate the degree to which its aims contribute to the generic and specific competences agreed in our context.

From this perspective, we have analysed the aims of the syllabuses of three Spanish universities that share a common teacher training model, in order to evaluate, through measurable indicators, their contribution to the above mentioned competences. As an example of indicators, we take the number of competences that appear in the syllabuses´ aims and the number of training hours dedicated to those aims.

We formulate some conclusions about the relevance of the three syllabuses. For example, they are relevant for the development of specific competences about curricular organization and the planning of mathematics topics for teaching; but they fail in taking into account the development of competences associated to the management of the mathematical contents in the classroom. Moreover, this work shows the usefulness of the competences framework as common language to state aims and to make clear the differences among syllabuses.

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WHY THE MATHEMATICS PERFORMANCE OF IRANIAN STUDENTS IN TIMSS WAS UNIQUE?

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The mathematics performance of Iranian students in the Third International Mathematics and Science Study (TIMSS), was way bellow the international average. However, only a few research studies and government-funded projects have been launched to investigate this matter.

In a study that we are reporting some of its results in this paper, we found out that the mathematics performance of grade 8 students was lower than grade 7 students in 25 questions that were administered to both groups of students. Before we go any further, we took these results, and contrasted them with the data from other countries participated in TIMSS' population 2. The results were astonishing, since this phenomenon was unique to Iranian students. In other words, Iranian students did worse when they got more formal instruction! In addition, the mathematics content of these 25 questions were mostly introduced to students at grade 8 and yet, they did better when they were not introduced to them at grade 7. We therefore, became certain that this unique phenomenon needs further scrutiny.

We thus, administered those 25 questions to four grade 7 and grade 8 students in the spring of 2003, to see if the same situation would happen again. Surprisingly again, the same thing happened for 17 out of 25 questions. To follow up on this matter, we interviewed 9 students from these four classes, and analysed those data to see what factors contribute to this unique result. The followings are some of our findings:

a. Most of these 25 questions used real life situations. Iranian students showed the lack of situated learning (Lave & Wenger, 1991) in their interviews.

b. At grade 7, students answered these questions using their common sense since they the mathematical content of them were not introduced to them formally. However, when they got the formal instruction for them, they were not able to use their common sense. This is what Hawson (1996) anticipated in advance, and called it the conflict between formal mathematics and commom sense/intuition.

c. Iranian students showed the lack of problem solving and metacognitive abilities in answering those questions.

CREATIVITY IN SCHOOL: INTERPRETATIONS AND THE PROBLEMATIC OF IMPLEMENTATION

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A number of studies have focused on mathematics teacher attitude and belief (e.g. Sztajn, P 1997, Gates, P. 2001); other studies have focused on professional development. Few studies have considered the effect of in-service teachers' attitudes and beliefs on two separate policy initiatives and the possibility of intersecting outcomes.

In 1999 a National Advisory Committee published a report: 'ALL OUR FUTURES: Creativity, Culture and Education'; this has been followed by a whole series of centrally led initiatives to promote creativity and creative teaching in schools. Key features of these initiatives include:

The development of a National Curriculum Project led by the Qualifications and Curriculum Authority (QCA) involving 120 teacher volunteers.

The development of a QCA Website 'Creativity: Find it - Promote it'.

The development of regional Creative Partnership agencies with a remit to link school curricula to external creative organizations.

Further to this in a parallel response QCA have developed a National Curriculum Strategy entitled 'Excellence and Enjoyment' for 5 –14 year olds, and the concept of a gifted and talented strand of students has been promoted, and guidance and policy frameworks again promulgated through QCA.

A part of my work in progress is to firstly explore secondary school mathematics teachers' beliefs and attitudes around concepts of creativity. From this I illustrate the difficulty of generating reforms in professional practice when issues of attitude, belief and efficacy are ignored. Findings will be presented which throw light on the importance in reform scenarios of considering the life history of the individuals concerned as well as the nature of reform and change in professional settings.

I also tentatively explore the intersection of the concept of 'gifted and talented' with teacher concepts of which students are able to be mathematically creative; this raises the possibility / probability of the hi-jacking of creativity for a minority of students.

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WORKSHOP ON DEVELOPING PROBLEM SOLVING COMPETENCY OF PROSPECTIVE TEACHERS

Shajahan Begum Haja

Abstract

This paper reports the outcome of a two-day state level workshop on Developing Problem Solving Competency of prospective teachers, which was conducted on 9th & 10th Nov 2002 in Tuticorin, India. As many as 60 pre service teachers (math & science) and 5 math educators from 7 colleges of education in Tamil Nadu participated in the workshop and the Vice Chancellor of Gandhigram Rural University, Tamil Nadu presided over the valedictory session of the workshop. Five mathematics educators including myself handled the 6 sessions of workshop.

I designed the workshop to meet the specific objectives of assisting the student teachers to identify the steps in problem solving, to ask structured questions during problem solving, to formulate and pose problems, to generate algorithms and heuristics, to use different techniques for analyzing and defining problems, and to use problem solving as an instructional method. A set of problems was given as pre-workshop assignment to ensure complete participation in the workshop.

A questionnaire was given at the end of the workshop to assess the attitude of problem solving behaviour of student teachers in terms of the variables: Sex, Educational Qualification (UG/PG), Most useful and rarely useful problem solving strategies, and Summated attitude score. The *most useful* problem solving strategy was found to be *Algebraic* and *Incubate* was speculated as *rarely useful* by the participants. Undergraduate male participants' attitude towards problem solving was lower than that of females while it was the reverse in the case of postgraduates. Evaluation Pro forma of workshop encapsulates the feedback of the participants that substantiates the objectives of the workshop are realised besides few deviant comments.

Key words: Workshop, Problem-solving Competency, Prospective Teachers, Pre workshop assignment, Questionnaire, Feedback.

ENHANCING MENTAL COMPUTATION IN YEAR 3

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The purpose of the study was to develop and investigate the effectiveness of an instructional program to enhance mental computation strategies (addition and subtraction) in a class of Year 3 students (approximately 8 years of age). The short instructional program made use of two models (100 board and empty number line) to support students' development of mental strategies. The students were encouraged to formulate and discuss mental computation strategies. Pre-instruction and post-instruction interviews were conducted to monitor students' progress. The interview items consisted of one-, two-, and three-digit addition and subtraction examples (1-digit items would be considered number fact items for Year 3 students).

At present, in Queensland, Year 3 students are taught written algorithms to solve 2-digit (with and without regrouping) and 3-digit (without regrouping) addition and subtraction examples. Mental computation of multidigit calculations does not feature; however, it will feature in a new syllabus that will be mandated in 2007 (QSA, 2003).

The students in this study had been introduced to the written algorithms for 2-digit addition and subtraction, with regrouping, and 3-digit addition and subtraction without regrouping. They were at varying levels of proficiency with the written algorithms, and they had not been taught any mental computation strategies.

In comparing the pre- and post-interview results, it became clear that most students had developed higher level mental strategies than they possessed before the program (see Reys, Reys, Nohda, & Emori, 1995 for explanation of mental computation strategies). In the short oral presentation, these strategic methods that the students developed will be discussed.

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THE EVOLUTION OF SECONDRY SCHOOL MOZAMBICAN TEACHERS'KNOWLEDGE ABOUT THE $\varepsilon - \delta$ DEFINITION OF LIMITS OF FUNCTIONS

Danielle Huillet

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This paper presents preliminary results of a research that aims to investigate how five high school mathematics teachers' knowledge of a mathematical concept evolve through their participation in a research group.

Building on Chevallard's anthropological approach (Chevallard, 1992), the participating teachers were put in contact with the limit concept through a new institution (the research group) which institutional relation to this concept was different from the relation of other Mozambican institutions where they had met it before (Secondary School, University). As a consequence, their personal relation to limits was expected to evolve.

The results presented here come from data gathered during two interviews, one at the beginning of their own research and one six month later, and a seminar dedicated to discussing the ε-δ definition and held before the second interview. The interviews and the seminar were audiotaped and transcribed.

The analysis of the data shows that, in the beginning of the research, the five teachers did not understand the ε-δ definition. They memorized it at school, but never used it in practice. The discussion during the seminar allowed them to reflect about this definition and identify critical points that make it difficult to understand. This reflection was especially challenging for one of the teachers, who discovered that he did not explain the definition properly to his students, reversing the order of ε and δ when doing a graphical representation.

As a consequence of the evolution of their personal relation to the definition, the two experienced teachers began questioning the institutional relation of Mozambican secondary school with this concept: is it worth teaching this definition to secondary school students, knowing that even teachers do not understand it?

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WHAT IS A MATHEMATICAL CONCEPT?

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The main goal of my ongoing research project is to find out how human beings form mathematical concepts. This is a very complicated matter, and has to be analyzed in several steps. The very first step is to clarify what a concept is, and then try to characterize the mathematical concepts. My starting point will be some observations of how various people use the word 'concept'.

THE CONCEPT OF A CONCEPT

I once gave this task to student teachers in connection with the didactics of mathematics in primary school: Pick out two mathematical concepts and describe how you would proceed to make your learners grasp the concept. The answers varied of course much in quality, but what is most interesting in connection with my project is the various choices of concepts to work on, and what this tells about how the word *concept* is commonly used. Here are some examples (given as $\text{concept}_1/\text{concept}_2$):

number/geometry; addition/subtraction; number/circle; estimation of quantity/shapes in two dimensions; cardinal number/ordinal number; comparing/sets; understanding of cardinality/classification; number/space and shape.

In the following discussions with the students, it became clearer to me than before that many people just mean a word or an expression when they use the word *concept*. This indicates a very poor understanding of what a concept is, and this may draw the attention away from what is essential in learning concepts.

I will discuss the didactical implications of the teacher's understanding of what constitutes a concept.

Mathematical concepts

It is very difficult to define what mathematical concepts are, in a way that separates them from all other concepts, and the necessity of this is questionable. It might still be possible to say something that could draw some limits. In addition, when we see examples as *geometry* or *shape* as proposed from the student teachers mentioned above, we realize that we have to deal with a *hierarchy* of mathematical concepts.

I will also discuss the difference between a mathematical concept as conceived by a mathematician and by a schoolchild, and the steps in forming the important concepts.

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STUDENTS' CONCEPTIONS OF LIMITS AND INFINITY

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Barbro Grevholm — Agder University College

The two concepts limit and infinity are crucial for mathematical analysis. Both concepts are complex but necessary for mathematics studies. The aim of this presentation is to discuss the students' explanations of their written solutions to limit tasks with special focus on infinity. The results presented are part of a larger study where the research question is: How do students deal with limits of functions at a basic mathematics course? (Juter, 2003).

Analyses of textbooks, curricula and student solutions to tasks were conducted to reveal how limits of functions and infinity were introduced to students and how the students solved problems in an analysis and algebra course at a Swedish university. The results are discussed in terms of *concept images* (Tall & Vinner, 1981).

Many students in the study calculated limits in various ways without problems. There were several tasks revealing students' reasoning. Some of the students gave correct answers with incorrect explanations to tasks. Infinity was often a reason for the students' mistakes. One task for example was to decide if the function f(x) = $\frac{x^3-2}{x^3+1}$ can attain the limit value $\lim_{x\to\infty}\frac{x^3-2}{x^3+1}$. It can not and as many as 26 % of the students answered "no", but with a motivation like: "x never attains ∞". Such situations give a false sense of security about the concept image's accuracy.

There were students showing other traces of incoherent or inadequate concept images as well. Some students were unable to solve tasks correctly since they had problems understanding the limit definition.

Textbooks used at upper secondary schools do not provide much theory or tasks about limits and infinity. Most new students at universities do not have a developed image of the concepts. The fact that textbooks do not deal with infinity in a thorough and explicit manner implies that students are expected to work with their possibly vague conceptions from childhood and school. The results of this study indicate that students need well founded conceptions of limits and infinity.

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DO COMPUTERS PROMOTE SOLVING PROBLEMS ABILITY IN ELEMENTARY SCHOOL?

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Our presentation describes a study in which the possibility of integrating problem solving assisted by computers in elementary school was investigated. The aim of the study was to find out if using Excel spreadsheets and promoting algorithmic thinking skills will yield better mathematical understanding, promote students' number sense, help them generalize, and motivate them to learn mathematics.

Kenney and Stanley (1993) claim that spreadsheets "offer … to approach from an analytic as well as visual and formal as well as informal, perspective. These permit… students to do mathematical modeling, to conduct computer exploration". Over recent years patterning activities have become a feature of the mathematics curriculum (Orton & Orton, 1996). In elementary school "… students should investigate numerical and geometric patterns and express them mathematically in words or symbols…analyze the structure of the pattern and how it grows or changes, organize this information systematically and use their analysis to develop generalizations about the mathematical relationships in the pattern" (NCTM, 2000).

Mathematical activities for elementary school students were written, presented in various ways: graphic, as a sequence, verbally or numerically, taking into consideration all the above. These problems promoted the use of different strategies leading to several algorithmic programs. Samples of activities will be presented during our presentation. Forty six students participated in the research. Lessons were observed, teachers and students were interviewed, students' journals were collected and two questionnaires were filled out by the students.

We found that writing algorithms using spreadsheets helps students solve problems using recursion, helps to predict how the pattern will continue, organize the information systematically and use their analysis to develop generalizations, confirming literature (Sassman, Olivier & Linchevski, 1999; Hershkovits & Kieran, 2001). Transition from arithmetic to algebra thus becomes easier.

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INTERPRETATION AND IMPLEMENTATION OF THE L97'S MATHEMATICS CURRICULUM

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The mathematical part of the curriculum, L97, in Norway reflects a constructivist view on learning and also a view based on socio-cultural theories. The curriculum describes different working methods in all subjects in general and in mathematics in particular. According to my interpretation of the curriculum, it encourages an investigative approach to teaching.

L97 stresses that the pupils shall be active in the learning process. They shall be experimenting and exploring and through collaboration with each other acquire new knowledge and understanding. A research study suggests that the curriculum is not implemented as intended. Studies comparing pupils' performance on tasks before and after Reform 97 show that both in grade 7 and in grade 10 pupils perform generally lower in 2001 and 2002 than in 1995 and 1994 respectively (Alseth et al, 2003). This is especially visible within *procedural knowledge.* There is no remarkable decline within what is described as students' *conceptual knowledge.* In my research I am working with 4 mathematics teachers to explore how they are interpreting the curriculum, both in terms of how they are thinking about it and expressing themselves in focus groups and interviews, and also in terms of what they actually do in the classroom. The relation between teachers' interpretation of the curriculum and their implementation of it is a main focus of my project. The methods I am using are fitting largely into an ethnographic approach (Bryman 2001). I immerse myself into mathematics classrooms and I make regular observations of the activities of teachers and students. I use field notes, mini disk recordings and also interviews to probe for beliefs about the nature of mathematics teaching seeking to identify what aspects of mathematics are important. This allows me to gain information that is not observable. I have also used a focus group method to get information about what the teachers said about L97 and how they related their teaching to what is said in the curriculum. According to Krueger (1994) focus groups are useful in obtaining information that might be difficult or impossible to obtain by using other methods. In my presentation I will present findings from analysis of my early data.

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RESEARCH ON THE PROCESS OF UNDERSTANDING MATHEMATICS: THE INCLUSION RELATION AMONG FRACTIONS, DECIMALS AND WHOLE NUMBERS

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In the research on the process of understanding mathematics, Koyama (1992) presented the so-called "two-axes process model" of understanding mathematics as a useful and effective framework for mathematics teachers. The model consists of two axes, i.e. the vertical axis implying levels of understanding such as mathematical entities, relations of them, and general relations, and the horizontal axis implying three learning stages of intuitive, reflective, and analytic at each level. By analyzing elementary school mathematics classes in Japan, Koyama (2000, 2002, 2003) suggested that a teacher should make a plan of teaching and learning mathematics in the light of "two-axes process model", and that she/he should play a role as a facilitator for the dialectic process of individual and social constructions.

The purpose of this research is to examine closely the 40 fifth-graders' process of understanding the inclusion relation among fractions, decimals and whole numbers in a classroom at the national elementary school attached to Hiroshima University. Up to the forth grade, these students had learned whole numbers, decimals and fractions. In order to improve their understanding of those numbers and promote their mathematical thinking, with a classroom teacher, we planned the teaching unit of "Fractions" and in total of 10 forty-five minutes' classes were allocated for the unit in the light of "two-axes process model". Throughout the classes we encouraged students to think mathematically the inclusion relation among fractions, decimals and whole numbers. The data were collected in the way of observation and videotape-record during these classes, and analyzed it qualitatively to see the change of students' thinking and the dialectic process of individual and social constructions through discussion among them with their teacher in the classroom. First, as a result of introducing the frame △/○ for their individual activities, by putting some whole numbers into the frame, students could make different fractions and classified them into three different categories. Second, as a result of the qualitative analysis of students' discussion, we found that students were interested in thinking the inclusion relation among fractions, decimals and whole numbers, and that they could explore the mathematical reason why some fractions were not changed into finite decimals or whole numbers.

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CLASSIFICATION AS A TOOL FOR BUILDING STRUCTURE[1]

Milan Hejný, Jana Kratochvílová

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Classification as a person's mental potency is an important characteristic of human thinking. The ability of primary teachers in this respect is very varied. The contribution is based on an experiment which was carried out during two two-day seminars with 60 primary teachers. The teachers were given the following two tasks:

Task 1. Twelve names are given: Alice, Audrey, Anthony, Boris, Brenda, Bernard, Cindy, Cedric, Clement, Dolly, Daniel, Deborah. Find three criteria so that the names are divided into: a) 4 classes of 3 names, according to the first criterion, b) 3 classes of 4 names, according to the second one, c) 6 classes of 2 names, according to the third one.

Task 2. Create a group of six objects (pictures, things) for your 6-7 year-old pupils and provide two classificatory criteria: a) the first one divides the set into 2 groups of 3 objects, b) the second one divides the set into 3 groups of 2 objects.

More than half of the solvers of Task 1b chose the following criterion: Each class consists of one name beginning with A, one with B, one with C and one with D. Some participants realized that this was distribution rather than classification. However, despite quite a long discussion there was a considerable number of teachers who could not see the difference between classification and distribution. Probably the most effective tool to address this problem is the following formulation of the classificatory criterion: Let us imagine classes as baskets with labels which uniquely determine if a randomly chosen object belongs to the basket or not.

