FLOWS WITH IMPOSED PERIODIC FORCING

1 *Mathematical Background*

The flow field is induced by an oscillatory jet of air emanating from a solid boundary. This oscillatory motion introduces a wave pattern that can be used as a timer for selective sampling of the flow field fluctuations. Thus, the weak organized wave motion can be extracted from a background field of finite turbulent fluctuations. The mathematical framework outlined here follows closely that originally presented by Hussain and Reynolds (1970, 1972). Some previous work in this area that may also be of interest are reported in Gatski and Liu (1980) and Obi, Ishibashi and Masuda (1997).

If such traveling waves exist within the fluctuating flow field, any fluctuating quantity $f(\mathbf{x}, t)$ can be decomposed into

$$f\left(\mathbf{x}, t\right) = \bar{f}\left(\mathbf{x}\right) + \tilde{f}\left(\mathbf{x}, t\right) + f'\left(\mathbf{x}, t\right), \tag{1}$$

where $\bar{f}\left(\mathbf{x}\right)$ is the (time) mean value, $\tilde{f}\left(\mathbf{x}, t\right)$ is the statistical contribution of the organized motion, and $f\prime\left(\mathbf{x}, t\right)$ the turbulence. Since the flow is not stationary, the ergodic hypothesis does not hold and the ensemble mean does not equal the long time average. Thus, a time average is defined as

$$\bar{f}\left(\mathbf{x}\right) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f\left(\mathbf{x}, t\right) dt \tag{2}$$

and a phase average is defined as

$$\langle f\left(\mathbf{x}, t\right)\rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} f\left(\mathbf{x}, t + n\tau\right) \tag{3}$$

where τ is the period of the imposed oscillatory motion. The phase average is the average at any point in space of the values of f that are realized at a particular phase ϕ in the cycle of the oscillating jet. The wave component is then given by

$$\tilde{f}\left(\mathbf{x}, t\right) = \langle f\left(\mathbf{x}, t\right)\rangle - \bar{f}\left(\mathbf{x}\right) \ , \tag{4}$$

so that

$$f\left(\mathbf{x}, \mathbf{t}\right) = \langle f\left(\mathbf{x}, \mathbf{t}\right)\rangle + f'\left(\mathbf{x}, \mathbf{t}\right), \tag{5}$$

These results show that at any point in the flow a time varying function $f(\mathbf{x}, t)$ can be partitioned into the three components parts defined in Eqs. (2), (3), and (4) given a reference oscillating signal at a given frequency (period). The following properties are associated with the triple-decomposition and prove useful in extracting information about both the mean field(s) and the statistical correlations of the fluctuation field. These relations include

$$\begin{gathered} \overline{\bar{f}g} = \bar{f}\bar{g}, \quad \left\langle \tilde{f} g \right\rangle = \tilde{f}\left\langle g\right\rangle, \quad \left\langle \bar{f} g\right\rangle = \bar{f}\left\langle g\right\rangle, \quad \overline{\langle f\rangle} = \langle \bar{f}\rangle = \bar{f}, \\ \bar{\tilde{f}} = 0, \quad \overline{f'} = 0, \quad \langle f'\rangle = 0, \quad \overline{\tilde{f}g'} = \overline{\left\langle \tilde{f}g' \right\rangle} = 0 \ . \end{gathered} \tag{6}$$

These relations show that the fluctuation f' is centered about both the time average and phase averaged means and that the background turbulence is uncorrelated with the organized motion.

2 *Phase-Averaged Navier-Stokes Equations*

In general, both pressure-velocity and density-velocity numerical solvers are used in RANS CFD codes. In this section, the phase-averaged formulations for both the incompressible and compressible forms of the Navier-Stokes equations will be presented. For contributors using time-accurate RANS-type formulations, the dependent variables calculated are the phase-averaged variables defined above and shown below to satisfy transport equations that are formaly equivalent to the usual RANS equations.

2.1 *Incompressible Phase-Averaged Equations*

With the decomposition and mathematical framework outlined in Section 1, it is possible to develop a set of phase-averaged Navier-Stokes equations that describe the behavior of the phase-averaged mean quantities as well as the corresponding equations for the phase-averaged turbulent quantities (Gatski and Liu, 1980).

In Section 2.1.1, the phase-averaged mean equations are presented and in Section 2.1.2 the phase-averaged turbulent quantities are presented.

