DDI-WA 5100; JPL P.O. AA-390672 JPL Task No. DDI-T-021
NAS 7-100

ON THE OPTIMUM REPRESENTATION OF A SIGNAL WITH A LIMITED SPECTRUM TRUNCATED BY THE KOTELNIKOV SERIES

By A. Yu. Lev and V. A. Kisel

Translated from Elektrosvyaz', No. 2, 1967, pp. 25-32
By Joseph L. Zygielbaum
Terra-Space Corporation, Malibu, California

1 / 07

N68-15370

The moments of gating of signals with a limited spectrum, which assure a maximum speed of convergence of a Kotelnikov series, are selected. A choice is also made of the discrete values of a signal with a limited spectrum. These values are in fact Fourier coefficients of spectral density which satisfy the zero boundary conditions along the phase.

INTRODUCTION

Discrete (harmonic) correctors (Fig. 1) are used widely for the compensation of distortions within the information transmitting channels (Ref. 1, 2). The calculation and adjustment of these correctors is made in accordance to the instant values $g(t_0 + k\Delta t)$ $(k = 0, \pm 1, \pm 2, \ldots)$ of the pulse reaction of the communication channel $g(t)$, which in the simplest case is a low frequency signal with a practically limited spectrum.

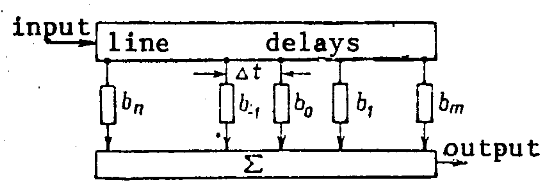

Fig. 1

The lattice function (discrete reaction) $g(t_0 + k\,\Delta t)$ (Fig. 2) is obtained by gating (selection) of the function $g(t)$ in accordance with the Kotelnikov theorem. A number of questions arise here, the sense of which is contained in the following. It is absolutely necessary to determine the value of t_0, at which the discrete reaction $g(t_0 + k\,\Delta t)$ describes in an ideal manner the continuous reaction $g(t)$, i.e. the value t_0, at which the rate of decrease of the selections of $g(t_0 + k\,\Delta t)$ with the growth of k is at a maximum and the spectral characteristic of $g(t)$ and $g(t_0 + k\,\Delta t)$ $(k = 0, 1, 2, \ldots)$ coincides with the frequency range in question. In addition it is necessary to point out the rule for numeration of the discrete values $g(t_o + k\,\Delta t)$, according to which these are in fact Fourier coefficients of spectral density which satisfy the zero boundary conditions along the phase.

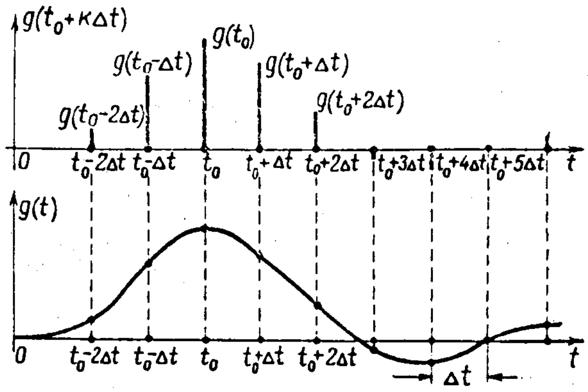

Fig. 2

The purpose of this paper is to give a reply to these questions.

STATEMENT OF THE PROBLEM

Let's assume that we have a function of time g(t), the spectral density $G(i\omega)$ of which is limited by a certain frequency ω_c $|G(i\omega)| = 0$ when $|\omega| > \omega_c$.

In other words g(t) is the pulse reaction of the low frequency channel, the transmission coefficient $G(i\omega)$ of which is limited sharply along the bend of the transmitting frequencies.

It is necessary to expand g(t) into a Kotelnikov series*:

$$g(t) \sim \sum_{-\infty}^{\infty} g(t_0 + \kappa\,\Delta t)\,\frac{\sin \omega_c\,[t-(t_0+\kappa\,\Delta t)]}{\omega_c\,[t-(t_0+\kappa\,\Delta t)]} \qquad (1)$$

in order to fulfill the following conditions:

1) the spectral density of the series in Equation (1) $\overline{G}(i\omega)$ is equal to the spectral density of the function g(t) throughout the section $[-\omega_c, \omega_c]$, including the boundary points $\pm\omega_c$: $\overline{G}(i\omega) = G(i\omega)$, $|\omega| \leq \omega_c$, and consequently the symbol for agreement in Equation (1) can be substituted by the symbol for equality;

*The symbol $\sim$ is the symbol of agreement which indicates that the spectral densities of the right and left parts are equal in the case of all ω, with the exception of specified points.

