Midterm Exam

Physics 105, Fall 04; Instructor: Petr Hořava
October 28, 2004, 8:10-9:30 a.m.

Problem 1.

Imagine a vertical circular hoop of radius R in three-dimensional Euclidean space. This hoop is rotating about a fixed axis with constant angular velocity Ω as shown in Figure 1. A (point-like) bead of mass m is threaded on the hoop, so that it can move without friction but is confined to move on the hoop. In addition, there is a spring whose one end is attached to the bead, while the other end is attached to a fixed point on the rotation axis, some fixed distance L away from the center of the hoop. The only role of the spring is to provide a linear force acting on the bead in the direction of the spring, with some fixed spring constant k. There are no other forces acting on the bead; in particular, there is no gravitational force.

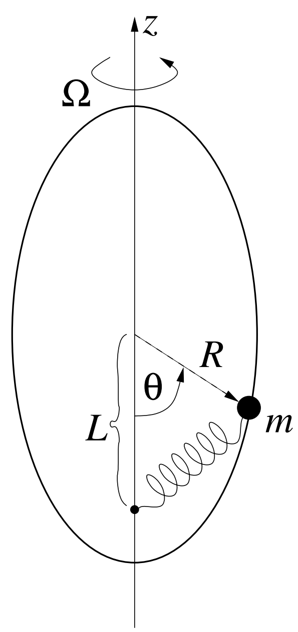

Fig. 1: The bead on a rotating hoop, in the presence of a central linear force centered on a point along the rotation axis, some distance L away from the center of the hoop.

1(a) What is the number of degrees of freedom of this problem? Find the Lagrangian and write down the Euler-Lagrange equations of motion for the bead.

Solution.

The problem has one degree of freedom, which can be conveniently parametrized by one generalized coordinate, θ. (Theta is a periodic coordinate, $\theta = \theta + 2\pi$, hence the configuration space is a circle, S^1.) The Cartesian coordinates (x, y, z) of the bead can be expressed as functions of the generalized coordinate θ and the time variable t as follows,

$$\begin{aligned} x &= R \cos \Omega t \sin \theta, \\ y &= R \sin \Omega t \sin \theta, \\ z &= -R \cos \theta. \end{aligned}$$

The Lagrangian is given by $\mathcal{L} = T - V$, where the kinetic energy is given by

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \Omega^2 \sin^2 \theta,$$

where we used the following expressions for $\dot{x}, \dot{y}$ and $\dot{z}$:

$$\begin{aligned} \dot{x} &= -R\Omega \sin \Omega t \sin \theta + R\dot{\theta} \cos \Omega t \cos \theta, \\ \dot{y} &= R\Omega \cos \Omega t \sin \theta + R\dot{\theta} \sin \Omega t \cos \theta, \\ \dot{z} &= R\dot{\theta} \sin \theta. \end{aligned}$$

The potential energy is given by $\frac{1}{2}k$ times the square of the distance between the two endpoints of the spring,

$$\begin{aligned} V &= \frac{1}{2} k \left(x^2 + y^2 + (z + L)^2 \right) = \frac{1}{2} k (R^2 + L^2 + 2Lz) \\ &= kLz + \text{constant} = -kLR \cos \theta + \text{constant}. \end{aligned}$$

This result should come as a pleasant surprise: Up to an additive constant, which we can always drop in the Lagrangian, the potential is just equivalent to the potential describing a uniform gravitational force pointing along the z direction! And this is of course a problem which already appeared on one of the homeworks.

To summarize, the Lagrangian in terms on the single degree of freedom θ is given by

$$\mathcal{L} = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} m R^2 \Omega^2 \sin^2 \theta + kLR \cos \theta.$$

Consequently, the Euler-Lagrange equation of motion is simply

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} \equiv mR^2 \ddot{\theta} - mR^2 \Omega^2 \sin \theta \cos \theta + kLR \sin \theta = 0.$$

1(b) Find the Hamiltonian H and energy E of the system. Is H a constant of the motion? Is E conserved? If you find a conservation law, identify the continuous symmetry responsible for it.

Solution.

