Analysis of Command Detector Signal-to-Noise Ratio Estimator

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We have investigated a specific technique for making $(SNR)^{1/2}$ estimates by using in-phase channel output averages to estimate signal and quadrature channel output averages to estimate noise. We have produced bounds to determine the accuracy of this technique when fluctuations of one standard deviation occur. Our results show the estimate is relatively independent of actual input signal-to-noise ratio (SNR) and can be improved only by increasing the number of samples in the averages.

I. Introduction

The present design of a command detector being developed for NASA calls for a signal-to-noise ratio (SNR) estimate that will be used for monitoring operations of the detector. Naturally, we wish to know for a given SNR and number of samples how good the estimate will be. The samples are derived from in-phase and quadrature outputs that are effectively data and error integrators. As is to be expected, the command detector circuitry must be as simple as possible. For this reason the absolute value of the in-phase and quadrature outputs rather than their squares will be sampled. Furthermore, since samples of the absolute value of the output provide an estimate of the square root of the SNR rather than the SNR itself, in this report we will be concerned with the relationship between SNR, number of samples, and the estimated square root of SNR. We will use the ratio of the mean plus standard deviation of the estimate to actual $(SNR)^{1/2}$ as a measure of the accuracy of the estimate. This ratio will be determined as a function of number of samples.

In Section II, we will model the probability distribution of the in-phase and quadrature outputs and develop an estimate for the square root of SNR. In Section III we will give bounds for the mean and variance of this estimate that converge to the exact values as the number of samples becomes large. Because these asymptotically tight bounds are easily calculated, they are presented rather than the exact values whose integral representations required numerical integration for evaluation. Finally, in Section IV we will discuss results and conclusions.

II. Probability Distribution of Samples and Development of Estimate

A. Data Integrator Output

The output of the in-phase or data channel is integrated for one symbol time T. The absolute value of the resulting integration becomes a single sample so that the ith sample X_i can be expressed as

$$X_i = \left| \int_{t_i}^{t_i+T} [d(t) + n_I(t)]\, dt \right| \qquad (1)$$

where $d(t)$ is the data assumed to be $\pm A$ over one symbol time, and $n_I(t)$ is the noise of the in-phase channel assumed to be white Gaussian. Thus,

$$X_i = \left| \pm AT + \int_{t_i}^{t_i+T} n_I(t)\, dt \right| \qquad (2)$$

The random variable

$$Z_{I,i} \equiv \int_{t_i}^{t_i+T} n_I(t)\, dt$$

is zero mean Gaussian with variance

$$\sigma^2 = \int_{t_i}^{t_i+T} dt \int_{t_i}^{t_i+T} ds\, \overline{n_I(t)\, n_I(s)} = \frac{N_0 T}{2} \qquad (3)$$

where $N_0/2$ is the power spectral density of $n_I(t)$. Since the noise is assumed white Gaussian and the samples X_i are taken from non-overlapping intervals, they are independent and identically distributed. From Eqs. (2) and (3) we have for the probability density

$$P_X(\alpha) = \begin{cases} \dfrac{1}{(2\pi\sigma^2)^{1/2}} \left\{ \exp\left[-\dfrac{1}{2\sigma^2}(\alpha - AT)^2 \right] + \exp\left[-\dfrac{1}{2\sigma^2}(\alpha + AT)^2 \right] \right\}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \qquad (4)$$

where $\sigma^2 = N_0 T/2$.

