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Simple Analytic Model for Refractive X-ray Focusing

The demonstration by Snigirev et al. [?] that refractive focusing of x-rays can be accomplished by making a line of small holes in low-Z material, led Dave Nygren [?] to propose that a much more effective system could be fabricated using LIGA to introduce lenticular shapes into plastic or beryllium. Initial calculations and simulations by Cederstrom [?] give encouraging results. While it is straightforward to do ray tracing for the system envisaged, it is instructive to examine a very simple model that illustrates the main features and from which we can identify the pertinent physical variables.

The optical properties of low-Z materials in the x-ray region determine the characteristics of the lens system. The real part of the dielectric constant at these frequencies is [?]

(1)

where ω_p is the plasma frequency, which for graphite is about 25 eV [?]. Thus the real part of the index of refraction is less than one, and

$$\delta \equiv 1 - n = \frac{1}{2}\frac{\omega_p^2}{\omega^2} \tag{2}$$

The absorption length, λ, of carbon at an energy E in the 1 keV to 30 keV region is about [?]

$$\lambda = 2\text{mm}\left(\frac{E}{10\text{keV}}\right)^3 \tag{3}$$

From ordinary geometrical optics we know that the focal length f of a single thin lens is determined by the difference δ between the index of refraction in the lens (here empty space) and that of the surrounding medium (here the low-Z material), and the radius of curvature R of the lens:

$$f = \frac{R}{2\delta} \tag{4}$$

Combining N thin lenses close together gives a device with total focal length

$$F = \frac{f}{N} = \frac{R}{2\delta N} \tag{5}$$

We shall make the greatly simplying assumptions that we consider only small rays and that they undergo a negligible lateral displacement in passing through the system. Consider the geometry shown in Fig. 1. The source is located a distance s_1 from the thin lens. The slot through which the x rays must pass is located a distance s_2 beyond the lens. The beam is focused at $s_2 + \Delta$.

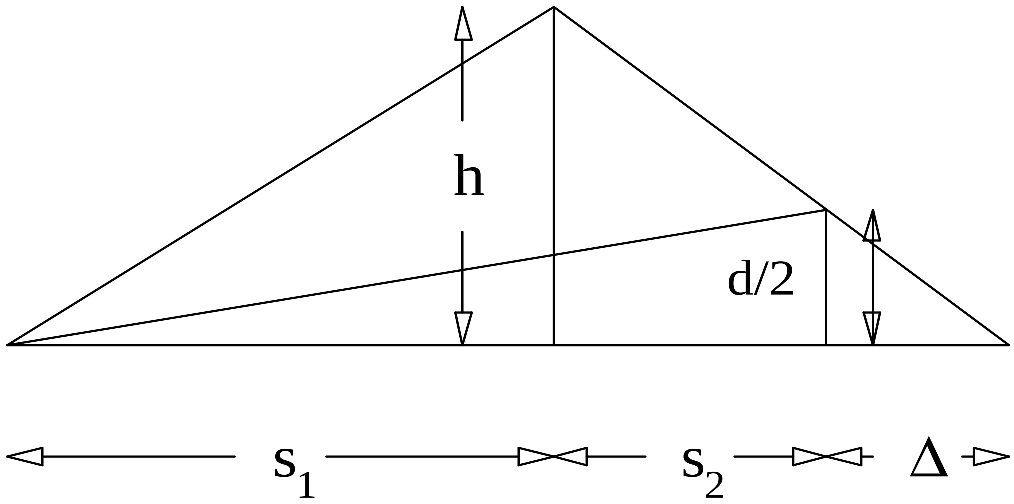

Figure 1: The geometry considered. The source is at s_1, the slot at s_2. The source is focused at $s_2 + \Delta$. The maximum lateral displacement of a ray that hits the slot is h.

It follows that

$$\frac{1}{s_1} + \frac{1}{s_2 + \Delta} = \frac{1}{F} \tag{6}$$

$$\frac{d/2}{\Delta} = \frac{h}{s_2 + \Delta} \tag{7}$$

In the small angle approximation, the maximal angle a ray can make with the horizontal and still strike the slot is

$$\theta = \frac{h}{s_1} = \frac{d/2}{s_1 s_2} \frac{1}{\epsilon} \tag{8}$$

where

$$\epsilon = \left| \frac{1}{s_1} + \frac{1}{s_2} - \frac{1}{F} \right| \tag{9}$$

The absolute value makes the relation valid even if the focus lies in front of the slot.