2.1.1 *Phase-averaged mean equations*

The mean flow equations are given by

$$\frac{\partial \langle u_i \rangle}{\partial x_i} = 0 \tag{7a}$$

$$\frac{D \langle u_i \rangle}{Dt} = \frac{\partial \langle u_i \rangle}{\partial t} + \langle u_j \rangle \frac{\partial \langle u_i \rangle}{\partial x_j} = -\frac{1}{\bar{\rho}} \frac{\partial \langle p \rangle}{\partial x_i} + \nu \frac{\partial^2 \langle u_i \rangle}{\partial x_j^2} - \frac{\partial \left\langle u_i' u_j' \right\rangle}{\partial x_j}. \tag{7b}$$

From these results, it is possible to obtain the time-independent mean values, for example

$$\bar{u}_i = \overline{\langle u_i \rangle}, \qquad \overline{u_i' u_j'} = \overline{\left\langle u_i' u_j' \right\rangle} \tag{8}$$

2.1.2 *Phase-averaged turbulent equations*

As an example, the phase-averaged formulation for a two-equation $\langle K \rangle$-$\langle \varepsilon \rangle$ model will be given. Other models under consideration can be formulated in terms of phase-averaged variables in a similar fashion.

The transport equations for the phase-averaged turbulent kinetic energy $\langle K \rangle$ ($= \langle u_i' u_i' \rangle /2$) and dissipation rate $\langle \varepsilon \rangle$ are given by

$$\frac{D \langle K \rangle}{Dt} = \langle u_i' u_k' \rangle \frac{\partial \langle u_i \rangle}{\partial x_k} - \langle \varepsilon \rangle + \frac{\partial}{\partial x_k} \left[\left(\nu + \frac{\nu_t}{\sigma_K} \right) \frac{\partial \langle K \rangle}{\partial x_k} \right] \tag{9}$$

$$\frac{D\langle\varepsilon\rangle}{Dt} = -C_{\varepsilon 1}\frac{\langle\varepsilon\rangle}{\langle K\rangle}\langle u_i' u_k'\rangle\frac{\partial\langle u_i\rangle}{\partial x_k} - C_{\varepsilon 2}\frac{\langle\varepsilon\rangle^2}{\langle K\rangle} + \frac{\partial}{\partial x_k}\left[\left(\nu + \frac{\nu_t}{\sigma_\varepsilon}\right)\frac{\partial\langle\varepsilon\rangle}{\partial x_k}\right] \tag{10}$$

where σ_K, σ_ε, $C_{\varepsilon 1}$, and $C_{\varepsilon 2}$ are closure coefficients, and an eddy viscosity relationship is assumed for the individual phase-averaged stress components so that

$$\langle u_i' u_j'\rangle = \frac{2}{3}\langle K\rangle\delta_{ij} - \nu_t\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\ , \tag{11a}$$

with

$$\nu_t = C_\nu\frac{\langle K\rangle^2}{\langle\varepsilon\rangle} \tag{11b}$$

and C_ν a closure coefficient.

2.2 *Compressible Phase-Averaged Equations*

In a compressible formulation, where Favre averaged variables are used, that is $\overline{\rho u_i}/\bar{\rho}$, the definition of the phase average has to be altered. In this case the compressible phase-averaged variables are defined as

$$\langle u_i\rangle = \frac{\langle\rho u_i\rangle}{\langle\rho\rangle}\ , \tag{12}$$

and the flow variables are partitioned as in Eq. (5). In Section 2.2.1, the phase-averaged mean equations are presented and in Section 2.2.2 the phase-averaged turbulent quantities are presented.

2.2.1 ***Phase-averaged mean equations***

In terms of phase-averaged Favre variables, the mass conservation equation can be written as

$$\frac{D\langle\rho\rangle}{Dt} = \frac{\partial\langle\rho\rangle}{\partial t} + \langle u_j\rangle\frac{\partial\langle\rho\rangle}{\partial x_j} = -\langle\rho\rangle\frac{\partial\langle u_j\rangle}{\partial x_j}\ , \tag{13a}$$

and the conservation of momentum as

$$\langle\rho\rangle\frac{D\langle u_i\rangle}{Dt} = -\frac{\partial\langle p\rangle}{\partial x_i} + \frac{\partial\langle\sigma_{ij}\rangle}{\partial x_j} - \frac{\partial\langle\rho\rangle\langle u_i' u_j'\rangle}{\partial x_j}. \tag{13b}$$

where

$$\langle\sigma_{ij}\rangle = 2\overline{\mu}\left[\frac{1}{2}\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right) - \frac{1}{3}\frac{\partial\langle u_k\rangle}{\partial x_k}\delta_{ij}\right]\ . \tag{13c}$$