2) the expansion should be carried out in such a manner that the coefficients in Equation (1) $g(t_o + k\,\Delta t)$ should decrease at a possibly maximum speed, i.e. that the series in Equation (1) should agree with the maximum speed, and when k is finite (for instance, $m \geq k \geq -n$) would approximate g(t) in the most favorable manner.

If, during the expansion of the function g(t), the conditions 1) and 2) are satisfied, then we consider that g(t) is represented in an optimum manner by the Kotelnikov series.

If the first condition is not satisfied, then it is impossible to adjust the harmonic corrector according to the discrete reply $g(t_o + k\,\Delta t)$ of the communication channel in such a manner that the amplitude-frequential and and phase-frequential characteristics of the channel would thereby become corrected along the entire 0 to ω_c range. Nonfulfillment of the second condition results in an unjustified complication of the corrector since it is necessary to take into consideration a large number of terms of the series in Equation (1).

The calculation and adjustment of the harmonic correctors is accomplished according to the discrete reaction $g(t_o + k\,\Delta t)$ without consideration of t_o, i.e. according to the spectral density $\bar{\bar{G}}(i\omega)$ of the function $g(t - t_o)$ (the t_o lag is not corrected).

According to a previously published paper (Ref. 3) the harmonic corrector compensates for the distortion of frequency characteristics $\bar{\bar{G}}(i\omega)$ along the entire frequency range [0 to ω_c], including the point ω_c, only in the case when $\bar{G}(i\omega)$ satisfies the boundary conditions along the phase

$$\arg G(i\omega_c) = 0. \tag{2}$$

A question arises in conjunction with this problem on the correct numeration of the discrete pulse reaction $g(t_o + k\,\Delta t)$, which might be formulated as follows. Let's assume that we have a finite sequence of the numbers a_o, a_1, a_2, ..., a_{n-1}, a_n, about which it is known that these are in fact instant values of the function g(t).

It is necessary to numerate correctly this sequence, i.e. to determine which of the a_k values are equal to $g(t_o)$, $g(t_o + \Delta t)$, $g(t_o - \Delta t)$, etc., so as to satisfy thereby the requirements of the second condition. The problem is reduced to the finding of the value $g(t_0)$, which is known as the basic reference value. The remaining values of $g(t_0 + k\,\Delta t)$ are obtained automatically. A nonfulfillment of these requirements leads to the fact that it might not be possible quite frequently to adjust the corrector according to the pulse reaction $g(t)$.

OPTIMUM REPRESENTATION OF A FUNCTION WITH A LIMITED SPECTRUM BY THE KOTELNIKOV SERIES

We shall expand the $G(i\omega)$ spectral density into a Fourier series along the section $[-\omega_c, \omega_c]$:

$$G(i\omega) \sim e^{-i\omega t_0} \sum_{-\infty}^{\infty} C_\kappa e^{i\omega\kappa\Delta t}, \quad \left(\Delta t = \frac{\pi}{\omega_c}\right), \tag{3}$$

where (Ref. 4)

$$C_{-\kappa} = \Delta t g(t_0 + \kappa\Delta t), \quad (\kappa = 0, \pm 1, \pm 2, \ldots). \tag{4}$$

Apparently,

$$\overline{G}(i\omega) = e^{-i\omega t_0} \sum_{-\infty}^{\infty} C_\kappa e^{i\omega\kappa\Delta t},$$

$$\overline{\overline{G}}(i\omega) = \sum_{-\infty}^{\infty} C_\kappa e^{i\omega\kappa\Delta t}.$$

It is known that a Fourier series of the $G(i\omega)$ function coincides with the values of the given function throughout the $|\omega| < \omega_c$ interval, provided that this is a continuous function within the indicated interval. During the fulfillment of this condition it is possible to put down the following:

$$\overline{G}(i\omega) = G(i\omega), \quad |\omega| < \omega_c. \tag{5}$$

Let us examine the behavior of the series in Equation (3) at the point of the saltus of the function for the following cases:

1. The module $|G(i\omega)|$ has a discontinuity at the point ω_k; arg $G(i\omega)$ is a continuous function of $|\omega| < \omega_c$. The Fourier series of such a function coincides at the point ω_k with the values of the module and the argument (Fig. 3):

$$|\overline{G}(i\omega_\kappa)| = \frac{|G(i\omega_\kappa + 0)| + |G(i\omega_\kappa - 0)|}{2},$$

$$\arg \overline{G}(i\omega_\kappa) = \arg G(i\omega_\kappa),$$

that is $\overline{G}(i\omega_k) \neq G(i\omega_k)$.

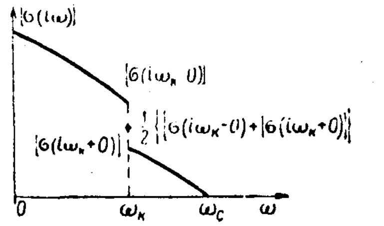

Fig. 3

2. The argument of the spectral density $G(i\omega)$ contains a discontinuity at the point ω_k, and the module $\left|G(i\omega)\right|$ is a continuous function of $(\left|\omega\right| < \omega_c)$. This case represents the greatest interest because it uncovers one of the characteristics of convergence of a Fourier series of a complex function. This characteristic is contained in the fact that the Fourier series of a complex function with a discontinuity of the argument coincides with the following values (Fig. 4a,b):

$$|\bar{G}(i\,\omega_K)| = \left|G(i\,\omega_K)\cos\frac{\varphi_1 - \varphi_2}{2}\right|, \tag{6}$$

$$\arg \bar{G}(i\,\omega_K) = \frac{\varphi_1 + \varphi_2}{2}, \tag{7}$$

where $\varphi_1 = \arg G(i\omega_k + 0)$, $\varphi_2 = \arg G(i\omega_k - 0)$.

Thus we observe at the point ω_k a defect of the function $\left|\bar{G}(i\omega)\right|$, which is contained in the fact that the value of $\left|\bar{G}(i\omega_k)\right|$ drops out of the general course of the curve $\left|G(i\omega)\right|$ (Fig. 4a). The discontinuity in the phase leads to a change of the amplitude of the spectral component ω_k.

The amplitude error depends on the magnitude of the saltus in the $\varphi_1 - \varphi_2$ phase and when $\varphi_1 - \varphi_2 = n\pi (n = 1,3,5,\ldots)$ $\left|G(i\omega_k)\right| = 0$, i.e. the ω_k harmonics is absent in the spectrum of $G(i\omega)$ and $\bar{G}(i\omega)$. In the case when the n are even $(n = 0,2,4,\ldots)$ $\left|G(i\omega_k)\right| = \left|\bar{G}(i\omega_k)\right|$ distortions of the amplitude of the ω_k component are also absent.

3. The module $\left|G(i\omega)\right|$ and the argument $\arg G(i\omega)$ contain discontinuities at the point ω_k. The Fourier series of such a function agrees at the point ω_k with the values:

$$|\bar{G}(i\,\omega_K)| = \frac{|G(i\,\omega_K + 0)| + |G(i\,\omega_K - 0)|}{2}\left|\cos\frac{\varphi_1 - \varphi_2}{2}\right|,$$

$$\arg \bar{G}(i\,\omega_K) = \frac{\varphi_1 + \varphi_2}{2}.$$

4. Let's assume that $G(i\omega)$ is continuous along the section $[-\omega_c, \omega_c]$. The equality (5) is thereby carried out for all values of $\omega < \omega_c$, with the exception of the boundary points $\pm\omega_c$, in the case of which the Fourier series in Equation (1) agrees, according to the module and the argument, with the values:

$$|\overline{G}(i\,\omega_c)| = |G(i\,\omega_c)\cos\varphi_c|, \quad \arg G(i\,\omega_c) = 0, \tag{8}$$

where $\varphi_c = \arg G(i\omega_c)$.

Consequently $\bar{G}(i\omega_c) \neq G(i\omega_c)$. The formulas in Equation (8) are obtained from the Equations (6) and (7), if we assume that $\varphi_1 = -\varphi_2$. It follows from Equation (8) that only after the boundary condition

$$\varphi_c = \arg G(i\,\omega_c) = 0 \tag{9}$$

is carried out, the equality $\bar{G}(i\omega) = G(i\omega)$ $(|\omega| < \omega_c)$ will be true, and the sign for correspondence in Equation (1) will be substituted by a sign for equality.

