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COMPUTATION OF CANTILEVER AIRPLANE WINGS.

By K. Thalau.

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COMPUTATION OF CANTILEVER AIRPLANE WINGS.*

By K. Thalau.

The so-called cantilever wings, often preferred in recent times, introduce, in conjunction with braced girder construction, high-grade statically indeterminate systems into the computation. We will first consider a cantilever wing with two spars and then a similar wing with three spars in an approximate computation.

The purpose of this treatise is, first of all, the determination of the effect of variously loaded spars on one another, since the neglect of this effect would present an economically very unfavorable computation method (See also the article by L. Ballenstedt, Technische Berichte, Vol. III, No. 4).

The system of spars and cross-bars alone (whether solid or built-up) does not matter at first, the original assumption being that the spars are rigidly braced by the cross-bars.

Two-Spar Wing

Our system can be regarded as a one-sided Vierendeel girder, on which the principal vertical components of the air forces act perpendicularly to the plane passing through the axes of the spars at the junction points (Fig. 1).

* From "Zeitschrift für Flugtechnik und Motorluftschiffahrt," May 26, 1924, pp. 103-109.

The magnitude and direction of the forces F and F' change according to the manner of loading, but are to be regarded as constant for a given load. Hence the load diagram of the air forces in this direction, contrary to their actual distribution over the width of the wing, is assumed to be rectangular. This assumption corresponds to the method of computation often employed. We are independent of variations in the dynamic pressure distribution, when we make the computation for the general case of a spar loaded with unit forces at its junction points. This produces certain π forces or moments, which we will designate by A'. An m-fold load $(m \times 1 = F)$ would accordingly produce an mA' force or moment. An n-fold loading of the rear spar $(n \times 1 = F')$ gives a corresponding symmetrical result of the form $A = mA' + nA'$.

We will now give the computation for a system with five fields of constant width λ and constant height 2h. We get our statically determinate main system in the form of two fixed girders by cutting through the cross-bars (ribs) and obtain, according to Vierendeel (Fig. 2), by combining the unknown lateral forces, the following new unknown quantities:

$$\left.\begin{aligned} X_1 &= \pi_1 \\ X_2 &= \pi_1 + \pi_2 \\ X_3 &= \pi_1 + \pi_2 + \pi_3 \\ X_4 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 \\ X_5 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 \end{aligned}\right\} \left\{\begin{aligned} \pi_1 &= X_1 \\ \pi_2 &= X_2 - X_1 \\ \pi_3 &= X_3 - X_2 \\ \pi_4 &= X_4 - X_3 \\ \pi_5 &= X_5 - X_4 \end{aligned}\right. \qquad (1)$$

In the case $\Sigma X = 0$, the front spar is acted on only by the external loads "1".

In the case $X_1 = -1$, all other X values (in the comparative group 1) equal 0 and we obtain $\pi_1 = -1$, $\pi_2 = +1$.

On following up the different cases of $X = -1$, we accordingly find that always two of the π forces are effective, which together give simple bending and torsion moment surfaces. An exception is formed by the case $X_5 = -1$, through which only $\pi_5 = -1$.

As follows from the subsequent figures, bending and torsional moments for the spars and bending moments for the cross-bars (ribs) are the most important in the elasticity equations.

Disregarding, therefore, the perpendicular and lateral forces, the general expression for the bending deflections is

$$\delta_{ik} = \int \frac{M_i\ M_k\ d_x}{E_H\ I_H} + \int \frac{\overline{M}_i\ \overline{M}_k\ d_x}{E_s\ I_s} + \int \frac{T_i\ T_k\ d_x}{G_H\ I_v{}'} \quad . \ . \ . \quad (2)$$

in which signify:

M_i, M_k, bending moments for the spars;

$\overline{M}_i, \overline{M}_k$, " " " " ribs;

T_i, T_k, torsion " " " spars;

E_H, elasticity modulus of a spar (same for all spars);

E_s, " " " " rib (" " " ribs);

I_H, inertia moment of a spar (" " " spars);

I_s, " " " " rib (" " " ribs);

G_H, gliding modulus of a spar (same for all spars);

$I_p{}'$, polar moment of inertia for a spar with reference to the deviation of its cross-section from the circular shape (same for all spars).

The relatively small angle of torsion of two cross-sections 1 cm apart (Hütte, Edition 22, Vol. I, p. 570) is, according to Grashof,

$$\theta = \zeta \frac{I_{pH}}{4\ I_{xH}\ I_{yH}} \frac{M_d}{G_H} = \frac{M_d}{I_{pH'}\ G_H}\ ;\quad \zeta \cong 1.2$$

If, starting from constructions at hand, we express I_{yH} by I_{xH}, then, with $I_{yH} = \frac{I_{xH}}{a}$ and $I_p = I_x + I_y$, we obtain

$$I_{pH} = I_{xH} + \frac{I_{xH}}{a} = I_{xH}\left(1 + \frac{1}{a}\right)$$

We then obtain

$$I_{pH'} = \frac{4\ I_{xH}\ I_{xH}}{a\left(1 + \frac{1}{a}\right) \zeta\ I_{xH}} = \frac{4}{1.2\ a\left(1 + \frac{1}{a}\right)} I_{xH};$$

$$\underline{I_{pH'}} = \frac{3.33}{a + 1} I_{xH} = \underline{b\ I_{xH}}$$

in which b is mostly a true fraction.

Moreover, if we put

$$I_s = c\ I_H;$$
$$G_H = d\ E_H;$$
$$E_s = e\ E_H;$$

so that, after substituting these values in equation (2), we find for S_{ik} only the elasticity modulus and the inertia moment of the spar

$$\delta_{ik} = \int \frac{M_i\ M_k\ d_x}{E_H\ I_H} + \int \frac{\overline{M}_i\ \overline{M}_k\ d_x}{e\ E_H\ c\ I_H} + \int \frac{T_i\ T_k\ d_x}{d\ E_H\ b\ I_H}$$

With $\frac{1}{e\ c} = c'$ and $\frac{1}{d\ b} = c''$ becomes

$$\underline{\delta_{ik} = \int \frac{M_i\ M_k\ d_x}{E_H\ I_H} + c' \int \frac{\overline{M}_i\ \overline{M}_k\ d_x}{E_H\ I_H} + c'' \int \frac{T_i\ T_k\ d_x}{E_H\ I_H}} \qquad (3)$$

The elasticity equations of our system now read

$$\left.\begin{aligned}
\delta_{01} &= X_1\ \delta_{11} + X_2\ \delta_{12} + X_3\ \delta_{13} + X_4\ \delta_{14} + X_5\ \delta_{15} \\
\delta_{02} &= X_1\ \delta_{21} + X_2\ \delta_{22} + X_3\ \delta_{23} + X_4\ \delta_{24} + X_5\ \delta_{25} \\
&\vdots \\
\delta_{05} &= X_1\ \delta_{51} + X_2\ \delta_{52} + X_3\ \delta_{53} + X_4\ \delta_{54} + X_5\ \delta_{55}
\end{aligned}\right\} \qquad (4)$$

By multiplying both sides of the elasticity equations by $E_H \times I_H$, these denominators are entirely eliminated.

We will now take up the representation of the moment areas and their evaluation for the different cases.

