

PRIMER ON THE CRAIG-BAMPTON METHOD

**AN INTRODUCTION TO BOUNDARY NODE FUNCTIONS,
BASE SHAKE ANALYSES,
LOAD TRANSFORMATION MATRICES,
MODAL SYNTHESIS AND MUCH MORE**

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October 4, 2000

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1. Introduction

The Craig-Bampton methodology is used extensively in the aerospace industry to re-characterize large finite element models into a set of relatively small matrices containing mass, stiffness and mode shape information that capture the fundamental low frequency response modes of the structure. The mode shape information consists of all boundary modes expressed in physical coordinates and a truncated set of elastic modes expressed in modal coordinates. These matrices are easily manipulated for a wide range of dynamic analyses. The method was first developed by Walter Hurty in 1964 (Ref. 1) and later expanded by Roy Craig and Mervyn Bampton in 1968 (Ref. 2).

The Craig-Bampton formulation is most often used for:

1. Efficient transmittal of spacecraft models to other organizations for a Coupled Loads Analysis (CLA); Craig-Bampton matrices are coupled with launch vehicle models and responses are determined for various flight events.
2. Base-shake analyses in which motion of the boundary degrees of freedom is specified for a model from a Coupled Loads Analysis and responses for various perturbations to the model may be determined without repeating the entire CLA;
3. Modal synthesis in which the models of two or more structures that have a common interface (each called a sub-structure) may be coupled together for an efficient analysis of the combined structure.

The purpose of this paper is to present a summary of the Craig-Bampton assumptions and methodology. Solutions to a few practical problems are outlined. Some familiarity with NASTRAN is assumed.

A secondary purpose is to present a standard set of notation that is based on NASTRAN DMAP terminology.

2. Development of Craig-Bampton Methodology

2.1 The Primitive Equation of Motion for a Structure

Complex linear elastic structures are universally analyzed today using finite element programs such as NASTRAN to generate mass and stiffness matrices that characterize the structure. The models are generally developed for static analyses and are thus very large, perhaps having several thousand degrees of freedom, since static analyses require a detailed set of grid points to map internal stresses and strains. However, dynamic analyses, which are based upon knowledge of fundamental frequencies and their associated mode shapes, are better performed with far fewer degrees of freedom in the formulation. Indeed the number of nodes needed to characterize the fundamental modes is relatively small. Furthermore, modes above 100 Hz are typically truncated since they contain too little energy to be physically significant.

When determining modes and mode shapes NASTRAN generates a set of critical degrees of freedom, called the A-set (analysis set), and uses Guyan reduction (see Appendix A) to generate equivalent mass $[M_{AA}]$ and stiffness $[K_{AA}]$ matrices that are associated with these freedoms. The analysis set typically contains a few hundred freedoms for a large finite element model that has several thousand degrees of freedom specified on GRID cards.

The corresponding displacements and accelerations for these degrees of freedom are contained in the matrices $[U_A]$ and $[\overset{\circ\circ}{U}_A]$. The applied forces are contained in the matrix $[F_A]$. The resulting undamped equation of motion for the free unconstrained structure is:

$$[M_{AA}] \left[\overset{\circ\circ}{U}_A \right] + [K_{AA}] [U_A] = [F_A] \quad \text{Eq. 2.1.1}$$

Since the Craig-Bampton method will require the use of boundary and interior points, it is convenient to partition these matrices into fixed, interfaced or supported boundary nodes, R, and the independent elastic nodes, L, as follows:

$$\{U_A\} = \begin{Bmatrix} U_R \\ U_L \end{Bmatrix} \quad \text{Eq. 2.1.2}$$

and Eq. 2.1.1 becomes

$$\begin{bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{bmatrix} \begin{bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{U}_L \end{bmatrix} + \begin{bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{bmatrix} \begin{bmatrix} U_R \\ U_L \end{bmatrix} = \begin{bmatrix} F_R \\ F_L \end{bmatrix} \quad \text{Eq. 2.1.3}$$

The boundary set includes not only the base degrees of freedom that might later be constrained but also the interface degrees of freedom that might later be coupled to another structure (modal synthesis). The boundary points can be released in later solutions.

The choice of notation R and L follows NASTRAN. The R-set are those degrees of freedom specified on a SUPORT card that remove rigid body motion from the structure and the L-set are those degrees of freedom that are left after removing the R-set from the A-set.

As a practical matter the effects of damping are considered when solving many dynamics problems. Damping is assumed to be proportional to the velocity of each point in the displacement set and the equation of motion becomes:

$$\begin{bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{bmatrix} \begin{bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{U}_L \end{bmatrix} + \begin{bmatrix} C_{RR} & C_{RL} \\ C_{LR} & C_{LL} \end{bmatrix} \begin{bmatrix} \overset{\circ}{U}_R \\ \overset{\circ}{U}_L \end{bmatrix} + \begin{bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{bmatrix} \begin{bmatrix} U_R \\ U_L \end{bmatrix} = \begin{bmatrix} F_R \\ F_L \end{bmatrix}$$

Eq. 2.1.4

where C_{LL} is typically the only non-zero term in the damping matrix.

2.2 The Craig-Bampton Transformation

There are two steps in performing the Craig-Bampton transformation. First and foremost, the set of elastic physical coordinates U_L for each mode is transformed to a set of modal coordinates Q_L . Thus the reduced finite element model discussed in Section 2.1 is transformed from a set of physical coordinates, U_A , to a hybrid set of physical coordinates at the boundary, U_R , and modal coordinates at the interior, Q_L . Identical solutions result from either formulation. The magic of the matrix Q_L is that each column, representing one mode shape, contains only one non-zero term.

Secondly, the set of modal solutions, Q_L , is truncated to some smaller set, say q_m . This is practical because in problems with multiple degrees of freedom the contribution of the higher frequency modes to the total response of a low frequency excitation is small and may be neglected. As a rule of thumb, the modal content of a given sub-structure should retain mode shapes with frequencies at least 1.5 times higher than that required in the composite structure in modal synthesis or 1.5 times higher than the excitation frequency.

The Craig-Bampton hybrid coordinates $\{U_R, q_m\}$ are related to the physical coordinates $\{U_R, U_L\}$ as follows:

$$\begin{Bmatrix} U_R \\ U_L \end{Bmatrix} = \{B \quad \Phi\} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix}, \quad m < L \quad \text{Eq. 2.2.1}$$

where $\{B\}$ has A rows and R columns and $\{\Phi\}$ has A rows and m columns. The vectors in $\{B\}$ are usually referred to as the Boundary Node Functions and the vectors in $\{\Phi\}$ are usually referred to as the Fixed Base Mode Shapes.

The Craig-Bampton transformation matrices $\{B\}$ and $\{\Phi\}$ may, in turn, be partitioned as:

$$\{B\} = \begin{Bmatrix} I \\ \phi_R \end{Bmatrix} \quad \{\Phi\} = \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix} \quad \text{Eq. 2.2.2}$$

where $[\phi_R]$ and $[\phi_L]$ are to be determined. The identity matrix $[I]$

has R rows and columns while matrix $[\phi_R]$ has L rows and R columns; the null matrix $[0]$ has R rows and m columns while the matrix $[\phi_L]$ has L rows and m columns. Thus

$$\{U_A\} = \begin{Bmatrix} U_R \\ U_L \end{Bmatrix} = \begin{Bmatrix} I & 0 \\ \phi_R & \phi_L \end{Bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} \quad \text{Eq. 2.2.3}$$

Note that the physical displacements of the interior points are computed by

$$[U_L] = [\phi_R][U_R] + [\phi_L][q_m] \quad \text{Eq. 2.2.4}$$

where $[\phi_R][U_R]$ are the rigid body displacements of the L degrees of freedom due to the R degrees of freedom and $[\phi_L][q_m]$ are the displacements of the L degrees of freedom relative to the displaced base.

Understanding the physical significance of the matrices $[\phi_R]$ and $[\phi_L]$ (or, alternatively, $\{B\}$ and $\{\Phi\}$) and learning how to compute and manipulate them readily is the essence of learning the Craig-Bampton method.

2.3 Boundary Node Functions (Constraint Modes), {B}

The Boundary Node Functions, {B}, are also known as Constraint Modes, Boundary Modes or Point Boundary Functions. The full set is composed of two sub-matrices, [I] and $[\phi_R]$. [I], with R rows and columns, is the identity matrix and is a mathematical statement of the obvious in Eq. 2.2.3, viz., the physical boundary points displace rigidly during rigid body motion. $[\phi_R]$, with L rows and R columns, is a transformation matrix that relates rigid body physical displacements at the interface, U_R , to physical displacements of the elastic degrees of freedom, U_L .