Any given function of $G(i\omega)$, which is continuous along the section $[\omega_c, \omega_c]$ can be converted in such a manner that it will satisfy the condition outlined in Equation (9), if we proceed as follows. We shall multiply $G(i\omega)$ by $e^{i\omega t_o}$, by selecting

$$t_0 = -\frac{\varphi_c}{\omega_c}, \tag{10}$$

which is equivalent to the subtraction of the shaded section from the characteristic of $\varphi(\omega) = \arg G(i\omega)$, as indicated in Fig. 5.

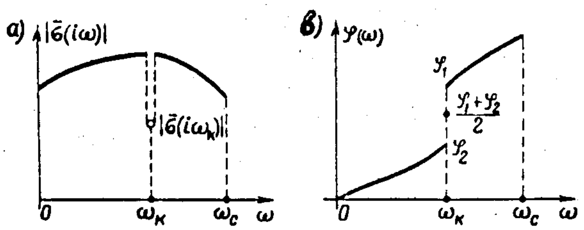

Fig. 4

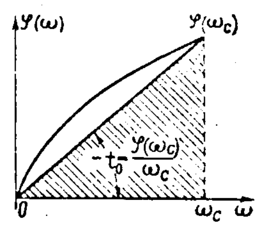

Fig. 5

The obtained function $\bar{\bar{G}}(i\omega) = G(i\omega)e^{i\omega t_o}$ satisfies the boundary condition of Equation (9), consequently the Fourier series of the function

$$\overline{\overline{G}}(i\,\omega) = \sum_{-\infty}^{\infty} C_\kappa\, e^{i\omega\kappa\Delta t} \tag{11}$$

coincides throuhout the $[-\omega_c, \omega_c]$ section with the values of the function $\bar{\bar{G}}(i\omega)$. By restating Equation (11) as follows:

$$\bar{G}(i\omega) = G(i\omega) = \bar{\bar{G}}(i\omega)e^{-i\omega t_0} = e^{i\omega t_0}\sum_{-\infty}^{\infty} C_\kappa e^{i\omega\kappa\Delta t}$$

and by taking the reciprocal Fourier conversion of both parts we obtain

$$g(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} G(i\omega)e^{i\omega t}d\omega = \sum_{-\infty}^{\infty} g(t_0 + \kappa\Delta t)\frac{\sin\omega_c[t-(t_0+\kappa\Delta t)]}{\omega_c[t-(t_0+\kappa\Delta t)]}.$$

The given Kotelnikov series represents accurately the initial g(t) function.

Consequently the series presented in Equation (1) defines accurately a function with a limited spectrum, provided that:

a) its complex density $G(i\omega)$ is continuous along the $\omega \leq \omega_c$ section;

b) the t_0 parameter is chosen in accordance with Equation (10).

It should be emphasized that of the four examined cases the greatest interest is presented by the last case, because practically all signals are characterized by a continuous spectral density. However the possibility of the presents of signals with a saltus in the phase and amplitude characteristics is not excluded. An example of the acquisition of such a signal is given in Fig. 6. A pulse sequence with a T-period of succession (Fig. 6a), the spectral density of which is a periodic function and contains dicontinuities at the points $\omega_c = \frac{\pi}{T}$ (Fig. 6b), is passed through a low frequency filter with a critical frequency of $\omega_c' > \omega_c$, in result of which we obtain a signal with a limited spectrum and a saltus of the spectral density (Fig. 6c).