<u>Case $\Sigma X = 0$.</u>- Only the external forces "1" act on the front spar (Fig. 3).

$$\left.\begin{aligned}
\eta_5 &= 1\ \lambda & P_5 &= \tfrac{1}{2}\lambda^2 \\
\eta_4 &= 1\ (2\lambda + \lambda) = 3\lambda & P_4 &= \tfrac{4}{2}\lambda^2 \\
\eta_3 &= 1\ (3\lambda + 2\lambda + \lambda) = 6\lambda & P_3 &= \tfrac{9}{2}\lambda^2 \\
\eta_2 &= 1\ (4\lambda + 3\lambda + 2\lambda + \lambda) = 10\lambda & P_2 &= \tfrac{16}{2}\lambda^2 \\
\eta_1 &= 1\ (5\lambda + 4\lambda + 3\lambda + 2\lambda + \lambda) = 15\lambda & P_1 &= \tfrac{25}{2}\lambda^2
\end{aligned}\right\} \ldots \qquad (5)$$

<u>Case $X_1 = -1$.</u>- We have $\pi_1 = -1$; $\pi_2 = +1$ (Fig. 4). Moments which generate pressure in the upper fibers of the girder are positive. The bending moment is the same for all ribs and equals $1 \times h$. At the same time, there is a constant torsional

moment for the spars, whereby it should be noted that the latter, with the exception of the case $X_5 = -1$, only acts within one field: with the case $X_1 = -1$, only for spar sections in field 2; with case $X_2 = -1$, only in field 3, etc.

The torsional moment areas, on account of their simplicity, were not plotted and are easy to follow.

Case $X_2 = -1$: $\pi_2 = -1$; $\pi_3 = +1$ (Fig. 5).

Cases $X_3 = -1$ and $X_4 = -1$ give analogous moment areas.

Case $X_5 = -1$.- This gives $\pi_5 = -1$ (Fig. 6). Here the influence of the torsional moment extends over the whole spar length with the magnitude $1 \times h$.

Regarding the evaluation of the simple moment areas, see Müller-Breslau, Vol. II, Chapter I, as also the handy formulas of Demel.*

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* Richard Schadek von Degenburg and Karl Demel "Hilfsmittel," etc., Berlin, 1915, p.7 - Published by Wilhelm Ernst and Sohn.

If, e.g., we wish to find the value of

$$\delta_{12} = \int \frac{M_1\ M_2\ d_x}{E_H\ I_H} + c' \int \frac{\overline{M}_1\ \overline{M}_2\ d_x}{E_H\ I_H} + c'' \int \frac{T_1\ T_2\ d_x}{E_H\ I_H}$$

the moment areas for the cases $X_1 = -1$ and $X_2 = -1$ are to be combined (Figs. 4-5). The integrals are to be extended only over the portions of the bars which are simultaneously subjected to moments. Hence

$$\int M_1\ M_2\ d_x = \underbrace{\int_0^\lambda (+\lambda)\ (+\lambda) d_x}_{\text{Field 1, behind spar.}} + \underbrace{\int_0^\lambda (-\lambda)\ (-\lambda)\ d_x}_{\text{Field 1, front of spar.}} + \underbrace{\int_0^\lambda \lambda\ x\ d_x}_{\text{Field 2, behind spar.}} + \underbrace{\int_0^\lambda (-\lambda)\ (-x)\ d_x}_{\text{Field 2, front of spar.}}$$

$$\int M_1\ M_2\ d_x = 2\lambda^3 + 2\frac{\lambda^3}{2} = \underline{\underline{3\lambda^3}}$$

$$c' \int \overline{M}_1\ \overline{M}_2\ d_x = \underbrace{2\ c \int_0^h (+x)\ (-x)\ d_x}_{\text{Cross-bar 2}} = \underline{\underline{-\frac{2}{3}\ c'\ h^3}}$$

(Continued bottom Page 7)

Demel's formulas give:

For the load magnitudes -

$$\left.\begin{aligned}
E\,I\,\delta_{01} &= \int M_0\,M_1\,d\,x = -\frac{101}{6}\lambda^3 &&= Z_1\\
E\,I\,\delta_{02} &= -\,23\,\lambda^3 &&= Z_2\\
E\,I\,\delta_{03} &= -\frac{157}{6}\lambda^3 &&= Z_3\\
E\,I\,\delta_{04} &= -\frac{82}{3}\lambda^3 &&= Z_4\\
E\,I\,\delta_{05} &= +\,100\,\lambda^3 &&= Z_5
\end{aligned}\right\} \quad \ldots . (6)$$

For the form magnitudes -

$$E\,I\,\delta_{11} = \frac{8}{3}\lambda^3 + \frac{4}{3}c'\,h^3 + 2\,c''\,h^2\,\lambda$$

$$E\,I\,\delta_{12} = 3\,\lambda^3 - \frac{2}{3}c'\,h^3$$

$$\left.\begin{aligned} E\,I\,\delta_{13}\\ E\,I\,\delta_{14}\end{aligned}\right\} = 3\,\lambda^3$$

$$E\,I\,\delta_{15} = -\frac{38}{3}\lambda^3 - 2\,c''\,h^2\,\lambda$$

(Continuation of footnote from Page 6)

$c'' \int T_1\,T_2\,d_x$ here equals zero. Hence

$$\underline{\underline{E_H\,I_H\,\delta_{12} = 3\,\lambda^3 - \frac{2}{3}c'\,h^3}}$$

The work of integration is rendered unnecessary by employing the above formulas. The moment areas are, due to the arrangement of the unknowns, almost all rectangular or triangular surfaces, the torsional moment areas being also rectangular surfaces. Trapezoidal surfaces are divided into triangles by means of diagonals. The signs must be given attention.

$$E\ I\ \delta_{21} = E\ I\ \delta_{12}$$

$$E\ I\ \delta_{22} = \frac{14}{3}\ \lambda^3 + \frac{4}{3}\ c'\ h^3 + 3\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{23} = 5\ \lambda^3 - 2\ c'\ \frac{h^3}{3}$$

$$E\ I\ \delta_{24} = 5\ \lambda^3$$

$$E\ I\ \delta_{25} = -\frac{56}{3}\ \lambda^3 - 2\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{31} = E\ I\ \delta_{13};\ E\ I\ \delta_{32} = E\ I\ \delta_{23}$$

$$E\ I\ \delta_{33} = \frac{20}{3}\ \lambda^3 + \frac{4}{3}\ c'\ h^3 + 2\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{34} = 7\ \lambda^3 - \frac{2}{3}\ c'\ h^3$$

$$E\ I\ \delta_{35} = -\frac{68}{3}\ \lambda^3 - 2\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{41} = E\ I\ \delta_{14};\ E\ I\ \delta_{42} = E\ I\ \delta_{24};\ E\ I\ \delta_{43} = E\ I\ \delta_{34}$$

$$E\ I\ \delta_{44} = \frac{26}{3}\ \lambda^3 + \frac{4}{3}\ c'\ h^3 + 2\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{45} = -\frac{74}{3}\ \lambda^3 - \frac{2}{3}\ c'\ h^3 - 2\ c''\ h^2\ \lambda$$

$$E\ I\ \delta_{51} = E\ I\ \delta_{15};\ E\ I\ \delta_{52} = E\ I\ \delta_{25};\ E\ I\ \delta_{53} = E\ I\ \delta_{35};$$

$$E\ I\ \delta_{54} = E\ I\ \delta_{45}$$

$$E\ I\ \delta_{55} = \frac{250}{3}\ \lambda^3 + \frac{2}{3}\ c'\ h^3 + 10\ c''\ h^2\ \lambda$$

We designate the load magnitudes by Z_1 to Z_5 and write the elasticity equations in the form of a matrix:

	X_1	X_2	X_3	X_4	X_5
Z_1	$\underline{\delta_{11}}$	δ_{12}	δ_{13}	δ_{14}	δ_{15}
Z_2	δ_{21}	$\underline{\delta_{22}}$	δ_{23}	δ_{24}	δ_{25}
Z_3	δ_{31}	δ_{32}	$\underline{\delta_{33}}$	δ_{34}	δ_{35}
Z_4	δ_{41}	δ_{42}	δ_{43}	$\underline{\delta_{44}}$	δ_{45}
Z_5	δ_{51}	δ_{52}	δ_{53}	δ_{54}	$\underline{\delta_{55}}$

(7)

If we wish to follow the effect of the loads on the different unknowns, we thus obtain, for any given unknown, the form

$$X_i = \beta_{1i}\ Z_1 + \beta_{2i}\ Z_2 + \beta_{3i}\ Z_3 + \beta_{4i}\ Z_4 + \beta_{5i}\ Z_5$$

For this purpose the corresponding matrix reads:

	Z_1	Z_2	Z_3	Z_4	Z_5
X_1	$\underline{\beta_{11}}$	β_{12}	β_{13}	β_{14}	β_{15}
X_2	β_{21}	$\underline{\beta_{22}}$	β_{23}	β_{24}	β_{25}
X_3	β_{31}	β_{32}	$\underline{\beta_{33}}$	β_{34}	β_{35}
X_4	β_{41}	β_{42}	β_{43}	$\underline{\beta_{44}}$	β_{45}
X_5	β_{51}	β_{52}	β_{53}	β_{54}	$\underline{\beta_{55}}$

(8)

Since $\delta_{ik} = \delta_{ki}$, then β_{ik} must equal β_{ki}, i.e., the β matrix, like the δ matrix, is symmetrical to the main diagonals (underlined). The β values may be found simply by the methods of Müller-Breslau, Vol. II, Chapter I, p. 178.