A phase-averaged conservation of total energy equation can be written as

$$\frac{\partial(\langle\rho\rangle\langle E\rangle)}{\partial t} + \frac{\partial(\langle u_j\rangle\langle\rho\rangle\langle H\rangle)}{\partial x_j} = \frac{\partial\langle u_i\rho E\rangle}{\partial x_j} - \frac{\partial\langle q_j\rangle}{\partial x_j}, \tag{14}$$

where the total energy and total enthalpy, $\langle\rho\rangle\langle E\rangle$ and $\langle\rho\rangle\langle H\rangle$, respectively, are

$$\langle E\rangle = c_v\langle T\rangle + \frac{\langle u_i\rangle\langle u_i\rangle}{2} + \frac{\langle u_i'u_i'\rangle}{2}, \tag{15a}$$

$$\langle H\rangle = \langle E\rangle + \frac{\langle p\rangle}{\langle\rho\rangle}, \tag{15b}$$

and

$$\langle q_j\rangle = -\left\langle k_T\frac{\partial T}{\partial x_j}\right\rangle \simeq -\overline{k}_T\frac{\partial\langle T\rangle}{\partial x_j}, \tag{16a}$$

$$\langle u_j\rho E\rangle = c_p\langle\rho\rangle\langle u_j'T'\rangle + \langle u_i\rangle\left(\langle\rho\rangle\langle u_i'u_j'\rangle - \langle\sigma_{ij}\rangle\right) + \frac{\langle\rho\rangle\langle u_i'u_i'u_j'\rangle}{2} - \langle u_i'\sigma_{ij}'\rangle\ . \tag{16b}$$

From these results, it is possible to obtain the time-independent mean values, for example

$$\bar\rho = \overline{\langle\rho\rangle},\quad \bar u_i = \overline{\langle u_i\rangle},\quad \bar E = \overline{\langle E\rangle},\quad \bar q_i = \overline{\langle q_i\rangle},\quad \overline{u_i'u_j'} = \overline{\langle u_i'u_j'\rangle} \tag{17}$$

2.2.2 ***Phase-averaged turbulent equations***

Once again, a phase-averaged formulation for a two-equation $\langle K\rangle$-$\langle\varepsilon\rangle$ model is used. Other models can be formulated in terms of phase-averaged Favre variables in a similar fashion.

The transport equations for the phase-averaged turbulent kinetic energy $\langle K\rangle$ ($=\langle u_i'u_i'\rangle/2$) and dissipation rate $\langle\varepsilon\rangle$ are given by

$$\langle\rho\rangle\frac{D\langle K\rangle}{Dt} = \langle\rho\rangle\langle u_i'u_k'\rangle\frac{\partial\langle u_i\rangle}{\partial x_k} - \langle\rho\rangle\langle\varepsilon\rangle + \frac{\partial}{\partial x_k}\left[\left(\mu+\frac{\mu_t}{\sigma_K}\right)\frac{\partial\langle K\rangle}{\partial x_k}\right] \tag{18}$$

$$\langle\rho\rangle\frac{D\langle\varepsilon\rangle}{Dt} = -C_{\varepsilon 1}\langle\rho\rangle\frac{\langle\varepsilon\rangle}{\langle K\rangle}\langle u_i'u_k'\rangle\frac{\partial\langle u_i\rangle}{\partial x_k} - C_{\varepsilon 2}\langle\rho\rangle\frac{\langle\varepsilon\rangle^2}{\langle K\rangle} + \frac{\partial}{\partial x_k}\left[\left(\mu+\frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial\langle\varepsilon\rangle}{\partial x_k}\right] \tag{19}$$

where σ_K, σ_ε, $C_{\varepsilon 1}$, and $C_{\varepsilon 2}$ are closure coefficients, and an eddy viscosity relationship is assumed for the individual phase-averaged stress components so that

$$\langle\rho\rangle\langle u_i'u_j'\rangle = \frac{2}{3}\langle\rho\rangle\langle K\rangle\delta_{ij} - \mu_t\left(\frac{\partial\langle u_i\rangle}{\partial x_j} + \frac{\partial\langle u_j\rangle}{\partial x_i}\right)\ , \tag{20a}$$

with

$$\mu_t = C_\mu\langle\rho\rangle\frac{\langle K\rangle^2}{\langle\varepsilon\rangle} \tag{20b}$$

and C_μ a closure coefficient.

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