Allowing motion at the boundary produces the set of Boundary Node Functions. To determine $[\phi_R]$ fix all boundary degrees of freedom and limiting consideration to the static problem ($U_L = U_R = 0$). Eq. 2.1.3, the governing equation for this case, reduces to:

$$[K_{LR}][U_R] + [K_{LL}][U_L] = 0 \quad \text{Eq. 2.3.1}$$

Then, release the first degree of freedom in the boundary set, U_R , and solve for the vector of structural displacements due to unit displacement of this point. Next, re-fix the first degree of freedom, release the second to unit motion and solve for its vector of structural displacements. Continue the process, in sequence, for each degree of freedom in the boundary set. Thus, the set of boundary displacements, U_R , becomes:

$$[U_R] = \begin{bmatrix} 1 & 0 & \dots & \dots \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} = [I] \quad \text{Eq. 2.3.2}$$

which is a column matrix indicating the ordered sequence of unit displacements.

By combining Eqs. 2.2.4 (with $q_m = 0$ because there is no elastic deformation of non-boundary points), 2.3.1 and 2.3.2, the set of internal displacements, U_L , may be solved as

$$[U_L] = -[K_{LL}]^{-1}[K_{LR}][U_R] = [\phi_R][U_R] \quad \text{Eq. 2.3.3}$$

where

$$[\phi_R] = -[K_{LL}]^{-1}[K_{LR}] \quad \text{Eq. 2.3.4}$$

Note the matrix inversion. This requires a non-singular stiffness sub-matrix and is often the source of computer round-off error.

The full set of displacements is now formed as:

$$\begin{bmatrix} U_R \\ U_L \end{bmatrix} = \begin{bmatrix} I \\ -K_{LL}^{-1} K_{LR} \end{bmatrix} [U_R] = \begin{bmatrix} I \\ \phi_R \end{bmatrix} [U_R] = [B][U_R] \quad \text{Eq. 2.3.5}$$

where {B} is the matrix of Boundary Node Functions. The sub-matrix [I] has R rows and columns and the sub-matrix [ϕ_R] has L rows and R columns.

NASTRAN NOTE: As a practical matter, [ϕ_R] is routinely computed in NASTRAN and is data block DM generated by module RBMG3 whenever SUPORT bulk data cards are present. In Craig-Bampton analyses, SUPORT cards are often used to define the R-set degrees of freedom because the "free-free" problem is still solved in dynamics. The SUPORT card gives a rigid body eigen-vector at 0 Hz in the direction of specified support code. It replaces the lowest frequency in the free-free run with this value. If more than 6 degrees of freedom are supported, real flexible eigen-vectors are over-written by the generated SUPORT'd eigen-vectors. The SUPORT card is used to find eigen-vectors along the coordinate axes for rigid body modes that already exist.

The rigid body modes are embedded in these functions for both statically determinant and statically indeterminate structures. The columns of [ϕ_R] are rigid body displacements at the L degrees of freedom due to a unit motion at one of the R degrees of freedom. Note, however, that for a structure with a statically determinant boundary (exactly 6 degrees of freedom), [B] is just the matrix of 6 rigid body modes at the boundary points. For this simple case, rigid body modes can be computed from just the locations and orientations of the degrees of freedom.

2.4 Fixed Base Mode Shapes (Constraint Modes), $\{\Phi\}$

The matrix of Fixed Base Mode Shapes, $\{\Phi\}$, is composed of two sub-matrices, $[0]$ and $[\phi_L]$. The matrix $[0]$, with R rows and m columns, is the null matrix and is a mathematical statement of the obvious in Eq. 2.2.3, viz., rigid body displacements are indeed rigid and not a function of the elastic mode shapes, q_m .

The matrix $[\phi_L]$, with L rows and m columns, is a transformation matrix that relates modal responses, q_m , to physical displacements of the elastic degrees of freedom, U_L . It is determined from the equations of motion with the interface degrees of freedom constrained $[U_R = U_R = 0]$ and with no forces acting on the interior points $[F_L = 0]$. The equation of motion for this case is

$$[M_{LL}][U_{LL}] + [K_{LL}][U_L] = 0 \quad \text{Eq. 2.4.1}$$

Assume harmonic response ($U_L = \phi_L q_m e^{i\omega_o t}$). Then

$$\{K_{LL} - \omega_o^2 M_{LL}\}[\phi_L] = 0 \quad \text{Eq. 2.4.2}$$

Eq. 2.4.2 represents unforced harmonic motion of the grounded structure. It may be solved using NASTRAN or some other computer program for the eigen-values, $[\omega_o^2]$, and mode shapes, $[\phi_L]$.

The generalized (modal) mass, $[\mu]$, is defined to be

$$[\mu] = [\phi_L^T][M_{LL}][\phi_L] \quad \text{Eq. 2.4.3}$$

where $[\phi_L]$ is diagonal since the mode shapes $[\phi_L]$ are normal. The matrix $[\mu]$ is almost always normalized to unity, that is, the mode shapes are scaled so that $[\mu] = [I]$.

The generalized stiffness is defined to be:

$$[\phi_L^T][K_{LL}][\phi_L] = [\mu][\omega_o^2] \quad \text{Eq. 2.4.4}$$

where $[\mu][\omega_o^2]$ is also diagonal.

NASTRAN NOTE: In NASTRAN the eigen-vectors $[\phi_L]$ are a non-standard output but may readily be computed in the READ module using appropriate DMAP. The data block is typically named PHIL. It is also common to mass normalize $[\phi_L]$ such that $[\mu] = [I]$, the identity matrix. Note that the units of the mode shapes are the same as the physical degrees of freedom (inches, meters, radians) and the units on the eigen-values are (radians/sec)².

Having solved for ω_o^2 and ϕ_L , the transformation back to physical displacements, U_L , is accomplished by

$$\{U_L\} = [\phi_L][q_m] \quad \text{Eq. 2.4.5}$$

where

$[\phi_L]$ is the matrix of eigen-vectors called normal mode shapes, normal because each mode shape is orthogonal to all others; it has L rows and m columns.

$[\omega_o^2]$ is the matrix of eigen-values and is diagonal. The eigen-values have units of radians per second squared. The natural frequencies of the fixed base structure in Hertz are computed as $\sqrt{\omega_o^2/2\pi}$.

$[q_m]$ is the column vector of generalized (modal) displacements. The generalized displacements are dimensionless so all units (inches, meters, radians, etc.) are contained in ϕ_L .

2.5 The Craig-Bampton Method

The Craig-Bampton method is based on a re-formulation of the equations of motion for a structure from the set of physical coordinates to a set of coordinates consisting of physical coordinates at some subset of boundary points and modal or generalized coordinates at the non-boundary points. Once transformed to modal coordinates, mode shapes representing higher frequency responses may be truncated without loss of information.

To apply the method, transform the coordinate system for the equation of motion for a linear, damped elastic structure

$$\begin{bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{bmatrix} \begin{bmatrix} \overset{\infty}{U}_R \\ \overset{\infty}{U}_L \end{bmatrix} + \begin{bmatrix} C_{RR} & C_{RL} \\ C_{LR} & C_{LL} \end{bmatrix} \begin{bmatrix} \overset{o}{U}_R \\ \overset{o}{U}_L \end{bmatrix} + \begin{bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{bmatrix} \begin{bmatrix} U_R \\ U_L \end{bmatrix} = \begin{bmatrix} F_R \\ F_L \end{bmatrix} \quad \text{Eq. 2.1.3}$$

using the Craig-Bampton transformation

$$\{U_A\} = \begin{Bmatrix} U_R \\ U_L \end{Bmatrix} = \{B \ \Phi\} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \begin{Bmatrix} I & 0 \\ \phi_R & \phi_L \end{Bmatrix} \begin{Bmatrix} U_r \\ q_m \end{Bmatrix} \quad \text{Eq. 2.2.1}$$

where $\{B \ \Phi\}$ are given in Eq. 2.2.2. This yields equations of motion in terms of truncated modal coordinates:

$$\{M_{AA}\} \{B \ \Phi\} \begin{Bmatrix} \overset{\infty}{U}_R \\ \overset{\infty}{q}_m \end{Bmatrix} + \{C_{AA}\} \{B \ \Phi\} \begin{Bmatrix} \overset{o}{U}_R \\ \overset{o}{q}_m \end{Bmatrix} + \{K_{AA}\} \{B \ \Phi\} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \begin{Bmatrix} F_R \\ F_L \end{Bmatrix} \quad \text{Eq. 2.5.1}$$

Multiply Eq. 2.5.1 by the transpose of the Craig-Bampton transformation matrix, $\{B \ \Phi\}^T$, to yield

$$\begin{Bmatrix} B^T M B & B^T M \Phi \\ \Phi^T M B & \mu \end{Bmatrix} \begin{Bmatrix} \overset{\infty}{U}_R \\ \overset{\infty}{q}_m \end{Bmatrix} + \begin{Bmatrix} B^T C B & B^T C \Phi \\ \Phi^T C B & \Phi^T C \Phi \end{Bmatrix} \begin{Bmatrix} \overset{o}{U}_R \\ \overset{o}{q}_m \end{Bmatrix} + \begin{Bmatrix} B^T K B & B^T K \Phi \\ \Phi^T K B & \mu \omega_o^2 \end{Bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \{B \ \Phi\}^T \begin{Bmatrix} F_R \\ F_L \end{Bmatrix}$$

Eq. 2.5.2

Equation 2.5.2 is the Craig-Bampton equation of motion. These equations may be readily solved for a large number of practical problems. The transformation is successful because the modes become uncoupled from each other, greatly reducing the manipulation required to solve the equations. Typically, modal displacements and accelerations are computed by numerical integration for a given set of initial conditions and forcing function time histories. Physical accelerations and displacements follow from the Craig-Bampton transformation matrix.