Using the definition of the Hamiltonian in terms of the Lagrangian, we get

$$H \equiv \dot{\theta}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \mathcal{L} = \frac{1}{2}mR^2\dot{\theta}^2 - \frac{1}{2}mR^2\Omega^2\sin^2\theta - kLR\cos\theta,$$

or, if one insists on writing the Hamiltonian in the phase-space variables θ and p (with $p \equiv \partial\mathcal{L}/\partial\dot{\theta}$),

$$H = \frac{p^2}{2mR^2} - \frac{1}{2}mR^2\Omega^2\sin^2\theta - kLR\cos\theta.$$

On the other hand, the total energy is given by the sum of the kinetic and potential energies,

$$E \equiv T + V = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{1}{2}mR^2\Omega^2\sin^2\theta - kLR\cos\theta.$$

Thus, E is *not* equal to H; instead, we have

$$E = H + mR^2\Omega^2\sin^2\theta.$$

Now, on to the conservation laws. Notice that the Lagrangian for our problem is time-independent, and recall that

$$\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}.$$

Hence, H is a constant of the motion. This also implies that the energy E is *not* a constant of the motion: if it were, the relation between E and H would imply that θ is always a constant, but that would be a contradiction. Thus, H is conserved, but E is not.

The symmetry responsible for the conservation of H is that of time translation invariance (=time independence) of the Lagrangian.

1(c) Identify all configurations of static equilibrium for this Lagrangian. For each point of static equilibrium, determine whether it is stable or unstable, by performing an explicit linearization of the problem in the Lagrangian formulation. Describe how the qualitative properties of the points of static equilibrium change as a function of the angular velocity Ω, radius R and the spring constant k. Find the frequency ω of the linear oscillation near each point of stable static equilibrium, as a function of k, R and Ω.

Solution.

Writing the Lagrangian as $\mathcal{L}(\theta,\dot{\theta}) = \frac{1}{2}mR^2\dot{\theta}^2 - V_{\text{eff}}(\theta)$, the points of static equilibrium will be the extrema of the effective potential $V_{\text{eff}}(\theta)$, i.e., solutions of

$$V'_{\text{eff}}(\theta) \equiv \frac{\partial \mathcal{L}}{\partial \theta} = R\sin\theta\left(kL - mR\Omega^2\cos\theta\right) = 0.$$

This equation can be satisfied in two ways: either by setting $\sin\theta$ equal to zero, which is satisfied for two possible values of θ:

$$\theta = 0, \qquad\qquad \theta = \pi,$$

or by setting $kL - mR\Omega^2\cos\theta = 0$. This second class of solutions will only exist if[1]

$$\left|\frac{kL}{mR\Omega^2}\right| \leq 1,$$

leading to

$$\theta = \pm\arccos\left(\frac{kL}{mR\Omega^2}\right) \equiv \pm\theta_0.$$

Thus, the qualitative behavior of the points of static equilibrium is as follows: For $kL > mR\Omega^2$, there are two points of static equilibrium: $\theta = 0, \pi$, while for $kL < mR\Omega^2$, there will be two additional points, $\theta = \pm\theta_0$. In the limit of $kL \to mR\Omega^2$ from below, the two extra points of static equilibrium smoothly approach $\theta = 0$.

Stability or instability of points of static equilibrium can be determined by linearizing the problem, i.e., by expanding the Lagrangian near the point of static equilibrium, and keeping only the quadratic part.

Near $\theta = 0$, we have

$$\mathcal{L} \approx \frac{1}{2}mR^2\dot{\theta}^2 - \frac{R}{2}(kL - mR\Omega^2)\theta^2,$$

indicating that $\theta = 0$ is stable or unstable for $kL > mR\Omega^2$ or $kL < mR\Omega^2$, respectively.

[1] From now on, without losing any generality, we shall assume that L is non-negative. The case of negative L can be mapped to the case of positive L by $z \to -z$.

Near $\theta = \pi$, we can write $\theta = \pi + \vartheta$, and get

$$\mathcal{L} \approx \frac{1}{2} m R^2 \dot{\vartheta}^2 + \frac{R}{2}(kL + mR\Omega^2)\vartheta^2.$$

Hence, independently of the value of the parameters (as long as they are all positive, as we have assumed throughout), $\theta = \pi$ is always unstable.