In the absence of the lens, the fraction of the x rays emitted by the source that would strike the slot would be

$$I_{no\ lens} = \frac{d}{s_1 + s_2} \tag{10}$$

With the lens present, but with no absorption of the x rays, these would be increased to

$$I_{with\ lens}^{no\ absorption} = \theta \tag{11}$$

though, this is valid only for small θ. To include absorption we simply incorporate a factor for the attenuation of the beam passing through the material. We assume the lenses have a radius of curvature R, so their surfaces have equations like

$$x = \frac{1}{2R} y^2 \tag{12}$$

A ray that has lateral displacement y traverses a length of absorbing material equal to

$$L = \frac{Ny^2}{R} \tag{13}$$

and is attenuated by a factor

$$e^{-\frac{Ny^2}{R\lambda}} \tag{14}$$

Thus the rms beam spread is

$$\sigma = \sqrt{\frac{R\lambda}{2N}} \tag{15}$$

Now the flux falling on the slot is given by an integral over the angle of the ray from the source:

$$
\begin{aligned}
I_{with\ lens}^{with\ absorption} &= \int_{-\theta}^{\theta} d\alpha \exp\left(\frac{-N s_1^2 \alpha^2}{R\lambda}\right) && (16)\\
&= \sqrt{\frac{\pi R\lambda}{N}} \frac{1}{s_1} \text{ erf}\left(\theta s_1 \sqrt{\frac{N}{R\lambda}}\right) && (17)\\
&= \sqrt{2\pi}\sigma \frac{1}{s_1} \text{ erf}\left(\frac{1}{2\sqrt{2} s_2 \epsilon} \frac{d}{\sigma}\right) && (18)
\end{aligned}
$$

Here we have used the error function

$$
\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-x^2) dx \tag{19}
$$

Finally, the gain is

$$
gain = I_{with\ lens}^{with\ absorption} / I_{without\ lens} = \sqrt{2\pi} \frac{s_1 + s_2}{s_1} \frac{\sigma}{d} \text{ erf}\left(\frac{1}{2\sqrt{2} s_2 \epsilon} \frac{d}{\sigma}\right) \tag{20}
$$

We can check some limits. First, in the absence of a lens, $\lambda = \infty$ and $F = \infty$. Thus σ becomes large and we use

$$
\text{erf}(z) \to \frac{2}{\sqrt{\pi}} z, \qquad z \to 0 \tag{21}
$$

and we find that the gain indeed goes to unity. The maximal gain occurs for perfect focus, i.e. $\epsilon \to 0$. Since

$$
\text{erf}(z) \to 1, \qquad z \to \infty \tag{22}
$$

we find that the maximal gain is

$$
gain_{max} = \sqrt{2\pi} \frac{s_1 + s_2}{s_1} \frac{\sigma}{d} \tag{23}
$$

If we take the source to infinity, we have

$$
s_2 \epsilon = 1 - \frac{s_2}{F} \tag{24}
$$

At some energy, E_0, the focus will be perfect and $\epsilon = 0$. At another energy, E, we have, according to Eqs.(**??**) and (**??**)

$$s_2 \epsilon = 1 - \frac{E^2}{E_0^2} \tag{25}$$

Defining $w = E/E_0$, we see that we can write, at least for some limited energy range,

$$\lambda = w^3 \lambda_0 \tag{26}$$
$$\sigma = w^{3/2} \sigma_0 \tag{27}$$

in an obvious notation.

Altogether for the case of the source at infinity, we have

$$gain = \sqrt{2\pi} \frac{\sigma_0}{d} w^{3/2} \text{ erf} \left(\frac{1}{2\sqrt{2}} \frac{w^{1/2}}{|w^2 - 1|} \frac{d}{\sigma_0} \right) \tag{28}$$

This is illustrated in Fig. **??**.