B. Error Integrator Output

The output of the quadrature or error channel is also integrated for one symbol time and the absolute value taken to form a single sample Y_i:

$$Y_i = \left| \int_{t_i}^{t_i+T} n_Q(t)\, dt \right| \qquad (5)$$

where $n_Q(t)$ is the noise of the quadrature channel also assumed to be white Gaussian with power spectral density $N_0/2$. Because the samples Y_i are taken from non-overlapping intervals, they are independent and identically distributed. Their probability density is obtained from Eq. (4) with $AT = 0$:

$$P_Y(\alpha) = \begin{cases} \dfrac{2}{(2\pi\sigma^2)^{1/2}} \exp\left[-\dfrac{\alpha^2}{2\sigma^2} \right], & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \qquad (6)$$

C. Development of Estimate

The output of the in-phase channel before the absolute value is taken is

$$\int_{t_i}^{t_i+T} [d(t) + n_I(t)]dt = \pm AT + Z_{I,i} \qquad (7)$$

The square of the mean of this output is $(AT)^2$ while its variance (Eq. 3) is $\sigma^2 = N_0 T/2$. Conventionally, the input SNR is defined as the bit signal energy divided by the one-sided noise spectral density or

$$\frac{A^2 T}{N_0} = \frac{1}{2} \frac{(AT)^2}{(N_0 T/2)}$$

Thus the signal-to-noise ratio estimator of the command detector will estimate the quantity

$$(\text{SNR})^{1/2} \equiv \left[\frac{(AT)^2}{2(N_0 T/2)} \right]^{1/2} = \frac{AT}{\sigma} \frac{1}{2^{1/2}} \qquad (8)$$

In Appendix A we show that the mean of the sample X_i (Eq. 1) is given by

$$\overline{X_i} = AT\left[1 + 2\epsilon\left(\frac{AT}{\sigma} \right) \right] \qquad (9)$$

where the function $\epsilon(AT/\sigma)$ is defined by Eq. (A-4) of Appendix A. This function is sufficiently complicated so that forming an unbiased estimate of AT using only the $\{X_i\}$ is not feasible. Nevertheless, for input SNRs greater than 3 dB (the design point input SNR for the command detector is 10.5 dB), $|\epsilon| \leq 0.0042$. Consequently we can neglect ϵ and employ M samples of the $\{X_i\}$ to form our estimate ν of AT:

$$\nu \equiv \frac{1}{M} \sum_{i=1}^{M} X_i \qquad (10)$$

From Appendix A by setting $AT = 0$, we find the mean of Y_i:

$$\overline{Y}_i = \left(\frac{2}{\pi}\right)^{1/2} \sigma = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{N_0 T}{2}\right)^{1/2} \qquad (11)$$

Thus, an unbiased estimate of σ would be

$$\Delta \equiv \frac{1}{M}\left(\frac{\pi}{2}\right)^{1/2} \sum_{i=1}^{M} Y_i$$

However, σ is in the denominator of Eq. (8), so Δ^{-1} will not provide an unbiased estimate of σ^{-1}. As we show in Appendix B as $M \to \infty$, $\overline{\Delta^{-1}} \to \sigma^{-1}$ so we will use Δ^{-1} as an estimate of σ^{-1} even though it is not unbiased. Our estimate W of $(\mathrm{SNR})^{1/2}$ is, therefore,

$$W \equiv \frac{\nu \Delta^{-1}}{2^{1/2}} = \left(\frac{1}{M}\sum_{i=1}^{M} X_i\right)\left(\frac{1}{M}\left(\frac{\pi}{2}\right)^{1/2} \sum_{i=1}^{M} Y_i\right)^{-1} \frac{1}{2^{1/2}} \qquad (12)$$

where we note that $\{X_i\}$ and $\{Y_i\}$ are statistically independent.