In the regime where $\theta = 0$ is unstable, i.e., for $kL < mR\Omega^2$, we can also linearize the problem near the additional two points of static equilibrium, $\theta = \pm\theta_0$. Writing now $\theta = \pm\theta_0 + \Theta$ and linearizing the problem for small Θ, we get

$$\mathcal{L} \approx \frac{1}{2} m R^2 \dot{\Theta}^2 - \frac{mR^2\Omega^2}{2}\left(1 - \frac{k^2L^2}{m^2R^2\Omega^4}\right)\Theta^2.$$

(The coefficient in front of Θ^2 can be easily obtained by evaluating $-\frac{1}{2}V''_{\text{eff}}(\theta_0)$, an easy calculation to perform, since we have already evaluated $V'_{\text{eff}}(\theta)$ – see above.) Clearly, in the entire range where these two extra points of static equilibrium exist, the coefficient of the Θ^2 term in the linearized Lagrangian is negative, and $\theta = \pm\theta_0$ are therefore points of stable static equilibrium.

(The borderline case of $kL = mR\Omega^2$ requires some extra care: this is the value for which $\theta = \pm\theta_0$ coincide with $\theta = 0$. In the expansion of the Lagrangian in the smallness of θ near $\theta = 0$, the quadratic piece $\sim \theta^2$ vanishes, and one has to go to higher orders in θ to determine whether $\theta = 0$ is marginally stable or marginally unstable. The expansion of the Lagrangian for this special case gives

$$\mathcal{L} \approx \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{kLR}{8}\theta^4,$$

implying that in the special case of $kL = mR\Omega^2$, $\theta = 0$ is marginally stable. However, this answer goes beyond what was required in the exam, and is mentioned here just for completeness.)

The frequencies of oscillation near points of stable static equilibrium can now be read off from the linearized Lagrangians; they are all of the form

$$\mathcal{L} = \frac{1}{2} m R^2 \dot{\theta}^2 - \frac{1}{2} c^2 \theta^2$$

for some constant c. In terms of m, R and c, the frequency is given by

$$\omega = \frac{c}{m^{1/2} R}.$$

This completes the solution of this problem.

1(d) Consider the same system as in Problems 1(a)-(c), except the other, fixed end of the spring has been moved some fixed distance D away from the rotation axis (see Figure 2). (This new location of the fixed end of the spring is constant with time, i.e., it is *not* co-rotating with the hoop.) Repeat the analysis of Problem 1(b) for this modified system, and determine which conclusions of 1(b) (if any) have changed.

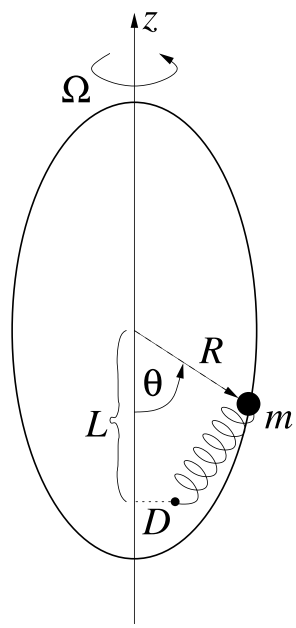

Fig. 2: The same bead on a rotating hoop as in Fig. 1, except now the source of the linear force has been moved some distance D away from the (vertical) axis of rotation.

Solution.

In this case, the Lagrangian is modified compared to the setup of 1(a)-(c), in the following way: L is still $T - V$, T is the same as in 1(a), and V is modified to

$$V(\theta, t) = \frac{1}{2}k\left((x-D)^2 + y^2 + (z+L)^2\right) = kLz - kDx + \text{constant}$$
$$= -kLR\cos\theta - kDR\cos\Omega t\sin\theta + \text{constant}.$$

Hence, the Lagrangian is now explicitly dependent on t, and the Hamiltonian will no longer be conserved, since there is no time translation invariance in this modified problem.

Problem 2.

It has been found that the experimentally determined interaction between the atoms of diatomic molecules can be described quite well by the Morse potential,

$$V_{\text{Morse}}(r) = A\left(e^{-2Br} - 2e^{-Br}\right),$$

where A and B are positive constants. Although this is a three-dimensional problem, we will treat it here in its simplified form, as a problem with one degree of freedom; hence, we assume that the Lagrangian is given by

$$L = \frac{1}{2}(\dot{r})^2 - V_{\text{Morse}}(r)$$

2(a) Identify the range of energies for which the orbits are bounded and for which the orbits are unbounded.