As a summary, we will give the solution diagram for this special case of five elasticity equations with five unknowns in in each equation. From the general case of the nine-membered elasticity equations, we obtain the following:

r	a_r	b_r	c_r	d_r	e_r	d_r'	c_r'	b_r'	a_r'	$\frac{1}{e_{Ir}}$	μ_r	ν_r	φ_r	ψ_r
1					δ_{11}	δ_{12}	δ_{13}	δ_{14}	δ_{15}	$\frac{1}{e_{I1}}$	μ_1	ν_1	φ_1	ψ_1
2				δ_{21}	δ_{22}	δ_{23}	δ_{24}	δ_{25}		$\frac{1}{e_{I2}}$	μ_2	ν_2	φ_2	
3			δ_{31}	δ_{32}	δ_{33}	δ_{34}	δ_{35}			$\frac{1}{e_{I3}}$	μ_3	ν_3		
4		δ_{41}	δ_{42}	δ_{43}	δ_{44}	δ_{45}				$\frac{1}{e_{I4}}$	μ_4			
5	δ_{51}	δ_{52}	δ_{53}	δ_{54}	δ_{55}					$\frac{1}{e_{I5}}$				

Taken in order, the coefficients are:

$$e_{I1} = \delta_{11}$$

$$\mu_1 = -\frac{\delta_{12}}{\delta_{11}}; \quad \nu_1 = -\frac{\delta_{13}}{\delta_{11}}; \quad \varphi_1 = -\frac{\delta_{14}}{\delta_{11}}; \quad \psi_1 = -\frac{\delta_{15}}{\delta_{11}}.$$

$e_{I2} = \delta_{22} + d_{I2}\,\mu_1$, in which

$$d_{I2} = \delta_{21}$$

$$\mu_2 = -\frac{d_{I2}\,\nu_1 + \delta_{23}}{e_{I2}};$$

$$\nu_2 = -\frac{d_{I2}\,\varphi_1 + \delta_{24}}{e_{I2}};$$

$$\varphi_2 = -\frac{d_{I2}\,\psi_1 + \delta_{25}}{e_{I2}};$$

$e_{I3} = \delta_{33} + c_{I3}\,\nu_1 + d_{I3}\,\mu_2$, in which

$$c_{I3} = \delta_{31}$$

$$d_{I3} = \delta_{32} + \delta_{31}\,\mu_1$$

$$\mu_3 = -\frac{c_{I3}\ \varphi_1 + d_{I3}\ \nu_2 + \delta_{34}}{e_{I3}}$$

$$\nu_3 = -\frac{c_{I3}\ \psi_1 + d_{I3}\ \varphi_2 + \delta_{35}}{e_{I3}}$$

$e_{I4} = \delta_{44} + b_{I4}\ \varphi_1 + c_{I4}\ \nu_2 + d_{I4}\ \mu_3,$ in which

$$b_{I4} = \delta_{41}$$

$$c_{I4} = \delta_{42} + b_{I4}\ \mu_1$$

$$d_{I4} = \delta_{43} + b_{I4}\ \nu_1 + c_{I4}\ \mu_2$$

$$\mu_4 = -\frac{b_{I4}\ \psi_1 + c_{I4}\ \varphi_2 + d_{I4}\ \nu_3 + \delta_{45}}{e_{I4}}$$

$e_{I5} = \delta_{55} + \delta_{51}\ \psi_1 + b_{I5}\ \varphi_2 + c_{I5}\ \nu_3 + d_{I5}\ \mu_4,$ in which

$$b_{I5} = \delta_{52} + \delta_{51}\ \mu_1$$

$$c_{I5} = \delta_{53} + \delta_{51}\ \nu_1 + b_{I5}\ \mu_2$$

$$d_{I5} = \delta_{54} + \delta_{51}\ \varphi_1 + b_{I5}\ \nu_2 + c_{I5}\ \mu_3$$

Beginning with the last β values, we proceed to make out the β tables.

$$\begin{aligned}
\beta_{15} &= \mu_1\ \beta_{25} + \nu_1\ \beta_{35} + \varphi_1\ \beta_{45} + \psi_1\ \beta_{55} \\
\beta_{25} &= \mu_2\ \beta_{35} + \nu_2\ \beta_{45} + \varphi_2\ \beta_{55} \\
\beta_{35} &= \mu_3\ \beta_{45} + \nu_3\ \beta_{55} \\
\beta_{45} &= \mu_4\ \beta_{55} \\
\beta_{55} &= \frac{1}{e_{I5}}
\end{aligned}$$

$$\begin{aligned}
\beta_{14} &= \mu_1 \beta_{24} + \nu_1 \beta_{34} + \varphi_1 \beta_{44} + \psi_1 \beta_{54} \\
\beta_{24} &= \mu_2 \beta_{34} + \nu_2 \beta_{44} + \varphi_2 \beta_{54} \\
\beta_{34} &= \mu_3 \beta_{44} + \nu_3 \beta_{54} \\
\beta_{44} &= \mu_4 \beta_{54} + \frac{1}{e_{14}} \\
\beta_{54} &= \beta_{45}
\end{aligned}$$

$$\begin{aligned}
\beta_{13} &= \mu_1 \beta_{23} + \nu_1 \beta_{33} + \varphi_1 \beta_{43} + \psi_1 \beta_{53} \\
\beta_{23} &= \mu_2 \beta_{33} + \nu_2 \beta_{43} + \varphi_2 \beta_{53} \\
\beta_{33} &= \mu_3 \beta_{43} + \nu_3 \beta_{53} + \frac{1}{e_{13}} \\
\beta_{43} &= \beta_{34} \\
\beta_{53} &= \beta_{35}
\end{aligned}$$

$$\begin{aligned}
\beta_{12} &= \mu_1 \beta_{22} + \nu_1 \beta_{32} + \varphi_1 \beta_{42} + \psi_1 \beta_{52} \\
\beta_{22} &= \mu_2 \beta_{32} + \nu_2 \beta_{42} + \varphi_2 \beta_{52} + \frac{1}{e_{12}} \\
\beta_{32} &= \beta_{23} \\
\beta_{42} &= \beta_{24} \\
\beta_{52} &= \beta_{25}
\end{aligned}$$

$$\begin{aligned}
\beta_{11} &= \mu_1 \beta_{21} + \nu_1 \beta_{31} + \varphi_1 \beta_{41} + \psi_1 \beta_{51} + \frac{1}{e_{11}} \\
\beta_{21} &= \beta_{12} \\
\beta_{31} &= \beta_{13} \\
\beta_{41} &= \beta_{14} \\
\beta_{51} &= \beta_{15}
\end{aligned}$$

The arrows indicate the order of the solutions. The unknowns can now be computed from 8, whereby no further difficulties interfere with following, in the manner indicated at the beginning, the effect of the variously loaded spars on each other.