III. Expressions for the Mean and Variance of Estimate

The mean and variance of our estimate W are given by

$$\overline{W} = \overline{\nu}\,\overline{\Delta^{-1}} \frac{1}{2^{1/2}} \qquad (13a)$$

$$2\sigma_W^2 = \overline{\nu^2}\,\overline{\Delta^{-2}} - (\overline{\nu})^2 (\overline{\Delta^{-1}})^2 = \sigma_\nu^2 \sigma_{\Delta^{-1}}^2 + \sigma_\nu^2 (\overline{\Delta^{-1}})^2 + (\overline{\nu})^2 \sigma_{\Delta^{-1}}^2 \qquad (13b)$$

The mean of ν is $\overline{X}_i$ and is given by Eq. (9), while the variance is $1/M$ times the variance of X_i; so from Appendix A:

$$\sigma_\nu^2 = \frac{1}{M}\sigma^2 \left\{1 - 4\left(\frac{AT}{\sigma}\right)^2 \left[\epsilon^2\left(\frac{AT}{\sigma}\right) + \epsilon\left(\frac{AT}{\sigma}\right)\right]\right\} \qquad (14)$$

In Appendix B we derive integral representations for the mean and variance of Δ^{-1}, but here we will display only easily calculated bounds for these quantities that are obtained from the integral representations:

$$\frac{1}{\sigma}\frac{M}{M+1} + \frac{1}{\sigma}\frac{4M}{(M-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2} < \overline{\Delta^{-1}} \le \frac{1}{\sigma}\frac{M}{M+1} + \frac{1}{\sigma}\frac{4M}{(M-1)(M+3)}\frac{2}{\pi} \qquad (15a)$$

$$\begin{aligned}\sigma_{\Delta^{-1}}^2 \le {} & \frac{1}{\sigma^2}\left\{\frac{64M(M+1)}{(M^2-4)(M+4)}\frac{1}{\pi^2} + \frac{8M}{(M+1)(M+4)}\frac{1}{\pi} - \frac{M^2}{(M+1)^2(M+2)}\right\} \\ & - \frac{1}{\sigma^2}\left\{\frac{8M^2}{(M^2-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2} + \left[\frac{4M}{(M-1)(M+3)}\right]^2\left(\frac{2}{\pi}\right)^{M+1}\right\}\end{aligned} \qquad (15b)$$

where the derivation requires $M > 3$. Since $(1/2)\,(AT/\sigma)^2$ is the actual SNR, we have the following bounds on our estimate W using Eqs. (9), (13), (14), and (15),

$$(\mathrm{SNR})^{1/2}\left[\frac{M}{M+1} + \frac{4M}{(M-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2}\right][1+2\epsilon] < \overline{W} \le (\mathrm{SNR})^{1/2}\left[\frac{M}{M+1} + \frac{4M}{(M-1)(M+3)}\left(\frac{2}{\pi}\right)\right]\left[1+2\epsilon\right] \qquad (16a)$$

$$\begin{aligned}\sigma_W^2 \le {} & \left\{\frac{64M(M+1)}{(M^2-4)(M+4)}\frac{1}{\pi^2} + \frac{8M}{(M+1)(M+4)}\frac{1}{\pi} - \frac{M^2}{(M+1)^2(M+2)} - \frac{8M^2}{(M^2-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2}\right. \\ & \left. + \left[\frac{4M}{(M-1)(M+3)}\right]^2\left(\frac{2}{\pi}\right)^{M+1}\right\}\left\{\frac{1}{2M}[1 - 8(\mathrm{SNR})(\epsilon^2+\epsilon)] + (\mathrm{SNR})[1+2\epsilon]^2\right\} \\ & + \left[\frac{M}{M+1} + \frac{4M}{(M-1)(M+3)}\left(\frac{2}{\pi}\right)\right]^2 [1 - 8(\mathrm{SNR})(\epsilon^2+\epsilon)]\frac{1}{2M}\end{aligned} \qquad (16b)$$

where $\epsilon = \epsilon([2(\mathrm{SNR})]^{1/2})$ is the function defined by Eq. (A-4) of Appendix A.