Eq. 2.5.2 may be re-written as:

$$\begin{Bmatrix} M_{BB} & M_{Bm} \\ M_{mB} & M_{mm} \end{Bmatrix} \begin{Bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{q}_m \end{Bmatrix} + \begin{Bmatrix} C_{BB} & 0 \\ 0 & C_{mm} \end{Bmatrix} \begin{Bmatrix} \overset{\circ}{U}_R \\ \overset{\circ}{q}_m \end{Bmatrix} + \begin{Bmatrix} K_{BB} & 0 \\ 0 & K_{mm} \end{Bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \begin{Bmatrix} F_R + \phi_R^T F_L \\ \phi_L^T F_L \end{Bmatrix}$$

Eq. 2.5.3

where

$$\begin{aligned} M_{BB} &\equiv B^T M B = \begin{Bmatrix} I \\ \phi_R \end{Bmatrix}^T \begin{Bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{Bmatrix} \begin{Bmatrix} I \\ \phi_R \end{Bmatrix} \\ &= M_{RR} + M_{RL} \phi_R + \phi_R^T M_{LR} + \phi_R^T M_{LL} \phi_R \end{aligned} \quad \text{Eq. 2.5.4a}$$

[M_{BB} is the structural mass matrix reduced to the boundary nodes in the same way a Guyan reduction would be done.]

$$\begin{aligned} M_{Bm} &\equiv B^T M \Phi = \begin{Bmatrix} I \\ \phi_R \end{Bmatrix}^T \begin{Bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{Bmatrix} \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix} \\ &= [M_{RL} + \phi_R^T M_{LL}] \phi_L \end{aligned} \quad \text{Eq. 2.5.4b}$$

$$\begin{aligned} M_{mB} &\equiv \Phi^T M B = \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix}^T \begin{Bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{Bmatrix} \begin{Bmatrix} I \\ \phi_R \end{Bmatrix} \\ &= \phi_L^T [M_{LR} + M_{LL} \phi_R] \end{aligned} \quad \text{Eq. 2.5.4c}$$

$$\begin{aligned}
M_{mm} &\equiv \Phi^T M \Phi = \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix}^T \begin{Bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{Bmatrix} \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix} \\
&= \phi_L^T M_{LL} \phi_L = [\mu]
\end{aligned}$$

Eq. 2.5.4d

[M_{mm} is the generalized mass and is almost always set equal to the identity matrix.]

$$\begin{aligned}
K_{BB} &\equiv B^T K B = \begin{Bmatrix} I \\ \phi_R \end{Bmatrix}^T \begin{Bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{Bmatrix} \begin{Bmatrix} I \\ \phi_R \end{Bmatrix} \\
&= K_{RR} + K_{RL} \phi_R + \phi_R^T [K_{LR} + K_{LL} \phi_R] \\
&= K_{RR} + K_{RL} \phi_R
\end{aligned}$$

Eq. 2.5.4e

[K_{BB} is the Guyan reduced stiffness matrix and is zero if the boundary is statically determinant.]

$$\begin{aligned}
K_{Bm} &\equiv B^T K \Phi = \begin{Bmatrix} I \\ \phi_R \end{Bmatrix}^T \begin{Bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{Bmatrix} \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix} \\
&= [K_{RL} + \phi_R^T K_{LL}] \phi_L = 0
\end{aligned}$$

Eq. 2.5.4f

$$\begin{aligned}
K_{mB} &\equiv \Phi^T K B = \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix}^T \begin{Bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{Bmatrix} \begin{Bmatrix} I \\ \phi_R \end{Bmatrix} \\
&= \phi_L^T [K_{LR} + K_{LL} \phi_R] = 0
\end{aligned}$$

Eq. 2.5.4g

$$\begin{aligned}
K_{mm} &\equiv \Phi^T K \Phi = \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix}^T \begin{Bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{Bmatrix} \begin{Bmatrix} 0 \\ \phi_L \end{Bmatrix} \\
&= \phi_L^T K_{LL} \phi_L = [\mu][\omega_o^2]
\end{aligned}$$

Eq. 2.5.4h

[Note that ω_0 is the natural frequency (radians/second) of the fixed base modes.]

$$C_{mm} \equiv \Phi^T C \Phi = 2\zeta\mu\omega_0 \quad \text{Eq. 2.5.4i}$$

[Note that ζ is the equivalent viscous damping defined as the ratio of damping, c , to critical damping, c_0 ; typically ζ is 0.01 but can vary for each mode. The amplitude of the response of a structure to a steady state excitation is inversely proportional to the damping. Thus, if the damping is doubled, the response is halved. The amplitude of the response of a structure to a transient excitation is far less dependent on damping and, quite often, the difference in response between 1 and 2% damping is negligible.]

As a practical matter, $B^T C B$, $B^T C \Phi$ and $\Phi^T C B$ are nearly always chosen equal to [0]. Damping of the boundary modes is non-standard and cannot be verified by test; only the sub-matrix $[2\zeta\mu\omega_0]$ has significance.

Note from Eqs. 2.5.4b and 2.5.4c that $[M_{Bm}] = [M_{mB}]^T$. Eq. 2.5.4e utilizes the relation found in Eq. 2.3.4. The fact that Eqs. 2.5.4f and 2.5.4g are null follows from Eq. 2.3.4. Finally, Eqs. 2.5.4d and 2.5.4h follow from Eqs. 2.4.3 and 2.4.4. Recall that the matrix ϕ_L is often mass normalized such that $[\mu] = [I]$, the identity matrix.

The generalized mass and stiffness matrices are defined to be $[M_{mm}]$ and $[K_{mm}]$, respectively.

Typically, in the analysis of spacecraft payloads, all forces applied to the structure come through the boundary points and there are no applied loads to the non-boundary points, i. e., only F_r is of concern and F_L is null. However, in the full-coupled loads analysis of payload and booster, applied loads do act at non-boundary points in the booster and they must be considered.

In summary for most practical problems, the generalized mass matrix is normalized, damping is ignored and only boundary forces are considered. For these conditions the dynamic equation of motion for the Craig-Bampton method given in Eq. 2.5.1 and restated in Eq. 2.5.3 can be stated simply as

$$\begin{bmatrix} M_{BB} & M_{Bm} \\ M_{mB} & I \end{bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} + \begin{bmatrix} K_{BB} & 0 \\ 0 & \omega_o^2 \end{bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \begin{Bmatrix} F_R \\ 0 \end{Bmatrix} \quad \text{Eq. 2.5.5}$$

where K_{BB} is zero for a statically determinant interface.

NASTRAN Note: A typical NASTRAN DMAP sequence that would be required to generate a Craig-Bampton model is shown in Appendix B.

2.6 Checks on the Craig-Bampton Model

There are several simple checks that should always be applied to Craig-Bampton models to guard against errors.

2.6.1 Rigid Body Check

Test the equations of motion by enforcing rigid body displacements at the boundary degrees of freedom, U_R , and observing that the non-boundary or elastic freedoms displace appropriately. Either by hand for models with a simple boundary or by computer for a more complex geometry, construct the matrix of six rigid body modes, $[R]$, for the boundary degrees of freedom that satisfy the equation:

$$[U_R] = [R][q_{\text{rigid body}}] \quad \text{Eq. 2.6.1}$$

where

$[q_{\text{rigid body}}]$ is the (6,1) vector of pure rigid body displacements and rotations for x, y, z, Rx, Ry and Rz motion at some convenient point and in global coordinates.

As a common example, the matrix of rigid body modes for a simple 6-degree of freedom interface at point A relative to some arbitrary point B is:

$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 & \Delta Z & -\Delta Y \\ 0 & 1 & 0 & -\Delta Z & 0 & \Delta X \\ 0 & 0 & 1 & \Delta Y & -\Delta X & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{Eq. 2.6.2}$$

where $\Delta X = X_B - X_A$, $\Delta Y = Y_B - Y_A$ and $\Delta Z = Z_B - Z_A$. Assuming no elastic motion, $[q_m] = \begin{bmatrix} \infty \\ q_m \end{bmatrix} = 0$, and applying unit rigid body displacement (one inch and one radian) or acceleration (one inch/sec/sec or one radian/sec/sec), the resulting boundary motion is:

$$[U_R] = [R][I] \quad \text{and} \quad \begin{bmatrix} \infty \\ U_R \end{bmatrix} = [R][I] \quad \text{Eq. 2.6.3}$$

A sample DMAP for an equilibrium check is shown in Appendix C.