Three-Spar Wing

From what has been said, the investigation of the three-spar system is now simple (Fig. 7). As before

$$\left.\begin{aligned} X_1 &= \pi_1 \\ X_2 &= \pi_1 + \pi_2 \\ X_3 &= \pi_1 + \pi_2 + \pi_3 \\ X_4 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 \\ X_5 &= \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 \end{aligned}\right\} \left\{\begin{aligned} \pi_1 &= X_1 \\ \pi_2 &= X_2 - X_1 \\ \pi_3 &= X_3 - X_2 \\ \pi_4 &= X_4 - X_3 \\ \pi_5 &= X_5 - X_4 \end{aligned}\right.$$

$$\left.\begin{aligned} X_6 &= \pi_6 \\ X_7 &= \pi_6 + \pi_7 \\ X_8 &= \pi_6 + \pi_7 + \pi_8 \\ X_9 &= \pi_6 + \pi_7 + \pi_8 + \pi_9 \\ X_{10} &= \pi_6 + \pi_7 + \pi_8 + \pi_9 + \pi_{10} \end{aligned}\right\} \left\{\begin{aligned} \pi_6 &= X_6 \\ \pi_7 &= X_7 - X_6 \\ \pi_8 &= X_8 - X_7 \\ \pi_9 &= X_9 - X_8 \\ \pi_{10} &= X_{10} - X_9 \end{aligned}\right.$$

The π forces are at first assumed to be produced by loading the front spar alone with the junction-point loads "1". The effect of the forces $F = m \times 1$ then equal mA'. The forces $F' = n \times 1$ in the rear-spar junction points again produce a symmetrical or bisymmetrical result.

If we now cause the loads "1" to act in like manner on the middle spar, then, because of the symmetry of the system, the unknowns, thereby produced, are $\pi_1 = \pi_6$, $\pi_2 = \pi_7$, etc. This will be manifested by the production of 10 correspondingly symmetrical δ values.

Since $X_1 = X_3$, $X_2 = X_7$, etc., only one of the two equation groups is necessary for making this part of the computation, whereby this part is immediately carried back to the case of the five equations with the five unknown quantities. If this last loading case, with the junction-point loads $r \times 1 = F''$, has an effect $r \times A'''$ on our system, then the total effect, when the forces F, F' and F'' act simultaneously and respectively on the front, middle and rear spars, will be

$$\underline{A = m\,A' + n\,A'' + r\,A'''}$$

We now have at our disposal 10 elasticity equations with 10 unknowns, as follows:

$$\delta_{01} = X_1\,\delta_{11} + X_2\,\delta_{12} + X_3\,\delta_{13} + X_4\,\delta_{14} + X_5\,\delta_{15} +$$
$$+ X_6\,\delta_{16} + X_7\,\delta_{17} + X_8\,\delta_{18} + X_9\,\delta_{19} + X_{10}\,\delta_{110}$$
$$\delta_{02} = X_1\,\delta_{21} + X_2\,\delta_{22} + X_3\,\delta_{23} + X_4\,\delta_{24} + X_5\,\delta_{25} +$$
$$+ X_6\,\delta_{26} + X_7\,\delta_{27} + X_8\,\delta_{28} + X_9\,\delta_{29} + X_{10}\,\delta_{210}$$
$$\vdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\delta_{010} = X_1\,\delta_{101} + X_2\,\delta_{102} + X_3\,\delta_{103} + X_4\,\delta_{104} + X_5\,\delta_{105} +$$
$$+ X_6\,\delta_{106} + X_7\,\delta_{107} + X_8\,\delta_{108} + X_9\,\delta_{109} + X_{10}\,\delta_{1010}.$$

The general structure of the form magnitudes δ_{ik} is the same as already given on a previous page. As Fig. 8 shows, the load magnitudes δ_{01} to δ_{05} equal zero, while δ_{06} to δ_{010} fully agree with the values of δ_{01} to δ_{05} already given.

The combination of the M_0 surfaces with the $M_1 - M_5$ surfaces make, as already mentioned, δ_{01} to δ_{05} equal to zero (Figs. 9-11).

We then have

$$E\ I\ \delta_{06} = -\frac{101}{6}\lambda^3\ ;$$

$$E\ I\ \delta_{07} = -\ 23\ \lambda^3\ ;$$

$$E\ I\ \delta_{08} = -\frac{157}{6}\lambda^3\ ;$$

$$E\ I\ \delta_{09} = -\frac{82}{3}\lambda^3\ ;$$

$$E\ I\ \delta_{0\,10} = +\ 100\ \lambda^3\ ;$$

The form magnitudes read:

$$E\ I\ \delta_{11} \div E\ I\ \delta_{15}$$

$$E\ I\ \delta_{16} = -\frac{4}{3}\lambda^3 + c''\ h^2\ \lambda$$

$$E\ I\ \delta_{17} = -\frac{3}{2}\lambda^3$$

$$E\ I\ \delta_{18} = -\frac{3}{2}\lambda^3$$

$$E\ I\ \delta_{19} = -\frac{3}{2}\lambda^3$$

$$E\ I\ \delta_{1\,10} = +\frac{19}{3}\lambda^3 - c''\ h^2$$

$$E\ I\ \delta_{21} \div E\ I\ \delta_{25}$$

$$E\ I\ \delta_{26} = -\frac{3}{2}\lambda^3\ (=\ \delta_{17})$$

$$E\ I\ \delta_{27} = -\frac{7}{3}\lambda^3 + c''\ h^2\ \lambda$$

$$E\ I\ \delta_{28} = -\frac{5}{2}\lambda^3$$

$$E\ I\ \delta_{29} = -\frac{5}{2}\lambda^3$$

$$E\ I\ \delta_{2\,10} = +\frac{28}{3}\lambda^3 - c''\ h^2\ \lambda$$

$$E\ I\ \delta_{31} \div E\ I\ \delta_{35}$$

$$E\ I\ \delta_{36} = -\frac{3}{2}\lambda^3\ (= \delta_{18})$$

$$E\ I\ \delta_{37} = -\frac{5}{2}\lambda^3\ (= \delta_{28})$$

$$E\ I\ \delta_{38} = -\frac{10}{3}\lambda^3 + c''\ h^2\ \lambda$$

$$E\ I\ \delta_{39} = -\frac{7}{2}\lambda^3$$

$$E\ I\ \delta_{310} = +\frac{34}{3}\lambda^3 - c''\ h^2\ \lambda$$

$$E\ I\ \delta_{41} \div E\ I\ \delta_{45}$$

$$E\ I\ \delta_{46} = -\frac{3}{2}\lambda^3\ (= \delta_{19})$$

$$E\ I\ \delta_{47} = -\frac{5}{2}\lambda^3\ (= \delta_{29})$$

$$E\ I\ \delta_{48} = -\frac{7}{2}\lambda^3\ (= \delta_{39})$$

$$E\ I\ \delta_{49} = -\frac{13}{3}\lambda^3 + c''\ h^2\ \lambda$$

$$E\ I\ \delta_{410} = +\frac{37}{3}\lambda^3 - c''\ h^2\ \lambda$$

$$E\ I\ \delta_{51} \div E\ I\ \delta_{55}$$

$$E\ I\ \delta_{56} = +\frac{19}{3}\lambda^3 - c''\ h^2\ \lambda\ (= \delta_{110})$$

$$E\ I\ \delta_{57} = +\frac{28}{3}\lambda^3 - c''\ h^2\ \lambda\ (= \delta_{210})$$

$$E\ I\ \delta_{58} = +\frac{34}{3}\lambda^3 - c''\ h^2\ \lambda\ (= \delta_{310})$$

$$E\ I\ \delta_{59} = +\frac{37}{3}\lambda^3 - c''\ h^2\ \lambda\ (= \delta_{410})$$

