IDS160B - Relativistic Astrophysics and Cosmology
Problem Set 6 1995 April

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1 Time Scales

In cosmology as in all of physics it is important to be able to estimate or calculate the time scales on which things happen. here are a few excercises to help strengthen that ability.

1.1 Schedule of Structure Formation

Events or things that happened during the evolution of the universe in terms of structure formation are listed in Table 1 with an estimate of the likely redshift. Fill in the estimated time assuming the Universe is 1.3×10^{10} years old at the present. (The age of a flat Universe is $2/(3\ H_0) = 6.67\ h^{-1}$ so that this corresponds to H_0 of 50 km/sec/Mpc for a flat universe.)

Table 1
Schedule of Structure Formation

Event	Redshift z	Time h^{-1} (yrs)
Gravitational potential perturbations	$\gtrsim 10^3$	
Spheriods of Galaxies	~ 20	
1st engines for active galactic nuclei	$\gtrsim 10$	
The intergalactic medium	~ 10	
Most distant known Quasar	5.89	
Dark matter	$\gtrsim 5$	
Dark halos of galaxies	~ 5	
Angular momentum of rotation of galaxies	~ 5	
1st 10% of heavier elements	$\gtrsim 3$	
Cosmic magnetic fields	$\gtrsim 3$	
Rich clusters of galaxies	~ 2	
Thin disks of spiral galaxies	~ 1	
Superclusters, walls, & voids	~ 1	

Hint: Make the approximation that for these times the Universe can be treated as matter dominated.

1.2 Times of Important Epochs

The Universe is radiation dominated for early times ($t < t_{eq}$), where $t_{eq} \cong 4.36 \times 10^{10}(\Omega_0 h^2)^{-1}$ sec is the epoch where the energy density in matter is equal to the energy density in highly relativitic particles. This would happend at a temperature of $kT_{eq} \cong 5.502(\Omega_0 h^2)$ eV.

As one goes back in time to higher temperatures/energies new particle species become important and add to the degrees of freedom. The relationship between temperature and time becomes

$$t = \frac{2.4207 \times 10^{-6}}{\sqrt{g_*}} (\frac{1 \text{ GeV}}{kT})^2 \text{sec} \tag{1}$$

where g_* is the relative number of degrees of freedom given by

$$g_* = \sum_{i=bosons} (\frac{T_i}{T})^3 g_i + \frac{7}{8} \sum_{i=fermions} (\frac{T_i}{T})^3 g_i \tag{2}$$

Figure 1 shows a plot of the relative number of degrees of freedom for the standard model of particle physics. It is roughly 3 in the current epoch rising sharply to about 11 at about 1 MeV (in the middle of Big Bang Nucleosynthesis) and eventually up to about 106.75 at the highly speculative GUT scale.

Fill in the following table:

Table 2
Critical Epochs in Early Universe

Event	$k\times$ Temperature (GeV, MeV, or eV)	Time (sec)
Planck Time		$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} \cong 5.3904 \times 10^{-43}$ sec
GUT Phase transition	$10^{13} - 10^{16}$ GeV	
ElectroWeak Symmetry Breaking	~ 100 GeV	
Baryosynthesis	~ 200 MeV	
neutrino (ν) decoupling	1 MeV	
$e^+ - e^-$ annihilation	0.5 MeV	
Big Bang Nucleosynthesis (BBNS)	0.07 MeV	
Matter-Radiation Equality	$5.502(\Omega_0 h^2)$ eV	$4.36 \times 10^{10}(\Omega_0 h^2)^{-1}$
Matter-Radiation Decoupling	0.26 eV	$5.6384 \times 10^{12}(\Omega_0 h^2)^{-1/2}$

Use $h = 0.5$ and $\Omega_0 = 1$.

2 Thermodynamics - Boson/Fermion & g_*

The density of states of a particle of species i is

$$\frac{dn_i(p)}{dp}dp = \frac{4\pi g_i}{h^3}\frac{p^2 dp}{exp[\frac{E-\mu_i}{kT}] \pm 1} = \frac{g_i}{2\pi^2\hbar^3}\frac{p^2\, dp}{e^{(E-\mu_i)/kT} \pm 1} \tag{3}$$

where $E^2 = p^2c^2 + (m_ic^2)^2$ and the plus or minus sign depends upon whether the particle is a fermion or boson respectively.