2.6.2 Rigid Body Mass Matrix

The Craig-Bampton equation of motion (Eq. 2.5.2) provides the (6,6) rigid body mass matrix

$$[R]^T [B^T M B][R] = \begin{bmatrix} m & 0 & 0 & 0 & mz & -my \\ 0 & m & 0 & -mz & 0 & mx \\ 0 & 0 & m & my & -mx & 0 \\ 0 & -mz & my & I_{xx} & I_{xy} & I_{xz} \\ mz & 0 & -mx & I_{yx} & I_{yy} & I_{yz} \\ -my & mx & 0 & I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \quad \text{Eq. 2.6.4}$$

where m is the total mass of the finite element model, x , y and z are the distances from the rigid body point to the center of mass, and the 'I' terms are the mass moments of inertia and products of inertia at the rigid body point.

2.6.3 Free Eigen-Value Problem

Since the Craig-Bampton equation of motion (Eq. 2.5.2) represents a free, unconstrained structure, it should yield natural frequencies and mode shapes for this condition. To check this, solve for the first few modes from the eigen-problem

$$\left\{ \begin{bmatrix} B^T K B & 0 \\ 0 & \mu \omega_o^2 \end{bmatrix} - \lambda^2 \begin{bmatrix} B^T M B & B^T M \Phi \\ \Phi^T M B & \mu \end{bmatrix} \right\} \{ \Phi \} = 0 \quad \text{Eq. 2.6.5}$$

The results should check the free modes from the original finite element model. If the boundary is statically determinate, $B^T K B = 0$ and the rigid body modes are zero by definition. However, for indeterminate boundaries, the eigen-value problem solution can give insight into the boundary modes.

To note the frequency range represented by the boundary functions, it is instructive to remove all fixed base modes and solve the eigen-problem:

$$\{ B^T K B \} - \lambda^2 \{ B^T M B \} \{ \Phi \} = 0 \quad \text{Eq. 2.6.6}$$

Print the eigen-values, λ , to see the range.

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2.6.2

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3. Common Applications of Craig-Bampton Methodology

3.1 Quasi-Static Analysis

It is often desirable to compute the boundary forces and internal displacements of a structure under a quasi-static g-field to check design load cases, set up initial conditions or look for potential errors in the analysis. The Craig-Bampton method is well suited for this computation.

Set up the boundary acceleration, $\begin{bmatrix} \infty \\ U_R \end{bmatrix}$, equal to the desired steady static acceleration, $[G]$, as though the structure were riding on a multi-degree of freedom elevator. Then

$\begin{bmatrix} \infty \\ U_R \end{bmatrix} = [G]$ is specified, $\begin{bmatrix} \infty \\ q_m \end{bmatrix} = 0$ because the load is static,

$\begin{bmatrix} \circ \\ U_R \end{bmatrix}$ is ignored, $\begin{bmatrix} \circ \\ q_m \end{bmatrix} = 0$ because $\begin{bmatrix} \infty \\ q_m \end{bmatrix} = 0$, and U_R does not need to be known since $[B^T K B]U_R = [K_{BB}][U_R] = 0$ for this case.

Consider the Craig-Bampton equation of motion, Eq. 2.5.2. The upper portion yields for boundary forces:

$$[F_B] = [B^T M B][G] = M_{BB} G \quad \text{Eq. 3.1.1}$$

and the lower portion yields for displacements:

$$[q_m] = -[\mu \omega_o^2]^{-1} [\Phi^T M B][G] = M_{mB} G \quad \text{Eq. 3.1.2}$$

3.2 Base-Shake Analyses

Base shake analyses are very common in structural dynamics and are ideally suited to the Craig-Bampton formulation. The input driving function in a base shake may be sinusoidal, constant, random or transient. Steady state sinusoidal inputs are used to simulate a base shake test that is a standard method of testing hardware. Transient inputs are often derived from a large coupled loads analysis involving the Shuttle and one or more payloads and then, as payload models are refined, the interface transients are applied to the component models rather than repeating the long and costly coupled loads analysis. The base shake analysis has proven to be a very efficient method for the iterative analysis of components without re-analyzing the entire structure.

The base-drive problem is characterized by applying known accelerations to the boundary and solving for the response of the complete structure. The Craig-Bampton formulation is particularly well suited for this analysis because the boundary is explicitly defined. $[F_R]$ is the set of forces that are exerted on the boundary and result in the base motions $[U_R]$. It should be emphasized that the forces, $[F_R]$, are usually not known initially; only the accelerations at the R degrees of freedom are known. The forces $[F_R]$ are really the forces of constraint that are required to produce the desired base motion.

Assume that $U_R(t)$ is known either as a transient or as steady state input. Then the Craig-Bampton equation of motion (Eq. 2.5.3) reduces to the equation for generalized response

$$\mu \ddot{q}_m(t) + 2\zeta\mu\omega_o \dot{q}_m(t) + \mu\omega_o^2 q_m(t) = -M_{mB} \ddot{U}_R(t) + \phi_L^T F_L(t) \quad \text{Eq. 3.2.1}$$

which is the standard form for dynamics equations and can readily be solved for q_m , \dot{q}_m and \ddot{q}_m . For the typical base shake analysis, $F_L(t)=0$ and the forcing function is just from the base acceleration, $-M_{mB} \ddot{U}_R(t)$.

The reactions at the boundary can also be found from Eq. 2.5.2 as functions of time:

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$$\mu \overset{\infty}{\ddot{q}}_m(t) + 2\zeta\mu\omega_o \overset{o}{\dot{q}}_m(t) + \mu\omega_o^2 q_m(t) = -M_{mB} \overset{\infty}{\ddot{U}}_R(t) + \phi_L^T F_L(t) \quad \text{Eq. 3.2.1}$$

which is the standard form for dynamics equations and can readily be solved for q_m , $\overset{o}{\dot{q}}_m$ and $\overset{\infty}{\ddot{q}}_m$. For the typical base shake analysis, $F_L(t)=0$ and the forcing function is just from the base acceleration, $-M_{mB} \overset{\infty}{\ddot{U}}_R(t)$.

The reactions at the boundary can also be found from Eq. 2.5.2 as functions of time:

$$[F_R(t)] = M_{BB} \overset{\circ\circ}{U}_R(t) + M_{Bm} \overset{\circ\circ}{q}_m(t) + C_{BB} \overset{\circ}{U}_R(t) + K_{BB} U_R(t) - \phi_R^T F_L(t) \quad \text{Eq. 3.2.2}$$

where, as explained in Section 6, C_{BB} is almost always null and K_{BB} is zero for statically determinant boundaries. Thus, for many problems Eq. 3.2.2 becomes:

$$[F_R(t)] = M_{BB} \overset{\circ\circ}{U}_R(t) + M_{Bm} \overset{\circ\circ}{q}_m(t) - \phi_L^T F_L(t) \quad \text{Eq. 3.2.3}$$

NASTRAN Note: The matrix M_{BB} is data block MR generated by NASTRAN module RBMG4 when SUPORT cards are used.

The matrix $F_L(t)$ is typically null since forces are applied only at the boundary. Also, for ease of manipulation Eq. 2.5.2, the equation of motion, is re-written to include the boundary acceleration as an output as well an input quantity. This results in the uncoupled linear equation:

$$\begin{bmatrix} I & 0 \\ 0 & \mu \end{bmatrix} \begin{bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{q}_m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2\zeta\mu\omega_o \end{bmatrix} \begin{bmatrix} \overset{\circ}{U}_R \\ \overset{\circ}{q}_m \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \mu\omega_o^2 \end{bmatrix} \begin{bmatrix} U_R \\ q_m \end{bmatrix} = \begin{bmatrix} I \\ -\phi_R^T MB \end{bmatrix} \overset{\circ\circ}{U}_R$$

Eq. 3.2.4

The physical acceleration response is computed from

$$\left\{ \overset{\circ\circ}{U}_A \right\} = \left\{ \overset{\circ\circ}{U}_R \right\} = \begin{bmatrix} I & 0 \\ \phi_R & \phi_L \end{bmatrix} \begin{bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{q}_m \end{bmatrix} \quad \text{Eq. 3.2.5}$$

Eq. 3.2.1 is readily solved using numerical integration routines (such as NASINT for transient response and SSR for steady state response in FLAME) for the modal displacements, q_m , and modal accelerations, $\overset{\circ\circ}{q}_m$.

3.3 Modal Participation Factors and Modal Masses for Base-Shake Analyses

It is very common in the aerospace industry to test structures on a shake table. In this test known accelerations are applied to the base of a structure. These displacements may be sinusoidal (typically a controlled sweep over a given frequency range or a sine burst at a given frequency) or random (typically with a controlled energy content). Base shakes are performed for a variety of reasons. Quite often one goal is to measure response levels and frequencies with accelerometers at several critical locations and then to use this data as a test basis for verifying the finite element model.

Finite element models can predict eigen-values and eigen-vectors for a structure that correlate very closely with measured test data. However they do not give a clear indication of which eigen-vectors will be important contributors in subsequent frequency response analysis. Knowing the relative importance of each mode in terms of how much mass is moving and in what direction is crucial for good design of structures.