$$E\ I\ \delta_{510} = -\frac{125}{3}\lambda^3 + 5\ c''\ h^2\ \lambda$$

$$\begin{aligned} EI\,\delta_{61} &= EI\,\delta_{16} \\ \delta_{62} &= \delta_{26} \\ \delta_{63} &= \delta_{36} \\ \delta_{64} &= \delta_{46} \\ \delta_{65} &= \delta_{56} \\ \delta_{66} &= \delta_{11} \\ \delta_{67} &= \delta_{12} \\ \delta_{68} &= \delta_{13} \\ \delta_{69} &= \delta_{14} \\ \delta_{6\,10} &= \delta_{15} \end{aligned}$$

$$\begin{aligned} EI\,\delta_{71} &= EI\,\delta_{17} \\ \delta_{72} &= \delta_{27} \\ \delta_{73} &= \delta_{37} \\ \delta_{74} &= \delta_{47} \\ \delta_{75} &= \delta_{57} \\ \delta_{76} &= \delta_{67} \\ \delta_{77} &= \delta_{22} \\ \delta_{78} &= \delta_{23} \\ \delta_{79} &= \delta_{24} \\ \delta_{7\,10} &= \delta_{25} \end{aligned}$$

$$\begin{aligned} EI\,\delta_{81} &= EI\,\delta_{18} \\ \delta_{82} &= \delta_{28} \\ \delta_{83} &= \delta_{38} \\ \delta_{84} &= \delta_{48} \\ \delta_{85} &= \delta_{58} \\ \delta_{86} &= \delta_{68} \\ \delta_{87} &= \delta_{78} \\ \delta_{88} &= \delta_{33} \\ \delta_{89} &= \delta_{34} \\ \delta_{8\,10} &= \delta_{35} \end{aligned}$$

$$\begin{aligned} EI\,\delta_{91} &= EI\,\delta_{19} \\ \delta_{92} &= \delta_{29} \\ \delta_{93} &= \delta_{39} \\ \delta_{94} &= \delta_{49} \\ \delta_{95} &= \delta_{59} \\ \delta_{96} &= \delta_{69} \\ \delta_{97} &= \delta_{79} \\ \delta_{98} &= \delta_{89} \\ \delta_{99} &= \delta_{44} \\ \delta_{9\,10} &= \delta_{45} \end{aligned}$$

$$\begin{aligned} EI\,\delta_{10\,1} &= EI\,\delta_{1\,10} \\ \delta_{10\,2} &= \delta_{2\,10} \\ \delta_{10\,3} &= \delta_{3\,10} \\ \delta_{10\,4} &= \delta_{4\,10} \end{aligned}$$

$$
\begin{aligned}
E\,I\,\delta_{10\,5} &= E\,I\,\delta_{5\,10} \\
\delta_{10\,6} &= \delta_{6\,10} \\
\delta_{10\,7} &= \delta_{7\,10} \\
\delta_{10\,8} &= \delta_{8\,10} \\
\delta_{10\,9} &= \delta_{9\,10} \\
\delta_{10\,10} &= \delta_{5\,5}
\end{aligned}
$$

If we again use a matrix for summarizing the ten elasticity equations, we then obtain, on designating the load magnitudes by Z_1 to Z_{10},

	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
1) 0	$\underline{\delta_{1\,1}}$	$\delta_{1\,2}$	$\delta_{1\,3}$	$\delta_{1\,4}$	$\delta_{1\,5}$	$\delta_{1\,6}$	$\delta_{1\,7}$	$\delta_{1\,8}$	$\delta_{1\,9}$	$\delta_{1\,10}$
2) 0		$\underline{\delta_{2\,2}}$	$\delta_{2\,3}$	$\delta_{2\,4}$	$\delta_{2\,5}$	$\delta_{2\,6}$	$\delta_{2\,7}$	$\delta_{2\,8}$	$\delta_{2\,9}$	$\delta_{2\,10}$
3) 0			$\underline{\delta_{3\,3}}$	$\delta_{3\,4}$	$\delta_{3\,5}$	$\delta_{3\,6}$	$\delta_{3\,7}$	$\delta_{3\,8}$	$\delta_{3\,9}$	$\delta_{3\,10}$
4) 0				$\underline{\delta_{4\,4}}$	$\delta_{4\,5}$	$\delta_{4\,6}$	$\delta_{4\,7}$	$\delta_{4\,8}$	$\delta_{4\,9}$	$\delta_{4\,10}$
5) 0					$\underline{\delta_{5\,5}}$	$\delta_{5\,6}$	$\delta_{5\,7}$	$\delta_{5\,8}$	$\delta_{5\,9}$	$\delta_{5\,10}$
6) Z_6						$\underline{\delta_{6\,6}}$	$\delta_{6\,7}$	$\delta_{6\,8}$	$\delta_{6\,9}$	$\delta_{6\,10}$
7) Z_7							$\underline{\delta_{7\,7}}$	$\delta_{7\,8}$	$\delta_{7\,9}$	$\delta_{7\,10}$
8) Z_8								$\underline{\delta_{8\,8}}$	$\delta_{8\,9}$	$\delta_{8\,10}$
9) Z_9									$\underline{\delta_{9\,9}}$	$\delta_{9\,10}$
10) Z_{10}										$\underline{\delta_{10\,10}}$

Herein it is now worth noting that (as the computation of the δ_{ik} values demonstrated) there is a symmetry in so far as

$$
\delta_{1\,1} = \delta_{6\,6}; \quad \delta_{2\,2} = \delta_{7\,7} \text{ etc.}
$$

$$
\delta_{1\,2} = \delta_{6\,7}; \quad \delta_{1\,3} = \delta_{8\,8} \text{ etc.}
$$

Hereby it is also possible for us to reduce the ten-membered elasticity equations to two groups of five equations with 5 unknowns. We first add equations 1 and 6, 2 and 7, 3 and 8, 4 and 9, 5 and 10, and then subtract them from one another. The resulting members are then designated briefly as follows:

$X_1 + X_6 = S_1$	$X_1 - X_6 = D_1$
$X_2 + X_7 = S_2$	$X_2 - X_7 = D_2$
. . . . . .	. . . . . .
$X_5 + X_{10} = S_5$	$X_5 - X_{10} = D_5$
$\delta_{11} + \delta_{16} = \delta_{11}^+$	$\delta_{11} - \delta_{16} = \delta_{11}^-$
$\delta_{12} + \delta_{17} = \delta_{12}^+$	$\delta_{12} - \delta_{17} = \delta_{12}^-$
. . . . . .	. . . . . .
$\delta_{15} + \delta_{110} = \delta_{15}^+$	$\delta_{15} - \delta_{110} = \delta_{15}^-$

and both groups of the elasticity equations are obtained in the form

	S_1	S_2	S_3	S_4	S_5
Z_6	δ_{11}^+	δ_{12}^+	δ_{13}^+	δ_{14}^+	δ_{15}^+
Z_7		δ_{22}^+	δ_{23}^+	δ_{24}^+	δ_{25}^+
Z_8			δ_{33}^+	δ_{34}^+	δ_{35}^+
Z_9				δ_{44}^+	δ_{45}^+
Z_{10}					δ_{55}^+

	D_1	D_2	D_3	D_4	D_5
$-Z_6$	δ_{11}^-	δ_{12}^-	δ_{13}^-	δ_{14}^-	δ_{15}^-
$-Z_7$		δ_{22}^-	δ_{23}^-	δ_{24}^-	δ_{25}^-
$-Z_8$			δ_{33}^-	δ_{34}^-	δ_{35}^-
$-Z_9$				δ_{44}^-	δ_{45}^-
$-Z_{10}$					δ_{55}

The solution of these equations has already been discussed. After obtaining the values S or D, we quickly find the unknowns X and from them, in turn, the π forces at the intersection points of the ribs. The latter computations make the least work.

We will now consider briefly the loading case of the middle spar. Here, as already stated, there are ten symmetrical load magnitudes (Fig. 14). We will also choose a symmetrical direction of the π forces (Fig. 15) corresponding to the bending of the system.