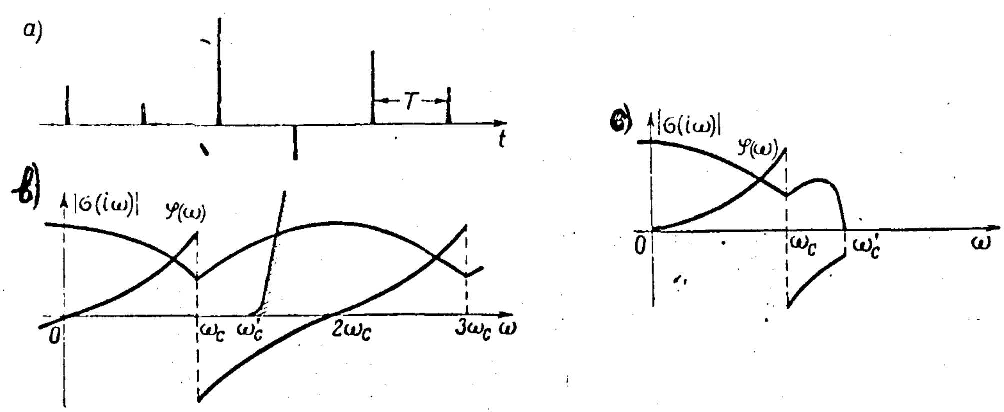

Fig. 6

We shall show now that by selecting t_o in accordance with Equation (10) the values of the lattice function $g(t_o+k\Delta t)$ decrease at a maximum possible rate with the increase of k, and consequently the series presented in Equation (1) converge at a maximum possible rate, and in the case of the given values of k ($-n \leq k \leq m$) the partial sum

$$g_1(t) = \sum_{-n}^{m} g(t_0 + \kappa \Delta t) \cdot \frac{\sin \omega_c [t - (t_0 + \kappa \Delta t)]}{\omega_c [t - (t_0 + \kappa \Delta t)]}$$

will approximate the g(t) function in the best way.

We shall separate the actual and imaginary parts of the function $\bar{\bar{G}}(i\omega)$ $\bar{\bar{G}}(i\omega) = A(\omega) + iB(\omega)$, where $A(\omega) = |\bar{\bar{G}}(i\omega)| \cos \varphi(\omega)$, $B(\omega) = |\bar{\bar{G}}(i\omega)| \sin \varphi(\omega)$. During the expansion of $\bar{\bar{G}}(i\omega)$ into a Fourier series the function $A(\omega)$ is represented by a series along the cosines and $B(\omega)$ is represented by a series along the sines:

$$A(\omega) = \sum_{0}^{\infty} a_\kappa \cos \kappa \Delta t \omega, \quad B(\omega) = \sum_{1}^{\infty} b_\kappa \sin \kappa \Delta t \omega,$$

where $a_k = C_k + C_{-k}$, $b_k = C_k - C_{-k}$.

If the $A(\omega)$ and $B(\omega)$ functions are continuous along the $\omega \leq \omega_c$ section and they satisfy the boundary conditions of $A(\omega_c) = A(-\omega_c)$, $B(\omega_c) = B(-\omega_c) = 0$, then the coefficients decrease at a rate of $\frac{1}{k^2}$, and the b_k coefficients decrease at a rate of $\frac{1}{k^3}$ (Ref. 5). The coefficients C_k decrease at a rate of not less than $\frac{1}{k^2}$, because $C_k = \frac{a_k+b_k}{2}$ and $C_{-k} = \frac{a_k-b_k}{2}$.

The boundary condition for $B(\omega)$ is fulfilled only through the selection of t_o in accordance with Equation (10), otherwise $B(\omega_c) \neq B(-\omega_c) \neq 0$. consequently the b_k coefficients decrease at a rate of $\frac{1}{k}$ and the rate of decrease of C_k drops to $\frac{1}{k}$ also, i.e. by k times.

The reasonings on the rate of decrease of the C_k coefficients are true also in respect to the lattice function $g(t_o + k\,\Delta t)$, on the strenght of the correlation (4).

The stated consideration shows that a correct choice of the t_o parameter raises the rate of decrease of the values of $g(t_o + k\,\Delta t)$ by k times. Therefore the indicated characteristic should be taken into consideration during the selection of a function with a limited spectrum. Thus the gating of a function with a limited spectrum must be accomplished not only synchronously, but also in phase with the highest ω_c frequency.

We shall mention one characteristic of the examined optimum gating. Since the coefficients in the series of Equation (1), $g(t_o + k\,\Delta t)$, drop to infinity at a rate of $\frac{1}{k}$ and the envelope functions

$$\frac{\sin \omega_c [t-(t_0+m\,\Delta t)]}{\omega_c [t-(t_0+m\,\Delta t)]} \quad (m=0,\ 1,\ 2\ldots)$$

drop at a rate of $\frac{1}{k^2}$ $(t = k\,\Delta t)$, then in the presence of a sufficiently large t, $(t \to \infty)$ the series in Equation (1) are described asymptotically by the function

$$g(t) \approx D \frac{\sin \omega_c (t-t_0)}{\omega_c (t-t_0)},$$

(D is a specified constant), the zeros of which are distributed at the $(t_o + k\,\Delta t)$ points. It is therefore possible to state that the optimum gating dictates such a selection of t_o, at which the gating is carried out at the zeros of the g(t) function in the presents of a sufficiently high t.