Solution.

The potential has one absolute minimum, located at the (only) solution of

$$V'_{\text{Morse}}(r) \equiv -2BAe^{-2Br}\left(1 - e^{Br}\right) = 0,$$

i.e., at $r = 0$ (cf. Figure 3).

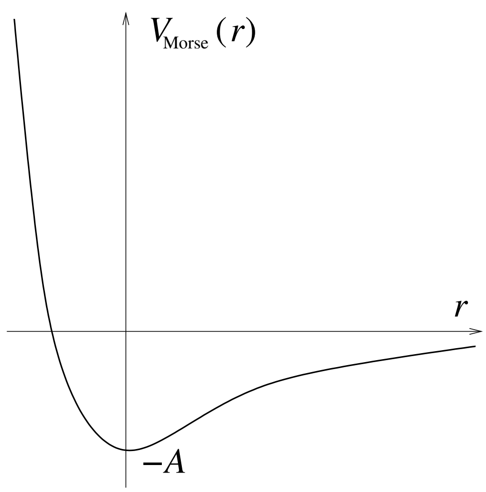

Fig. 3: The Morse potential.

The value of the potential energy at this absolute minimum is $V_{\text{Morse}}(0) = -A$. At $r \to +\infty$, we have $V(r) \to 0$. Hence, by the conservation of energy and the shape of the potential, there are two classes of orbits:

- bounded for $-A \leq E < 0$,
- unbounded for $E \geq 0$.

2(b) Draw the phase portrait of the system.

Solution.

The phase portrait is supposed to be a picture that captures the qualitative behavior of different classes of trajectories of the system in phase space. For systems whose energy is conserved (such as our problem here), the lines of the phase portrait are the lines of constant energy. In addition, it is customary to place an arrow on each line of constant E in the phase portrait, indicating the direction of time flow along the trajectory.

For our problem, the lines of constant energy satisfy

$$E = \frac{1}{2}(\dot{r})^2 + A\left(e^{-2Br} - 2e^{-Br}\right),$$

and the phase portrait of the system looks qualitatively as follows:

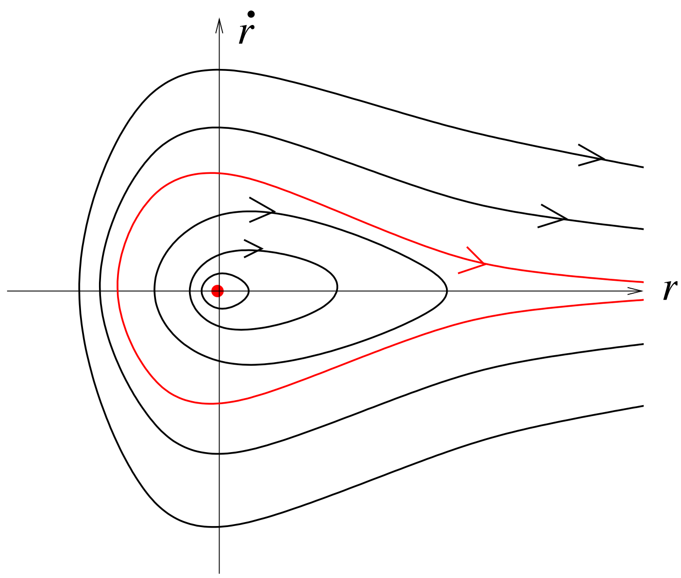

Fig. 4: The phase portrait of the one-dimensional motion in the Morse potential.

The red line (which asymptotes to $r = +\infty$ and $\dot{r} = 0$ as $t \to \pm\infty$) is the separatrix, separating the bounded from unbounded orbits. The origin $r = 0$, $\dot{r} = 0$ is a point of stable static equilibrium.

Problem 3.

Consider a system described by the following Lagrangian,

$$L = \frac{1}{2}m(\dot{x})^2 - \frac{1}{2}\omega^2 x^2,$$

with m and ω (positive, real) constants. Imagine that for $t < 0$, the system is in its point of static equilibrium, $x = 0$. At time $t = 0$, we turn on an outside driving force, whose time dependence is given by

$$F(t) = \begin{cases} F_0 t, & t \in [0, T], \\ 0, & t > T, \end{cases}$$

with F_0 a fixed constant. Using the method of Green's functions, determine the time interval T for which the force has to stay turned on if we wish the final configuration at $t > T$ to be again in the point of static equilibrium,

$$x(t) = 0 \qquad \text{for} \quad t > T.$$

Solution.