About a third of solvers of Task 2 confused classification with association, e.g. a five-member group of teachers created the following six objects: ant, butterfly, flower, grass, squirrel, and tree. The classification into two classes "plants versus animals" is correct, however, linkages "squirrel – tree", "ant – grass", "butterfly – flower" do not have classificatory, but associative character. Some solvers could not see their mistake in this case either. One of the solvers suggested putting a label "nest" on the basket, as she understood it as a linkage between the animal and the plant on which it lives.

In the ongoing research, some other mental functions which play an important role in structuring mathematical knowledge are analysed: hierarchization, schematization, the usage of isomorphism, chaining, the creation and usage of generic models in concept creation processes. The contribution will address them, too.

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CHARACTERISTICS OF THE PROJECT AS AN EDUCATIONAL STRATEGY[1]

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From our long-term research in the field of exercitation of students' projects in teaching of mathematics, it is clear that well chosen and well scheduled projects can become one of the means to evoke optimal environment for qualitative changes in the development of student's personality and to create such an educational setting which presupposes positive aims. Students who work on a project avouch for solving the problems connected with the assigned topic, they take part in planning the individual steps in the progress of the project, they suggest their ways to solve the core problem of the project. This is how they naturally become included in the teaching and learning process.

A well prepared and appropriate project provides enough space to develop the learning strategies of the individual students and simultaneously sufficient space (mainly from the time perspective) for solving the projects and obtaining their results. This leads students to a more active approach to their learning. In such context, we consider students' projects to be a specific educational strategy and we see it as a component of the educational process which includes dynamism. Thus the subject of our investigation is not only the final result but mainly the student's way towards it.

In detail, we defined the characteristics of the project as an educational strategy based on the active approach of students towards their own learning. These are: realization of students' needs and interests; the development of their competencies and capacities; self-regulation in learning; motivation; role changes in the class; implicitness of the teacher; orientation on presenting the results; team cooperation; updating the school stimuli in relation to space, time and content; interdisciplinarity; social relevance; a change in the conception of school. Some phenomena connected with these will be demonstrated by concrete evidence in the presentation.

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RETENTION EFFECT OF RME-BASED INSTRUCTION IN DIFFERENTIAL EQUATIONS

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The current research literature offers a number of different learning methods or instructional tools for student retention in many subjects (e.g., Billington, 1994). In this context, this research investigated the retention effect of an RME-based instruction comparatively. For the purpose, the instrument to measure students' understanding was developed and administered to two different groups (i.e., RME-DE and TRAD-DE) at the end of the fall semester 2002 and at the end of the fall semester 2003.

The TRAD-DE is based on the traditional lecture-based instructional method in which students are provided with a set of analytic methods for solving differential equations. On the contrary, our project class, the RME-DE, integrated technology with symbolic, graphical, numerical, and qualitative methods for analyzing a wide variety of differential equations of real-world concern. In the RME-DE, the students were encouraged to discuss key concepts embedded in given context problems and the professor orchestrated the students' interaction to lead to the emergence and the establishment of taken-as-shared mathematical meaning.

The result of the statistical analysis has shown that the RME-DE got statistically higher marks than the TRAD-DE. Especially, the RME-DE was successful in mathematical modeling and in the conceptual questions related to graphical representations. The RME-DE gained not only significantly higher mean scores than the TRAD-DE but also earned the nearly same marks to those that they earned in the test of the previous year.

Based on the result of the analysis, we argue that the RME-DE course can help to increase retention for students' understandings in differential equations. Especially, the ability to form graphical images of mathematical relationships related to the better retention of strategies. In addition, it can be said that the RME-DE students learned and retained the way how to link differential equations with real world situations for mathematical modeling as they were actively working on the relationship between a behavior of a solution to a differential equation and rate of change. These findings provide empirical evidence to adapt the instructional design perspective of RME to mathematics education at a university level.

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THE CONNECTION BETWEEN ENTRANCE EXAMINATION PROCEDURES AND PRE-SERVICE ELEMENTARY TEACHERS' ACHIEVEMENT IN MATHEMATICS

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The paper presents results of a project funded by the Academy of Finland on pre-service elementary teachers' views of mathematics. Over the course of the project we will study a total of 269 students from three Finnish universities (Helsinki, Turku and Lapland). Our aim is to investigate students' views of mathematics and mathematical abilities at the beginning of their studies and to compare the effect of different teaching methods on students' views of mathematics.

In Finland the number of applicants for elementary teacher education is about ten times greater than the number accepted, and the students admitted are generally of high quality. Yet, the mathematics proficiency of pre-service elementary teachers in Finland is not always satisfactory (cf. Merenluoto & Pehkonen 2002).

In this article, we focus on the implications of the entrance examination procedures at the three institutions. Particularly crucial to a consideration of the entrance examination is content validity, which is concerned with the strength of the similarity between what is measured in the selection device and the essential elements of the job (Rose & Baydoun 1995). In this study the research problem is "What is the connection between the entrance examination procedures and students' achievement in mathematics at the beginning of their studies?"

The findings show that the mathematics component of the entrance examination at the University of Turku had a distinct effect on the quality of applicants: the mathematics achievement of students admitted to the elementary teacher program was better on average than that of students accepted at the University of Helsinki and the University of Lapland as measured by the students' upper secondary school grades, their grades on the Matriculation Examination (ME) and their scores on the achievement test administered at the beginning of their university studies. Nearly half of those whose performance fell in the lowest quartile on the achievement test had not even done mathematics as part of their ME. One alternative to having an entrance examination for mathematics would be to take students' grades on the ME into account in the second phase of the admissions process.

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PRE-SERVICE ELEMENTARY TEACHERS' SITUATIONAL STRATEGIES IN DIVISION

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Here we will present some preliminary results of our research project on pre-service primary teachers' views of mathematics (the project financed by the Academy of Finland; project #8201695). We have collected survey data of 269 pre-service primary teachers in the beginning of their mathematics studies. Here we will concentrate on teacher students' understanding of division. Division is an essential arithmetical operation, and there are many misconceptions connected to it. These might be: "You must always divide the bigger number by the smaller one" (e.g. Hart, 1981) or "You can operate with the digits independently: 84÷14=81 because 8÷1=8 and 4÷4=1" (Anghileri, Beishuizen & van Putten, 2002).

Understanding of division with decimal numbers was measured by task 16.8÷2.4. About half of the students (51 %) could do the calculation. Students used different approaches in solving the problem. Using quotitive division and "trial and error" or "repeated addition" had usually led to the right answer. "Operating with the digits independently" had caused the most common wrong answer 8.2 and "dividing in half" led to answer 8.4.

Task "Solve 7÷12 by using long division algorithm" measured among other things, whether students divided the numbers in right order. 16 % of students divided 12 by 7. In task "Write a word problem to task 6 ÷ 24 and solve it" students seemed to be able to write a word problem little better than to calculate the task. The most common wrong answer was 4. Another usual answer was 0.4. Although tasks 7÷12 and 6÷24 were much alike students solutions varied. Situational strategies in solving division tasks seem to depend at least on numbers, problem structure and different concrete situations (cf. De Corte & Verschaffel, 1996).

Our data suggest that mathematical understanding that students have in division is inadequate for teaching division for understanding. During teacher education it is also important that elementary teacher students become conscious of the difficulty of division and of the different misconceptions people have in division so that they can better teach division for their pupils.

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DEVELOPING OF MATHEMATICAL PROBLEMSOLVING AT COMPREHENSIVE SCHOOL

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A problem solving course was planed for the purposes of the study. The aim was to develop the students´ problem solving skills and change their attitudes towards mathematical problems. The lessons were integrated with various school subjects.

INTRODUCTION

The teaching of problem solving is often overshadowed by routine tasks. Only fast and talented students have time to complete the problem tasks as extra tasks without the teacher's guidance. The problem solving course was planned with materials and lessons plans for the study. It consists of 29 lessons over the period 22.10.-2.12.2003. The teaching of problem solving was integrated with mathematics, Finnish, natural sciences, handicraft and technology.

METHOD

The research was carried out on grade 6 students (11-years). First the experimental class and the control class took part in an initial measurement; the problem solving test which included various kinds of problem tasks. An experimental class was taught the problem solving course. The course attempted to influence students' attitudes towards the problem tasks. They were taught to use the solving map method as a help to solve problem tasks. After the course the experimental class and the control class took part in a final measurement. After the course the students of the experimental class were interviewed.

RESULTS

In the problem solving tests the experimental class clearly improves to performance compared to the control classes. In the experimental class the boys were better than the girls and the boys improved their results more than the girls. According to the interviews, the majority of the students feel that their attitudes towards mathematics and problem solving became more positive during the course. The students' problem solving processes become more effective when they adopted solving map-methods. This was clearly visible from the entries in the notebooks which every student made during the problem solving course.

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MATHEMATICS TEACHING AS INVISIBLE WORK

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The point of departure for this research is the notion that significant features of mathematics teaching are invisible. "Invisible" refers here not only to the features of teaching that are physically invisible: these include such practices as the planning work that goes on in the mind of the teacher, or the relational work that the teacher engages as a medium for learning. But in this work, "invisible" connotes the features of teaching that are actually physically visible but nonetheless go unnoticed by observers of teaching. For example, the teacher often works hard to knit together student ideas in her efforts to build collective understandings in whole class discussions. These efforts, though physically present and easy to discern, often evade the notice of observers. Teachers likewise deploy mathematical knowledge to move a problem along or engage a particular student. Knowing what piece of mathematical information, or what question to ask at a given moment, is the essence of good mathematical teaching practice. And yet even practiced observers of teaching miss these important moves made in the flow of instruction.

This research is based on data collected in an elementary mathematics methods course for prospective teachers. The prospective teachers engaged in a modified form of lesson study (Lewis, 2002; Stigler and Hiebert, 1999) in which the prospective teachers and their instructor planned, executed, and analyzed lessons for elementary mathematics students. All data for this modified lesson study were collected for analysis, including videotapes of the planning, teaching, and reflection sessions on the lessons, student work, children's written work in class, and the instructor's field notes. This short oral presentation will include categories of teaching practice that were generally found to be "invisible" to prospective teachers, and will detail the efforts made to make such features of teaching more visible. The usefulness of lesson study in this context will be considered. Finally, this presentation will touch briefly on the theoretical framing for why teaching work is often "invisible." This theoretical framework is drawn from scholarship in such diverse fields as mathematics, anthropology, women's studies, organizational psychology, and sociology.

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CABRI-GEOMETRE: TWO WAYS OF SEEING IT AND USING IT

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At the PME25 I gave a talk about a preliminary analysis of two case studies of my ongoing PhD. At this PME I would like to discuss and share the final analysis of them as I have completed my doctoral studies.

Here, both theoretical and empirical support to a view on secondary mathematics teachers' use of Cabri is presented. In particular, I argue that the use of a software package for teaching is not only linked to the school curriculum but also strongly linked to what a teacher *sees* in it. By treating software packages as *texts* and teachers as *readers* of such texts from an *anti-essentialist* viewpoint of technology (Lins 2002, Grint and Woolgar 1997), this paper discusses the final analysis of two the case studies – *The Cabri of Anthony* and *The Cabri of Camilla* of my PhD studies.

Anthony and Camilla, both teachers from a state school of Bristol (UK), were interviewed in front and away from a computer, talking about and describing *her*/*his* Cabri. The teachers also had two of their lessons within a Cabri environment observed. Methodological issues will be discussed in the talk.

The doctoral studies aimed to look at *what was being said* by the teachers about Cabri; and to investigate to what extent this was *linked* to the teachers' use of Cabri in and out of the classroom. Here, to look at 'what is being said' means to look at what meanings are being produced by the teachers for Cabri. One of my assumptions is that the software package which reaches the classroom environment is not *the* software that once had been designed but rather *a* software: the one the teacher has constituted. The Cabri in a classroom is *a* Cabri: *the Cabri of the teacher*.

One of the said powerful features of Cabri is the *drag-mode* that allows deformation of figures, where ideas of (in)dependence can be explored by establishing relationships among points on the figures. From the two case studies, *seeing* and *using* Cabri as such has shown not to be the case. The drag-mode has nothing to do with The Cabri of each teacher by the time they were interviewed. This does not imply it will never be. New meanings can or will be produced by them for Cabri, as meaning production is to be understood as process rather than something static. The point is the importance of the *awareness* of the Cabri of the teacher in order to understand *how* and *why* Cabri is being taken and used in a classroom in such a way.

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FIRST GRADERS' AND KINDERGARTEN CHILDREN'S KNOWLEDGE OF GRAFIC SYMBOLE SYSTEM OF NUMBERS AND ADDITION AND SUBTRRACTION

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The initial process of teaching mathematical numbers and addition and subtraction is undertaken using a graphic symbol system.

The graphic symbol system, as well as other symbolic systems, is characterized by being arbitrary, accepted and transferred information. It also has its special "symbolic components" such as the symbols of the numbers figures and its "model of using symbols" i.e. the right order of writing the exercise symbols in mathematics (Lesh & Doerr, 2000).

Before children enter school, they have already acquired some informal knowledge about the mathematical graphic symbol system. when they start school, they begin a formal learning of this symbol system (Bialystok, 2000).

This study has examined the knowledge of kindergarten children and first graders concerning the "symbolic components" and "models of using the symbols" of numbers and addition and subtraction exercises using a graphic symbol system. A special item was developed for the purpose of the study. The study includes 154 respondents (48 kindergarten children and 106 first graders). Each child had an individual session in which he was asked to produce and identify natural numbers and addition and subtraction exercises using a graphic symbol system.

It was found that when first graders could not produce or identify the numbers or the addition and subtraction exercises, it was mainly a result of lack of knowledge or misconceptions about the "symbol usage model".

Conversely, the causes of the difficulties in those tasks for the kindergarten children involved both "symbolic components" and the "symbol usage model". These findings may cast light on the development stages of two central aspects in young children's knowledge development of graphic representation of numbers and addition and subtraction exercises. In the presentation, I provide examples of items and details from the results.

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STUDENTS' ABILITY TO COPE WITH ROUTINE TASKS AND WITH NUMBER-SENSE TASKS IN ISRAEL AND IN KOREA

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Despite the importance of number sense, it seems that emphasis of the mathematics curriculum in elementary school is on computational algorithms and procedures. Number sense can be developed (e.g., Markovits and Sowder, 1994) but in order to do this students should explore a variety of tasks requiring number sense as an integral part of their mathematics studies. Reys et al. (1999) compared number sense proficiency of students aged 8 to 14 years in Australia, Sweden, United States and Taiwan. They found, as expected, that the performance levels on the items varied across the countries, but also that regardless of country variable, students exhibited low performance on the number sense tasks. Reys and Yang (1998) investigated the relationship between computational performance and number sense among sixth and eighth grade students in Taiwan. They found that students' performance on questions requiring written computation was significantly better than on similar questions relying on number sense.

In our study we compare the way 250 sixth grade students from Korea and Israel cope with routine tasks and with tasks requiring number sense. The comparison turns to be interesting since Korean students keep doing very well on international tests, while Israeli students are ranked much lower on the list (e.g., Mullis, et al., 2000). A written questionnaire containing 24 open-ended tasks with 12 routine tasks and 12 number sense tasks was developed and given to the students.

Preliminary results show that Israeli students performed better on the routine tasks than on the number sense tasks. Many students did not use number sense on the routine tasks, thus ending with incorrect responses. For example, on 2372 : 8 students got more than 2372, without paying attention to this unreasonable answer. Korean students also performed better on the routine tasks. In comparison with Israeli students, they performed well both on the routine and on the number sense tasks. However, Korean students had a tendency to use standard algorithms on the number sense tasks, as long as they were applicable, which might result in correct answers without good number sense.

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LEARNING SCHOOL MATHEMATICS *VERSUS* BEING MATEMATICALLY COMPETENT – A PROBLEMATIC RELATIONSHIP

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It is not clear why the very same person seems to be mathematically competent in one setting (e.g. metalwork) but suggests enormous difficulties in learning school mathematics (Abreu, Bishop and Presmeg, 2002). Instead of addressing this issue as a matter of 'transfer of knowledge', in this study it was decided to approach the problem from a situated learning point of view, assuming learning as an integral part of social practice (Lave & Wenger, 1991; Wenger, 1998).

In order to analytically describe youngsters' practices (not socially defined as mathematical) and try to understand how those practices relate to the school mathematics curriculum, a group of secondary school students' participation in mathematics and metalwork classes within a course on metalwork, was observed during three months.

'Becoming' (a metalworker) turned to be the driving force for participation of students in the activities as it shaped their way of addressing the tasks and their alignment within both school mathematics' practice and metalwork. The results of this study point to a problematic interpretation of the idea of mathematical competence which is nowadays being spread among mathematics teachers and mathematics educators (Fernandes & Matos, 2003).

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TEACHERS CREATE MATHEMATICAL ARGUMENTATION

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The purpose of the present study is threefold: (a) to compare the kinds of mathematical argumentation that different teachers create in mathematics classrooms; (b) to explore how teachers' mathematical argumentation develops over the school year; and (c) to examine the way teachers change their mathematical argumentation under different conditions.

The importance of enhancing mathematical argumentation in the classrooms has recently been emphasized by researchers and educators (e.g., NCTM, 2000). There is also rich research showing how students construct their mathematical argumentation. In addition, a few studies reported the changes in students' mathematical argumentation under different instructional conditions (e.g., Kramarski & Mevarech, 2003). Yet, little is known at present on issues relating to the kinds of mathematical argumentation that teachers create in mathematics classrooms. Given the important role of the teachers in developing students' mathematical reasoning, it is essential to explore how teachers create mathematical argumentation, and how they change (if at all) their mathematical argumentation over the year, or when they implement different instructional methods.

To address the above issues, we observed two mathematics teachers over one academic year. Both teachers are female, having the same level of education (B.Ed.), and similar years of experience (about ten years). They both taught eighth grade classrooms in the same school. Once a week, we video-taped each teacher over one study period. During the year, from time to time, as teachers thought appropriate, they implemented metacognitive instruction method called IMPROVE (Mevarech & Kramarski, 1997) to which they were introduced during an in-service training. The video-tapes were analyzed using qualitative methods. Teachers' mathematics argumentation was classified into categories on the basis of these data.

The findings show interesting differences between the two teachers in the way they create mathematical argumentation. Furthermore, the teachers were quite consistent in the way they use mathematical argumentation in the classroom. Some differences were found, however, when they implemented the IMPROVE method. The advantages and limitations of such studies will be discussed at the conference, as well as the theoretical, methodological, and practical implications.