Evidently if g(t) is an even function in respect to a specific moment of time, t', then it is necessary to select $t_o = t'$ and this will assure an optimum g(t) gating.

SELECTION OF A BASIC REFERENCE PULSE

We shall construct a trigonometric polynomial $Q = \sum_{\kappa=0}^{n} a_\kappa e^{-i\omega\kappa\Delta t}$ which, when correctly numerated, represents a partial sum of the Fourier series of spectral density $\bar{\bar{G}}(i\omega)$ in Equation (11) and has therefore the characteristic $\arg Q = 0$ $(\omega = \omega_c)$. This characteristic is included included in the numeration principle of the sequence $\{a_k\}$. The a_k valuesshould be numerated in such a manner that they will represent coefficients of the Fourier series of the function which satisfies the boundary condition of Equation (9). By changing the ω frequency from $\omega = 0$ to $\omega = \omega_c$ the Q-vector will describe an angle equal zero within the complex plane.

The angle described by the Q-vector will equal zero only in the case when the Q-polynomial is multiplied by the $e^{\omega i n_1 \Delta}$ multiplier

$$Q_1 = Q\, e^{i\omega n_1 \Delta t} = \sum_{\kappa=0}^{n} a_\kappa e^{-i\omega(\kappa - n_1)\Delta t}$$

or by changing the numeration of the a_k values:

$$Q_1 = \sum_{\kappa=-n_1}^{n-n_1} a'_\kappa \, e^{-i\omega\kappa\Delta t}, \quad (a'_\kappa = a_{\kappa+n_1}),$$

where n_1 is the number of roots of the algebraic polynomial $P(x) = \sum_{\kappa=0}^{n} a_\kappa x^\kappa$ along the module. These roots are lesser than a unit (i.e. they are located within the z complex plane inside of a unit radius).

We consider that the P(x) polynomial does not contain any roots which are located on the circle of a unit radius, and this corresponds to the case when $|G(i\omega)| \neq 0 \quad (|\omega| \leq \omega_c)$ (Ref. 6). By comparing the coefficient of the Q_1 polynomial with the values of $g(t_o + k\,\Delta t)$ we obtain:

$$g(t_0) = a_{n1};\ g(t_0+\Delta t) = a_{n1}+1;\ g(t_0+l\Delta t) = a_{n1}+l.$$

Thus the selection of a basic reference value is reduced to the finding of the number of roots of the P(x) polynomial, the module of which is less than a unit.

EXPERIMENTAL VERIFICATION

The harmonic correctors are calculated and adjusted according to the discrete values of the pulse reaction of the low frequency communication channel. Such reactions represent for all practical purposes signals with a limited spectrum and therefore the preceding considerations apply in their case.

