Equivalency of Two "Cramér Conditions"

by

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Here, we prove the following for real v and any real random variable $K \geq 0$.

Lemma 1: *The condition* $\liminf_{v\to\infty} \left|1-\mathbb{E}e^{ivK}\right| > 0$ *holds if and only if* $\limsup_{v\to\infty} \left|\mathbb{E}e^{ivK}\right| < 1$.

In the renewal theory literature, the condition $\liminf_{v\to\infty} \left|1-\mathbb{E}e^{ivK}\right| > 0$ appears in at least one paper by C. Stone [2]. Asmussen refers to $\limsup_{v\to\infty} \left|\mathbb{E}e^{ivK}\right| < 1$ as the Cramér Condition [1, p. 142], and he cites Stone in his references. Thus, Stone was probably aware of Lemma 1, but I have been unable to turn up any direct reference proving the equivalence of the two conditions. If anyone knows of such a reference, please email me.

Proof of Lemma 1: (**Sufficiency**) If $\limsup_{v\to\infty} \left|\mathbb{E}e^{ivK}\right| < 1$, then $\liminf_{v\to\infty} \left|1-\mathbb{E}e^{ivK}\right| \geq \liminf_{v\to\infty} \left(1-\left|\mathbb{E}e^{ivK}\right|\right) > 0$.

(**Necessity**) If $\limsup_{v\to\infty} \left|\mathbb{E}e^{ivK}\right| = 1$, then there are two sequences $\{v_n\} \uparrow \infty$ and $\{\theta_n\} \subseteq [0, 2\pi]$ such that $\lim_{n\to\infty} \mathbb{E}e^{i(v_n K - \theta_n)} = 1$. Because $[0, 2\pi]$ is compact, we can select a subsequence $\{\theta_{j(n)}\}$ with a limit point $\theta := \lim_{n\to\infty} \theta_{j(n)}$. Select the subsequence $\{v_{j(n)}\}$, and renumber it so that $\lim_{n\to\infty} \mathbb{E}e^{i(v_n K - \theta)} = 1$.

The idea behind the following proof is that $v_n K - \theta$ becomes concentrated on integer multiples of 2π. Thus, the differences $(v_n K - \theta) - (v_m K - \theta) = (v_m - v_n) K$ concentrate there as well if $\min\{m, n\} \to \infty$.

For each integer n, choose a larger integer $m(n)$ so that the sequence $\{v_{m(n)} - v_n\}$ is strictly increasing to infinity. For brevity, define the random variables $A := A(n) := v_n K - \theta$ and $B := B(n) := v_{m(n)} K - \theta$, which satisfy $\lim_{n\to\infty} \mathbb{E}e^{iA(n)} = \lim_{n\to\infty} \mathbb{E}e^{iB(n)} = 1$.

As a preliminary, we prove the plausible statement that

$$\text{(1.1)} \qquad \lim_{n\to\infty} \mathbb{E}e^{iX_n} = 1 \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{E}\left|1 - e^{iX_n}\right| = 0$$

for any sequence of real random variables $\{X_n\}$. The Chebyshev and Cauchy-Schwarz inequalities yield

$$\text{(1.2)} \quad \left\{\mathbb{E}\left|1 - e^{iX_n}\right|\right\}^2 \le \mathbb{E}\left\{\left|1 - e^{iX_n}\right|^2\right\} = 2\left(1 - \mathbb{E}\cos X_n\right) \le 2\left|1 - \mathbb{E}e^{iX_n}\right| \le 2\mathbb{E}\left|1 - e^{iX_n}\right|.$$

The final inequality is a standard inequality on norms. Eq (1.1) therefore follows from Eq (1.2).

The inequality $\left|1 - e^{i(B-A)}\right| = \left|1 - e^{-iA} + e^{-iA}\left(1 - e^{iB}\right)\right| \le \left|1 - e^{iA}\right| + \left|1 - e^{iB}\right|$ yields $\mathbb{E}\left|1 - e^{i\{B(n)-A(n)\}}\right| \le \mathbb{E}\left|1 - e^{iA(n)}\right| + \mathbb{E}\left|1 - e^{iB(n)}\right|$. Because of Eq (1.1), $\lim_{n\to\infty} \mathbb{E}\left|1 - e^{iA(n)}\right| = \lim_{n\to\infty} \mathbb{E}\left|1 - e^{iB(n)}\right| = 0$, so $\lim_{n\to\infty} \mathbb{E}\left|1 - e^{i\{B(n)-A(n)\}}\right| = 0$. Eq (1.1) again shows that $\lim_{n\to\infty} \mathbb{E}e^{i\{B(n)-A(n)\}} = 1$. Because $B(n) - A(n) := \left(v_{m(n)} - v_n\right)K$,

$\lim_{n\to\infty} \mathbb{E}e^{i\left(v_{m(n)}-v_m\right)K} = 1$ with $\left\{v_{m(n)} - v_n\right\} \uparrow \infty$. Accordingly, $\liminf_{v\to\infty} \left|1 - \mathbb{E}e^{ivK}\right| = 0$, proving Lemma 1**.**

References

[1] Asmussen, S. Applied Probability and Queueing. New York: Wiley 1987

[2] Stone, C.J.: On moment generating functions and renewal theory. Annals of Mathematical Statistics **36** 1298-1301 1965