If m and ω were both equal to one, the Green's function would be just $\sin t$ (for $t > 0$). The dependence of the Green's function on m and Ω can be easily restored, giving

$$G(t) = \begin{cases} \frac{1}{\omega} \sin\left(\frac{\omega}{\sqrt{m}} t\right) & \text{for } t > 0, \\ 0 & \text{for } t < 0. \end{cases}$$

In terms of this Green's function, the solution of our driven oscillator problem is given (for $0 < t < T$) by

$$x(t) = \int_0^t G(t - t') F_0 t' dt' = \frac{1}{\omega} \int_0^t \sin\left(\frac{\omega}{\sqrt{m}}(t - t')\right) F_0 t' dt'.$$

The integral is elementary, leading to

$$x(t) = \frac{F_0\sqrt{m}}{\omega^2} \left\{ t - \frac{\sqrt{m}}{\omega} \sin\left(\frac{\omega t}{\sqrt{m}}\right) \right\}.$$

We now have to match it at $t = T$ to $x(T) = 0$, $\dot{x}(T) = 0$. These conditions give

$$\dot{x}(T) \equiv \frac{F_0\sqrt{m}}{\omega^2} \left\{ 1 - \cos\left(\frac{\omega T}{\sqrt{m}}\right) \right\} = 0,$$

$$x(T) \equiv \frac{F_0\sqrt{m}}{\omega^2} \left\{ T - \frac{\sqrt{m}}{\omega} \sin\left(\frac{\omega T}{\sqrt{m}}\right) \right\} = 0.$$

The first of these equations is satisfied only if

$$\frac{\omega T}{\sqrt{m}} = 2k\pi, \qquad k \in \mathbf{Z}.$$

Upon substituting this into the second equation, we get an additional condition, $2k\pi = (-1)^k$. The only solution of this condition is $k = 0$. Hence, the only solution of our problem is $T = 0$: the only way to restore the initial condition of static equilibrium is not to turn on the driving force at all!

Problem 4.

Imagine a point-like particle of mass m whose motion is confined to the surface of an infinite cylinder in three-dimensional Euclidean space with Cartesian coordinates x, y, z. This cylinder is extended along the y coordinate, and its cross-sections by planes of constant y are circles of constant radius R, $x^2 + z^2 = R^2$. Besides this constraint, there are no other forces acting on this particle (in particular, there is no gravitational force).

4(a) Write down the Lagrangian. Identify two continuous symmetries (besides time translation invariance), and find the corresponding conserved quantities using the Noether theorem. What is the physical interpretation of these symmetries and conservation laws from the perspective of the ambient three-dimensional space?

Solution.

This system has two degrees of freedom. We can parametrize the configuration space (=the cylinder) by two coordinates y and θ, with the latter defined such that

$$x = R\cos\theta, \quad z = R\cos\theta.$$

In these coordinates, the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}mR^2\dot{\theta}^2.$$

Note that θ is a periodic coordinate with periodicity 2π. This Lagrangian has two continuous symmetries: translations in y,

$$y \to y + y_0,$$

and translations in θ:

$$\theta \to \theta + \theta_0.$$

The corresponding conserved quantities I_y and I_θ can be extracted from the Noether theorem, leading to

$$I_y = m\dot{y}, \qquad I_\theta = mR^2\dot{\theta}.$$

This is the usual connection between translation invariance along a direction, and the corresponding conservation of the momentum associated with that direction.

In terms of the original three-dimensional space, translations along y are spatial translations, while the translations of θ are interpreted as spatial rotations around the y axis. Hence, what appears from the point of view of the generalized coordinate θ as conservation of linear momentum, is reinterpreted in the physical three-dimensional space as the conservation of the corresponding component of the angular momentum.

4(b) Using the variational principle of least action, show that between any two points P_1 and P_2 on the cylinder, the particle travels along a helical trajectory. Given the initial and final point, does this variational problem have more than one solution? If so, explain.