Hereby, in the cases X_6 to $X_{10} = -1$, the signs of the M_6 to M_{10} surfaces alternate, so that the combinations of the M_0 surfaces with the M_6 to M_{10} surfaces all have the same sign, negative in this case. The first five equations are all correspondingly to be provided also with a Z member.

In conclusion, we note that the computation difficulties lie almost exclusively in the solution of the many-membered equations. From this side also, attention is called to the fact that only work with the calculating machine offers, in many places, promise of good results.

The usual utilization of determinants is inexpedient in the present instance, where the number of equations and unknown quantities exceeds four (Compare Müller-Breslau, Vol. II, Chapter I).

After the decisive forces have been determined, they naturally give the bending moments along with the torsional moments, which now, however, extend their effect over the whole length of the spars. For strong torsional effects, such as may be produced at the base of long spars, the stresses produced in the longitudinal direction of the spars must be considered (See Goetzke, Z.d.V.d.I., 1909, p.935).

This treatise is intended, first of all, to combine, in the example discussed, the most essential theoretical considerations with the practical viewpoints so decisive in airplane construction. It is, of course, possible to employ more refined methods of calculation, such as, e.g., the introduction of the torsional moments and the transverse forces in the girder plane at the intersection points of the ribs, as further independent unknowns, but the work of computation would then be entirely disproportionate to the practical advantages gained.

A numerical evaluation of the above may be given in a future article.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

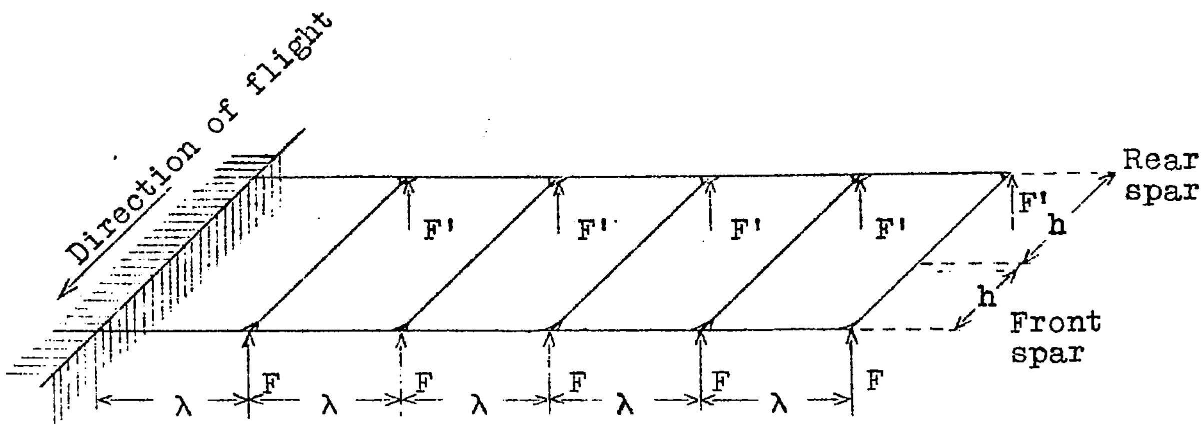

Direction of flight
Rear spar
F'
F'
F'
F'
F'
h
h
Front spar
F
F
F
F
F
λ
λ
λ
λ
λ

Fig. 1.

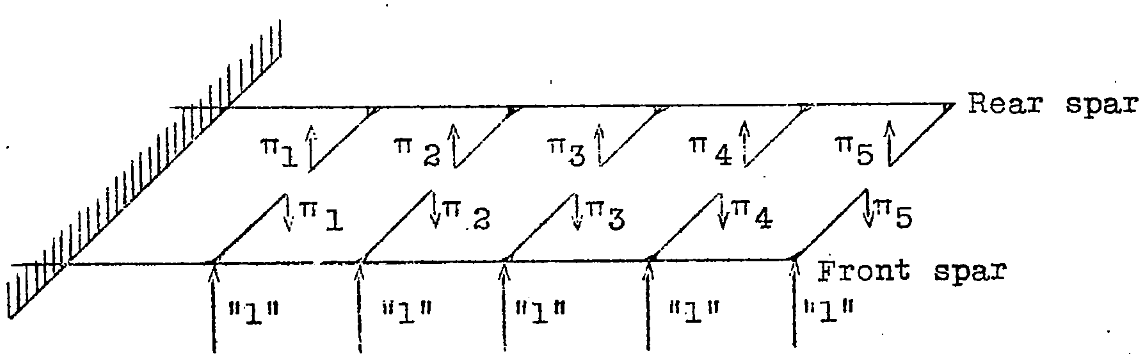

Rear spar
π1
π2
π3
π4
π5
π1
π2
π3
π4
π5
Front spar
"1"
"1"
"1"
"1"
"1"

Fig. 2.

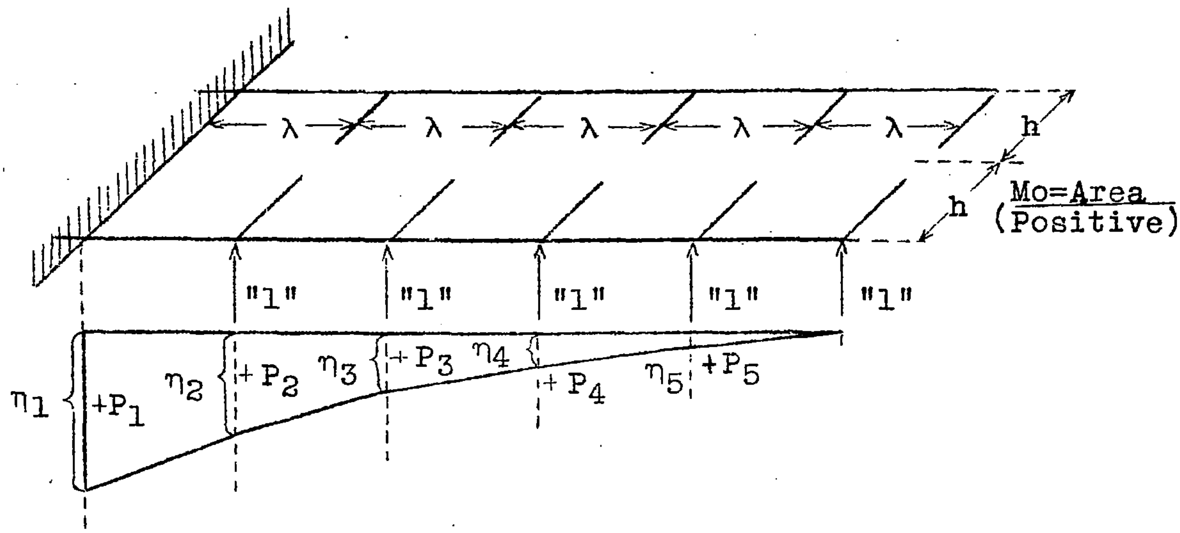

λ
λ
λ
λ
λ
h
h
Mo=Area
(Positive)
"1"
"1"
"1"
"1"
"1"
η1
+P1
η2
+P2
η3
+P3
η4
+P4
η5
+P5

Fig.3.