IV. Discussion of Results and Conclusions

We can measure the accuracy of the estimate by considering the ratio

$$\frac{\overline{W}_{ub} + \sigma_W}{(\mathrm{SNR})^{1/2}} \tag{17}$$

where $\overline{W}_{ub}$ is the upper bound on the mean $\overline{W}$ of Eq. (16a) and σ_W is the upper bound on the standard deviation of Eq. (16b). This ratio should be close to unity if our estimate is good and measures roughly how closely the estimate approximates $(\mathrm{SNR})^{1/2}$ when statistical fluctuations of one standard deviation occur. We have plotted this ratio in Fig. 1 as a function of M, the number of samples, for input SNRs (as defined by Eq. 8) of 0, 5.25, and 10.5 dB. The design point input SNR for the command detector is 10.5 dB so the 0 dB curve represents performance of the estimate when the input SNR is 10.5 dB below design.

From Fig. 1 we notice immediately two features of the estimate. First, the accuracy of the estimate is quite insensitive to actual input SNR: the ratios are within 0.15 dB of each other for $M \geq 16$ for input SNRs differing by 10.5 dB. Furthermore, increasing input SNR above 8 dB has virtually no effect on the estimate. Second, the accuracy is extremely dependent on the number of samples: an estimation accurate to within 1.5 dB requires about 45 samples while one good to within 0.5 dB requires more than 500 samples. The number of samples M must be increased to improve the estimate, and since the standard deviation of the estimate decreases as $M^{-1/2}$ for M large, an extremely large number of samples is required for very accurate estimates. Figure 1 gives quantitative support to these concluding remarks.

Finally, we should note that previous analyses (Refs. 1–7) have arrived at essentially the same conclusions. The estimate in this work is for $\mathrm{SNR}^{1/2}$ and is obtained from averages of absolute values of in-phase and quadrature channels. In the previous works the estimates were for SNR and were obtained from averages of squares of the relevant channel outputs. Intuitively, we might expect this work to agree qualitatively with the other analyses, but, in fact, it agrees quantitatively as well. For given input SNR and number of samples, averages of either absolute values or squares provide estimates of approximately the same accuracy (Ref. 7).

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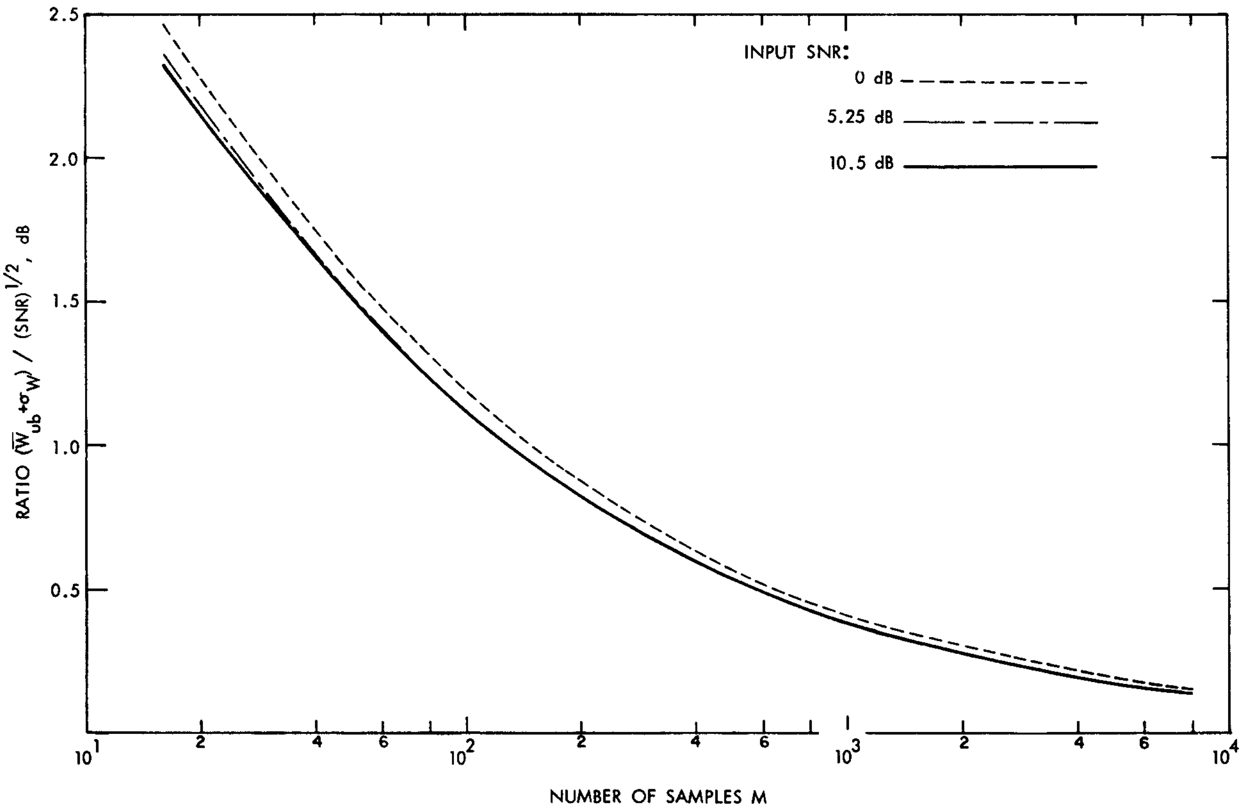