Solution.

The variational principle says that the first variation of the action functional S,

$$S = \int_{P_1}^{P_2} L(y, \theta, \dot{y}, \dot{\theta}) dt,$$

should be zero. In the process of evaluating the first variation, we also determine appropriate boundary conditions; in our case, we intend to keep the values of the coordinates at some initial time t_1 and some final time t_2 fixed. These initial and final values of the coordinates y and θ (i.e., the coordinates of the two points P_1 and P_2) will be denoted by y_1, θ_1 and y_2, θ_2, respectively.

The equation for the first variation of the action gives

$$0 = \delta S = \delta \int_{P_1}^{P_2} \left\{ \frac{1}{2} m\dot{y}^2 + \frac{1}{2} mR^2 \dot{\theta}^2 \right\} = \int_{P_1}^{P_2} \left\{ m\dot{y}\delta\dot{y} + mR^2 \dot{\theta}\delta\dot{\theta} \right\}.$$

Further integration by parts leads to

$$0 = \delta S = -\int_{P_1}^{P_2} \left\{ m\ddot{y}\delta y + mR^2 \ddot{\theta}\delta\theta \right\} + m(\dot{y}\delta y + R^2 \dot{\theta}\delta\theta)\Big|_{P_1}^{P_2}.$$

The last term vanishes by our assumption that the intial and final points P_1 and P_2 are kept fixed. Hence, the solution of the problem is linear motion,

$$y(t) = v_y t + y_0, \qquad \theta(t) = v_\theta t + \theta_0,$$

with v_y, v_θ, y_0 and θ_0 all constants to be determined from matching the boundary conditions. Setting $t_1 = 0$ without loss of generality, these boundary conditions are:

$$\begin{aligned} y(0) &= y_1, \qquad & y(t_2) &= y_2, \\ \theta(0) &= \theta_1, \qquad & \theta(t_2) &= \theta_2 + 2\pi n, \quad n \in \mathbf{Z}. \end{aligned}$$

Here we have taken into account the periodicity of θ. This set of boundary conditions has one solution for each integer value of the "winding number" n. Hence, the variational problem has a countably infinite number of solutions for each pair of points P_1 and P_2. We have also shown that they all correspond to linear motion along the cylinder, i.e., a helical motion in physical three-dimensional space.

4(c) Imagine the same system as in 4(a)-(b), except now the radius of the circular cross-section of the constraining surface depends exponentially on y:

$$x^2 + z^2 = R_0^2 e^{2y},$$

with R_0 a constant. Write down the Lagrangian for this system, derive its Hamiltonian, and write down explicitly the Hamilton (=canonical) equations of motion.

Solution.

We will use the same generaized coordinates y, θ. In terms of those, we now have

$$x = R_0 e^y \cos\theta, \qquad z = R_0 e^y \sin\theta.$$

The Lagrangian is now given by

$$\mathcal{L} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2} m (R_0^2 e^{2y} + 1)\dot{y}^2 + \frac{1}{2} m R_0^2 e^{2y} \dot{\theta}^2.$$

The canonical momenta are

$$p_y \equiv \frac{\partial \mathcal{L}}{\partial \dot{y}} = m(R_0^2 e^{2y} + 1)\dot{y},$$
$$p_\theta \equiv \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m R_0^2 e^{2y} \dot{\theta},$$

leading to the following Hamiltonian,

$$H = \frac{p_y^2}{2m(R_0^2 e^{2y} + 1)} + \frac{p_\theta^2}{2m R_0^2 e^{2y}}.$$

Notice also that since all terms in the Lagrangian are quadratic in the generalized velocities, the Hamiltonian is equal to the Lagrangian.

The canonical Hamilton equations of motion are:

$$\dot{y} = \frac{\partial H}{\partial p_y} \equiv \frac{p_y}{m(R_0^2 e^{2y} + 1)}, \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} \equiv \frac{p_\theta}{m R_0^2 e^{2y}},$$

and

$$\dot{p}_y = -\frac{\partial H}{\partial y} \equiv \frac{R_0^2 p_y^2 e^{2y}}{m(R_0^2 e^{2y} + 1)^2} + \frac{p_\theta^2}{m R_0^2 e^{2y}}, \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \equiv 0.$$

This concludes the solution of the exam.