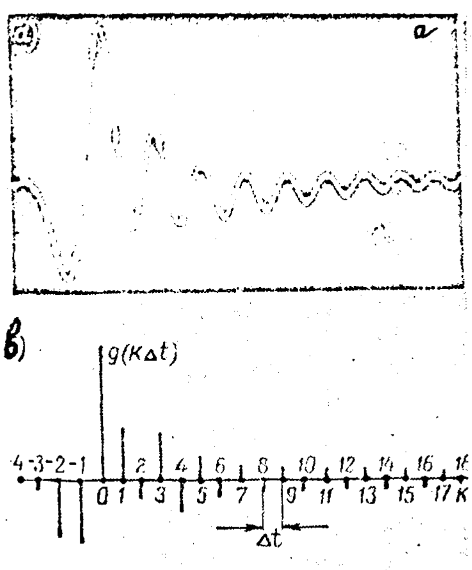

Fig. 7

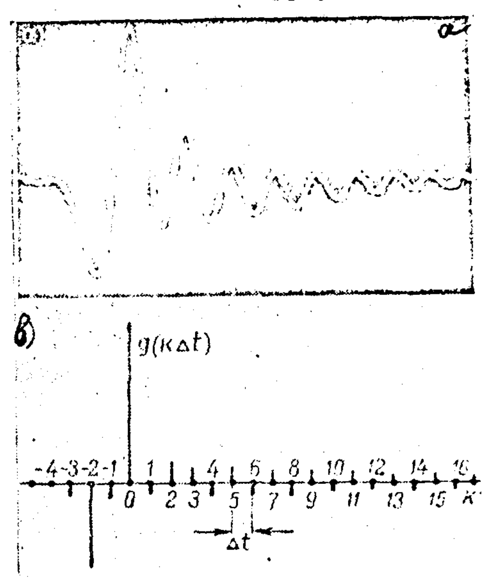

Fig. 8

In Fig. 7a, 8a and 9a are given the oscillograms of the pulse reaction of a low frequency filter of the m type. The bright areas on the graph correspond to the Kotelnikov intervals $\Delta t = \frac{\pi}{\omega_c}$ (ω_c is the cutoff frequency of the filter), in which the selection of the g(k Δt) discrete values accomplished. The figures differ from each other only by the positions of the bright areas on the graphs, i.e. by the positions of the moments of selection.

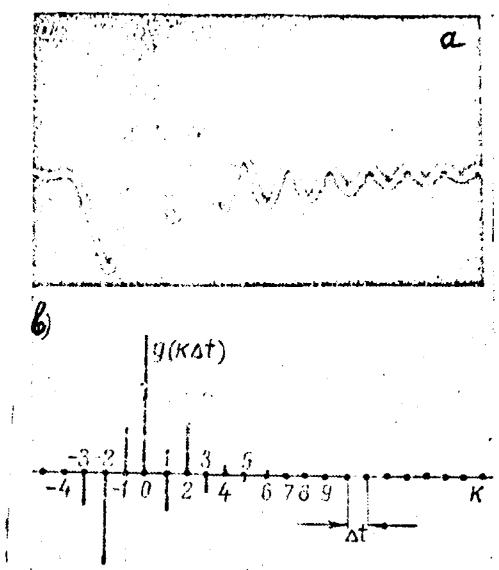

Fig. 9

In Fig. 7b, 8b and 9b are shown respectively the descrete reactions of g(k Δt) which are obtained at various positions of the selection moments. The optimum position of the selection moments is given in Fig. 9. When t is sufficiently large ($t > 7\Delta t$) the selection moments coincide with the zeros of the g(t) pulse reaction. The thereby obtained g(k Δt) descrete reaction contains a maximum rate of decrease of the g(k Δt) values. As is evident in the oscillograms g(k Δt) = 0 when $k \geq 7$, i.e. it is sufficient to have 10 values of g(k Δt) ($-3 \leq k \leq 7$) in order to describe fully the continuous g(t) reaction in accordance with the Kotelnikov theorem, as shown in Equation (1). In the case of a nonoptimum position of the gating moment (Fig. 7 and 8), at least a 2 to 2.5 times larger number of g(k Δt) ($-3 \leq k \leq 17$) values is required in order to describe satisfactorily g(t).

Fig. 8 corresponds to a selection of gating moments at which the obtained basic reference value g(0) coincides with the g(t) maximum. As can be seen by comparing Fig. 8 and 9, such a selection in the case of asymmetric pulse reactions does not constitute an optimum.

LITERATURE

1. I. M. Linke. A variable time equalizer for videofrequency waveform correction. "Proc. Inst. Electr. Engns," 1952, v. 99, p.IIIa,№18

2. R. M. Lucky. Automatic equalization for digital communication. Bell Syst. techn. J. , 1965, № 4.

3. V. A. Kisel'. Elementy rascheta garmonicheskikh korrektorov. (Calculation elements of harmonic correctors), Elektrosvyaz', 1964, № 7.

4. S. Goldman. Teoriya informatsii (Information theory), Izd. Inostr. Lit. 1957

5. A. N. Krylov. Lektsii o priblizhennykh vychislenii (Lectures on approximate calculations), Gostekhizdat, 1954.

6. V. A. Kisel', Kriterii korrektiruyemosti kanala po impulsnoy reaktsii. Trudy uchebnykh institutov sviyazi. (Criteria for corrections in accordance with the pulse reaction. Proc. of Educational Institutes for Communication), No. 17, 1963.