One can find the total number density, n_i , energy density, $\epsilon_i = \rho_i c^2$, pressure P_i, and entropy density, s_i, due the the species by simply integrating up over the occupation number.

$$\begin{aligned}
n_i = \int f(p) d^3p &= \int \frac{4\pi g_i}{h^3}\frac{p^2\, dp}{e^{(E-\mu_i)/kT} \pm 1} \\
\epsilon_i = \int E f(p) d^3p &= \int \frac{4\pi g_i}{h^3}\frac{p^2\, E(p)\, dp}{e^{(E-\mu_i)/kT} \pm 1} \\
P_i = \int \frac{1}{3}\frac{p^2}{E} f(p) d^3p &= \int \frac{4\pi g_i}{3h^3}\frac{p^4/E(p)\, dp}{e^{(E-\mu_i)/kT} \pm 1} \\
s_i &= \frac{(P_i + \epsilon_i)}{T}.
\end{aligned}$$

By a miracle of math (or show that)

$$\int_0^\infty \frac{x^m\, dx}{e^x + 1} = (1 - \frac{1}{2^m})\int_0^\infty \frac{x^m\, dx}{e^x - 1} \tag{4}$$

and from integral tables

$$\int_0^\infty \frac{x^2\, dx}{e^x - 1} = \Gamma(m+1)\zeta(m+1) \tag{5}$$

where Γ is the Gamma (factorial) function and ζ is the Riemann zeta function

$$\Gamma(3)\zeta(3) \cong 2 \times 1.202$$
$$\Gamma(4)\zeta(4) = 6 \times \frac{\pi^4}{90}.$$

(a) Now find the expressions for n_i and energy density $\epsilon_i = \rho_i c^2$ as well as P_i and s_i for massless (highly relativistic approximation) particles with chemical potential $\mu_i = 0$. (Here we neglect the chemical potential setting $\mu_i = 0$, but not below.)
(b) Next we do a calculation relevant to baryosynthesis. Show that for relativistic ($kT_i >> m_ic^2$), degenerate ($\mu_i >> kT_i$) fermions

$$\rho_i \cong \frac{1}{8\pi^2\hbar^3}g_i\mu^4, \quad n_i = \frac{1}{6\pi^2\hbar^3}g_i\mu_i^3, \quad P = \frac{1}{24\pi^2\hbar^3}g_i\mu_i^4 \tag{6}$$

find an expression for s_i.
(c) For relativisitic particles (bosons or fermions) with $\mu_i < 0$ and $|\mu_i| < kT$ show that

$$\rho_i = \frac{3g_i}{\pi^2\hbar^3}(kT_i)^4 e^{\mu_i/kT_i}, \quad n_i = \frac{g_i}{\pi^2\hbar^3}(kT_i)^3 e^{\mu_i/kT_i}, \quad P = \frac{g_i}{\pi^2\hbar^3}(kT_i)^4 e^{\mu_i/kT_i}, \tag{7}$$

and find an expression for the entropy s_i.

3 Relic Degenerate Neutrino

A massless, degenerate ($|\mu_\nu| >> kT$) will contribute an energy density

$$\rho_\nu \cong \frac{\mu_\nu^4}{8\pi^2\hbar^3} \tag{8}$$

as shown in the previous problem. Use the constraint $\Omega h^2 \lesssim 1$ to show that

$$\frac{\mu_\nu}{kT_\nu} \lesssim 53, \quad \text{and} \quad \frac{n_\nu - n_{\overline{\nu}}}{n_\gamma} \lesssim 3.7 \times 10^3. \tag{9}$$

4 Charge of the Universe

Can a $k = +1$ (closed) Friedmann universe contain net electric charge?

Consider an open Friedmann model with net charge density n_Q, assumed to be distributed homogeneously (e.g. small difference in electron and proton charge). Show that the constraint $\Omega h^2 \lesssim 1$ implies $n_Q/s \lesssim 10^{-27}$. What limit does that give on the difference in charge between an electron and proton, if they exist in equal numbers?