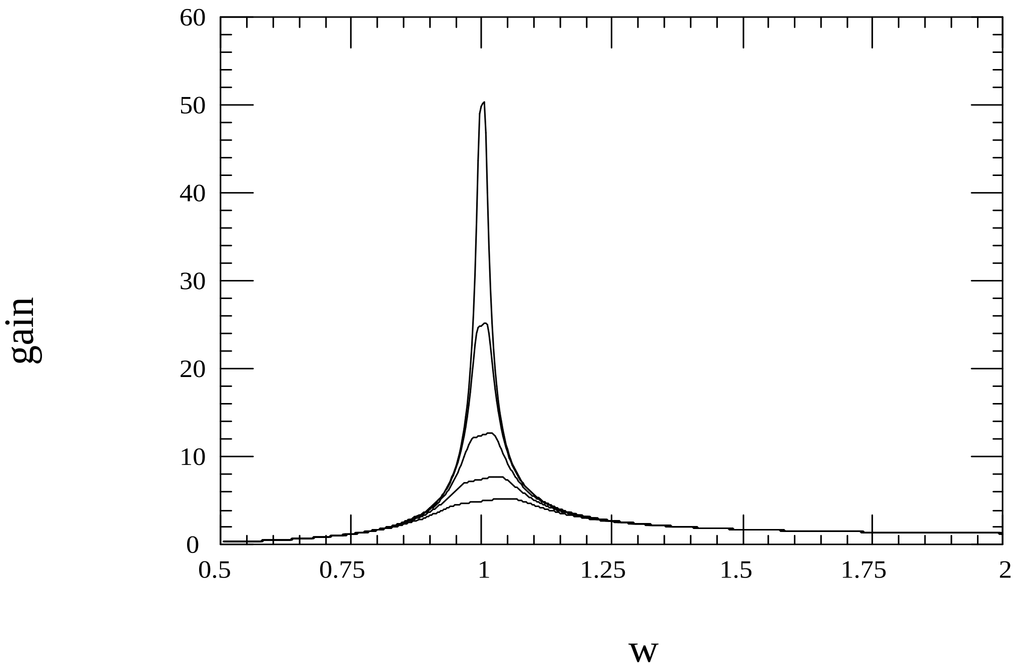

Figure 2: The gain as a function of the ratio of the energy E' to the energy at which the system is perfectly in focus. The curves are for values of the ratio $\sigma/d = 20, 10, 5, 3, 2$, with the highest ratio giving the highest curve.

A similar exercise in geometry for a point source a distance y above the horizontal axis yields the result

$$gain = \sqrt{\frac{\pi}{2}}\frac{s_1+s_2}{s_1}\frac{\sigma}{d}\left[\mathrm{erf}\left(\frac{d}{2\sqrt{2}\sigma\epsilon s_2}+\frac{y}{\sqrt{2}\sigma\epsilon s_1}\right)-\mathrm{erf}\left(-\frac{d}{2\sqrt{2}\sigma\epsilon s_2}+\frac{y}{\sqrt{2}\sigma\epsilon s_1}\right)\right] \tag{29}$$

A finite source size is thus easily simulated by integrating this over a suitable range in y.

A comparison of a point source with finite sources in shown in Fig. **??**.

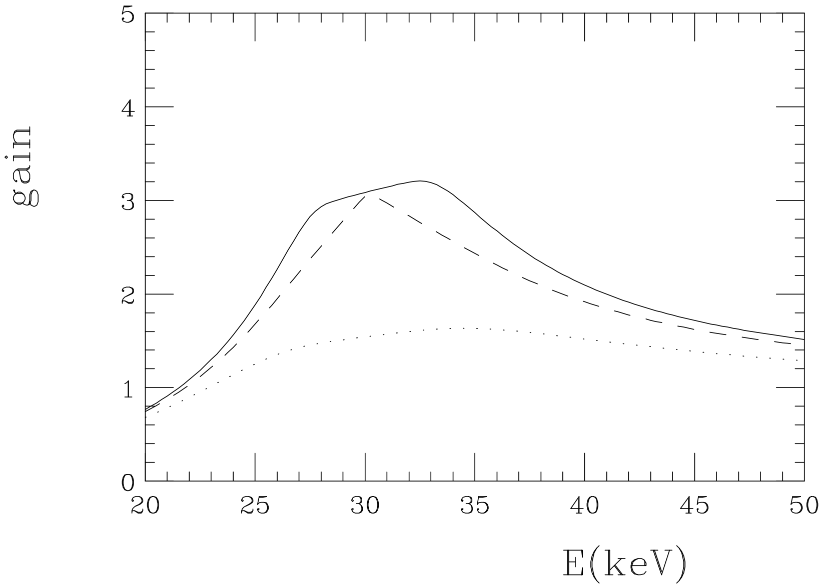

Figure 3: The gain as a function of the the energy for a lens system optimized for 30 keV. The lens is made of PMMA and has 1290 lenses, each with a radius of curvature of 150 microns. The source is 60 cm from the lens. The detector is 30 cm further and has a slot size of 50 microns. The curves shown are for a point source (solid), a source of 100 microns (50 microns above the horizontal axis and 50 microns below, dashes) and a source of 200 microns (dotted).

References

[1] A. Snigirev, V. Kohn, I. Snigireva, and B. Lengeler,"A Compound Refractive Lens for Focusing High Energy X-rays," *Nature*, **374**, 49 (1996).

[2] D. Nygren, private communication.

[3] B. Cederstrom, unpublished note, "X-Ray Focusing Using Refractive Optics in Low-Z Materials," May, 1997.

[4] See, for example, J. D. Jackson, *Classical Electrodynamics*, Second Edition, J. Wiley, New York, 1975, p. 288.

[5] J. D. Jackson, op. cit., p. 321.

[6] Particle Data Group, Review of Particle Properties, 1996, *Phys. Rev.* **D54** 1 (1996), p. 141.