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PROFESSIONALISM IN MATHEMATICS TEACHING IN SOUTH AFRICA; ARE WE TRANSFORMING?

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The National Teacher Education Audit (1995) and the survey on Mathematics and Science Teachers in 1997 found that there was lack of professionalism in most teacher education institutions , especially previously disadvantaged colleges of education. Unfortunately, the problem was more than 80% of mathematics teachers were trained at colleges of education.

Breen (1994), argues that the model used for professionalism in mathematics teachers during apartheid years, called for absolute and uncritical compliance. In advocating a new era professionalism for mathematics teachers, Hindle (1997) asserts that what the country desperately needs is professionalism in which each individual takes responsibility for their own personal development, a critical practitioner who takes a lead in a particular field, in this case , mathematics education.

Interventions to transform the quality of mathematics education in the country were implemented. For example, the impact of Mkhize's 1999 intervention is captured by the learners' remarks: " At last we gained something from a math lesson and I enjoyed the lesson." This study aimed to investigate the impact of one of the interventions on high school mathematics teachers. Preliminary results will be discussed.

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IN THE TRANSITION FROM ARITHMETIC TO ALGEBRA: MISCONCEPTIONS OF THE EQUAL SIGN

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Students in elementary grades encounter the equal sign but many misunderstand its meaning (Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999), Following early algebra researchers' recommendations of smoothing the transition from arithmetic to algebra, we focus on the understanding of the equal sign as one of the keys in the development of mathematical and particularly algebraic thinking. This understanding enables to employ relational thinking where students look across the equal sign at relationships between numbers to solve equations, as in 8 + 4 = _ + 5.

In our study, we analyzed the understanding of the equal sign in two groups of 3^{rd} graders (n=15) and $5^{th}/6^{th}$ graders (n=26), paying attention to any evidence of the use of relational thinking. Open number sentences, such as 14 + _ = 13 + 4 and _ = 25 - 12, were proposed to the students. In the 3^{rd} grade group, different reactions depending on the structure of the sentence were encountered. The necessity of equivalence between both sides of the equal sign was not recognized and the equal sign was mainly considered as an order to perform an operation. A wide variety of responses was encountered in the sentences where this conception was more difficult to apply, (e.g. 12 + 7 = 7 + _ and 14 + _ = 13 + 4). The kind of wrong responses given, students' comments and the multiple answers to the sentence 12 + 7 = 7 + _ suggested that students were not using relational thinking.

In the $5^{th}/6^{th}$ grade group, most responses were correct and some use of relational thinking may be inferred from the wrong answers. Notable difficulties were observed in the subtraction sentence which could result from the greater complexity of applying relational thinking in subtraction contexts. The better performance of this group, not observed in other studies, may be a consequence of some (Socratic) instruction these students received from a university mathematics teaching group during the month previous to the study.

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MATHEMATICAL PROCESS: AN ANALYSIS OF THE STUDENT COMMUNICATION ON OPEN-ENDED PROBLEM

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The purpose of this study was to analyze the mathematical process on communication of sixth grade students during solving a mathematical open-ended problem in small group of four children with different mathematical backgrounds. With a constructing triangle activity as a task for group discussion, the cognitive-metacognitive framework from Artzt and Armour-Thomas was employed to examine the effects of small-group communication on data interpretation by sixth graders. The categories of problem-solving behaviors within a small group were defined as read, understand, explore, analyze, plan, implement, verify, and watch and listen. The communication and gesture of group discussion were audiotaped and videotaped and transcribed into protocols. The analysis illustrated a picture of group dynamic of communication and individual's talk.

The result of this analytical descriptive study highlighted in demonstration the types of communication, the significance of the interplay of cognitive and metacognitive behaviors was revealed, which supported the work of Artzt and Armour-Thomas (1992) and Curcio and Artzt (1998). The study also showed students of different mathematical background exhibited their nature of different communication. Inclusively, the implications for instruction were presented.

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STUDENT AUTHORITY IN MATHEMATICS CLASSROOM DISCOURSE

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In order to achieve diffusion of authority in mathematics classroom discourse, it is necessary to investigate the authorship (Burton, 1999) of knowledge, that is how mathematical learners come to create and negotiate their meanings. In an epistemological perspective of author/ity (Povey and Burton, 1999) meaning is understood to be negotiated. Teachers and learners who are sharing this perspective work with an understanding that they are members of a knowledge-making community.

In the ongoing research project to be presented, author/ity in the mathematics classroom discourse is studied. My research participants are six student teachers in a programme of teacher education for primary and lower secondary school, and two mathematics teachers, at the university college, who are responsible for the compulsory 30 credits course in mathematics didactics in the programme.

Central research questions are: What is the nature of student author/ity in mathematics classroom discourse? In what ways is student teachers' school-based learning helpful in order to develop a shared perspective of author/ity between teachers and learners?

Data has been and will be gathered by means of video recordings and field notes from whole-class lessons and student teachers' group work in mathematics at the university college, and from school-based learning. I also intend to conduct semi-structured interviews with student teachers. I am planning an inductive approach to analysis of data from observation and interviews.

Sociocultural theories of learning (Lave and Wenger, 1991) are underpinning the project, because of the emphasis that knowledge is constructed through interaction and in a context. To know is in sociocultural theories closely related to participation in communities of practice.

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PROFESSIONAL IDENTITY AND PROFESSIONAL KNOWLEDGE: BEGINNING TO TEACH MATHEMATICS

Helio Oliveira

This communication is based on a research concerning the construction of the professional identity of four beginning mathematics teachers, according to their biography, professional knowledge, relational processes, schooling context and the social conditions of post-modern society.

The research follows the interpretative paradigm. Four study cases concerning beginning mathematics teachers have been constructed, mainly based in the analysis of a set of interviews with biographical character realized in the course of three school years.

This study permits to confirm that identity is an idiosyncratic, complex and multidimensional process. During the first years of teaching there is a big concern for issues concerning the professional knowledge, and simultaneously the development of a "subjective educational theory" assumes great relevance in professional identity construction (Calderhead, 1997; Kelchtermans, 1993). For these beginning teachers, the subject matter they teach is a fundamental aspect in their identity, reconfirming their previous vocational choice. Besides that, two of these teachers broadened their professional identity integrating a fundamental role as educators.

All these teachers express sympathy with a progressist view of mathematics education, in line with their teacher education experience, however this has different meanings for them. This study gathers evidence that teacher education is interpreted from different point of views depending on the biography and self of the student teacher.

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CELEBRATING DIVERSITY: THE ROLE OF MATHEMATICS IN A CURRICULAR ALTERNATIVE TO PROMOTE INCLUSION

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In Portugal, the goals of basic schooling stress the central role of attitudes, values and practices that contribute to engage students as legitimate participants in democratic learning communities. In the last decade, teachers who already had inclusive practices engaged in several types of projects, in which the different agents of the educational community had a more active role, giving voice to those who are usually less heard: pupils, parents, school staff (other than teachers) and other partners of the educational setting.
In this action-research project an alternative curriculum was developed in a class (14 pupils) during the 5th and 6th grades, in a school from a poor and multicultural area in Lisbon. This curriculum was conceived as a mediation tool for inclusive participation (Oliveira e César, submitted). A follow-up was implemented to assess its impact. Data were gathered through participant observation, interviews, questionnaires and several documents. The curriculum was organised around problems and issues that, through recurrent debate with the students, integrated their daily experiences and contributed to respond to various concerns. In this context, mathematics education assumed a vital role in bringing dynamism to many learning situations and in some cases it was actually the starting point for planning activities and defining strategies, in an integrated and collaborative teaching perspective.
The results show that 10 pupils finished the 6th grade and 8 of them went on to the 7th grade. In all of them several competencies were developed. Outcomes highlight that if we change our practices and teaching strategies during compulsory education many students attain both achievement and a better socialization. The data illuminate that to develop a curricular flexibility we need: a) Interdisciplinarity methodologies; b) Students' engagement in the educational process as legitimate participants; c) Teachers' critical perspective of the curriculum; d) Mathematics seen not as a selective subject but allowing for the interpretation of students' daily experiences; e) Inclusive schooling principals and practices as a way to prepare critical citizens.

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MIGHT STUDENTS BE KNOWLEDGE PRODUCERS?

Paulo Oliveira

Student's investigative work in the mathematics' classroom has been supported through multiple perspectives. In this talk I propose to analyse the epistemological legitimacy of those investigations realized by students.

Mathematical knowledge as well as the investigative praxis of the professional mathematician is viewed according to post-foundacional epistemologies. In this sense relevance is given to scientific, cultural, social, historical and institutional elements in the analysis of mathematical praxis and knowledge. So the praxis of the professional mathematician and the knowledge he produces in investigative settings is prototypical of the praxis and knowledge production by students when engaged in investigative tasks.

I will discuss an epistemological model that conceptualises the mathematical classroom as a strong epistemological space in which students might produce knowledge epistemologically relevant. This relevancy is implicated in the way I conceive the epistemological possibilities of the mathematical classroom. I characterize the epistemological structure of the mathematics classroom according to seven categories namely:

1) value and limits of the knowledge produced; 2) knowledge validation criteria; 3) epistemological obstacles; 4) the relationship between the one who produces knowledge and knowledge itself; 5) the identification of conditions favourable and unfavourable to knowledge production; 6) the aesthetic dimension of knowledge and 7) the characterization of 'new' mathematical knowledge.

I will briefly illustrate the applicability of this epistemological model to the case reported by Michael Keyton (Keyton, 1997).

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WHOLE SCHOOL REFORM IN MATHEMATICS

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Nanaikapono Elementary School is situated in a working class neighborhood on the island of Oahu. The school currently records the following percentage ethnicities among its 966 students: 61% Hawaiian or part-Hawaiian, 12% Samoan, 10% Filipino, and 17% Caucasian, Asian, and other Pacific Islanders. The school is struggling to meet the requirements for progress set by the No Child Left Behind Act (NCLB) (2001).

This struggle began with the implementation of state standards set in 2000. (HCPS II, 2000). Three university faculty were informally invited in 2000 to help with that implementation and were more formally engaged to help raise performance levels of the school in mathematics and language to enable the school to meet the requirements of NCLB. We here report the approaches to changing mathematics teaching taken so far and their effectiveness to date.

In mathematics we have adapted the notions of *quantitative literacy* (Steen, 1997) to use as the backdrop to our interventions.

History of intervention

We are now in the third phase of intervention. Earlier phases were informal contacts with faculty, then whole school workshops, and invited visits to individual classrooms. Most recently we are now recently working directly with teachers by grade level on topics of their choice.

Results to date

SAT-9 pre- and post-test differences taken in 2002, and the recent 2003 test results show consistent gains by almost all grade levels. The school has continued to work on developing its own curriculum initiative and was recently recognized in its district as out-performing other schools. More detail will be available during the presentation.

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ELEMENTARY STUDENT TEACHERS' SELF-CONFIDENCE AS LEARNERS OF MATHEMATICS

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The paper will describe some of first results of the research project ”Elementary teachers' mathematics”, financed by the Academy of Finland for the years 2003–06 (project #8201695). The project concentrates on the development of elementary student teachers' view of mathematics in three Finnish universities (University of Helsinki, University of Lapland, and University of Turku).

View of mathematics is a large entity of a student's knowledge, beliefs, conceptions, attitudes and emotions. In the view of mathematics, one may distinguish at least two components: The view of oneself as a learner and teacher of mathematics, and the view of mathematics and its teaching and learning. More on view of mathematics one may find e.g. in the paper Pehkonen & Pietilä (2003). Self-confidence that pertains to the first component, has a central role in the formation of view of mathematics (cf. McLeod 1992).

Participants of the project are 269 elementary student teachers. In Helsinki there are two separate groups of students. All participants were measured in autumn 2003 with a 'view of mathematics' indicator, where one part was a self-confidence scale containing 10 items from the Fennema-Sherman attitude scale (Fennema & Sherman 1976).

Students' views of themselves as learners of mathematics differ from each other in different universities in the beginning of basic studies in mathematics. About one fifth of the students have a weak self-confidence. The normal student groups in Turku and Helsinki have the highest self-confidence, and the other two the weakest one. The difference is statistically significant only between the female students of Helsinki normal group and the additional group. Furthermore, there are small differences between the male students in Lapland and Turku, and between the female students of Turku group and the Helsinki additional group. In the case of men, the differences were not statistically significant – probably because of their small numerus.

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UNDERSTANDING AND TRANSFORMING OUR OWN PRACTICE BY INVESTIGATING IT

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In their practice, mathematics teachers face many complex problems. This includes pupils' failure regarding the curriculum objectives and socialization and enculturation aims; the inadequacy of curricula regarding the needs and characteristics of pupils; the incomprehension of a large part of society, specially mass media, to the adverse conditions in which teachers work. Instead of waiting for solutions provided from the outside, educators are increasingly researching directly such problems. A similar phenomenon occurs also in fields such as teacher education, health, social work, and rural development.

There are many reasons to carry out such research: (i) it contributes to get solutions for the problems; (ii) yields the professional development of the actors involved and helps to improve their organizations; and (iii), in some cases, it may contribute to the development of professional culture in this field of practice and even to the knowledge of society in general (Jaworski, 2001; Krainer & Goffree, 1999).

This paper presents the experience of a group of mathematics teachers and teacher educators that worked together for an extended period of time writing a book about their experiences researching their own practices. This activity led the group to an extended reflection about their experiences and enabled a deeper interaction with the professional community. It sketches the issues that were researched by the participants and the main features of their methodology. Finally, the paper ends up summarizing some of the theoretical and practical issues involved in this activity: (i) Paradigmatic problems, concerning the relationship of this research with well established research traditions (positivistic, interpretative, critical); (ii) Epistemological problems regarding credibility and criteria of quality of such work; (iii) Methodological problems regarding the construction of research objects, given the particular relationship between the researcher and the research problem; (iv) Ethical problems regarding the different roles and responsibilities of professionals regarding their clients and concerning the inner dynamics of cooperative research groups and the role of leadership.

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IS IT TIME TO LET GO OF CONSERVATION OF NUMBER?

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Since the publication of Piaget's work on children's conceptions of number (Piaget 1952), conservation of number as been discussed as a key aspect of this conception. Indeed, Piaget (1953) claimed that children are not usually able to conserve number until the age of 6 or 7 and that 'children must grasp the principle of conservation of quantity before they can develop the concept of number'. Until recently the mathematics curriculum for children in preschool settings in the UK went along with Piaget's teaching, focusing almost entirely on 'pre-number' concepts of sorting, matching, one-to-one correspondence and conservation of number. Although the curriculum in the UK has now changed (QCA 1999), in practice many teachers are unhappy with a more number-based curriculum having been trained 'with Piaget'.

There has been much work that questions Piaget's conclusions. As long ago as 1978 Margaret Donaldson and colleagues (e.g. Donaldson 1978) suggested that children 'fail' such tests because they try to make 'human sense' of them, rather than because they believe that a set is more numerous if it is spread out. Similarly, Schubauer-Leoni and Perret-Clermont (1997) argue that Piaget saw such tests as 'purely cognitive', while they should be seen as being 'socio-cognitive', and many standard books on teaching young children mathematics recognize the limitations of Piaget's work (e.g. Thompson 1997).

My study of preschool English children (3:5 - 4:11 years) using a slightly adapted version of the Piagetian task (unpublished) showed that while all the children gave the standard response, they could all conserve number when asked different questions. My subsequent study of 4-6 year old children learning addition in the classroom offers some insight into children's thinking: how they can understand number conservation in context, why number conservation might be problematic in other contexts, and the role of language in this.

The presentation will examine whether it is now time to let go of the concept as a way of explaining children's understanding of number, and to acknowledge young children's complex, if as yet incomplete, understanding of cardinal number.

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CONNECTIONS BETWEEN SKILLS IN MATHEMATICS AND ABILITY IN READING

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The aim of the study presented in this short oral communication, is to investigate developmental differences between pupils with and without difficulties in mathematics and/or reading, and to investigate the manner in which these differences are reflected in their level of performance on different tasks in mathematics.

In research which is primarily concerned with reading, difficulties in mathematics are often seen merely as a result of reading difficulties (Miles & Miles, 1992). However, this cannot be the whole truth since approximately half of the pupils with difficulties in mathematics do not have additional difficulties in reading (Ostad, 1998). Reading ability seems to influence growth in mathematical achievement (Jordan, Hanich, & Kaplan, 2003).This and other findings point to the value of differentiating between mathematics difficulties with normal reading ability, and mathematics difficulties with co-morbid reading difficulties (Geary & Hoard, 2001; Rourke & Conway, 1997).

The sample in the study comprised 1038 pupils in the age cohort 8-15 years. These pupils were classified into four achievement groups based on their performance on standardized achievement tests in mathematics and reading: those pupils with difficulties in mathematics but not in reading (MD-only), those pupils with difficulties in both mathematics and reading (MDRD), those pupils with difficulties in reading but not in mathematics (RD-only) and those pupils without difficulties in any of the areas (NMRD). The study is ongoing, and uses cross-sectional and longitudinal data to examine how the pupils in the different achievement groups differ from each other in developing ability to solve: word problems, written calculations, calculations without pen and paper and approximate arithmetic.

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A STUDY OF FOURTH-GRADE STUDENTS' EXPLORATIONS INTO COMPARING FRACTIONS

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This paper investigates the growth of mathematical understanding in a class of fourth grade students as they build models to compare fractions as part of a year-long teaching experiment. This research builds upon the work of Steencken (2001)) and Bulgar (2003) who examined other components of this teaching experiment.

This paper examines the development of mathematical reasoning in a class of twenty-five fourth grade students as they build models with Cuisenaire Rods™ to compare fractions. These students were part of a year-long teaching experiment led by Carolyn A. Maher and assisted by other researchers from Rutgers University Graduate School of Education in New Brunswick, NJ. The purpose of the teaching experiment was to study the ways that young children develop mathematical ideas when challenged with tasks that invite mathematical dialog. Researchers encouraged the students to clarify their thinking and justify their solutions through model building, discussions and reflections upon their work. In the sessions examined for this study students built multiple models with the Rods to determine which of a set of two fractions was larger and by how much. Students were asked to explore the relationship among the models and to study the models for clues that would help them build another model to compare the same fractions. This paper will examine how a group of seven of these students was able to find the difference between fractions with unlike denominators and how they were able to discover a generalized solution that enabled them to build multiple models.

