The Cramér Condition and Roots Near the Imaginary Axis

by

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In the complex t-plane, let $t := u + iv$ so $u := \mathrm{Re}\, t$ and $v := \mathrm{Im}\, t$. (The symbol "$:=$"denotes a definition.) If $K \geq 0$ be a random variable, it is said to be "strongly non-lattice", if and only the "Cramér Condition" $\liminf_{v\to\infty} \left|1 - \mathbb{E}e^{ivK}\right| > 0$ holds [1, p.142]. Lemma 1 indicates how the Cramér Condition affects the placement of the roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ near the imaginary axis.

Lemma 1 assumes that $\mathbb{E}e^{\tilde{r}K} < \infty$ for some $\tilde{r} > 0$. In addition, r and ρ satisfy $0 < 4r \leq \rho < \tilde{r}$ but are otherwise arbitrary. (The factor of 4 is consistent with notation in my other articles.)

Lemma 1: If K *is strongly non-lattice, then there exists some* $r > 0$, *such that the only root of* $\mathbb{E}e^{\zeta K} - 1 = 0$ *within the closed strip* $\{t : u_1 \leq u \leq u_2\}$ *is* $\zeta_0 = 0$.

In fact, the converse holds, although we do not prove it here.

Proof of Lemma 1: The Cramér Condition $\liminf_{v\to\infty} \left|1 - \mathbb{E}e^{ivK}\right| > 0$ implies that K is non-lattice. (If K had span σ, $\lim_{n\to\infty} \mathbb{E}e^{iv_n K} = 1$ for $v_n = 2\pi n \sigma^{-1}$.) In addition, $3\eta := \liminf_{v\to\infty} \left|1 - \mathbb{E}e^{-ivK}\right| = \liminf_{v\to\infty} \left|1 - \mathbb{E}e^{ivK}\right| > 0$. Consequently, there exists some $w_0 > 0$ so that $\left|1 - \mathbb{E}e^{ivK}\right| \geq 2\eta$ within the two rays $\{t : u = 0 \text{ and } |v| \geq w_0\}$ on the imaginary axis.

A Taylor expansion of $\mathbb{E}e^{tK} = \mathbb{E}e^{(u+iv)K}$ about $t_0 = iv$ yields the inequality $\left|\mathbb{E}e^{tK} - \mathbb{E}e^{ivK}\right| \le 4r\mathbb{E}\left(Ke^{\rho K}\right)$ for $t \in S\overline{(0,4r)}$. Reduce $r > 0$ if necessary, so that $4r\mathbb{E}\left(Ke^{\rho K}\right) \le \eta$. By the triangle inequality $\left|\mathbb{E}e^{tK} - 1\right| \ge \left|\mathbb{E}e^{ivK} - 1\right| - \left|\mathbb{E}e^{tK} - \mathbb{E}e^{ivK}\right| \ge \eta > 0$, no roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ lie within the two semi-infinite strips $\left\{t : 0 \le u \le 4r \text{ and } |v| \ge w_0\right\}$.

Because of the Cramér Condition, K is not identically 0. Thus, the set of roots ζ of $\mathbb{E}e^{\zeta K} - 1 = 0$ within the compact rectangular region $\left\{t : 0 \le u \le 4r \text{ and } |v| \le w_0\right\}$ can have no limit point. (Otherwise, $\mathbb{E}e^{\zeta K} \equiv 1$ identically [2, p. 149].) Thus, the set of roots is finite: $\zeta_0 := 0, \zeta_1, ..., \zeta_m$. Because K is non-lattice, the only root of $\mathbb{E}e^{\zeta K} - 1 = 0$ actually on the imaginary axis is $\zeta_0 = 0$ [8, p. 500]: for any other root, $\operatorname{Re}\zeta > 0$. Reduce $r > 0$ if necessary, to make $4r < \min\left\{\operatorname{Re}\zeta_1, ..., \operatorname{Re}\zeta_m\right\}$, thereby excluding $\zeta_1, ..., \zeta_m$ from the rectangular region $\left\{t : 0 \le u \le 4r \text{ and } |v| \le w_0\right\}$.

We have therefore displayed $r > 0$, such that the only root of $\mathbb{E}e^{\zeta K} - 1 = 0$ within $\left\{t : u_1 \le u \le u_2\right\}$ is $\zeta_0 = 0$, completing the proof.

References

[1] Asmussen, S. Applied Probability and Queueing. New York: Wiley 1987

[2] Levinson, N. and Redheffer, R.M. Complex Variables. San Francisco: Holden-Day 1970