Fig. 1. Plots of ratio of upper bound of estimate W_{ub} plus one standard deviation σ_W to actual $SNR^{1/2}$ as a function of number of samples M. The three plots are for input SNRs of 0, 5.25, and 10.5 dB defined by Eq. (8) of the text.

Appendix A

In this appendix we will derive the mean and variance for a random variable X whose probability density is given by

$$P_X(\alpha) = \begin{cases} \frac{1}{(2\pi\sigma^2)^{1/2}}\left\{\exp\left[-\frac{1}{2\sigma^2}(\alpha - AT)^2\right] + \exp\left[-\frac{1}{2\sigma^2}(\alpha + AT)^2\right]\right\}, & \alpha \geq 0 \\ 0, & \alpha < 0 \end{cases} \quad \text{(A-1)}$$

I. Mean of X

The mean of X is given by

$$\begin{aligned} \overline{X} &= \int_{-\infty}^{\infty} \alpha P_X(\alpha)\, d\alpha \\ &= \int_{-AT/\sigma}^{\infty} (\sigma u + AT) \exp\left[-\frac{1}{2}u^2\right]\frac{du}{(2\pi)^{1/2}} \\ &\quad + \int_{AT/\sigma}^{\infty} (\sigma u - AT) \exp\left[-\frac{1}{2}u^2\right]\frac{du}{(2\pi)^{1/2}} \\ &= \left(\frac{2}{\pi}\right)^{1/2} \sigma \exp\left[-\frac{1}{2}\left(\frac{AT}{\sigma}\right)^2\right] \\ &\quad + AT\left[Q\left(-\frac{AT}{\sigma}\right) - Q\left(\frac{AT}{\sigma}\right)\right] \end{aligned} \quad \text{(A-2)}$$

where we use the function

$$Q(x) = \int_{x}^{\infty} \exp\left[-\frac{1}{2}u^2\right]\frac{du}{(2\pi)^{1/2}}$$

Since $Q(-x) = 1 - Q(x)$, we have

$$\overline{X} = AT\left[1 + 2\epsilon\left(\frac{AT}{\sigma}\right)\right] \quad \text{(A-3)}$$

where the function $\epsilon(y)$ is defined by

$$\epsilon(y) = \exp\left[-\frac{1}{2}y^2\right]\frac{1}{y}\frac{1}{(2\pi)^{1/2}} - Q(y) \quad \text{(A-4)}$$

and vanishes as $\exp[-y^2/2](1/y^3)$ as y becomes large.