Modal participation factors, which are a property of the structure just as generalized mass and stiffness are, can be calculated from the mode shapes of the structure. Consider a slightly more general form of the Craig-Bampton equation of motion for a damped structure on a shaker table, Eq. 3.3.10, where the mode shapes are not necessarily mass normalized:

$$[\mu][\ddot{q}_m] + 2[\mu][\xi][\dot{q}_m] + [\mu][\omega_o^2][q_m] = -[M_{mB}][\ddot{U}_R] \quad \text{Eq. 3.3.1}$$

Note that the coefficients of the generalized motions are all diagonal matrices and that $[\mu] = [\phi_L]^T [M_{LL}][\phi_L]$ from Eq.

2.3.3 and $M_{mB} = \phi_L^T (M_{LR} + M_{LL} \phi_R)$ from Eq. 2.2.7.

Each individualized modal equation, i , is:

$$\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = -(1/m_i) \phi_{iL}^T (M_{LR} + M_{LL} \phi_R) \ddot{U}_R \quad \text{Eq. 3.3.2}$$

where ϕ_{iL} is the row of ϕ_L associated with the mode i and where m_i , the generalized mass for mode i , equals $[\phi_{iL}]^T [M_{LL}][\phi_{iL}]$.

Define an $L \times R$ matrix of factors, $[p]$, with rows

$$p_{iR} = (l/m_i) \phi_{iL}^T (M_{LR} + M_{LL} \phi_R) \quad \text{Eq. 3.3.3}$$

Each row of [p] contains as many terms as there are motions for the shaker, or equivalently, as many degrees of freedom as are defined in the R-set. In a base shake, motions are applied either at only one point or at a number of points which are rigidly connected to a single point so the maximum number of freedoms in the R-set should not exceed 6, i. e., the problem is statically determinant. Each column of [p] contains the factors for all the modes associated each degree of freedom in the R-set. Each term in [p] is interpreted as a modal participation factor since the solution to Eq. 3.3.2 for some specified base motion at one degree of freedom will be proportional to the corresponding term in [p].

An important observation from Eq. 3.3.3 is that, for a determinant system that has been mass normalized, the matrix M_{mB} is in reality the matrix of modal participation factors.

Modal participation factors are useful in predicting the modal amplification of a sinusoidal input at resonance. Assume a shaker motion that is a steady state sinusoidal input at some frequency ω_j and amplitude \bar{U}_R such that

$$U_R = \bar{U}_R e^{i\omega_j t} \quad \text{and} \quad \overset{\infty}{U}_R = -\omega_j^2 \bar{U}_R e^{i\omega_j t} \quad \text{Eq. 3.3.4}$$

The equation of motion for each mode becomes:

$$\begin{aligned} \overset{\infty}{q}_i + 2\xi_i \omega_i \overset{o}{q}_i + \omega_i^2 q_i &= -p_{iR} \overset{\infty}{U}_R \\ &= p_{iR} \omega_j^2 \bar{U}_R e^{i\omega_j t} \end{aligned} \quad \text{Eq. 3.3.5}$$

Use the method of undetermined coefficients to solve for the generalized motions. Assume $q_i = C e^{i\omega_j t}$, $\overset{o}{q}_i = i\omega_j C e^{i\omega_j t}$ and $\overset{\infty}{q}_i = -\omega_j^2 C e^{i\omega_j t}$. Substituting these values into Eq. 3.3.4 and noting that at resonance $\omega_i = \omega_j$, $C = \left(i p_{iR} \bar{U}_R / 2\xi_i \right)$ and

$$\overset{\infty}{q}_i = -i \left[p_{iR} Q_i \bar{U}_R \right] e^{i\omega_j t} \quad \text{Eq. 3.3.6}$$

where $Q_i = (1/2\xi_i)$ is the modal amplification of mode i . Eq. 3.3.6 indicates that, at resonance, the modal acceleration is lagging the base acceleration by 90 and that modal acceleration is proportional to the product of the modal participation factor, modal amplification and the magnitude of the shaker input.

Modal mass is an analytical measure of how much mass is moving in each translational and rotational direction for each mode. The total translational mass in each direction equals the total analytical mass in that direction. The total rotational inertia about each axis equals the total rotational inertia about the support point. By scanning the modal mass values in each direction, the analyst may accurately decide at what frequency the mode shapes may be truncated without losing appreciable model accuracy.

Modal masses are related to the constraint forces. Consider the constraint forces previously derived for a base shake analysis (Eq. 3.2.2) and note that for a determinant boundary $[K_{BB}] = [0]$. Then

$$\begin{aligned} [F_R] &= [M_{BB}] \begin{bmatrix} \infty \\ \infty \\ \infty \end{bmatrix} + [M_{Bm}] [q_m] \\ &= [M_{BB}] \begin{bmatrix} \infty \\ \infty \\ \infty \end{bmatrix} + [(M_{RL} + \phi_R^T M_{LL}) \phi_L] \begin{bmatrix} \infty \\ \infty \\ \infty \end{bmatrix} \end{aligned} \quad \text{Eq. 3.3.7}$$

Neglect the component due to shaker motion and consider only the elastic component of $[F_R]$. The constraint force for each mode i becomes:

$$F_{iR} = (M_{RL} + \phi_R^T M_{LL}) \phi_{Li} q_i \quad \text{Eq. 3.3.8}$$

The complex amplitude of this force, using Eq. 3.3.5, is:

$$\begin{aligned} F_{iR} &= (M_{RL} + \phi_R^T M_{LL}) \phi_{Li} \left(i p_{iR} Q_i \bar{U}_R \right) \\ &= i (M_{LL} \phi_R + M_{LR})^T \phi_{Li} (1/m_i) \phi_{Li}^T (M_{LL} \phi_R + M_{LR}) Q_i \bar{U}_R \\ &= -i \bar{M}_{RR}^i Q_i \bar{U}_R \end{aligned} \quad \text{Eq. 3.3.9}$$

where

$$\bar{M}_{RR}^i = (M_{LL} \phi_R + M_{LR})^T \phi_{Li} (1/m_i) \phi_{Li}^T (M_{LL} \phi_R + M_{LR}) \quad \text{Eq. 3.3.10}$$

is the modal mass matrix for mode i.

In practice, $[\bar{M}_{RR}]$ may be readily computed from the modal participation factor, $[p_{iR}]$, defined by Eq. 3.3.3. The modal mass for mode i is:

$$M_{RR}^i = [p_{iR}]^T m_i [p_{iR}] \quad \text{Eq. 3.3.11}$$

It is interesting to note that summing for all the modes (m=L) yields \bar{M}_{RR} , a square matrix of size R, where:

$$\begin{aligned} \bar{M}_{RR} &= (M_{LL} \phi_R + M_{LR})^T M_{LL} (M_{LL} \phi_R + M_{LR}) \\ &= (M_{RL} + \phi_R^T M_{LL}) (\phi_R + M_{LL} M_{LR}) \\ &= M_{RL} \phi_R + \phi_R^T M_{LL} + \phi_R \phi_R^T M_{LR} + M_{RL} M_{LL} M_{LR} \end{aligned} \quad \text{Eq. 3.3.12}$$

If there are no masses at the R-set degrees of freedom, then $\bar{M}_{RR} = M_{BB}$, the rigid body mass matrix with respect to the boundary.

It should be noted that modal participation factors and modal masses could be readily computed from the physical model using NASTRAN rigid solution 3 with just a few lines of DMAP. This is presented in Appendix D.

As a final note, base shake problems are easily solved using matrix manipulation programs such as FLAME. However, if the problem is to be solved in NASTRAN, base accelerations are typically applied by adding a fictitious seismic mass to the R-set degrees of freedom and then applying the appropriate force to the base such that the desired accelerations result.

3.4 The Load Transformation Matrix (LTM)

The Load Transformation Matrix (LTM) is used to recover physical parameters such as accelerations, displacements, forces and stresses from the original finite element model using the generalized accelerations and, for redundant interfaces, the physical displacements of the boundaries. Although the use of the LTM is standard practice in the aerospace industry, several definitions of the LTM exist, depending on company practice and the nature of the problem being solved. Swales Aerospace uses an LTM based on recovering A-set displacements from the physical model; this approach is well-suited for NASTRAN since all element forces and stresses are derived from the displacement field.

3.4.1 Displacement LTM's

The functional relationship between A-set displacements and generalized motions (including boundary displacements) can be established as follows. Recall that the primitive equation of motion for a structure in finite element analysis is:

$$\begin{Bmatrix} M_{RR} & M_{RL} \\ M_{LR} & M_{LL} \end{Bmatrix} \begin{Bmatrix} \overset{\circ\circ}{U}_R \\ \overset{\circ\circ}{U}_L \end{Bmatrix} + \begin{Bmatrix} K_{RR} & K_{RL} \\ K_{LR} & K_{LL} \end{Bmatrix} \begin{Bmatrix} U_R \\ U_L \end{Bmatrix} = \begin{Bmatrix} F_R \\ F_L \end{Bmatrix} \quad \text{Eq. 2.1.2}$$

where it will be assumed that no interior forces act ($F_L = 0$).