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QUALITY IN MATHEMATICS TEACHERS TRAINING SYLLABUSES

Rico, L. (U. Granada); Gil, F.; Moreno, M. F.; Romero, I. (U. Almería); González; M. J. (U. Cantabria); Gómez, P.; Lupiañez, J. L. (U. Granada).

The purpose of this study is to design instruments to measure the quality of prospective mathematics teachers training plans, within the Spanish context. We conceive the quality of a syllabus as a multidimensional concept that can be articulated by means of *dimensions* and *competences*.

We consider three dimensions for evaluating the quality of a teacher training syllabus: its *relevance*, as the measure of the degree in which the syllabus suits the requirements and expectations of the participants' social setting; its *effectiveness*, as the measurement of the degree in which the syllabus achieves its aims; and its *efficiency*, measurement of the expenditure of time and resources required for accomplishing the tasks involved in the program syllabus.

Each one of these dimensions is organized by means of factors. Each factor represents an area of interest related to the training aims. We use the notion of *competence*, developed in the Tuning Project (Gónzalez & Wagenaar, 2003) to establish three groups of factors: *Generic Competences*, which analyses the generic training for any university graduate; *Professional Knowledge*, which evaluates the general foundations that constitute the theoretical referents for the mathematics teacher; and *Professional Competences*, which measures the ability of the prospective teacher to perform practical tasks.

By crossing the three dimensions and the three factors, we obtain a 3x3 table. For each cell of this table, we have elaborated instruments for collecting, codifying, and analysing information, which produce *quality indicators*. For example, an indicator for the cell “relevance/generic competences” is the degree of presence of the Tuning generic competences in the specific aims of the training syllabus.

In order to develop and validate this model of quality evaluation, we take three prospective mathematics teachers syllabuses as referents. These syllabuses are currently been implemented in three Spanish Universities: Almería, Cantabria y Granada.

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ANALYSIS OF TEACHERS' ACTIVITY FOR CONTINUED DESIGN OF COMPUTER-BASED TEACHING SUPPORT. THE CASE OF DIAGNOSING STUDENTS' ALGEBRA COMPETENCES

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We propose to present the main lines of a study about teachers' assessment practices and an analysis of theirs interacting with a software —PEPITE— designed for helping to assess competences in elementary algebra. Previous observations of teachers using PEPITE (Delozanne, Grugeon & Jacobini, 2002) indicated that they were interested in the type of questions asked in the test (Pepitest) but they were not prone to use the students' profiles proposed from an automated analysis of students' answers to the test (Pepiprofile).

Two hypotheses are proposed for explaining such observations and were tested in an exploratory study: firstly, the didactic analysis of algebra competences which underlied the elaboration of the profiles departs from the general approach proposed in textbooks and generally followed by teachers; secondly, Pepiprofil is designed for individual student's assessment, while the French context of teaching is mainly oriented toward assessing classroom progression in algebra, while individual evaluation is a global assessment of the student's level in mathematics.

Two methods were used: 1) analysis of assessment practices from "open" interviews about their algebra teaching, before and after a continued training session about algebra teaching, involving Pepite; 2) observation of the same teachers working for the first time with Pepite during the training session. The main features of the analyses will be illustrated, and results concerning teachers with contrasted professional experience will be presented. They confirm and precise our two hypotheses. Consequences are derived concerning, on the one hand, the continued design for the software Pepite, and, from the other hand, the training requirements (Artigue, 1998) for teachers being able to use Pepite as an assessment support system.

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PROFILES IN LOGIC AND MATHEMATICAL PERFORMANCE IN CALCULUS TASKS BY GRADUATE STUDENTS

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The role of proof in maths education has been stressed in numerous studies (Hanna, 2000). In a previous study (Rogalski & Rogalski, 2001) we showed the difficulties students encountered when dealing with implications with a false premise, a case that appears in advanced mathematics. Durand-Guerrier (2003) also emphasises the specific complexity of an implication P(x)=>Q(x), where property P can be or not be satisfied by the objects under analysis. This is often the case in calculus with quantified assertions. Studies concerning this level strongly suggest that there are strong relationships between students' logical and mathematical competence.

The study which will be exposed (in a graphical form) aimed to go further on this point. It is based on the answers of 178 graduate students to a test about reasoning in everyday domains and in mathematics. In (Rogalski & Rogalski 2001) we defined four profiles of students from their behaviour when confronted to implications with false premises. Such profiles correlated with performances in several reasoning tasks (Rogalski & Rogalski 2001, 2003). Now we will present data showing that students' with "logical" or "pertinent" profiles succeed better than the two other profiles in tasks involving property of rational numbers or behaviour of real sequences. In fact, three factors appear to affect graduate students' behaviour in calculus tasks: logic-based profiles, availibility of calculus knowledge, and strategies for managing somehow complex mathematical tasks. Perspectives for further research are proposed.

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THE IMPORTANCE OF TEACHERS' ATTITUDES FOR THE USE OF NEW TECHNOLOGIES IN MATHEMATICS CLASSROOMS

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We will present results from an on-going research studying the implementation of technological tools in middle-school (children 12-15 yrs. old) mathematics classrooms, focusing in particular on the importance of the teachers' role for students learning. This research is part of a national project sponsored, since 1997, by the Mexican Ministry of Education. The project, known as EMAT (Teaching of Mathematics with Technology), by incorporating results from international research in computer-based mathematics education to the practice in the "real world", has implemented and researched the use of several open tools (e.g. Spreadsheets, Cabri-Géomètre, and Logo) with hundreds of teachers, over several years. Much of the philosophy and pedagogy underlying the design of mathematical microworlds (Noss & Hoyles, 1996) was incorporated into the project. A pedagogical model was designed putting emphasis on changes in the classroom structure and teaching approach and an extensive amount of computer-based activities were developed for each of the tools, covering the different themes of the middle-school curriculum.

The research evaluating the project has been carried out on many levels. In particular we evaluated: (i) The ways in which the student and teacher materials are used; (ii) teacher's attitudes, use of the proposed pedagogical model and ability to link the technology-based activities with mathematical knowledge; (iii) children's attitudes (though direct observation and interviews) and (iv) children's mathematical performance both in standardized tests and through their academic scores. Although the project has been successful on many levels (see SEP-ILCE, 2000), some teachers have difficulties in adapting to the proposed pedagogical model, lack experience working with technology, and often lack adequate mathematical preparation. These teacher deficiencies have shown to have an impact on students' learning. In one research phase, we correlated case studies of 12 teachers during Logo sessions with the data collected from over 1000 of their students. We found a strong correlation between teachers' performance and children's mathematical scores and appreciation of the technological tools. These results are not surprising –Noss & Hoyles (1996) emphasise the influence of the setting on children's performance and mathematical behaviour– but they provide solid evidence of the importance of teachers' attitudes and pedagogical implementation of the materials, for students' learning.

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THIRD GRADERS GENERATE THEIR OWN WORD PROBLEMS

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Documents from different educational organizations had long advocated a synergistic method for the teaching and learning of mathematics (Everybody Counts, NRC, 1989; Professional Standards for School Mathematics, NCTM, 1991; A Call for Change, MAA, 1991; Principles and Standards for School Mathematics, NCTM, 2000). This method advocates a conceptual and collaborative interaction between teacher and students that brings to the fore the complementarities between partnership and individuality, inter-subjectivity and subjectivity, conventionality and idiosyncrasy, teaching and learning. The paper analyzes the method a teacher used with third graders to elicit the generation of story problems throughout the school year, and the students' solution strategies based on their own diagrams and their conceptualization of number. Students' diagrams and symbolizations are analyzed from the Peircean semiotic perspective.

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TEACHERS' MANAGEMENT OF THE EPISTEMOLOGICAL FEATURES OF MATHEMATICS: SEARCHING FOR LINKS WITH PUPILS' MATHEMATICAL KNOWLEDGE

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It is widely accepted today that the mathematical meaning constructed by the pupils is largely determined by the way, in which the subject matter knowledge is processed and filtered in the mathematics classroom. Within this perspective, a number of studies have focused on the consequences of teachers' management of the epistemological features of mathematics for the subject matter knowledge generated in the classroom (e.g., Steinbring, 2001, Tzekaki et al., 2001). However, there are no studies systematically examining the impact of this management on the epistemological features of the mathematical knowledge formulated by the pupils.

In previous studies (e.g., Kaldrimidou et al., 2000), we examined how teachers handle the epistemological features of mathematics in various contexts. The main conclusion drawn from these studies was that teachers tended to deal with these features in a unified manner, mixing them with morphological and procedural elements of the mathematical knowledge. In this presentation, we report on a systematic investigation of the epistemological features of the mathematical knowledge held by a sample of 1,055 pupils, whose teachers' instructional practices were studied and led to the above conclusion. The data came from the subjects' written answers to a number of items, which aimed to identify their understanding of basic mathematical facts. The items with similar epistemological characteristics of the mathematical knowledge targeted were clustered together and the corresponding answers were analysed. The results of this analysis showed that the students' poor performance was due to their almost exclusive reliance on morphological and procedural elements of the mathematical knowledge, which, as argued above, dominated their teachers' instructional practices. This suggests that it might be valuable to examine more systematically the role of the epistemological features emerging in the mathematics classroom in relation to the meaning attached to the mathematical concepts and processes by the students.

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FARM MATHEMATICS

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In terms of out-of-school mathematics, life in the countryside can surprise and amuse us more than we expect. Mathematics is not just cloistered in technology and advanced problems but also and in rural countryside activities. Rural life is the pillar of the primary sector of any society we can thing of. Regardless the development and technology of first world countries, rural life plays a little role in the base of its structure.

Activities such as feeding animals, shearing, vaccinating, preparing the land to be cultivated, preparing the animals for reproduction, recollecting eggs, or simply identifying animals are tasks from the farmers' daily routines which include lots of mathematical processes. I believe there is a need to identify the mathematical content and procedures outside the formal environment of Mathematics in order to understand better the "lifelong learning" phenomena in which adults are immersed. For instance, a farm is a simple example of a non-formal location where to discover and analyse the processes and the mathematical knowledge involved in them.

People who learn to solve problems 'on the job' often have to do it differently from people who learn in theory, at school. I would like to show the evidence of the knowledge utilized by subjects with little schooling in mathematics for developing certain work-related tasks in a farm.

The research is being conducted in two farms from Lleida, Catalunya, in the north of Spain. One of them is a calf's farm with 184 females and 96 males. It is ruled and controlled by two farmers, who are the owners. They also have over 200 lambs and chickens. The second farm is a farm that produces goods for self-consumption and for small selling. They have chickens and rabbits and they sell wine, olives, and other small homemade products. The owners are an old non-literate couple.

The phenomena that I am researching is the identification of rural mathematical mental processes within these two farms, what the farmers know and how mathematical problem solving can be related to real-life situations. Those farmers have already developed mathematical strategies (particularly arithmetical), and it is important to become aware of what these strategies may be.

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THE EFFECTS OF TECHNOLOGY ON MAKING CONJECTURES: LINKING MULTIPLE REPRESENTATIONS IN LEARNING ITERATIONS

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Numerous studies have suggested that different technologies have different effects on students' learning of mathematics, particularly in facilitating students' graphing skills and preferences for representations. For example, there are claims that students who prefer algebraic representations can experience discomfort in learning mathematics concepts using computers (Weigand and Weller, 2001; Villarreal, 2000) whilst students using calculators preferred graphical representation (Keller and Hirsch, 1994).

Although, arguably, the teaching of mathematics has traditionally centred more on procedural skills, it is possible that students' understandings, preferences and difficulties in relating different representations might be explained by analysing students' thought processes in terms of the making of conjectures.

Within the topic of iteration, this study investigated how using graphical calculators, and PC-based graphing software changed A-level mathematics students' conjectures in relation to: 1) students' understanding of the concepts of iteration, and their discovery of the properties of particular iterations; 2) students' preferences for representations; and 3) the way the students express their conjectures.

Students were observed tackling iteration questions using graphical calculators, and, later, graphing software. The students' written inferences were collected using two parallel worksheets and were subsequently analysed using a coding scheme developed based on previous studies in the literature, and focusing on students' conjectures as a unit of analysis.

The investigation found similar results to those of previous studies in terms of graphing difficulty, linking different representations and preferences for representations. However, the results also hinted that the computer positively influences students' understanding of iteration and their movement between representations more than the graphical calculator; and this possibility requires further research.

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ATTITUDES TOWARDS MATHEMATICS AND MATHEMATICS TAUGHT WITH COMPUTERS: GENDER DIFFERENCES

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The increasing presence of technology to support the teaching of mathematics in Mexican middle-school (12-15 years old) leads to the need of monitoring eventual changes in students' attitudes towards mathematics. In particular we are interested in studying gender differences in this process. The AMMEC (Attitudes towards Mathematics and Mathematics Taught with Computers) scale was designed to detect students' attitudes toward mathematics (11 items); towards mathematics taught with computers (11 items); and self-confidence in mathematics (7 items). The reference scales used were the Computers Attitude Questionnaire (Knezek and Christiansen, 1995a, b), the Fennema and Sherman Mathematics Attitudes Scales (1986) and Forgasz's (2002) survey questionnaire on gendered beliefs on computers for learning mathematics. The scale was applied to 228 girls and 211 boys with 1, 2 and 3 years experience in using computers for mathematics. A general tendency to a positive attitude towards mathematics and towards mathematics taught with computers was detected. There were no significant gender differences in this. Significant differences were found in self-confidence: more boys than girls were not sure about their own capabilities to work in mathematics (42.7% boys vs. 32.9% girls), but significantly more girls than boys considered that they were definitively not good for mathematics (48.7% girls vs. 38.4% boys). In contrast, more girls than boys obtained higher marks in mathematics. A positive correlation between high marks and a positive attitude towards mathematics taught with computers was found for boys while a positive correlation between high marks and self-confidence was found for girls.

In relation to a longer experience in using computers for mathematics, there was, for both boys and girls, a more positive attitude toward mathematics and towards mathematics taught with computers. However, self-confidence in mathematics decreased for both boys and girls.

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COMPUTER GRAPHICS TRAINING OF SPATIAL ABILITY WITH DIRECT MANIPULATION

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Computer graphics with the possibility of direct (user-controlled) manipulation can visualise or simulate mental operations required for solving problems by describing the various aspects of spatial abilities. We shall base our investigation on the so-called "pictoralistic thesis", which states that mental image processing operations are based, among others, on the visual experience of objects and object manipulations. The reseach problems are: How should a learning software be structured to support the process of solving item-like spatial perception problems by user-controlled direct manipulations? (Software development problem) How effective is user-controlled direct manipulation in computer-supported solving of item-like spatial perception training problems as compared to the conventional way of solving such problems in printed form, which relies on mental operations only, or compared to a combination comprising of computerised and pencil-and-paper item-like problems which are presented consecutively? (Software evaluation problem)

The development of the modular training program "Fold –Rotate – Tilt – Cut" was based on corresponding standardised tests, which fit to certain facettes of spatial ability. This will provide suitable criteria for analysing the effectiveness of our training software.

The main results from an investigation using a control group design with about eighty 9-graders per treatment, which is first completed for the modules Fold and Rotate: All the treatments are of significant efficiency. The module Fold and it's combination with paper and pencil tasks are equally effective, but these two treatments are significantly more efficient than the treatment paper and pencil tasks. The treatments module Rotate and paper and pencil tasks are equally efficient, but the combination of module Rotate with paper and pencil tasks is more efficient than these two treatments etc. These results have corresponding interpretations ...

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HOW PRE-SERVICE MATHEMATICS TEACHERS UNDERSTAND PERCENTAGE PROBLEMS

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The frequent use of percentages in everyday life and in the sciences mandates a sound knowledge of the subject. An 8th grade teacher posed a percentage problem to students, who offered four different answers, correct and incorrect. Stemming from the lesson and the ensuing discussions, this study investigates percentage problem solving, concepts and misconceptions held by pre-service secondary school mathematics teachers.

Percent is one-hundredth. Percentages describe part of a quantity and are not numbers such as fractions. Fractions have many functions, only one of which is describing part of a quantity. A percentage can be replaced by a fraction and vice versa, but only when the fraction describes part of a quantity. Elementary percentage problems are of three principal types: calculating the value of the percentage, calculating the percentage and calculating the fundamental size. Monteiro (2003) pointed out that prospective elementary mathematics teachers have difficulties with the concepts of ratio and proportion. Aware that percentage problems are part of proportion we expected to find such difficulties too.

The study objectives were to determine whether pre-service teachers (N=17) are able to identify correct and incorrect solutions to the percentage problem and how they go about solving this problem. The research tools were composed of two questionnaires and personal interviews.

Findings: Most pre-service teachers were unable to identify the correct and incorrect solutions. Like Hershkowitz's findings (1988) problems of calculating the value of the percentage were easier to solve and were solved correctly by most of the pre-service teachers.

The answers of both the 8th grade students and those of the pre-service teachers are presented here. The misconceptions held by pre-service teachers as shown by our results are linked to Fischbein's (1987) intuition theory.

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CATEGORY OF STUDENTS' JUSTIFICATION AND ITS RELATION TO THE STRUCTURE OF ARGUMENTATION: AN ANALYSIS OF DISCOURSE IN SYSTEMS OF LINEAR DIFFERENTIAL EQUATION

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This paper focuses on types of justifications the students used, their characteristics and relationships, and the tendency in the justification use in an inquiry-oriented differential equations course of a university in Korea. In this research, the term 'justification' is used broadly to include efforts to convince one and/or to persuade others. Recently, with re-evaluation of proof, there has formed a consensus that the functions of proof should be much more comprehensive to include explanation and communication. In this respect, we will call justification as proof playing these various kinds of functions to support conjectures. The current research literature offers a number of different frameworks for classifying students' justifications (e.g., Knuth & Elliot, 1998; Harel & Sowder, 1998). We adapted these frameworks, in particular, Krummheuer (1995)'s elaboration of Toulmin (1958)'s argumentation scheme for our analysis.

We have collected video-recordings of all the class session, which were transcribed for discourse analysis. In addition, data such as student interviews, the students' worksheets and reflective journals were collected to supplement the result of the discourse analysis.

The discourse analysis has shown that students' justifications in our project class can be categorized into three types; justification using external resource, justification using empirical resource, and justification using formal resource. These categories can be divided into ten subcategories. The students use a different type of justification to offer an alternative approach a problem or to extend their explanations. Moreover, the analysis of students' justifications in the structure of argumentation shows that there is no particular hierarchy in the types of students' justification. All categories of students' justifications are distinguished each other and have their specific roles.