II. Variance of X

The second moment of X is

$$\begin{aligned} \overline{X^2} &= \int_0^{\infty} \alpha^2 \exp\left[-\frac{1}{2}\left(\frac{\alpha - AT}{\sigma}\right)^2\right]\frac{d\alpha}{(2\pi\sigma^2)^{1/2}} \\ &\quad + \int_0^{\infty} \alpha^2 \exp\left[-\frac{1}{2}\left(\frac{\alpha + AT}{\sigma}\right)^2\right]\frac{d\alpha}{(2\pi\sigma^2)^{1/2}} \end{aligned} \quad \text{(A-5)}$$

If we substitute $\beta = -\alpha$ in the second integral, we have

$$\overline{X^2} = \int_{-\infty}^{\infty} \alpha^2 \exp\left[-\frac{1}{2}\left(\frac{\alpha - AT}{\sigma}\right)^2\right]\frac{d\alpha}{(2\pi\sigma^2)^{1/2}} = \sigma^2 + (AT)^2 \quad \text{(A-6)}$$

which is the second moment of a Gaussian random variable with mean AT and variance σ^2. The variance of X is

$$\sigma_X^2 = \overline{X^2} - (\overline{X})^2 = \sigma^2\left\{1 - 4\left(\frac{AT}{\sigma}\right)^2\left[\epsilon^2\left(\frac{AT}{\sigma}\right) + \epsilon\left(\frac{AT}{\sigma}\right)\right]\right\} \quad \text{(A-7)}$$

where $\epsilon(AT/\sigma)$ is given by Eq. (A-4).

Appendix B

In this appendix we wish to give integral representations and bounds for the mean and variance of Δ^{-1}, where

$$\Delta = \frac{1}{M}\left(\frac{\pi}{2}\right)^{1/2} \sum_{i=1}^{M} Y_i$$

I. Integral Representations

We will give a representation for $\overline{\Delta^{-L}}$, where $M > L + 1$, which insures convergence of the integral. From the probability density for Y_i (Eq. 6 of the text), we have

$$\overline{\Delta_M^{-L}} = \int \cdots \int d\alpha_1 \cdots d\alpha_M \exp\left[-\frac{\alpha_1^2 + \cdots + \alpha_M^2}{2\sigma^2}\right] \times \left(\frac{2}{\pi\sigma^2}\right)^{M/2} \left[\frac{1}{M}\left(\frac{\pi}{2}\right)^{1/2} \sum_{i=1}^{M} \alpha_i\right]^{-L} \qquad \text{(B-1)}$$

where the subscript M indicates M samples. We will utilize the following identity to obtain a representation for $[\alpha_1 + \cdots + \alpha_M]^{-L}$:

$$[c + \alpha_1 + \cdots + \alpha_M]^{-1} = \int_0^\infty d\beta \exp\{-\beta[c + \alpha_1 + \cdots + \alpha_M]\} \qquad \text{(B-2)}$$

Differentiating L times with respect to c and setting $c = 0$ gives

$$[\alpha_1 + \cdots + \alpha_M]^{-L} = \frac{1}{(L-1)!}\int_0^\infty \beta^{L-1} \exp\{-\beta[\alpha_1 + \cdots + \alpha_M]\} \qquad \text{(B-3)}$$

Substituting (B-3) into (B-1) and interchanging the order of integration gives

$$\overline{\Delta_M^{-L}} = \left(\frac{2}{\pi}\right)^{L/2} \frac{M^L}{(L-1)!}\int_0^\infty \beta^{L-1}\left[\int_0^\infty d\alpha \exp\left[-\frac{1}{2\sigma^2}(\alpha^2 + 2\sigma^2\beta\alpha)\right]\left(\frac{2}{\pi\sigma^2}\right)^{1/2}\right]^M d\beta$$
$$= \left[\left(\frac{2}{\pi}\right)^{1/2}\frac{M}{\sigma}\right]^L \frac{1}{(L-1)!}\int_0^\infty x^{L-1}\left[2\,Q(x)\exp\left(\frac{1}{2}x^2\right)\right]^M dx \qquad \text{(B-4)}$$

where

$$Q(x) \equiv \int_x^\infty \exp\left[-\frac{y^2}{2}\right]\frac{dy}{(2\pi)^{1/2}}$$

The mean and variance of Δ^{-1} are obtained from (B-4) using $L = 1,2$.