Thus,

$$[M_{LR}][\overset{\circ\circ}{U}_R] + [M_{LL}][\overset{\circ\circ}{U}_L] + [K_{LR}][U_R] + [K_{LL}][U_L] = [0] \quad \text{Eq. 3.4.1}$$

and

$$\begin{aligned} [U_L] &= -[K_{LL}]^{-1} \left\{ [M_{LR}][\overset{\circ\circ}{U}_R] + [M_{LL}][\overset{\circ\circ}{U}_L] + [K_{LR}][U_R] \right\} \\ &= -[K_{LL}]^{-1} [M_{LR}][\overset{\circ\circ}{U}_R] - [K_{LL}]^{-1} [M_{LL}][\overset{\circ\circ}{U}_L] + \phi_R [U_R] \end{aligned} \quad \text{Eq. 3.4.2}$$

since, from Eq. 2.4.3, $[\phi_R] = -[K_{LL}]^{-1} [K_{LR}]$.

Using the Craig-Bampton transformation (Eqs. 2.2.1 and 2.2.2),

$$\begin{aligned}
 [U_L] &= \{ [K_{LL}^{-1} M_{LR}] \quad [-K_{LL}^{-1} M_{LL}] \} \begin{Bmatrix} I & 0 \\ \phi_R & \phi_L \end{Bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} + [\phi_R][U_R] \\
 &= \{ [K_{LL}^{-1} M_{LR} - K_{LL}^{-1} M_{LL} \phi_R] \quad [-K_{LL}^{-1} M_{LL} \phi_L] \} \begin{Bmatrix} I & 0 \\ \phi_R & \phi_L \end{Bmatrix} \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} + [\phi_R][U_R]
 \end{aligned}$$

Eq. 3.4.3

Merge U_R and U_L to form the A-set displacements:

$$[U_A] = \begin{bmatrix} 0 & 0 & I \\ -K_{LL}^{-1} [M_{LR} + M_{LL} \phi_R] & -K_{LL}^{-1} M_{LL} \phi_L & \phi_R \end{bmatrix} \begin{bmatrix} U_R \\ q_m \\ U_R \end{bmatrix}$$

Eq. 3.4.4

Eq 3.4.4 forms the basis for the LTM development. Appendix E presents a NASTRAN DMAP sequence for solving this equation. Element forces and stresses follow from knowledge of the displacement field. The desired components of the LTM are requested in the NASTRAN output file. A post-processor then reads this data and reformats it appropriately. The LTM has units element forces (or stresses) per modal acceleration (or unit interface displacement). Typically stress equations are also coded in matrix form with units stress per element force (or interface displacement). The analytical flow is as follows:

$$\begin{aligned}
 [\text{Stress}(\text{time})] &= [\text{Stress/element force}] \times \\
 &\quad [\text{Element force/modal acceleration (or IF disp)}] \times \\
 &\quad [\text{Modal acceleration}(\text{time}) \text{ or IF disp}(\text{time})]
 \end{aligned}$$

Note that quite often only acceleration histories are of concern and interface displacements due to redundancy are ignored. Stresses induced by these displacements are often small enough to be ignored.

3.4.2 Interface Force LTM

Sometimes only the interface forces and net center of gravity forces are of interest. Using the upper part of Eq. 2.5.5, the boundary (interface) forces can be written in terms of the Craig-Bampton acceleration and boundary displacements as

$$[M_{BB} \quad M_{Bm}] \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} + [K_{BB}] \{U_R\} = \{F_B\} \quad \text{Eq. 3.4.5}$$

Eq. 3.4.5 can be re-arranged to form a single matrix that relates the boundary forces to the Craig-Bampton coordinates as:

$$[M_{BB} \quad M_{Bm} \quad K_{BB}] \begin{Bmatrix} U_R \\ q_m \\ U_R \end{Bmatrix} = \{F_B\} \quad \text{Eq. 3.4.6}$$

The matrix $[M_{BB} \quad M_{Bm} \quad K_{BB}]$ is the Interface Force LTM that relates Craig-Bampton accelerations and boundary displacements to the interface forces.

The Net CG Acceleration LTM can be created from Eq. 3.4.6 by transforming the interface forces to the spacecraft CG (using the transformation matrix T_{B-CG}) and then dividing through by the 6x6 generalized mass matrix. This can be written as follows:

$$[M_{BB} \quad M_{Bm}] \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \{F_B\} = [M_{BB}] [T_{B-CG}] \begin{Bmatrix} U_{CG} \end{Bmatrix} \quad \text{Eq. 3.4.7}$$

Since $T_{CG-B}^T = T_{CG-B}$

$$[T_{CG-B}^T \quad M_{BB} \quad T_{B-CG}]^{-1} [T_{CG-B}] [M_{BB} \quad M_{Bm}] \begin{Bmatrix} U_R \\ q_m \end{Bmatrix} = \begin{Bmatrix} U_{CG} \end{Bmatrix} = [Net \ CG \ Accel] \quad \text{Eq. 3.4.8}$$

where the transformation matrix T_{B-CG} may be computed in NASTRAN or by hand as shown in Section 3.6. Note that the forces due to

the boundary displacements do not contribute to the Net CG Acceleration and thus the boundary displacements are not included in the formulation even for a redundant interface.

The Interface Force LTM and the Net CG Accel LTM can be combined into a single LTM as follows:

$$\begin{bmatrix} [Net\ CG\ LTM] & [0] \\ [M_{BB} & M_{Bm}] & [K_{BB}] \end{bmatrix} \begin{Bmatrix} U_R \\ q_m \\ U_R \end{Bmatrix} = \begin{bmatrix} [Net\ CG\ Accelerations] \\ [Interface\ Forces] \end{bmatrix} \quad \text{Eq. 3.4.9}$$

3.5 Application of the Craig-Bampton Method to Modal Synthesis

The Craig-Bampton method was developed expressly to provide a way to couple (synthesize) two or more structures that are defined in terms of their modal properties. The two-structure case defined here is easily extended to the multi-body case; experience has shown, however, that the synthesis of many structures should be broken down into a sequence of two-structure steps.

Let the superscripts A and B distinguish the two structures that will be coupled. By simple overlay of Eq. 2.1.4, the equation of motion for the system is:

$$\begin{bmatrix} M_{AA}^A & 0 \\ 0 & M_{AA}^B \end{bmatrix} \begin{bmatrix} \overset{oo}{U}_A^A \\ \overset{oo}{U}_A^B \end{bmatrix} + \begin{bmatrix} C_{AA}^A & 0 \\ 0 & C_{AA}^B \end{bmatrix} \begin{bmatrix} \overset{o}{U}_A^A \\ \overset{o}{U}_A^B \end{bmatrix} + \begin{bmatrix} K_{AA}^A & 0 \\ 0 & K_{AA}^B \end{bmatrix} \begin{bmatrix} U_A^A \\ U_A^B \end{bmatrix} = \begin{bmatrix} F_A^A \\ F_A^B \end{bmatrix} \quad \text{Eq. 3.5.1}$$

The structures are connected by equating the boundary displacements of B to those of A by

$$[U_R]^B = [e][U_R]^A \quad \text{Eq. 3.5.2}$$

Matrix [e] is the necessary transformation made by the analyst to correct for differently ordered boundary points or for different coordinate locations. It may be as simple as a matrix of direction cosines or even just the identity matrix. If the boundary points are not exactly at the same location, [e] must also contain rigid links to put the points together. Usually, the matrix for just re-ordering the boundary points looks like:

$$[e] = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad \text{Eq. 3.5.3}$$

If there are boundary points in A that do not connect to B, just leave a zero column in [e]. The matrix [e] will not be square for this case.

The system displacement vector, $[U^s]$, is:

$$[U^s] = \begin{bmatrix} U_R^A \\ q_m^A \\ q_m^B \end{bmatrix} \quad \text{Eq. 3.5.4}$$

To transform from the subsystem displacements to this system displacement set, we must use:

$$[U^s] = \begin{bmatrix} U_A^A \\ U_A^B \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ \phi_R^A & \phi_L^A & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & \phi_R^B & \phi_L^B \end{bmatrix} \begin{bmatrix} U_R^A \\ q_m^A \\ U_R^B \\ q_m^B \end{bmatrix} \quad \text{Eq. 3.5.5}$$

$$= \begin{bmatrix} I & 0 & 0 & 0 \\ \phi_R^A & \phi_L^A & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & \phi_R^B & \phi_L^B \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ e & 0 & 0 \\ 0 & 0 & I \end{bmatrix} [U^s] \quad \text{Eq. 3.5.6}$$

$$= [E][U^s] \quad \text{Eq. 3.5.7}$$

The big transformation matrix E is defined by comparing the last two equations. In practice, however, the matrix E never has to be formed because, when it is used, the results fall out in terms of data that has already been computed.