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ADAPTED LEARNING WITH SPECIAL FOCUS ON THE ASPECT OF LANGUAGE IN MATHEMATICS

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This contribution has its origin from a project concerning "The first years at school and adapted teaching", consisting of five sub-projects. The project is composed of an observation and an action part. In the observation part, two different observers will each year observe ten classes from the first to the fourth grade. Each observer stay in three different classes each year, one week in each class. We are observing the teacher, a selected pupil (a new one each day) and the class. We use structured observation form and qualitative observation, as well as interview with the teacher. In the action part we are going to collaborate with the involved teachers in developing the first years at school.

The project will last for three years. No final conclutions are so far established. In this short presentation I will present some results from the first (02/03), and second (going on winter – spring 04) field study, and eventually try to establish some temporary conclutions based on these results.

My part of the project focus on the question: "Which areas of knowledge are emphasized, and how are these being used in mathematics in the first years at school?" In the observation part I try to observe how the work in mathematics is adapted to the individual pupil's abilities, and which kind of working methods that are used. I try to identify which kinds of knowledge that are stimulated, for example: facts, skills, conceptions, structures of conceptions, strategies, processes or attitudes. Is the teaching prepared for wondering, exploration and experimenting? Are the pupils encouraged to express their thoughts and meanings through different modes of expression? Are they invited or challenged to explain their way of thinking?

The evaluation of the Norwegian compulsory school, "Reform 97" shows that drill, focusing on rules and algorithms, and fragmentary activities, still is attached importance (Alseth et al. 2003 and Haug 2003). This is one of the reasons why I, in the action part of the project, will focus on working methods concerned with the pupils' activity, problem solving and working methods stimulated by the use of language.

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PROSPECTIVE TEACHERS' UNDERSTANDING AND REPRESENTATIONS OF MULTIPLICATION OF FRACTIONS

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There is a growing consensus that teachers' knowledge is an important element in student learning. Since Shulman (1987) classifies pedagogical content knowledge as typical knowledge teachers should have, there is a lot of research which is interested in what a teacher should know and be able to do.

However, many studies show that teachers possess a limited knowledge of mathematics, including the mathematics they teach (Ma, 1999; Ball, 1990;Behr, and Lesh, 1991; Simon, 1993). A majority of teachers are good at performing computation, but few are able to explain the conceptual basis of the procedures in performing computation.

The purpose of this paper is to explore how prospective teachers understand multiplication of fractions with word problems and how they explain and justify the meaning behind their coumputation steps with multiple ways of representations.

Sixty prospective teachers were participated. All participants were majoring in elementary mathematics but have different grade level. The task was written task. It consisted of three problems and took about 10-20 minutes.

This study showed that prospective tea chers have limited knowledge of multiplication of fractions. First, sixty eight percent of prospective teachers have recognized the word problem as one where multiplication of fractions could be applied. Second, seventy seven percent of total prospective teachers represented the word problem using representations. Overall, eighty nine percent of prospective teachers recognizing the word problem as multiplication of fractions represented the word problem using graphical representations. Most prospective teachers who recognized the word problem as a multiplication of fractions could explain their thinking using the graphical representations. As representations, fraction bars, area model, number line, part-whole model, and set model were used. Fraction bar was used most frequently.

The major finding from this study was that misconceptions prospective teachers might have when solving multiplication of fractions using word problem. This study categorized prospective teachers' misconceptions into three and found possible three reasons. This study has implications for both teacher educators and assessment developers.

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LESSON STUDY PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS

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Ongoing professional learning is an essential part of development as a teacher since pre-service courses can only provide teachers with the skills and understandings necessary to begin their career. In New South Wales, Australia, the Department of Education began a program of professional learning based on the *Lesson Study* program of Japan. This report is an evaluation of the *Lesson Study* project and the writers believe its uniqueness comes from the fact that the program is self-contained within schools, whereas in other places, the adaptations have included the leadership of an external 'expert'. In New South Wales, the schools planned, implemented and evaluated their own work and progress.

Lesson Study is designed to assist teachers to both produce quality lesson plans and to gain a better understanding of student learning. From the beginning of 2003 over 150 schools were involved. The authors of this paper carried out an evaluation of the program in 2002.

The *Lesson Study* process involves a *Lesson Study* team, which is a small group of volunteer teachers coordinated by a volunteer team leader. The team meets regularly (1-2 periods a week) to plan, design, implement, evaluate and refine lessons for a unit of work selected by the teachers in the team.

The Third International Mathematics and Science Study (TIMSS) suggested that student learning would not improve to any great extent until teachers were given the opportunity and the support to further develop and increase the effectiveness of their teaching skills (Lokan, Ford & Greenwood, 1996). The *Lesson Study* program aims to do just that. This evaluation focussed on the 5 critical levels of professional development evaluation as proposed by Guskey (2000, p.2).

Surveys were completed by the team leaders and also by the other teachers both before embarking on the process and later. The analysis of these surveys and data from other sources indicated that the most valued parts of the process were the collegiality that developed, the greater motivation of the students and the value of visiting other classrooms. These are expanded in the full paper.

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CHANGING STUDENTS' ATTITUDES TO MATHEMATICS THROUGH SMALL-GROUP COLLABORATION?

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In recent years there has been great interest in small-group learning in mathematics. From the range of studies in this area some have focused on the nature of different types of verbal interactions on achievement (eg.Webb, 1991); others have focused directly on insights from the discourse itself (eg.Pirie, 1991). Few studies have considered the effects of small-group work on students' attitudes to mathematics and their perceptions of how mathematics might be taught.

The authors have been engaged in a large four-year study investigating the effects of small-group work on student attitude, attainment and working practices in English, Mathematics and Science in the early secondary years (age 11 - 14). A recent review of this 'core' curriculum in England and Wales indicates that students enjoy group work but are given few opportunities to learn in this way. With regard to mathematics in particular, Nardi and Steward (2003) found that the perceived isolated nature of school-mathematics and limited opportunities for peer collaboration are factors in further alienating students from the subject.

At the beginning of the current academic year attitude questionnaires were administered to just over 1000 students in twenty different schools. Using 5 point Likert scale items, the questionnaire probed general attitudes to small-group work and to liking the core subjects. General results from this survey round show:

- declining attitudes towards mathematics during the early secondary years
- a significant gender difference in (not) liking mathematics (girls more negative)
- very positive attitudes towards group work in general

Our work in schools in the three subject areas also suggests that small-group work is less likely to be used in mathematics teaching compared to the other 'core' subjects. Within the twelve mathematics classes in the project, similarities and differences in 'pre-disposition to group-work' are evident. Individual student scores have also been calculated to highlight those who are very positive or very negative about group-work. The team intends to examine these more 'extreme' classes and pupils through observation and student interviews. At the end of this year the same questionnaire will be administered to examine whether there are significant shifts in students' attitudes towards learning style in mathematics.

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BELIEFS AS AN INFLUENCE ON MATHEMATICAL REASONING

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Upper secondary students' task solving reasoning was analysed with a focus on strategy choices and implementations. Beliefs were identified and connected with the reasoning that took place.The preliminary results indicates a close relationsship between beliefs and strategy choice.

Earlier research shows that students tend to mainly or only use superficial strategies when solving problematic situations in mathematical tasks (Bergqvist et al, 2003; Lithner, 2000). Superficial reasoning may solve tasks, but is insufficient for long-term learning. There were also little evidence of students using reasoning based on mathematical properties in the problematic situation, Plausible Reasoning (PR). In this on- going study, I try to answer the question why students reason in a specific way. One of the central factors that influence your mathematical problem solving ability is beliefs. The main question here is "How do beliefs affect reasoning?" and the study is based on following research questions: What type of beliefs exists when a student is facing a common problematic situation?, How do they affect the student's action and reasoning? and What are the characteristics beliefs in superficial reasoning situations, as opposed to PR situations? Another purpose with the study is to develop a structure for analysing the relationship between beliefs and reasoning. The teoretical framework concerning reasoning is the same as Bergqvist et al (2003). The framework about beliefs is heavily influenced by Schoenfeld (1985). Data was collected by video recording task solving sessions, interviews and a questionnaire.

Some preliminary results indicates a close relationsship between beliefs and strategy choice.

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PERCEPTIONS OF ORDER: THE CASE OF DYNAMIC BEHAVIOR IN DGE

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The sequential organization of actions necessary to produce a figure in any Dynamic Geometry Environment (DGE) introduces an explicit order of construction. In a complex figure this sequential organization produces what is, in effect, a hierarchy of dependencies as each part of the construction depends on something created earlier. (Jones, 2000) This hierarchy of dependencies is one of the main factors that determine DB within DGE (Jackiw & Finzer, 1993; Laborde, 1993).

The longer DGEs are in use and under study the more we learn about their contribution to the learning of geometry but also about the obstacles they are liable to pose to such learning. (Chazan & Yerushalmy, 1998; Healy & Hoyles, 2001)

As a part of a larger study on the complexities involved in understanding DB, Junior high students and graduate students in math education were asked to predict the DB of points that were part of a geometric construction they had executed using a DGE according to a given procedure, and to explain their predictions.

This presentation focuses on user's perceptions of dragging, and their accordance to the order of construction and hierarchy of dependencies, which derive from this order.

The study reveals that while hierarchy in geometric constructions in a DGE is mirrored by the DB, user actions often indicate a **reverse–order** perceptions, specifically, that the DB of certain elements from among a set of those constructed by the (n) initial steps is affected by the elements constructed by step (n+1).

The presentation concludes with a brief discussion of some implications for learning activities and software design.

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COLOURFUL MATHS: FROM FICTION TO REALITY…

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Alike many other European countries, Portugal has a growing number of ethnic and foreigner communities that are impelled to enter our educational system. However in a school that is required to be more and more inclusive, it is not possible to ignore the *leading actors* of this *movie* set in a small – though not that small – scenario that constitutes Mathematics' Education.

The nature of the tasks proposed and the didactic contract assume an extremely relevant role in pupils' performances which can be crucial to the inclusion process (César, 1998). The social marking of tasks constitute an advantage, allowing them to become truly significant to students, motivating their active participation, namely when they are rooted in their own culture (Favilli, Oliveras and César, 2003).

This study is included in *Interaction and Knowledge* project, whose main goal is to understand and promote peer interactions in classrooms. This part of the study was based on a hands-on handicraft activity – the Batiks. Basically, Batiks is a pure cotton wrap tainted with colours where a drawing is contrasted. So, this activity was also explored in other subjects. During some mathematics classes, students made batiks that were used later on to explore some 8th grade contents, namely proportionality, areas, translations and statistics. These tasks were proposed to 84 students (8th grade) from 4 different classes and schools, working collaboratively for the first time during this school year. Data were collected through participant observation, protocols, interviews, and teachers' and external observers' reports.

The students' engagement in the task, their attitudes, participation and increased interest, were strong affective responses that illuminate the potentialities of collaborative work and of these handicraft activity. Even some of the pupils that still experienced low achievement in mathematics were legitimate participants and their knowledge appropriation was better in the contents related to this task. The analysis of some video recording illustrates these phenomenon and also how inclusive ideals can be put into practice. In a multicultural and heterogeneous society it is urgent to find ways of promoting inclusivity, celebrating and respecting the diversity of *actors* that coexist in this *scenery*.

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PROSPECTIVE TEACHERS' KNOWLEDGE OF EXISTENCE THEOREMS

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A large body of research describes high-school students, prospective teachers and teachers' knowledge of universal theorems (e.g., Healy & Hoyles, 2000; Martin & Harel, 1989). This paper describes the characteristics of prospective teachers' knowledge of existence arithmetic statements. The main aims of the study were: (1) To examine prospective teachers' competence in constructing proofs of existence theorems; (2) To probe prospective teachers' views of given, written proofs of such theorems; (3) To examine the relationship between the prospective teachers' competence in constructing proofs of existence theorems and the prospective teachers' views of given, written proofs and refutations; (4) To examine the similarities and differences between elementary school and middle school prospective teachers' competence in constructing proofs of existence theorems and their views of given, written proofs of such theorems.

Ninety-three prospective teachers from several major teachers colleges in Israel participated in the study. They were given two questionnaires that were developed for this study: "The Constructing Proofs and Refutations Questionnaire" and "The Judging Proofs and Refutations Questionnaire".

About 50% of the prospective elementary teachers and 20% of the prospective, middle-school teachers incorrectly argued that the existence theorems that were included in the questionnaire are false. Furthermore, about two thirds of the prospective elementary teachers and one half of the prospective middle school teachers argued that numerical examples that fulfill the statements are just examples and could not be regarded as mathematical proofs. These responses suggest that some participants developed a general view that a mathematical statement is true only if it holds for "all cases", a view that is adequate for universal theorems but not for existence theorems.

In the presentation we shall provide typical prospective teachers' responses and draw some educational implications.

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HOW DO MATHEMATICS EDUCATION PROFESSORS DECIDE WHAT TO TEACH IN GRADUATE LEVEL CURRICULUM COURSES?

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Even though there are many research studies related to the teaching of K-12 mathematics, few studies in the mathematics education literature relate to university-level teaching. In the U.S., undergraduate teaching has gained increased attention in the past few years due to calls from the National Research Council (1996) and the National Science Foundation (1996). However, from our review of the literature, studies related to graduate-level teaching are very few and they are not particular to mathematics education. Herzig (2002), for example, studied mathematics doctoral students' understanding of mathematical culture and its possible implications for instruction. She indicated that graduate students education is important since these students will be the next generation of teachers for undergraduate and graduate level courses. In a "domino" effect, their education impacts mathematics education at all levels. Therefore, there is a need to study how professors teach graduate students.

In this short oral presentation, we will report our investigation on how three professors of mathematics education in the United States (two male, one female) decide what to teach in their graduate-level mathematics curriculum courses. Curriculum courses are our special interest because they capture three important aspects of professors' beliefs: their views of mathematics, their views of mathematics teaching, and their views of mathematics learning for K-12 students.

We have interviewed the professors individually for an hour and investigated artifacts related to the course (such as syllabi and personal notes). The analysis of the interview transcripts is the backbone of the report. We found that professors' designs of their courses change depending on: 1) their views of mathematics curriculum for K-12; 2) their views of graduate students' contributions to classroom atmosphere; and 3) the changes they want to see in graduate students. From our analysis, professors' research agenda was the main contributor to how they interpreted the three aspects listed above and why they implemented very different mathematics curriculum courses for graduate students.

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PROBLEM-SOLVING SKILLS TO OPTIMISE GEOMETRIC CONCEPT DEVELOPMENT

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How do students develop geometric concepts? Does a web-based lexicon with representations of objects, supported by a step-by-step problem-solving approach and Cinderella software, initiate a process of concept development?
'Pictures' (photos), the first level in the instructional design theory described by Seel and Winn (1997), are directly related to the simulated object. The second level, called 'figures', consists of drawings that help create or transform mental images and activates visual reasoning as in the description of external representations (Gutiérrez, 1996). Theoretical representations are components of the third level, named 'symbols'. The natural growth in this conceptual process of thinking is determined by empirical abstraction focusing on objects and their properties, pseudo-empirical abstraction focusing on actions, and reflective abstraction focusing on properties and the logical deduction (Tall, 2004). Cinderella software activates the development of a pseudo-empirical abstraction process because it enables student's further exploration. Problem-solving skills are based on the Heuristic Mathematics Education approach (Van Streun, 1989).

Data were analysed by categorising catchwords in three levels of argumentation: colloquial language, formalized language, and logic (Van Hiele, 1986). Progress in concept development was difficult to establish because of the type of problems (only application problems). The number of correct step-by-step solutions has increased (process). Animations at the second level took too much time, so students didn't use the lexicon sufficiently (product). Teachers didn't support students by indicating other representations at the same level, nor at another level through switching (teachers' role). Students preferred alternating computer-based collaborative work with teacher-centred learning activities.

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STUDIES OF THE EARLY MATH STRATEGY: A LONGITUDINAL STUDY OF TEACHER DEVELOPMENT

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Studies of the Early Math Strategy is an examination of a province-wide implementation of a teacher development strategy designed by the Ontario Ministry of Education. This program is designed to increase the elementary teacher' understanding of mathematics and to model and develop effective instructional practices. This paper presents a discussion of the preliminary stages of a large-scale research project aimed at studying this implementation process.

Mathematics education poses substantial challenges for elementary teachers, who often lack the knowledge of mathematics required to effectively implement reform-oriented mathematics programs (Ball, 1990; Ball, Lubienski & Mewborn, 2001). Teachers need a sound understanding of mathematics and of how children learn mathematics to probe student thinking and to recognize the important concepts that are inherent in students' mathematical activity (Ball, 1990; Kahan, Cooper & Bethea, 2003). Recognizing and developing teachers' understanding of mathematics and of how children develop mathematics concepts is essential to improving student learning. The important role of teachers is a message that resonates in the implementation plan of this initiative. The implementation plan aims to develop teacher expertise and to provide opportunities for teachers to connect new understanding with work in their own classrooms.

Our research project gathers both quantitative and qualitative data through questionnaires, interviews, analysis of training, and case studies. It is aimed at understanding the teaching of mathematics, the instructional strategies that teachers use, and the types of professional development and resources they find useful. As this project is in its initial stages, we plan to present a research report that will discuss the research design in further detail and will provide preliminary results of the analysis of the initial questionnaire and training sessions.

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UNIVERSITY STUDENTS' EXTENSION OF LINEAR MODELS TO NON-LINEAR SITUATIONS

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This communication presents a study of university students' extension of linear models to non-linear situations. By linear models we mean the model $y=a.x+b$, particular representations of direct proportionality and the diagram for the rule of three. This phenomenon has been studied at the primary and secondary levels, and has been called "illusion of linearity" or "linear misconception" (Behr, Hare, Post, & Lesh, 1992; De Bock, Von Doorem, Janssens, & Verschaffel, 2002; De Bock, Von Doorem, Verschaffel, & Janssens, 2001; De Bock, Verschaffel & Janssens, 1998). There exists agreement in describing the "linear misconception" as persistent and resistant to change. Studies about this phenomenon among university students are not frequent, even though its presence and persistence have been observed in diverse types of problems and contexts at that level. That situation led us to carry out an exploratory study to document, describe and analyze the presence of such phenomenon among Argentinean 18-20-year old students, which were studying agronomy in the University of Córdoba, and were attending their first calculus course. Using students' written productions, we analyzed the types of problems that were solved by extension of a linear model, the students' strategies,and the difficulties of interpretation that could be associated with the statement of the problems. The students' errors were not understood as failures, but rather as a construction based on their mathematical conceptions that have epistemological value. We also analyzed the particular teaching environment of the calculus course in order to raise some conjectures to explain the phenomenon from the teaching perspective.