5 Nuclear Density

Compute the redshift at which the density of the universe is comparable to nuclear density. Use the BBNS value for

$$\eta = \frac{N_{baryon}}{n_\gamma} = 2.67 \times 10^{-8} (\Omega_{baryon} h^2 = 0.024) \quad \text{where} \quad n_\gamma \cong 415 \text{cm}^{-3} \tag{10}$$

How big was the presently observable universe (our current horizon) at that epoch? What was the temperature at this epoch? What time was it? Should the equation of state be the ideal gas law?

6 Big Bang Nuclear Synthesis residuals

When the nucleons merge to form He^4 about 25 MeV of energy is released. Assuming that this energy is soon thermalized and turned into photons, estimate (order of magnitude) the increase in the radiation temperature. Hint: compare to the increase in temperature for electron-positron annihilation.

7 Jean's Length and Mass

7.1 Dimensional Analysis

Use the arguments about time scales (t_{exp}, t_{grav}, & $t_{pressure}$) to show that the Jean's length and mass are of order

$$\lambda_J = \sqrt{\pi}\frac{v_s}{\sqrt{G(\rho + P/c^2)}} \quad \text{and} \quad M_J = \frac{4\pi}{3}(\frac{\lambda}{2})^3\rho \tag{11}$$

7.2 Jean's Mass for Weakly Interacting Dark Matter

What power of the scale factor is the Jean's mass proportional to for the relativistic phase? For the non-relativitic but radiation dominated phase? For the matter dominated phase. Hint: Since the particles are postulated to be weakly interacting we can assume that they are collisionaless and the quantity pa (momentum times scale factor) is a constant. It is also useful to remember that

$$v_s^2 = \frac{\partial P}{\partial \rho} \tag{12}$$

7.3 Jean's Mass Near Decoupling

Now we consider the baryon Jean's mass near decoupling. The energy density, pressure, and entropy density before decouopling are:

$$\epsilon = \rho c^2 = n_B m_H c^2 + \frac{4}{c}\sigma_{SB}T^4, \quad P = \frac{4}{3c}\sigma_{SB}T^4, \quad s = \frac{16}{3c}\sigma_{SB}T^3 \tag{13}$$

Adiabatic expansion of the universe keeps s/n constant so that to calculate the sound speed the $\partial P/\partial \rho$ has to be evaluated for constant s/n. Show that the answer is

$$v_s^2 = \frac{\partial P}{\partial \rho}|_{s/n=cons} = \frac{c^2}{3}\frac{Ts}{Ts + n_B m_H c^2} = \frac{c^2}{3}\frac{kT}{kT + \eta' m_H c^2} \tag{14}$$

where $\eta' \equiv k n_B/s$, yielding

$$M_J = \frac{\pi^{5/2}c^{1/2}}{9\sigma_{SB}^{1/2}m_H^2 G^{3/2}}(\frac{s}{n_B})^2(1 + \frac{sT}{n_B M_H c^2})^{-3} \tag{15}$$

As the temperature drops to about 3000 K, the combination of electrons with the ionized hydrogen and helium nuclei to neutral atoms is nearly complete. It is at this point that the matter and radiation effectively decouple. After decoupling, radiation pressure is unimportant and the gas of neutral atoms behaves as a monoatomic gas with $\gamma = 5/3$ so that the various quantities are:

$$\epsilon = \rho c^2 = n_b m_H c^2 + \frac{3}{2}\frac{n_B kT}{c^2}, \quad P = n_B kT_B, \quad v_s^2 = \frac{5}{3}\frac{kT_B}{m_H}. \tag{16}$$

The Jean's mass becomes

$$M_J = 4(\frac{\pi}{3})^{3/2}(\frac{5kT_B}{G})^{3/2} n_B^{-1/2} m_H^{-2} \tag{17}$$

Just after recombination the temperature of the matter is equal to the temperature of the radiation. since it is collisional, how does the temperature drop subsequently? What is the ratio of the Jean's mass from shortly before decoupling to just afterwards? What is the value for one or both?