II. Bounds for Mean and Variance

The bounds we will obtain depend upon the inequality of Ref. (8), which for our application states:

$$\left(\frac{2}{\pi}\right)^{1/2}[x + (x^2+4)^{1/2}]^{-1} < Q(x)\exp\left[\frac{1}{2}x^2\right] \le \left(\frac{2}{\pi}\right)^{1/2}[x + (x^2 + 8/\pi)^{1/2}]^{-1} \qquad \text{(B-5)}$$

Applying this to (B-4) gives

$$\frac{1}{(L-1)!}\left(\frac{M}{\sigma}\right)^L 2^M\left(\frac{2}{\pi}\right)^{(M+L)/2}\int_0^\infty x^{L-1}[x + (x^2+4)^{1/2}]^{-M} dx < \overline{\Delta_M^{-L}} \le \frac{1}{(L-1)!} \times \left(\frac{M}{\sigma}\right)^L 2^M\left(\frac{2}{\pi}\right)^{(M+L)/2}\int_0^\infty x^{L-1}[x + (x^2 + 8/\pi)^{1/2}]^{-M} dx \qquad \text{(B-6)}$$

A. Bound for the Mean

Consider (B-6) when $L = 2$. The integrals involved can be done by letting $x = a \sinh(t)$, so

$$\int_0^\infty x\,[x + (x^2 + a^2)^{1/2}]^{-M}\,dx = \frac{1}{M^2 - 4}\,\frac{1}{a^{M-2}} \tag{B-7}$$

Thus, (B-6) becomes

$$\frac{1}{\sigma^2}\,\frac{4M^2}{M^2 - 4}\left(\frac{2}{\pi}\right)^{(M+2)/2} < \overline{\Delta_M^{-2}} \leq \frac{1}{\sigma^2}\,\frac{4M^2}{M^2 - 4}\left(\frac{2}{\pi}\right)^2 \tag{B-8}$$

We can obtain the bound on $\overline{\Delta_M^{-1}}$ by relating $\overline{\Delta_M^{-1}}$ to $\overline{\Delta_M^{-2}}$ Consider (B-4) for $L = 2$ and integrate by parts with $u = (Q(x))^M$ and $dv = \exp[-(1/2)x^2 M]\; x\,dx$:

$$\overline{\Delta_M^{-2}} = \left[\frac{M}{\sigma}\left(\frac{2}{\pi}\right)^{1/2}\right]^2 2^M \left\{\left[Q^M(x)\exp\left[\frac{1}{2}x^2 M\right]\frac{1}{M}\right]_0^\infty + \int_0^\infty Q^{M-1}(x)\exp\left[\frac{1}{2}x^2(M-1)\right]\frac{dx}{(2\pi)^{1/2}}\right\}$$

$$= -\frac{M}{\sigma^2}\,\frac{2}{\pi} + \frac{M^2}{M-1}\,\frac{1}{\sigma}\,\frac{2}{\pi}\,\overline{\Delta_{M-1}^{-1}} \tag{B-9}$$

Using (B-9) with inequality (B-8), we find:

$$\frac{1}{\sigma}\,\frac{M}{M+1} + \frac{1}{\sigma}\,\frac{4M}{(M-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2} < \overline{\Delta_M^{-1}}$$

$$\leq \frac{1}{\sigma}\,\frac{M}{M+1} + \frac{1}{\sigma}\,\frac{4M}{(M-1)(M+3)}\,\frac{2}{\pi} \tag{B-10}$$