Transforming from the component sets for the various sub-structures to the system set of coordinates results in the following system equation of motion:

$$[M_{AA}^S] \begin{bmatrix} \ddot{U}^S \\ \dot{U}^S \\ U^S \end{bmatrix} + [C_{AA}^S] \begin{bmatrix} \dot{U}^S \\ U^S \\ 0 \end{bmatrix} + [K_{AA}^S] [U^S] = [F^S] \quad \text{Eq. 3.5.8}$$

where the synthesized mass is

$$[M_{AA}^S] = [E]^T \begin{bmatrix} M_{AA}^A & 0 \\ 0 & M_{AA}^B \end{bmatrix} [E]$$

Eq. 3.5.9

which expands to

$$[M_{AA}^S] = \begin{bmatrix} (B^T M_{AA} B)^A + e^T (B^T M_{AA} B)^B e & (B^T M_{AA} \Phi)^A & e^T (B^T M_{AA} \Phi)^A \\ (\Phi^T M_{AA} B)^A & \mu^A & 0 \\ (\Phi M_{AA} B)^B e & 0 & \mu^B \end{bmatrix}$$

Eq. 3.5.10

and where the synthesized damping is

$$[C_{AA}^S] = [E]^T \begin{bmatrix} C_{AA}^A & 0 \\ 0 & C_{AA}^B \end{bmatrix} [E] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (2\xi\mu\omega_o)^A & 0 \\ 0 & 0 & (2\xi\mu\omega_o)^B \end{bmatrix}$$

Eq. 3.5.11

and where the synthesized system stiffness is

$$[K_{AA}^S] = [E]^T \begin{bmatrix} K_{AA}^A & 0 \\ 0 & K_{AA}^B \end{bmatrix} [E]$$

Eq. 3.5.12

which expands to

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3.5.3

$$[K_{AA}^S] = \begin{bmatrix} (B^T K_{AA} B)^A + e^T (B^T K_{AA} B)^B e & 0 & 0 \\ 0 & (\mu\omega_0^2)^A & 0 \\ 0 & 0 & (\mu\omega_0^2)^B \end{bmatrix} \quad \text{Eq. 3.5.13}$$

and, finally, the synthesized system forcing functions are

$$[F_A^S] = [E]^T \begin{bmatrix} F_A^A \\ F_A^B \end{bmatrix} \quad \text{Eq. 3.5.14}$$

which will not be expanded because it is typically more efficient to include the forced degrees of freedom in the LTM.

3.6 Reduction of a Redundant Interface to a Single Point

The Craig-Bampton equation of motion for a structure (or sub-structure) on a 6 degree of freedom boundary and unity generalized mass with no internal loads is given by:

$$[\overset{\circ\circ}{q}_m] + 2[\xi][\omega_o][\overset{\circ}{q}_m] + [\omega_o^2][q_m] = -[M_{mB}][\overset{\circ\circ}{U}_R] \quad \text{Eq. 3.2.1}$$

where $[\overset{\circ\circ}{U}_R]$ represents motion of the interface degrees of freedom. If this set of interface DOFs is indeterminate, it may be advantageous under certain circumstances to tie these points together for application of a single or "average" set of interface motions. For example, when solving for the steady state response of a structure with an indeterminate boundary, it is easier to apply the motions to a single point and it may be even more appropriate since the mounting of a structure to a shaker table rigidizes the interface. As another example, it may be easier to visualize the excitation of several redundant degrees of freedom by first averaging the excitation in each direction and then applying them to one point that is rigidly connected to the others. The single point need not be (and typically isn't) one of the original boundary points. Obviously any elasticity in the interface (such as ovaling of a ring) is lost, as the interface points are all rigidly connected to the new point. Therefore, this technique should be used only when the interface is stiff relative to the rest of the structure or when stiffness is to be added to the interface.

Consider a structure with three boundary points, B1, B2 and B3, each with 6 DOFs for a total of 18 interface DOFs. These points are to be replaced by the single point A. The coordinates of point Bi are {Xbi, Ybi, Zbi, RXBi, RYBi, RZBi} and the coordinates of point A are {XA, YA, ZA, RXA, RYA, RZA}. Let the displacements of points Bi be {Uix, Uiy, Uiz, Uijx, Uijy, Uijz} and the displacements of point A be {ux, uy, uz, ujx, ujy, ujz}. Further, let the set of all displacements at the boundary be [U] where [U] is an 18 x n matrix (n is the number of time steps) and let the set of displacements at point A be [u] where [u] is a 6 x n matrix. The displacement vectors [U] and [u] are related by the transformation matrix [C] that may be determined from simple geometric considerations. This relation is given as:

$$[U] = [C][u] \quad \text{Eq. 3.6.1a}$$

or, in expanded notation for any given point in time:

$$\begin{bmatrix} U1_x \\ U1_y \\ U1_z \\ U1_{\theta_x} \\ U1_{\theta_y} \\ U1_{\theta_z} \\ U2_x \\ U2_y \\ U2_z \\ U2_{\theta_x} \\ U2_{\theta_y} \\ U2_{\theta_z} \\ U3_x \\ U3_y \\ U3_z \\ U3_{\theta_x} \\ U3_{\theta_y} \\ U3_{\theta_z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \Delta Z_1 & -\Delta Y_1 \\ 0 & 1 & 0 & -\Delta Z_1 & 0 & \Delta X_1 \\ 0 & 0 & 1 & \Delta Y_1 & -\Delta X_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \Delta Z_2 & -\Delta Y_2 \\ 0 & 1 & 0 & -\Delta Z_2 & 0 & \Delta X_2 \\ 0 & 0 & 1 & \Delta Y_2 & -\Delta X_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & \Delta Z_3 & -\Delta Y_3 \\ 0 & 1 & 0 & -\Delta Z_3 & 0 & \Delta X_3 \\ 0 & 0 & 1 & \Delta Y_3 & -\Delta X_3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \\ u_{\theta_x} \\ u_{\theta_y} \\ u_{\theta_z} \end{bmatrix}$$

Eq. 3.6.1b

where

$$\begin{aligned}
\Delta X_i &= X_{Bi} - X_A \\
\Delta Y_i &= Y_{Bi} - Y_A \\
\Delta Z_i &= Z_{Bi} - Z_A
\end{aligned}$$

Eq. 3.6.2

The pattern of Eq. 3.6.1b may be repeated for any arbitrary number of redundant interface points.

To reduce the number of interface DOFs in Eq. 3.2.1 replace $[\overset{\infty}{U}_R]$ with $[C][\overset{\infty}{u}]$ and solve as usual for modal responses. Note, however, that when transforming modal responses back to physical responses using the Craig-Bampton transformation (Eq. 2.2.1), the matrix of Boundary Mode Functions, $[B]$, must be replaced with the matrix $[B']$ where $[B'] = [B][C]$.

It is also interesting to note that the 6x6 rigid body mass matrix of the structure about point A may be computed from the product $[C]^T [M_{BB}] [C]$.

To obtain the "average" motion $[u]$ of a single point in each direction when motions $[U]$ act at several boundary points, one cannot simply average the motions of all the boundary points in each direction. Continuing with the example above, the correct approach is to determine the transformation matrix $[T]$ such that:

$$[u] = [T][U] \quad \text{Eq. 3.6.3}$$

where $[T]$, a 6 x 18 array, is the 'inverse' of $[C]$. Begin by multiplying Eq. 3.6.1a by $[C]^T$, the transpose of $[C]$. Then

$$[C]^T [u] = [C]^T [C][u] \quad \text{Eq. 3.6.4}$$

where $[C]^T [C]$ is a square matrix with a defined inverse. It follows that

$$[u] = \{[C]^T [C]\}^{-1} [C]^T [u] = [T][U] \quad \text{Eq. 3.6.5}$$

so that the desired inverse transformation is

$$[T] = \{[C]^T [C]\}^{-1} [C]^T \quad \text{Eq. 3.6.6}$$

4. REFERENCES

1. Hurty, Walter C., "On the Dynamic Analysis of Structural Systems Using Component Modes." AIAA Paper No. 64-487, First AIAA Annual Meeting, Washington, DC, June 29-July 2, 1964.
2. Craig, Roy and Bampton, Mervyn. "Coupling of Substructures for Dynamic Analyses." AIAA Journal, Vol. 6, No. 7, July 1968, pp. 1313-1319.
3. Haile, William B., "Modal Synthesis Boundary Reaction Convergence Test." Engineering Memorandum S&M 354, Space Telescope Program. Lockheed Missiles and Space Co., Inc., Sunnyvale, California, December 1, 1982.
4. Wilson, Thomas. "A NASTRAN DMAP Alter for the Coupling of Modal and Physical Coordinate Substructures."

APPENDIX A

GUYAN REDUCTION

Guyan Reduction is a methodology used to reduce the number of degrees of freedom in a large finite element model, typically generated for static analyses, to some much smaller number of freedoms that can be much more readily solved for dynamic analyses. In NASTRAN this reduced set of freedoms is called the ASET or analysis set.