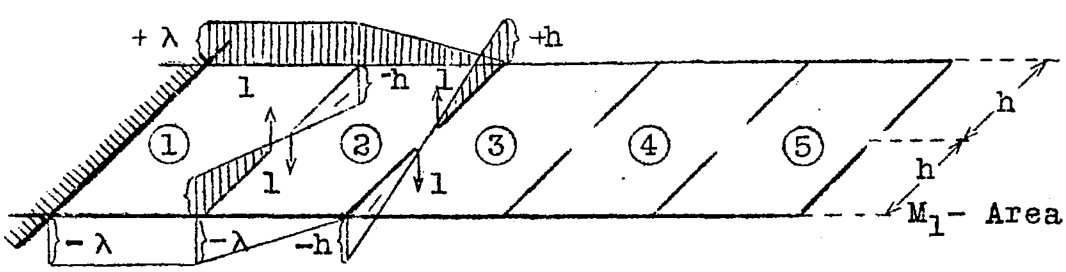
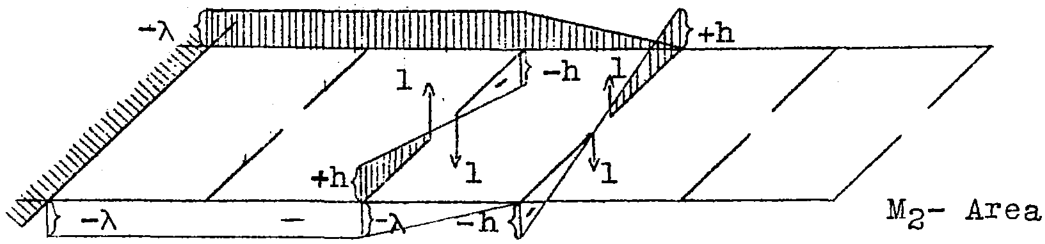
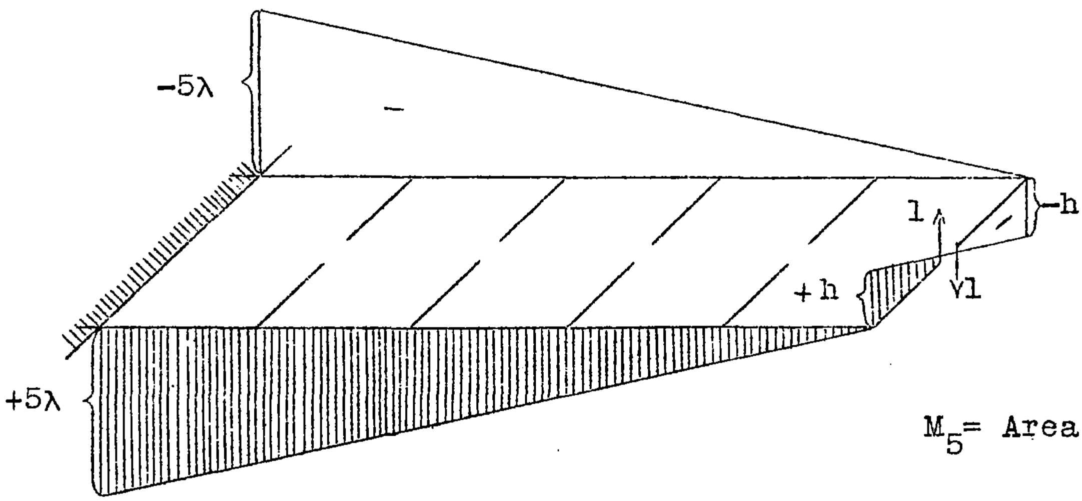

+ λ
+h
-h
1
1
1
1
1
2
3
4
5
h
h
M1- Area
- λ
-λ
-h

Fig.4.



$-\lambda$
$+h$
$-h$
l
l
$+h$
l
l
$-\lambda$
$-$
$-\lambda$
$-h$
M_2- Area

Fig. 5.



-5λ
$-$
$-h$
l
$+h$
l
$+5\lambda$
M_5= Area

Fig. 6.

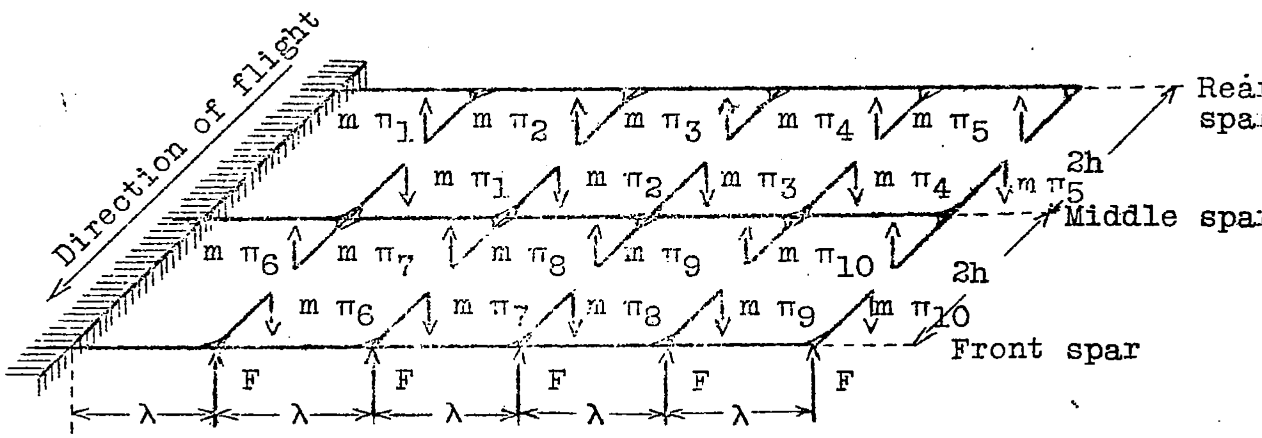

Direction of flight
Rear spar
Middle spar
Front spar
2h
2h
F
λ

Fig. 7.

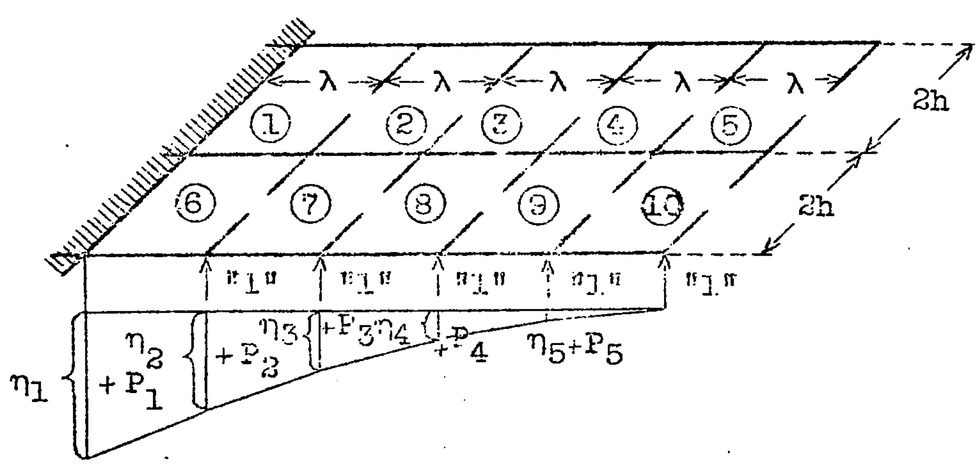

λ
2h
2h
"1"

Fig. 8.

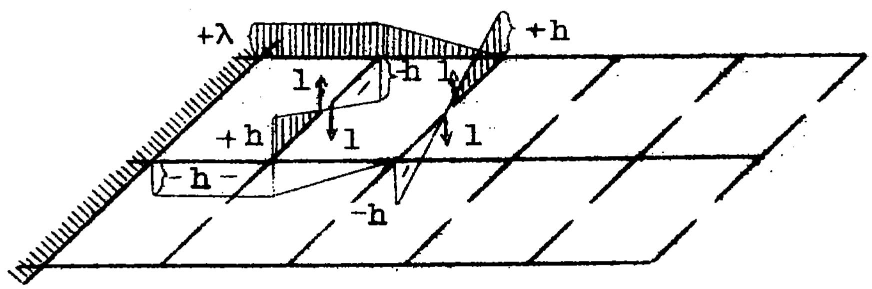


Fig.9.