B. Bound for the Variance

Consider (B-6) when $L = 3$. Again letting $x = a \sinh(t)$ simplifies the integrals giving:

$$\int_0^\infty x^2\,[x + (x^2 + a^2)^{1/2}]^{-M}\,dx = \frac{2M}{(M^2 - 1)(M^2 - 9)}\,\frac{1}{a^{M-3}} \tag{B-11}$$

So, (B-6) becomes

$$\frac{1}{\sigma^3}\,\frac{8M^4}{(M^2-1)(M^2-9)}\left(\frac{2}{\pi}\right)^{(M+3)/2} < \overline{\Delta_M^{-3}}$$

$$\leq \frac{1}{\sigma^3}\,\frac{8M^4}{(M^2-1)(M^2-9)}\left(\frac{2}{\pi}\right)^3 \tag{B-12}$$

We can relate $\overline{\Delta_M^{-3}}$ to $\overline{\Delta_M^{-2}}$, $\overline{\Delta_M^{-1}}$ by using $L = 3$ in (B-4) and integrating by parts with $u = x(Q(x))^M$ and $dv = \exp[(1/2)x^2 M]\; x\,dx$:

$$\overline{\Delta_M^{-3}} = \frac{1}{2}\left(\frac{M}{\sigma}\left(\frac{2}{\pi}\right)^{1/2}\right)^3 2^M \left\{\left[xQ^M(x)\exp\left[\frac{1}{2}x^2 M\right]\frac{1}{M}\right]_0^\infty - \frac{1}{M}\int_0^\infty \frac{d}{dx}[xQ^M(x)]\exp\left[\frac{1}{2}x^2 M\right]dx\right\}$$

$$= \frac{1}{2}\left(\frac{M}{\sigma}\left(\frac{2}{\pi}\right)^{1/2}\right)^3 2^M \left\{\frac{1}{(2\pi)^{1/2}}\int_0^\infty \left[Q(x)\exp\left(\frac{1}{2}x^2\right)\right]^{M-1} x\,dx - \frac{1}{M}\int_0^\infty \left[Q(x)\exp\left(\frac{1}{2}x^2\right)\right]^M dx\right\}$$

$$= \frac{1}{\pi\sigma}\,\frac{M^3}{(M-1)^2}\,\overline{\Delta_{M-1}^{-2}} - \frac{1}{\pi}\,\frac{M}{\sigma^2}\,\overline{\Delta_M^{-1}} \tag{B-13}$$

Using (B-13) with inequality (B-12) we find a bound for the variance $\sigma^2_{\Delta^{-1}}(M-1) \equiv \overline{\Delta_{M-1}^{-2}} - (\overline{\Delta_{M-1}^{-1}})^2$

$$\sigma^2_{\Delta^{-1}}(M-1) \leq \frac{1}{\sigma^2\pi^2}\,\frac{64M(M-1)}{(M+1)(M^2-9)} + \frac{1}{\sigma}\left(\frac{M-1}{M}\right)^2 \overline{\Delta_M^{-1}} - (\overline{\Delta_{M-1}^{-1}})^2 \tag{B-14}$$

Using bound (B-10) for $\overline{\Delta_M^{-1}}$, $\overline{\Delta_{M-1}^{-1}}$, we finally obtain

$$\sigma^2_{\Delta^{-1}}(M) \leq \frac{1}{\sigma^2}\left\{\frac{64M(M+1)}{(M^2-4)(M+4)}\,\frac{1}{\pi^2} + \frac{8M}{(M+1)(M+4)}\,\frac{1}{\pi} - \frac{M^2}{(M+1)^2(M+2)}\right\}$$

$$- \frac{1}{\sigma^2}\left\{\frac{8M^2}{(M^2-1)(M+3)}\left(\frac{2}{\pi}\right)^{(M+1)/2} + \left[\frac{4M}{(M-1)(M+3)}\right]^2\left(\frac{2}{\pi}\right)^{M+1}\right. \tag{B-15}$$