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TEACHERS' CONCEPT OF SLOPE AS REPEATED ADDITION

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We present an analysis of the complexities surrounding one teacher's real-time presentation of her emergent understanding of slope as repeated addition. Our lens on the social collective as a learning group positions the teacher within the community of learners collaboratively building on prior experiences. This research is situated in a three-year professional development program for twenty-five practicing elementary teachers preparing to work as numeracy coordinators in one school district. Teachers were asked to determine which is steeper, a slope of 1/2 or a slope of 2/3, and to provide two convincing representations to support their conclusions. They had been using Cuisenaire rods in an earlier investigation, and the rods remained available for teacher use. Specific videotaped episodes, in which teachers articulated, inscribed, or kinesthetically presented slope, were identified as critical events and analyzed through the use of open coding and constant comparison (Powell, Francisco, & Maher, 2003).

Teachers with prior knowledge of slope plotted points by counting rise over run. Lyn, a teacher without prior knowledge of slope, began to use the Cuisenaire rods to literally build a presentation of her emerging understanding of slope. Lyn presented 2/3 slope to the entire class by forming a stair-step arrangement of light green Cuisenaire Rods which she called "threes". Similarly, Lyn built a representation for 1/2 slope using the red rods, which she called "twos". Several other teachers questioned Lyn, as was the custom in the collaborative learning community.

Lyn:	One, two…so I just go two,two,two (*stacking the red rods)*
Christine:	Why do you think those are twos though?
Lyn:	Cause they're just twos (*referring to the red rods*), I'm using the twos, I'm just using twos as halves.
Linda R:	How come you only went over 1 (*referring to the slope of 2/3*)?
Brenda:	(*responding instead of Lyn*) That's two-thirds one time and two-thirds two times so you're adding two-thirds to it each time.
Lyn:	I've never done slope in my life.

Lyn's representation of slope was grounded in the recursively defined functions she studied previously. Lyn built her representation for slope as a process of repeated addition before thinking about rise-over-run. Teachers with limited formal mathematical backgrounds tried to make sense of both traditional and non-traditional representations rather than rush to use standard formulas in building their knowledge of slope.

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CAN STUDENTS DEFINE ABSTRACT CONCEPTS: USING GENERALIZATION PRINCIPLE IN LEARNING ALGEBRAIC STRUCTURES

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Generalization as a mathematics activity takes significant place within the research on the learning and teaching of mathematics. It is necessary to say that though the generalizing process in didactics of mathematics (as well as the method of generalization in mathematics) is quite known (Mason et al, 1985), having various directions for research and using in teaching, it has been little used for teaching algebraic structures to students. Our theoretical position is grounded in the theory of active learning processes in mathematics (Hiebert, 1992; Wang, Haertel and Walberg, 1993). We would like to consider the possibility of the using methods of active learning combining both didactic ideas and research methods in mathematics itself. It may make clear how the process of teaching mathematics can be constructed similarly to the process of mathematical research and how this kind of teaching contributes to the development of students' mathematical thinking. While studying algebraic structures using generalization principle, on the first stage of the interview 78 first year undergraduate students were given a questionnaire having 21 tasks of different kinds for two aspects of generalizing process in order to find out directions and priorities of students' mathematical thinking, first of all their abilities to define abstract concepts while studying this theme under given conditions. We took into consideration that to support mathematically thinking one needs a questioning, challenging and reflective atmosphere (Mason, 1985, p.153). 61 students were able to define the concept of a group on their own having meanwhile constructed the local theory of the group S_n. 13 students independently came to the concept of a semigroup studying generalized permutations. The second stage of the interview concerned students' motivation in learning under the given conditions. 11 questions were proposed to the same students to find out their priorities and dislikes in learning this theme.

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PROBLAB: A COMPUTER-SUPPORTED UNIT IN PROBABILITY AND STATISTICS

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ProbLab is a computer-based middle-school curricular unit in probability and statistics designed to enrich student thinking in the domain. The ProbLab unit is part of the Connected Probability project (Wilensky, 1997) and includes a suite of interactive models[1] authored in the NetLogo (Wilensky, 1999) modeling-and-simulation environment and using the HubNet Participatory Simulation technological infrastructure (Wilensky & Stroup, 1999). ProbLab's design rationale and interactive materials reflect our view of the domain as constituted on three interrelated pillars: theoretical probability, empirical probability, and statistics. Students explore connections between these pillars by constructing and experimenting with domain *bridging tools* (Abrahamson, 2004), such as the *9-block*, a 3-by-3 array of squares, each of which is either green or blue. A 9-block is at once one of all 512 permutations in its combinatorial sample space (theoretical prob.), a randomly generated compound event (empirical prob.), and a sample of out of a population of squares (statistics).

Figure 1 (from left): One of 512 9-blocks; 6th-grade students create and assemble the combinatorial space; the resultant Combinations Tower; an empirical experiment that dynamically builds frequency distributions of randomly generated 9-blocks; on his laptop, a student takes 9-block and 1-block samples from a hidden population.

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[1] All ProbLab models are available for free download at http://ccl.northwestern.edu/.

SYNOPTIC AND EPISTEMOLOGICAL VISION OF POINTS IN A FIGURAL TASK ON THE CARTESIAN PLANE

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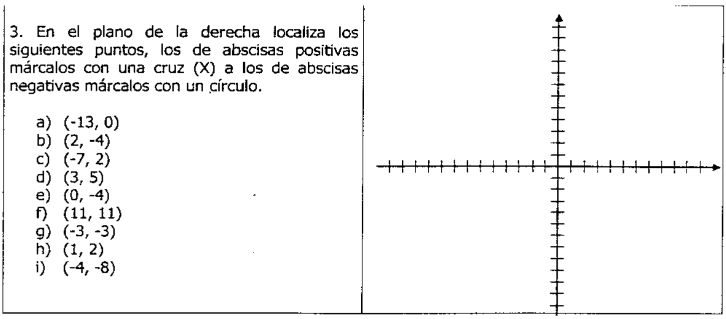
This research is based on two cognitive functions related to vision: epistemological vision (EV), associated with local perception, and synoptic vision (SV), associated with global perception, Duval (1999). Although point-by-point procedure is in general, enough to build a graph it not the same for to **see** the final configuration in the spatial sense. I proposed 101 students this task that claim:



In the plane on the right, you plot the following points indicated; those of them with positive abscissas should be marked with an "X", and those with negative abscissas marked with a circle.

Plotting points needs spatial orientation in order to choose the adequate direction (right or left, up or down for each case), but it is not necessarily evident that this task will give to students a general criterion for making the spatial orientation of points on the plane. In marking task, they must decide on the right direction, as well as select a suitable mark to put on the point.

The results say us that the use of EV for plotting points does not imply the employment of SV which students need in order to mark points on the plane. And considering that both types of vision play an important role in understanding graphs completely, it is necessary to have practice in order to develop both types of vision.

- Duval R. (1999), Representation, Vision y Visualization, Cognitive Function in Mathematical Thinking, *Proceedings of the Twenty First Annual Meeting PME-NA* v 1, pp 2-26

GREEN RESPONSE AND GREEN DIALOG - COMMUNICATION AMONG TEACHERS AND STUDENTS

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Nils Johan Kjøsnes – associate professor at Sør-Trøndelag University College

Since the middle of the nineties we have worked with response and dialog among teachers and students. Traditionally teachers receive exercises and homework from the students, correct them with a red pen with C for the correct and W for the wrong answers. Our experience is that this method is of little help to the students. In stead we have introduced green response and green dialog in the communication between teachers and students. Green response is a metaphor for giving feedback to the students from the stage or level they are at. We can both talk about correcting with a green pen, giving green response and talking to the students with a "green voice".

When you as teacher give green response or participate in a green dialog you have to try to figure out and understand the students' way of thinking. Our experience tells us that behind the most meaningless or insane answer there is a sort of logical thinking. An aspect of giving green response is to find that logical thinking.

Our examples to illustrate green response and green dialog are both from pupils at different ages and students in teacher education. We will present works from a 7-year old boy working with subtraction or "take away" like this: 5-4=5, 4-1=4, 1-1=1 3-0=3 etc. And we will present a 14-year old girl who can add and subtract large numbers mechanically without really understanding the position value of our number system. We will also present a 13-year old girl who solves an exercise so cleverly that the teacher suspects her of cheating.

Many of our examples are taken from students in teacher education who work with mathematical symbols that give no meaning to them. The students work with the numerical system without the Arabic numbers, using symbols that are logical, but not the ones they are acquainted with. We will present green dialogs between the teacher and the students and among the students connected to these examples.

Our examples will be presented on a poster together with an explanation of what we mean by green response and green dialog. And during the presentation we want to discuss this idea in a way that can give us green response to our idea and work. We hope that our presentation will become a real green dialog between the participants and the two of us.

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STUDENTS´ UNDERSTANDING OF AREA AND DEFINITE INTEGRAL CONCEPTS WITHIN AN ENHANCED COMPUTER LEARNING ENVIRONMENT[1]

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This study investigates what first year engineering university students showed after they had taken a calculus course in which they used systematically Derive Software to work on a series of tasks that involve numerical, graphic and algebraic approaches. In particular, students had opportunity to use a special designed Utility File as a means to approximate areas of bounded curves (via the use of rectangles, trapezoids, and parabolic regions). Fundamental research questions included: *To what extent do students display relationships between graphic, algebraic, and numerical representations in their problem solving approaches? And what type of difficulties do students experience as a result of using Derive and Utility File?*

Basic ideas that helped frame the study recognize that the use of CAS functions as a cognitive tool for students not only to solve problems but also to make sense and understand mathematical ideas; besides, this type of tools provides students the opportunity to generate new mathematical representations that help them investigate relationships associated with a situation or phenomenon in study and to appreciate the balance between formal and informal mathematics (Heid, 2002). Thus, the use of representations plays a fundamental role in students´ construction of concepts (Goldin, 1998). Thirty-one first year engineering university students participated in the study. The study was carried out during one term meeting six hours a week plus two hours of computer lab session. Results indicated that some students relied on the use of the software as a means to validate their paper and pencil work, other group of students used the software to graph and calculate approximated areas and a third group of students combined both paper and pencil and software approaches to solve problems but often failed to connect concepts that appeared in the study of the definite integral with basic ideas (and procedures) previously studied (area of simples figures). Although the use of the software provided an interesting window for students to free them from memorizing formulae or calculations procedures it is also important to recognize that students need time to mature and develop a robust conceptual understanding of the definite integral.

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THE ILLUSION OF LINEARITY: A LITERATURE REVIEW FROM A CONCEPTUAL PERSPECTIVE

Dirk De Bock [1 2], Wim Van Dooren [1 3], Dirk Janssens [1] and Lieven Verschaffel [1]

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Students' tendency to apply the linear model also in situations where it is not applicable – often called the 'illusion of linearity' – has been frequently described and illustrated in the mathematics education literature. During the last decade, systematic empirical research contributed to our understanding of this phenomenon by providing data on its scale and persistence in different experimental settings (De Bock, 2002). However, the phenomenon of students' improper linear reasoning still suffers from conceptual vagueness, due to the absence of serious attempts to integrate the different aspects of this phenomenon in a broader theoretical framework. Different aspects are, e.g., students' well-known improper 'k times A, k times B' reasoning when dealing with various numerical relations, their tendency to associate the missing-value format of a word problem with a strategy of setting up and solving a proportion, but also their tendency to represent functional relations preferentially by straight lines and their overreliance of the linear properties $f(a + b) = f(a) + f(b)$ and $f(ka) = k\,f(a)$ while simplifying algebraic expressions.

This poster will report the results of a systematic literature review about the illusion of linearity, aimed at providing more conceptual clearness in this phenomenon. Various examples were identified in the literature, showing how the illusion of linearity plays trick to people of different ages and in different cultures, working in diverse domains of mathematics (such as arithmetic, algebra, geometry, probability and calculus) and science. On the poster, we will first show a series of famous historical examples (such as the duplication of the square in Plato's dialogue Meno or the probability problems of de Méré) and discuss how these examples were interpreted by leading scholars in the field. Second, we will elaborate on various examples of utterances of the illusion of linearity mentioned in reports of empirical research and in practical-oriented publications. We will relate the different utterances of this phenomenon not only to mathematical domains or educational levels, but also to the different aspects of the concept of linearity itself and how they may lure people into the linearity trap.

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SONA DRAWINGS: A DIDACTICAL SOFTWARE

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A *lusona* is a sand drawing consisting of a rectangular net of dots and one or more closed polygonals enclosing the dots. The polygonals are drawn respecting a few easy rules.

Sona drawings have many possible didactical uses, at different school level, as it has been shown by Gerdes. Among them the introduction of the notion of GCD.

To make more evident the link between indigenous and scientific knowledge, the past and present, a software has been implemented that allows a PC to draw (in a continuous motion) a *lusona*, once its dimensions (that is the rectangle sides) have been chosen.

The software aims also to provide teachers with a new didactical tool from which pupils could benefit in their appropriation of one of the less beloved mathematical concepts. It is often too hard for them to understand what the GCD of two natural numbers represents: its computation is just a technical and harid exercise! On the contrary, *Sona* software allows users to consider the GCD as the solution to a geometrical problem: how many polygonals are necessary to enclose a set of dots ordered in a PxQ rectangle?

The piloting of the software is part of a didactical proposal, consisting in a module under implementation in a few Italian schools. The proposed module starts from teacher drawing on the blackboard a couple of full *sona* and soon after a partial one, with no explanations. Pupils, working in small groups, are then asked to investigate the rules needed to make the *sona* and to draw (pencil-and-paper) a few of them. Soon after, the software can be used to have more examples in very short time. Pupils are also asked to focus their attention in the number of the lines N needed to enclose the dots in the rectangle, to make a record of all the data (the input numbers P, Q and the output N) and to conjecture the possible relations among them. Each step is followed by a discussion among the pupils, who, probably thanks to the teacher's guidance, should allow the class to better depict the underlying notion of common divisor, first, and to fully appropriate the notion of Greatest Common Divisor, after.

The module validation is obtained by the comparison of the results of a final test on the GCD submitted to pupils in classrooms where the module has been implemented and pupils in classrooms where the concept of GCD has been introduced in the usual, standard, way.

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ON THE POSSIBILITIES OF SUCCESS OR FAILURE OF A TEACHING MODEL. ALGEBRAIC BLOCKS FOR LOW-SCHOOL-PERFORMANCE STUDENTS

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This is an empirical study focusing on the ambiguity between the negative numbers and the subtraction operation in algebraic tasks. It is based on Filloy´s theory for the empirical observations in mathematics education. Filloy (1991) stated that cognitive tendencies appear in teaching situations. In this study, the teaching situation refers to the Algebraic Blocks model. Gallardo (2002) found four levels of acceptance of negatives: subtrahend, signed, relative and isolated numbers, which are identified as cognitive tendencies in this study. Research question: Can the Algebraic Blocks model be used to analyze the ambiguity between the subtraction and the negatives? Is there a teaching model that is the best of all? This is a case study of three low-performance 12-13 subjects: S1, S2, S3. The main results. Intermediate senses of negative numbers appeared: subtrahend, signed, relative and isolated numbers. This tendency was extreme when S3 only accepted subtrahends and did not recognized the signed numbers. Inhibiting mechanisms emerged: S3 focused on the binary sign and ignored the unary signs; S2 and S3 had difficulties with general numbers in open sentences. S1, S2, S3 avoided negative solutions in equations. The effect of obstructions derived from semantics on syntax was extreme in S3. It has been proven that it is necessary to give up the search of the good model, due to extreme cognitive tendencies. Examples of these facts are exhibited in a poster presentation.

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EXPLORING THE MATHEMATICAL KNOWLEDGE OF GRADE 1 AND GRADE 2 CHILDREN WHO ARE VULNERABLE IN LEARNING MATHEMATICS

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As part of the Early Numeracy Research Project (Clarke, Cheeseman, Gervasoni, Gronn, Horne, McDonough, Montgomery, Roche, Sullivan, Clarke, & Rowley, 2002), clinical interviews were used to determine the mathematical knowledge constructed by Australian children in the first three years of schooling. This knowledge was linked to a research-based framework of growth points in four number domains (Counting, Place Value, Addition and Subtraction, and Multiplication and Division), and this framework provided a means of identifying children who were vulnerable in learning mathematics.

In 2000, 576 of 1497 Grade 1 children and 659 of 1538 Grade 2 children were identified as vulnerable in at least one of the four number domains. The diversity of domains and combinations of domains in which children were vulnerable is striking. This is demonstrated by the diagrams on the poster. Clearly, children who are vulnerable in learning school mathematics have diverse learning needs that call for particular instructional responses from teachers. It is likely that teachers need to make individual decisions about the instructional approach for each child. The results indicate that there is no single ‘formula’ for describing children who are vulnerable in learning school mathematics, or for describing the instructional needs of this diverse group of students. Further, the diversity of children’s mathematical knowledge in the four domains suggests that knowledge in any one domain is not necessarily prerequisite for knowledge construction in another domain. This finding has implications for the way in which the school mathematics curriculum is introduced to children. It seems likely that children may benefit from concurrent learning opportunities in all number domains, and that experiences in one domain should not be delayed until a level of mathematical knowledge is constructed in another domain.

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EMOTION AND AFFECT IN MATHEMATICAL EDUCATION EXPLORING A THEORETICAL FRAMEWORK OF INTERPRETATION

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The reconceptualización of affective control in the present decade is marked by two essential features. One is the attempt to consolidate a satisfactory theoretical framework for its interpretation, the other to place it within the social context in which learning takes place (Gómez-Chacón, 2000a and 2000b).

In this communication we present a model for the study of the interactions between cognitions and affects in the learning of Mathematics. The model will be used for describing emotional responses, their origin as well as for surveying their evolution in the subjects under consideration.

Certain dimensions related to affects and cognitions are specifically focused, namely, affect itself, meta-affects and belief systems. Attention is drawn to the importance of taking into consideration these dimensions in investigations of this nature, particularly in the case of school-failing students and in multicultural contexts. The study has sought to analyse if it is possible to interpret the emotional responses of the young from a perspective of social identity and of cultural identity

Two groups of study will be presented: One is a group of students in secondary education in Spain and the other is a group of students in secondary education in Belgium. The study in Spain was first accomplished.