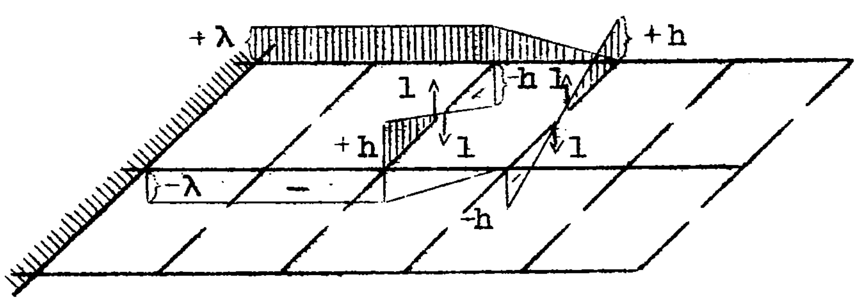


Fig.10.

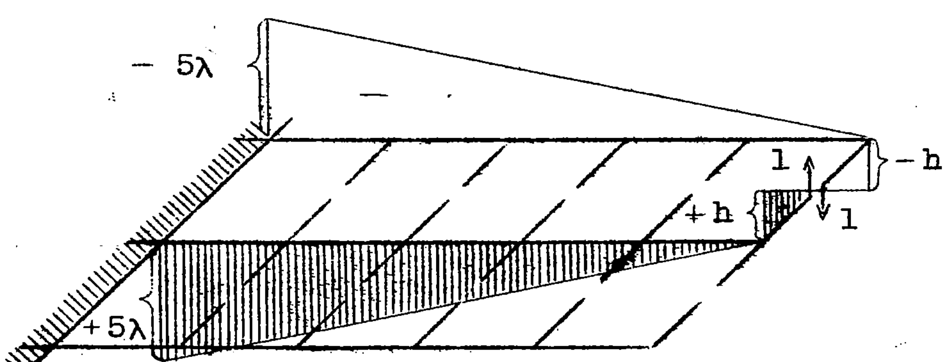


Fig.11.

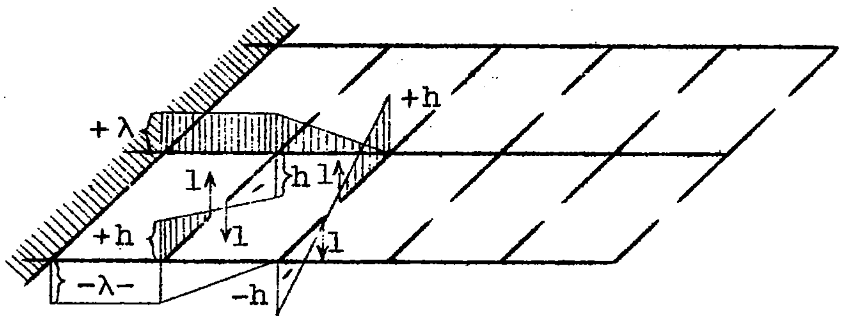


Fig.12.

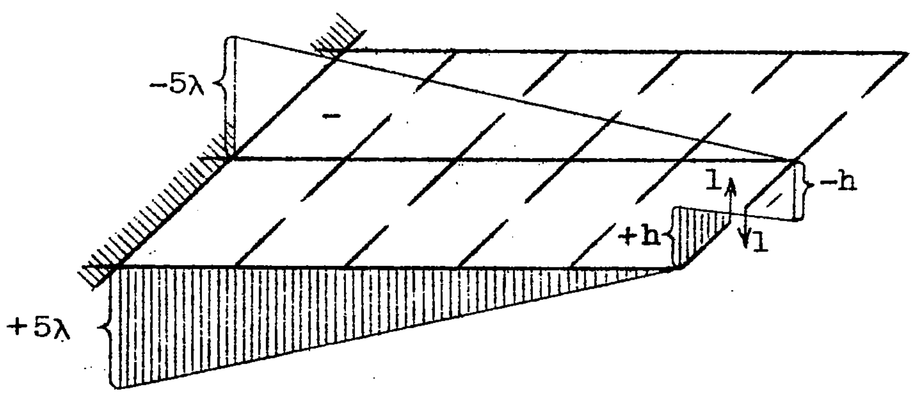


Fig.13.

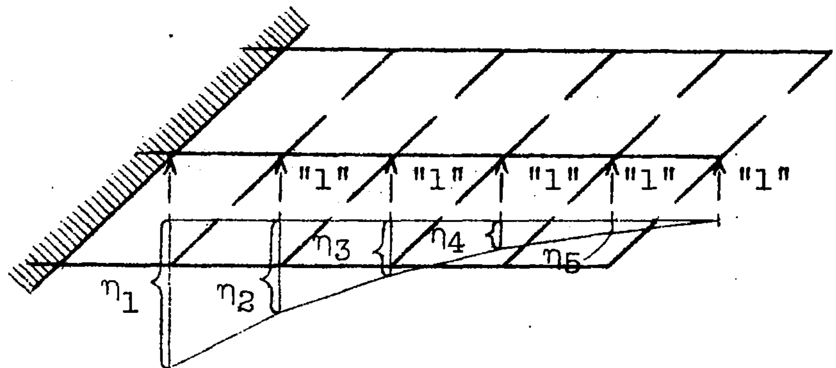

"1"
"1"
"1"
"1"
"1"
η_1
η_2
η_3
η_4
η_5

Fig.14.

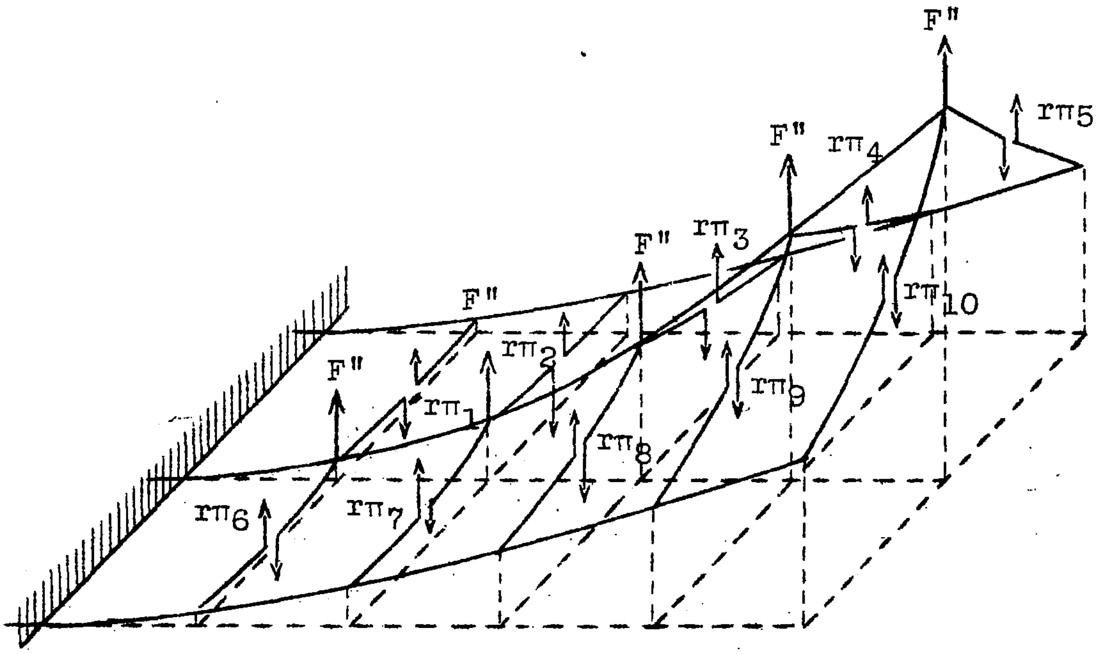

F"
F"
F"
F"
F"
$r\pi_1$
$r\pi_2$
$r\pi_3$
$r\pi_4$
$r\pi_5$
$r\pi_6$
$r\pi_7$
$r\pi_8$
$r\pi_9$
$r\pi_{10}$

Fig.15.