The study in Belgium has been carried out with students of Portugese origin living in Brussells, in their 7th to 12th levels of education, from February to June 2003. The data collected has been attained in various forms: in clases of Portuguese language and Portugese culture which are complementary to the ordinary curriculum; in schools where there is a Portuguese population but there is lack of intercultural focus (during clases in mathematics and in a weekly meeting only with those students of Portugese origin); in schools where there is an intercultural focus in the field of mathematics.

The investigation has been qualitative in character, combining methods proper to etnology in case-studies.

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ABSTRACTION IN THE LEARNING OF MATHEMATICS BY FIFTH-GRADERS IN RUSSIA

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In this communication the problems of abstraction in the learning of mathematics at 5th grade of lower secondary school are considered. In particular, we investigated the process of abstracting during the introduction of the following concepts: natural numbers, ordinary fractions, a segment, direct line, a ray, scales and coordinates.

Nemov (2000) wrote in his "General foundations of psychology": "One can abstract not only properties, but also operations, in particular ways of problem solving. Their use and transfer to other conditions are possible only when the chosen way of a solution is realized and interpreted regardless to a concrete problem". In a school course of mathematics the means of abstraction is more often applied at forming mathematical skills rather than mathematical concepts. In 5th grade, during the introduction of variables and at the beginning of work with the formulas, the pupils pass to the second, higher level of abstraction, when the abstraction from concrete numerical data happens. Note that in a text-book-companion of Shevrin et al. (1989) for the study of each arithmetical operation basic types of corresponding word problems are selected and the patterns of solving such tasks are presented as schemes.

The next important step in the raise of a level of abstraction of mathematical objects is a symbolic representation of properties of addition and subtraction. The introduction of a symbolic representation influences also the method of presenting a subject matter. If properties of addition and subtraction studied before the acquaintance with a symbolic representation have been introduced by a concrete and inductive method, the properties of multiplication and division of natural numbers, which are studied later, are being introduced by an abstract and deductive method.

Note that the operation of rounding off and fulfilling mathematical operations with the rounded values of numbers by its external form and structure is very close to the operation of abstraction. Rounding a number off, we distract from digits insignificant for us which we throw off, we work only with remaining, "essential" digits.

Important for the further study of mathematics are skills connected with measurement of magnitudes. In 5th grade the measurement of the length of a segment and the degree measurement of an angle is fulfilled. Abstraction is again present here during the development of skills of use of measuring gears.

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BENCHMARKS IN AN EARLY NUMBER CURRICULUM

Kath Hart

UK

The United Nations has stated that in the not too distant future all children should be attending school. What will a child, the first in his family to ever be educated, need to learn in mathematics? In many parts of the world the process of universal education has begun, although the schools are ill-equipped, over –crowded and staffed by untrained [or under-trained] teachers. Malawi, Ghana and Zambia in sub-Saharan Africa are three countries with multiple problems. Each however has a national mathematics curriculum which is long and shows little regard for the general conditions in which it is taught. Schools, communities and international donors are continually disappointed by the seeming lack of achievement of children in their first few years in school. In Malawi, 40 per cent of pupils leave school at the end of grade one

The poster is to present a suggested progression of achievable Number concepts based on discussion with educators, interviews and tests of children [N = 1000] in grades one to four in Malawi, Zambia and Ghana. It is proposed as a start for a curriculum which could be taught in very basic conditions and which could be built on when confidence has returned.

Interviews and questions have been informed by the work of Clarke and his team [2001],Fuson [1992] ,Wright [1996], Davis[1992] and Hart and Yahampath [1999]. All of these had investigated some aspect of what appeared to be cognitively demanding in 'Early Number' but nobody has a detailed progression supported by evidence. In spite of the fact that any mathematics syllabus purports to be a progression of increasingly sophisticated concepts.

The poster will show four stages of Whole Number Learning, illustrated by some suggestions for assessment, based on evidence [although limited] from disadvantaged pupils. It is hoped that conference participants will engage in discussion and hypothesis further steps.

THE APPROPRIATION OF NOTIONS OF REFLECTION BY VISUALLY IMPAIRED STUDENTS

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This poster presentation will describe a study being developed in São Paulo, Brazil with the objective of investigating the processes by which visually impaired students appropriate aspects related to the geometrical transformation reflection. We are interested in the forms by which these learners, during instructional events, incorporate into their own vocabulary the mathematical voice, as well as the conceptual changes that the appropriation of such vocabulary might provoke. To this end, we are elaborating activities and interventions seeking to create the conditions necessary for the emergence of a symbolic space (ZPD) in which face to face interactions between the participants involved in an instructional event might motivate the production of new meanings.

We opted to use Vygotsky's method of double stimulation (Vygotsky 1998), in which the subject is faced with a task which considerable exceeds her actual cognitive state. In our study the first stimulus is given in the form of a tool to explore geometrical objects (to be illustrated in the poster) and the second stimulus involves the researchers interventions. We intend to present some of our analyses of the dialogs occurring during interviews with two visually impaired students as they worked on a series reflection and symmetry tasks adapted for tasks previously used in studies of the conceptions of sighted learners.

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DYNAMIC MODELS OF LINEAR FUNCTIONS

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In a connected algebra environment the class is typically subdivided into numbered groups, where each student has a two-number identity that can then serve as "personal parameters," a Group Number and a Count-Off Number. Students then create mathematical objects – in the cases discussed here, they are linear Position vs. Time functions that drive animated screen objects. The functions depend in some critical way on students' respective personal parameters either on a hand-held device or on a computer. Students' work is aggregated (via a wireless network), organized and selectively displayed using MathWorlds for the PC and then discussed using carefully constructed questions and features that control what is shown (e.g. individual construction vs. group based construction). Classroom discussion then becomes an "algebraic activity" where students compare their mathematical construction relative to each other and thus generalizations and abstractions becomes a social activity. We will demonstrate a genre of activities that develop core algebraic ideas of parametric variation through this new classroom infrastructure. Figure 1 below illustrates one example where we have 2 groups of 4 students. The problem statement asks a student to construct a motion algebraically that starts at a position ahead of a target motion (Y=2X) which is equal to their group number but travels at a speed equal to their count-off number. Aggregation thus enables a graphical gestalt, described by some students in our studies as "fans" prior to us showing the whole collection. Animating "families of motions" also leads to developing dynamic models of linear functions and parametric variation.

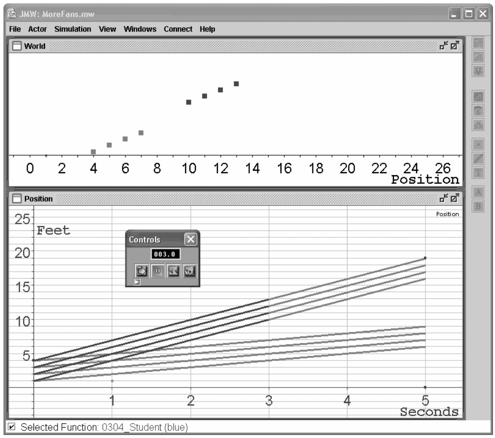

Figure 1:Making "fans" from two groups

STUDENT'S ATTEMPT TO SOLVE SEVERAL-STEP PROBLEMS IN PROBABILITY

Kjærand Iversen,

Agder University College/Nord-Trøndelag University College

The main focus of the research is on student's meaning-making process when working with several-step problems in probability. These problems can be introduced in connection with stochastic phenomenon with *simultaneous stochastic objects* or in *causal stochastic situations.* The throwing of two dice exemplifies the first, while the device Binostat exemplifies the latter. The participants are students in lower secondary school. In a survey by Green (1983) a total of 1620 students from lower secondary school participated. The written test used consisted of problems in probability. One of the problems was a three-step problem with a robot (the Robot problem). Only 9% of the participants were able to solve this problem. In a research by Fischbein (1975) a total of 42 students from lower secondary school used a set of experimental devices when working with different problems in probability. Two three-steps problems were included. As many as respectively 58% and 69% had a correct response to these problems. As a part of my research 168 students undertook a written test where the Robot-problem was included. A significant higher percentage of these students had a correct response to the Robot-problem than in Green survey. In the main part of my research pairs of students works in an ICT-environment with both simultaneous and causal stochastic problems. This part of the research is not finished but the analysis of the data collected so far indicate that students in lower-secondary school are quite able to solve the chosen causal problems. The simultaneous several-step problems seem to be much harder for the students. Earlier research by Pratt (1999) and Stohl (2000) show that even young children construct important intuitions about simultaneous two-step problems. I wants my research to dig deeper into students understanding of the multiplicative relationship involved in several-step problems in probability. The presentation will show the software that is developed so far aimed at being helpful for the students in their meaning-making process. This includes the software *Flexitree* and *Spinners*.

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CONSTRUCTIVIST APROACHES IN THE EDUCATION OF FUTURE TEACHERS, CASE OF GEOMETRY

Darina Jirotková & Nada Stehliková

Charles University in Prague, Czech Republic

Much of the existing research relating to students training to teach particularly in Primary schools often shows that they have a weak mathematical background and lack of understanding of the subject. This often results in them disliking and having a fear of the subject and later teaching pupils facts and algorithms to commit to memory and reproduce them in solutions to standard problems. We feel strongly that this just continues the vicious circle we see happening in many schools and which culminates in our students disliking and fearing mathematics and then passing on these emotions to the many pupils they will teach throughout their career. This phenomenon does not just apply to student teachers but to many teachers already working in the school system. Therefore it is necessary to prepare courses which can be used in initial and in-service education of teachers to address this problem.

The authors' institution has re-designed several of the mathematics and mathematical education courses offered to our primary and secondary student teachers so that they are taught using constructivist teaching principles. It is hoped that by getting them to work in the way in which they could work with their pupils, they will be convinced of the benefits of the constructivist principles and apply them in the classroom.

The poster will take examples from the area of geometry, using grid-paper as a vehicle for learning, and show how to develop a simple initial idea into a range of sophisticated geometrical ideas. The poster will show how the tutor engenders the students' natural curiosity by posing relevant problems. This in turn gets the students to raise problems in response to the work they are doing. The tutor does not pass on ready-made knowledge but asks further questions and guide classroom discussion. It will also be shown how the students have taken more responsibility for their own learning and how they have experienced the excitement and joy of gaining new knowledge. This approach changes the students' attitude towards mathematics and their way of thinking about the subject. We intend to include examples which show the unpredictability of working in this way and how different groups have reacted differently to the same initial stimulus. The authentic episodes from the University classrooms in the poster will illustrate the constructivist principles outlined above.

The contribution was supported by a research project Cultivation of Mathematical Thinking and Education in European Culture, No. J13/98:114100004.

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WOW! IT WOULD BE FUN TO LEARN MATH BY PLAYING A GAME! : Number Concept and Mathematical Strategy in the Game *Yut-Nori*

Ho-Kyoung Ko

There are many traditional Korean games that are exciting to play. Of all the Korean board games, *Yut-Nori* is one of the most effective games for enhancing children's numerical ability and mathematical strategy.

To play *Yut-Nori*, one needs a game board (*Yut pan*), playing pieces (*Mals*), and four sticks that have a round side and a flat side. A player determines the number of spaces that a piece can be moved by counting the number of flat sides that are turned up after the sticks are thrown. *Yut-Nori* is a race to get all four playing pieces around and eventually off the board. Luck is involved when the player throws the sticks, and strategy is involved when the player moves the pieces around the game board.

This study aimed at finding how *Yut-Nori* helps the development of children's numerical ability—counting, addition, and ordinal numbers—and children's mathematical strategies—ordering capacity, number-conservation, part-part-whole, and shortcuts.

Key Words: *Yut-Nori*, number concepts, mathematical strategy.

THE IMPACT OF THE NATIONAL NUMERACY STRATEGY IN ENGLAND ON PUPILS' CONFIDENCE AND COMPETENCE IN EARLY MATHEMATICS

Chris Kyriacou and Maria Goulding
University of York, Department of Educational Studies

This paper outlines how the systematic review approach was used to explore the impact of the National Numeracy Strategy in England on pupils' confidence and competence in early mathematics. An Evidence for Policy and Practice Information (EPPI) Mathematics Education Review Group in the United Kingdom was established in October 2003 with funding by the UK government to carry out systematic research reviews on teaching and learning mathematics covering the years of compulsory schooling in the UK (i.e. ages 5-16) and the 16-19 age group. The review group is coordinated by the Department of Educational Studies at the University of York. Membership of the review group comprises researchers, teacher educators, policy makers and teachers drawn from across the UK and aborad. The Review Group is funded to identify one review question each year and to conduct a systematic review of the literature to address this question. The first review question established by the group is: "Has the Daily Mathematics Lesson, in the context of the National Numeracy Strategy, helped pupils to develop confidence and competence in early mathematics?" This paper will outline the stages involved in conducting a systematic review and the findings of the review for this question. The left hand side of the poster will display a flow diagram indicating the key stages involved in conducting a systematic review. The right hand side of the poster will display the list of studies identified for the in-depth analysis together with the key findings which have emerged from the analysis.

TACTILE MANIPULATION & COMMUICATION

Graham H Littler & Darina Jirotková
University of Derby, UK; Charles University on Prague, CZ

The authors have researched pupils' tactile perception of three-dimensional solids over a period of time. (Jirotková, 2001, Jirotková & Littler, 2002, 2003). We have done this by designing a series of tasks that the pupils undertake in pairs and during which they are unable to see the shapes they are handling. We were able to video their tactile manipulation during the tasks and to record their communication between the pairs as they progressed through each task.

In 2003 three tasks, in particular, led to interesting links being established between the pupils' tactile manipulation and their communicative skills. These results also linked to the Van Hiele/Pegg's levels of insight into 3-D geometry. (Van Hiele, 1986, Pegg, 1997). Each pupil had a set of 14 different solids, which were hidden behind screens. In the first task pupil A was given a solid by the researcher and by only tactile manipulation, had to describe it to pupil B. Pupil B then had to find the described solid in his/her set of solids by tactile perception only. They were allowed to give and ask for as much information as they needed. The second task only differed from the first in that pupil A chose a solid for him/herself from amongst his/her 14 solids and pupil B could only ask for more information without being specific. The final task involved pupil A choosing a solid tactilely with pupil B determining the chosen solid tactilely by asking questions to which pupil A could only answer 'Yes' or 'No'.

The analysis of this research led to the defining of three types of tactile manipulation for the primary pupils with whom we worked. These levels matched closely to the first three Van Hiele levels which relate to the same age group of pupils. Similarly the level of communication skills which each pupil possessed linked closely with the type of tactile manipulation they used. This link enables the teacher to identify weaknesses both in tactile manipulation and communicative skills, especially the meaning of the words the pupil uses. Clearly being able to observe the manipulation gave insights into the strategies the pupil used to solve the tasks.

The poster will give the types of tactile manipulation and give examples of pupils working through the tasks, their communication and pictures of their manipulation. The link between the types of manipulation and Van Hiele's levels will be illustrated.

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A FRAMEWORK FOR ACTION FOR TEACHER DEVELOPMENT

Kay McClain

Vanderbilt University

This poster presentation provides an analysis of data taken from a four-year teacher collaboration conducted in the context of design research (cf. Brown, 1992). The analysis delineates aspects of an approach to teacher development that takes as its starting point the importance of teachers' practice being grounded in a deep understanding of the mathematics they teach while placing students' current ways of reasoning at the forefront of instructional decisions. In particular, the analysis entails a focus on iterative task-analysis cycles (cf. McClain, 2003) employed throughout the teacher collaboration in order to generate a Framework for Action. diSessa and Cobb (2004) note that Frameworks for Action "play a critical role in organizing research around instruction" (p. 82). This Framework, therefore, provides a structure for understanding and analyzing both 1) the collaborative interactions and 2) how teachers' participation in cycles of task analysis supported their development of more sophisticated ways of conceptualizing their instructional practice.

A graphic of the Framework for Action is used as the focal point of the poster presentation and supported by examples from related analyses of each aspect of the Framework. The poster presentation pushes beyond a focus on the *form* of teacher collaborations (e.g. use of student work, lesson study, use of cases) to examine the *functions* of resulting interactions, how they became constituted in interaction, and what they afforded in the context of the professional development of mathematics teachers. To this end, the visual articulation of the Framework for Action provides the mechanism for clarifying how particular features of the teacher collaboration were related to the goals of both supporting and sustaining mathematics teacher learning. Analysis of the data therefore serves to clarify the role of a focus on teacher content knowledge and students' thinking as a basis for teacher change and the importance of a framework to guide the interactions and subsequent analysis.

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DEVELOPING EFFECTIVE 'RATIO' TEACHING IN PRIMARY SCHOOL: RESULTS FROM A CASE STUDY

Christina Misailidou,

University of Manchester

This poster provides results from a case study concerning the teaching of 'ratio': Jack, a primary school teacher, in collaboration with the author planned and taught an introduction of the topic 'ratio' in his class of 29 pupils, aged 10 to 11.

For effective mathematics teaching, Bell et al. (1985) suggest that first, pupils' errors can be identified through 'diagnostic tests' and then these errors can be resolved through 'conflict discussion'. Accordingly, our 'ratio diagnostic test' (Misailidou and Williams, 2003) was administered to Jack's class to identify his pupils' errors. Then two teaching sessions were planned based on the test results and the 'tools for teaching' suggested by the author i.e. combinations of arguments, 'models' and teaching interventions that have been found to enhance pupils' proportional reasoning (Misailidou and Williams, under review). Each session was build around a central task derived from the diagnostic test; a 'mixing paint' task and a 'sharing bread' task were used. During the sessions the pupils worked individually, in small groups and as a whole class. The aim - set by the teacher - was to persuade each other through reasonable arguments about methods and answers concerning the tasks. Pictorial representations and coloured counters on an overhead projector were used for modelling the tasks. The pupils were advised to use drawings to communicate their thoughts and methods and at the end of each session they were asked to produce reports stating their final decisions. These reports indicate that the pupils can learn to reason proportionally through discussions with their peers and aided by appropriate models.

The poster presents the 'ratio diagnostic test' and an overview of the two teaching sessions, including the tasks and the models that were used. It provides video snapshots and the final products of pupils' work, and tracks the development of their proportional reasoning throughout the sessions contrasted with their performance on the diagnostic test.

Acknowledgement: The project was funded by the Economic and Social Research Council (ESRC)

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DEVELOPMENT OF MATHEMATICS LESSON PLANS USING ICT BY PROSPECTIVE ELEMENTARY SCHOOL TEACHERS[1].

JeongSuk Pang, SeongHee Kim, & MinSung Kwon

Korea National University of Education

In the era of information and technology, mathematics teachers need to take advantage of ICT(Information & Communication Technology) in their lessons. The purpose of this study was to foster professional development of pre-service elementary school teachers in terms of incorporating ICT in mathematics lesson plans in a way to achieve their instructional goals. The subjects were about 170 juniors who took a course of elementary mathematics teaching method for fall semester in 2003.

We emphasized that lesson plans with ICT should consider the characteristics of elementary mathematics teaching and learning, student-centered activities, development of mathematical power, effective use of various multimedia materials, connection to real-life contexts. Whereas previous studies tended to focus mainly on developing teaching activities per se, this research underlined not only activities but also plans in order to improve overall instructional design ability of prospective teachers.

We developed six instructional models using ICT that are appropriate to elementary mathematics teaching. Learning the different aspects of the models, the pre-service teachers developed their own lesson plans, which were reviewed by teacher educators as well as in-service teachers and were partly applied to mathematics classrooms. Given the various feedbacks, 229 lesson plans were finally developed and have been serviced through EDUNET, the most comprehensive educational information service on-line system in Korea. This poster presents exemplary lesson plans and instructional activities with each model.

[1] This study was supported by the Korea Education & Research Information Service Grant in 2003.

MATHEMATICS TEXTBOOKS AND THEIR USE IN SECONDARY CLASSROOMS IN ENGLAND, FRANCE AND GERMANY: CONNECTIONS, QUALITY AND ENTITLEMENT

Birgit Pepin

Oxford Brookes University, UK

Linda Haggarty

The Open University, UK

In this presentation it is argued that lower secondary mathematics textbooks in England, France and Germany place different emphases on 'connectivity' of mathematical knowledge. The research on which this presentation is based (Economic and Social Research Council, Grant Number R000223046) investigates similarities and differences of mathematics textbooks at lower secondary level in the three countries. The aim is to understand the range of ways in which commonly taught topics in secondary mathematics are addressed in textbooks, in order to widen our understanding of how mathematics is perceived in the different contexts, of the pedagogical ‘intentions’ of mathematics textbooks, and of how mathematics is 'prepared' and presented for understanding.

In order to analyse mathematics textbooks, popular selling textbooks from each country were statistically sampled and analysed on the basis of a schedule which draws on the range of ideas in the literature and which had been devised particularly for this project. (In addition, observations and interviews revealed teachers' views on the use of mathematics textbooks, but this strand of the study will not be the focus of the presentation.)

In this poster presentation the findings with respect to the mathematical area of 'Directed Numbers' are being reported, and with respect to 'connectedness' involving connections between topics in and out of mathematics, between mathematics and particular contexts, the progression of mathematical ideas, and connections between textbooks used in subsequent years.

What became apparent from the data collected was that textbooks in different countries provide different (mis)representations of mathematics for their students in schools. Whereas in some countries students are inundated with skills, procedures and disconnected mathematical knowledge, in others students are allowed to develop an appreciation of its interconnectedness and generalisable nature.

Looking at textbooks and different representations of knowledge in different countries helps to sharpen the focus of analysis by suggesting new perspectives. Further, the former approaches may encourage teachers, and in particular student teachers, to prepare and present mathematical knowledge for pupil understanding in a non-fragmented and interconnected way.

GIRLS' PARTICIPATION IN SOME REALISTIC MATHEMATICS: REFLECTIONS FROM STUDENT TEACHERS

Hilary Povey and Corinne Angier

Sheffield Hallam University

This poster will report on a project designed to engage some initial teacher education students in action research in the context of activity days organised for school girls from a disadvantaged area of Northern England.

The teacher education students are in their first year of a two year route into secondary mathematics teaching. They will draw on ideas from the Realistic Mathematics Education tradition from the Netherlands (see, for example, Treffers 1993): that 'mathematics must be connected to reality, stay close to children's experience and be relevant to society … offering the students problem situations which they can imagine' (van den Heuval-Panhuizen, 2000: 3f). The teacher education students will plan and develop a series of activity days on the theme of engineering mathematics, drawing on a wide range of published material (see, for example, Buxton, 1991; Gibbs, 1999; SMILE, nd). These days will contribute to a European funded project intended in general to widen the participation of disadvantaged groups in higher education and in particular to increase girls' participation in mathematically based subjects.

The students will collect data relating to the girls' experiences of the activity days. They will attempt to evaluate the girls' engagement in mathematics using an action research model. They will write individual reports of the project for their module assessment.

The poster will describe and display the work undertaken on the activity days. It will offer reflections from both the girls and from the teacher education students.

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SCALING IN ELEMENTARY SCHOOL: UNDERSTANDING AND LEARNING TROUGH A WEB-BASED 'SCALING WORKSHOP'

Toril Eskeland Rangnes

The Norwegian Teacher Academy

A classroom study I did with 11 year olds indicate that many of the students know where scaling is used in everyday life (for instance, in maps and when building small models). They lacked, however, a more formal knowledge.

In a project developing a web-based learning program www.matemania.no, a 'scaling workshop' was developed. We (the developers) wanted this to give the students the possibility to do experiments and give them some experience with scaling, using objects they know well - their bedroom and their own height.

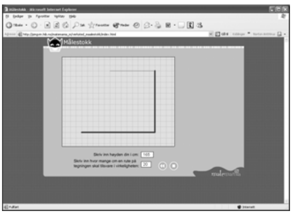

Fig 1

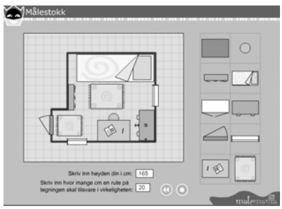

Fig 2

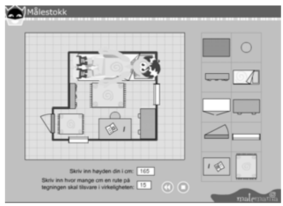

Fig 3

Fig 1: The student has to write her/his height, in this case 165 cm. He/she must decide how many cm one side in one square is in the real word, in this case 20cm. Now the student can draw the walls of his/her own bedroom.

Fig 2: The furniture will appear and the student can pull them into the room, place them, stretch them to the right size and rotate them if necessary.

Fig 3: When finished, they can click a button and a boy will glide into the bed in right scale. If the student has made a mistake with the scale of the furniture, the boy will be too big or too small for his bed. (The scale in fig 3 is not the same as in fig 1 and 2.)

Topics for further studies could be to find ways in which this program can help the students in their understanding of scaling, and study whether limitations of the program (e.g., the fact that one side in the square is not necessarily one cm on the screen) cause obstacles in learning scale. I will do a classroom-study on these issues during spring 2004 and hope to be able to present some results at PME28.

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FEEDING CALVES, A MATHEMATICAL ACTIVITY

Laia Saló Nevado
University of Helsinki, Finland

The poster was designed after the observation and field work in a farm of Lleida (Spain) of the feeding processes of calves. We decided to take a closer look to this daily routine. We where interested in the mathematical processes developed in the daily activities of the farmers of the farm.

There were 280 calves between 2 months and a year old and two people were in charge of all the animals and their daily feeding. Feeding activities include the reorganisation of the animals in smaller groups depending on the age and the quantity of food they needed to receive. The younger ones were fed with artificial milk. The older ones were given feed and straw.

The farmers deal with big quantities of food supplies such as 50000 kilos of feed, 500000 of fresh pressed corn ("Ensilat de blat de moro") or 200000 kilos of ryegrass, which should be distributed during the year for those 280 calves.

The farmyards were 3 fenced quadrilateral areas; and the feeding had a concrete schedule. It was very interesting to observe the flux of animals between the different spaces as a strategy for feeding since the spaces had an interesting geometrical shape. All three spaces had different size and they were not too big in consideration of the amount of calves they had.

The organisation of the farmers in order to efficiently feed everyday all the animals was very smooth, and gave space for further reflection over the abstract procedure of animal movements, since the spaces where limited to the inside area of the farmyards.

The main reflection over the research was directly connected to ethnomathematics. We consider that an ethnomathematical research is a good starting point to approach a social study of the rural mathematics, in concrete, farm mathematics.

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SUPPORTING VISUAL SPATIAL REPRESENTATIONS IN BUILDING EARLY NUMERACY

Cynthia Nicol, Carole Saundry, Heather Kelleher

University of British Columbia, Canada

Over the past decade there has been increased interest in the role of imagery in developing mathematical understanding. Studies (e.g. Cruz, Febles, & Díaz, 2000) explore the use of visual imagery in solving mathematical problems, both numeric and geometirc, and challenge learning models that do not adequately account for mathematical conceptions that are more spatially based. As recognition of this, various assessments of young children's early number development have incorporated the assessing and supporting of visual image-making (e.g. The Early Numeracy Research Project of Victoria, Australia; The British Columbia, Canada Early Numeracy Project).

Our work with the British Columbia Early Numeracy Project [ENP] provides us with a valuable context to analyze: 1) children's developing visual mental imagery and 2) how teachers support their students' numeracy development through visual imagery tasks. The ENP has involved sixteen teachers and their students from five school districts, including rural and urban schools, across the province to develop assessment tasks and instructional resources on four aspects of early numeracy: mathematical disposition, number skills, number concepts, and spatial thinking.

Analyzing data collected through teachers' one-to-one performance assessment interviews with approximately 200 students (twenty-one of these video-taped) we map out student responses and representations to two of these tasks. Through interviews with all 16 project teachers and other teachers now using the ENP assessment and resource materials we find that teachers are particularly surprised and intrigued at learning more about how their students are using visual images, making mental representations, and designing transformation to solve problems. Teachers report being able to recognize their students' mathematical thinking in a new way. However, although interested, only a few teachers have reported actively exploring how they might use what they've learned about their students' visual image making to support numeracy development.

Results of this study are significant in that they provide us with portraits of how young children use visual mental representation to solve problems and what teachers' think about their students' strategies together with the implications this has for practice. Digital video images of students' working on the tasks, visual mappings of student responses, and excerpts from teachers' analysis of their students and their participation in the project will be displayed.

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ABOUT THE DIMENSIONALITY OF THE SETS OF NUMBERS

Mihaela Singer, Institute for Educational Sciences, Romania

Cristian Voica, Department of Mathematics, University of Bucharest, Romania

This study draws on insights gleaned from recent educational research on children's intuitions and representations about the sets on numbers. We explore to what extent there is a correlation between the dimensions of some mathematical concepts and children's representations about these concepts. We discuss findings concerning students' conceptions, strategies and beliefs about two-dimensional sequences. A two-dimensional sequence supposes two recursive alternate rules.

The methodology we used consists in a series of specific questionnaires applied to students in grades 1 to 4, followed by interviews with selected students. The items contained two-dimensional sequences represented both in a mono-dimensional way (as a "linear" sequence) and in a two-dimensional manner (where the two rules are visualized on the two dimensions of the plane). Besides the content questions, the questionnaires also contained some meta-cognitive questions mostly asking students to express their preference regarding specific tasks.

We remarked that the students generally understand the set of natural numbers through sequences in which the recursion follows a unique rule, while the set of rationals is perceived through sequences in which the recursion follows a double rule. We identified various testimonies for students' linear perception of **N** and for the bi-dimensional perception of **Q**. For instance, many students have chosen a translation as a way to continue a geometrical sequence. When the same type of task was represented both mono-dimensionally and two-dimensionally, some students used the translation for the linear representation and, in the same time, the correct rule to continue the pattern, for the bi-dimensional representation. Moreover, when asked about their preferences, most of the students decided that the sequences with a plane distribution are preferred or are used as a base for reasoning in the "linear" cases.

From these considerations, a possible hypothesis is derived: the mental representation of children's algebraic concepts depends on the dimensionality of these concepts. In addition, children tend to use geometrical bi-dimensional drawings to underlie some algebraic ideas. This is why those concepts that have a natural multidimensional (2 or 3) structure (e.g.: the set of rational numbers, the functions) seem to be easier internalized at an intuitive level than the ones with a one-dimensional structure (e.g.: the set of natural numbers, the geometrical line). Therefore, the dimensionality of mathematical objects needs to be taken into account more seriously in curriculum development and teaching practice.

INVERSE RELATIONS BETWEEN DIVISION TERMS: A DIFFICULTY CHILDREN ARE ABLE TO OVERCOME

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A problem children experience with the concept of division is the difficulty in understanding the inverse relations between the number of parts into which a particular whole was divided and the size of these parts (Correa, Nunes & Bryant, 1998; Kornilaki & Nunes, 1997; Squire 2002). This relation is considered crucial for the comprehension of the invariant principles of division (Nunes & Bryant, 1996). An intervention study was carried out with 34 third grade Brazilian children who exhibited such difficulties. A control group and an experimental group divided the children equally. Each of them was subjected to a pre-test and a post-test. The experimental group received an intervention that sought to make explicit the inverse relations between the division terms. The intervention involved discussions on situations that (i) required the child to comprehend the effect of increasing /diminishing of the divisor over the dividend, and (ii) explored the inverse relations between the number of parts and the size of the parts by employing problems in which the dividend was kept constant. In the division problems presented, the dividend referred to either the number of parts or to the size of the parts. The pre-test and post-test were analyzed for the number of correct answers and justifications, which varied from inadequate justifications to comprehension of the inverse relations. On the pre-test, the two groups presented the same level of difficulty. Comparisons between pre- and post-test showed that the experimental group not only exhibited a larger number of correct responses, but were also able to offer justifications expressing comprehension of the inverse relations between the division terms. The control group exhibited no improvement when comparing the two testing occasions. The conclusion was that the intervention helps children to overcome their difficulties regarding the inverse relations between division terms. The nature of the intervention and its educational implications are discussed.

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THE INTERNET-BASED PROJECT MADIN (MATHEMATICS EDUCATION ON THE WEB)

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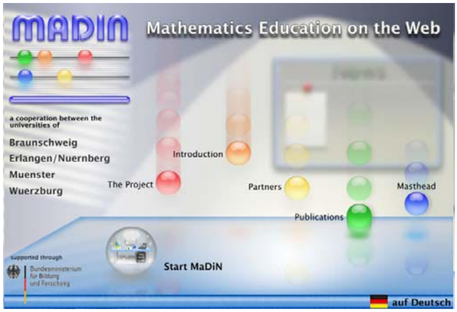
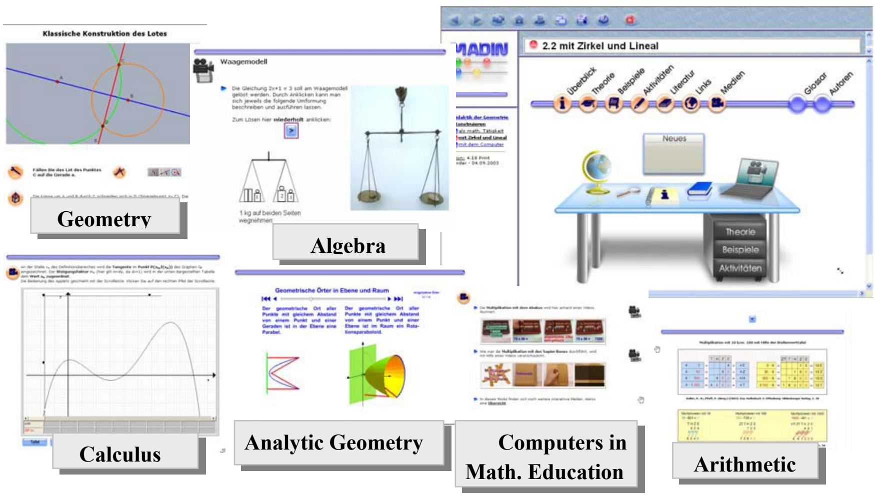


The goal of the MaDiN project is the development of an Internet-based teaching and learning system for university teaching of mathematics teacher students. It is a joint project of four German universities (Nuernberg, Braunschweig, Muenster and Wuerzburg) are conducting the project in partnership, which runs from 2001 to 2004. It is funded by the German government. The system supports the lecturer to prepare and give lessons for teacher students in mathematics education (didactics of arithmetic, geometry, algebra, calculus, stochastics, new technologies) and in elementary mathematics (arithmetic, geometry). It is used *in addition to* the lecture and *not instead* of the lecture: more specifically, the system consists of knowledge-based modules, which are designed for *the lecturer*, helping him/her to prepare and give the lecture and providing him with examples, problems, supplementary texts, pictures, videos, animations and constructions, and *for the student*, helping him/her to prepare and to repeat contents of the lecture, but also to study supplementary contents on his own. The starting point of a MaDiN-module is a "desktop". Here is an overview of the contents integrated into the system.



A STUDY OF DEVELOPING “SCHOOL-BASED” MATHEMATICS TEACHING MODULE ON “TIME”

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The purpose of this one-year case study was to develop a “school-based” mathematics teaching module. Based on the “OCTL” model (Yao, 2001), a practical teaching module of fourth grade was developed, named “Time”. The objectives of this “Time” module included constructing mathematical concepts related time, preparing abilities of problem solving, emphasizing the connection between mathematics and daily life of students. The mathematics knowledge contained in this “Time” teaching module was to understand the time-related concepts, the relations among days, hours, minutes, seconds and the conversions of these time units, to know how to measure time, to develop the sense of time and so on. The teaching module was made up of five instructional activities including “a day of Maruko’s school life”, “my holiday life”, “a tour of scenic and historic spots”, “weekend plan” and “ a round-the-island tour”, all of which are based on the topic of “Time”. Both qualitative and quantitative research techniques, such as field notes, interviews, observations, related document, and surveys, were applied to collect data for investigating responses of students during the period of conducting modules in the cooperative teacher’s mathematics class. There were three main findings reported in this study. One was the result of paper test emphasizes on students’ learning of Time-related concepts, another was students’ affective feelings about the activities designed in this Time module through questionnaires, the other was participating teacher’s opinion about this instructional module. These findings were useful for researcher to reflect and to modify the design of this Time module. As for the procedures of developing mathematics instructional module, according to the researcher and participating teacher’s one-year investigation, the better gateway is to adopt collaboration: confirming subject→gathering relevant information→taking use of the wisdom from a supporting group to outline the initial design→developing entire structure of the module→composing details of lesson plan→designing learning sheet and related assessment→pre-testing module→revising module→putting the module into practice→collecting and analyzing data→revising module once again→concluding the development of this instructional module. Finally, some suggestions were also presented in this article for developing, and using the “school-based” mathematics teaching modules.

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