Consider the following equation in which the 1 space represents the omitted dof's that are assumed to carry no force and the 2 space represents the ASET dof's:

$$\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ f_2 \end{bmatrix} \quad \text{Eq. A.1}$$

Note that the ASET contains both boundary and interior freedoms. Then

$$k_{11} x_1 + k_{12} x_2 = 0 \quad \text{Eq. A.2}$$

$$x_1 = -k_{11}^{-1} k_{12} x_2 \quad \text{Eq. A.3}$$

$$\left[-k_{21} k_{11}^{-1} k_{12} + k_{22} \right] x_2 = f_2 \quad \text{Eq. A.4}$$

$$\bar{K}_{22} x_2 = f_2 \quad \text{Eq. A.5}$$

where \bar{K}_{22} is the ASET stiffness.

Note that Eq. A.3 relates the omitted freedoms to the ASET freedoms and gives rise to the following transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ I \end{bmatrix} x_2 \quad \text{Eq. A.6}$$

where $T_1 = -k_{11}^{-1} k_{12}$.

This is equivalent to modal reduction. Consider the eigenvalue problem for the overall structure and transform the displacements as above:

$$K \Phi = M \Phi \Lambda \quad \text{Eq. A.7}$$

$$T^T K T \Phi = T^T M T \Phi \Lambda \quad \text{Eq. A.8}$$

$$K^* \Phi = M^* \Phi \Lambda \quad \text{Eq. A.9}$$

$$\begin{bmatrix} T_1 & I \end{bmatrix} \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ I \end{bmatrix} x_2 = \begin{bmatrix} T_1 & I \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \bar{K}_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ I \end{bmatrix} x_2 = \bar{K}_{22} x_2 = f_2 \quad \text{Eq. A.10}$$

The corresponding mass matrix may be derived from the laws of motion:

$$f = M \ddot{x} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} \quad \text{Eq. A.11}$$

Applying the transformation in Eq. A.6 and noting that the matrix T is not a function of time, the acceleration matrix may be transformed as follows:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ I \end{bmatrix} \ddot{x}_2 \quad \text{Eq. A.12}$$

$$f = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ I \end{bmatrix} \ddot{x}_2 = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad \text{Eq. A.13}$$

where f has forces both in 1 and 2 space.

Finally, transform the forces as follows

$$f_2 = [T_1 \ I] f = [T_1 \ I] \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ I \end{bmatrix} \ddot{x}_2 = \bar{M}_{22} \ddot{x}_2 \quad \text{Eq. A.14}$$

where

$$\bar{M}_{22} = [T_1 \ I] \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} T_1 \\ I \end{bmatrix} \quad \text{Eq. A.15}$$

APPENDIX B

NASTRAN DMAP for Generating a Craig-Bampton Model. (This DMAP is for MSC Version 64. It assumes that the mode shapes are mass normalized and that the boundary degrees of freedom are specified on SUPORT cards).

```
SOL 3,0
ALTER 416,417
$ Fixed Interface Modal Model
1.  READ      KLL,MLL,,,EED,USET,CASECC/LAMA,PHIL,MI,OEIGS/
      MODES/S,N,NEIGV $
2.  OFP       LAMA,OEIGS// $
3.  UMERGE    USET,PHIL,/PHILA/A/L/R $
4.  PARAM     // NOP/V,N,MRP=-1 $
5.  MATGEN    ,/AMR/1/REACT $
6.  DIAGONAL  MI/MIDIAG/SQUARE $
7.  EQUIV     MIDIAG,MI/MP $
8.  UMERGE    USET,DM,AMR/PHIRB/A/L/R $
9.  PARAM     //C,N,ADD/V,N,NORPN/V,N,REACT/V,N,NEIGV $
10. MATGEN    ,/MP/6/NORPN/REACT/NEIGV $
11. MERGE     PHIRB,,PHILA,,MP,/PHIX/1 $
12. MPYAD     MLL,DM,MLR/MTP1////2 $
13. MPYAD     PHIL,MTP1,/MHB/1////2 $
14. TRNSP     MHB/MBH $
15. MERGE     MR,MHB,MBH,MI,MP,/MRRGN $
16. MATPRN    MI,MRRGN// $
17. SPMYAD    PHIL,KLL,PHIL,,,/KW/3////1 $
18. MATPRN    KW
19. DIAGONAL  KW/KWW/SQUARE $
20. MATPRN    KWW// $
21. MPYAD     KLR,DM,KRR/KBB/1////2 $
22. MATPRN    KBB// $
23. MERGE     KBB,,,KWW,MP,/KRRGN $
24. MATPRN    KRRGN// $
25. OUTPUT4   KRRGN,MRRGN,PHIX,,-1/21 $
      ENDALTER
```

NOTES:

1. READ (Extract eigen-values from a real symmetric matrix) solves Eq. 2.4.2. It takes the constrained mass (MLL) and stiffness arrays (KLL) and calculates the cantilevered modes (PHIL) and eigen-values (LAMA). It also computes the modal mass matrix, MI, which is given as [] in Eq. 2.4.3. NEIGV is the number of eigen-vectors found.
2. OFP (Output file processor) prints LAMA, the real eigenvalue table, and OEIGS, the real eigen-value summary table.

3. UMERGE (Merge two matrices based on USET) forms PHILA, the fixed base mode shape matrix Φ defined by Eq. 2.2.2. The mode shapes are brought to A-set size by merging zeros into the boundary coordinates.
4. PARAM (Parameter processor) defines the parameter MRP used by EQUIV in step 7.
5. MATGEN (Matrix generator) generates AMR, an RxR identity matrix.
6. DIAGONAL (Strip diagonal from matrix) forms MIDIAG, a square matrix whose diagonal terms are taken from the matrix MI. (MIDIAG and MI are equivalent; MIDIAG has small off-diagonal terms that are set to zero in MI).
7. EQUIV (Data block name equivalence) makes MIDIAG and MI equivalent.
8. UMERGE forms PHIRB, the Boundary Mode Function matrix [B] defined by Eq. 2.2.2. It merges the identity matrix AMR with the rigid body transformation matrix DM that is defined as ϕ_R in Eq. 2.3.4.
9. PARAM defines the parameter NORPN to be the sum of REACT (the number of SUPORT degrees of freedom) and NEIGV (the number of eigen-vectors retained in the Craig-Bampton analysis).
10. MATGEN forms a column matrix, MP, with (R+m) rows. The first R rows contain zeros and the remaining m rows contain ones.
11. MERGE forms PHIX, the [Ax(R+m)] Craig-Bampton transformation matrix, $[B \ \Phi]$, defined by Eq. 2.2.1. Note that the order of the A rows is in the NASTRAN sequencing order with the L and R freedoms inter-dispersed.
12. MPYAD (Matrix multiply and add) forms $[M_{LR} + M_{LL} \phi_R]$, part of Eq. 2.5.4c.
13. MPYAD forms $[M_{mB}]$ (Eq. 2.5.4c).
14. TRNSP (Matrix transpose) forms MBm, the transpose of MmB.
15. MERGE forms MRRGN, the full Craig-Bampton mass matrix defined in Eq. 2.5.3.
16. MATPRN prints MI and MRRGN.

17. SMPYAD (Matrix series multiply and add) computed ω_o^2 from Eq. 2.4.4 and calls it KW.
18. MATPRN prints the matrix of eigen-values KW.
19. DIAGONAL forms KWW, a square matrix whose diagonal terms are taken from matrix KW. (KW and KWW are equivalent; KW has small off-diagonal terms that are set to zero in KWW).
20. MATPRN prints matrix KWW.
21. MPYAD computes KBB, as given by Eq. 2.5.4e.
22. MATPRN prints matrix KBB.
23. MERGE forms KRRGN, the full Craig-Bampton stiffness matrix defined by Eq. 2.5.3.
24. MATPRN prints matrix KRRGN.
25. OUTPUT4 writes matrices KRRGN, MRRGN and PHIX onto Fortran unit 21.

APPENDIX C

NASTRAN DMAP for Computing
an Equilibrium Check on the Free-Free Stiffness Matrix
(This DMAP is for MSC Version 64).

Appendix C presents the NASTRAN DMAP sequence for performing an equilibrium check on the stiffness matrices of a physical model. The purpose of this check is to test for grounding in a structure that will generate erroneous internal forces. The check shown is for the free-free stiffness (F-set) matrix but the approach is applicable to the other sets as well. The rigid body transformation matrix KRBF multiplies the stiffness matrix KFF. This product is then normalized to its maximum value to yield the array KRBFN. Large terms in KRB indicate grounding; the location of the ground is shown by KRBFN.

```
SOL 3,0
ALTER 125
1  VECPLOT,,BGPDT,EQEXIN,CSTM,,/RBGLOBAL/V,Y,MPFPNT=0//4 $
2  VEC      USET/VRB/G/F/COMP $
3  PARTN   RBGLOBAL,VRB,/RBFSET,,,/+1 $
4  TRNSP   RBFSET/RBTF $
5  MPYAD   KFF,RBTF,/KRB/ $
6  MPYAD   RBFSET,KRB,/KRBF/ $
7  NORM    KRB/KRBFN/ $
8  MATPRN  KRBF// $
9  MATGPR  GPL,USET,SIL,KRBFN//F///1.-2 $
ENDALTER
```

NOTES:

1. VECPLOT generates RBGLOBAL, the global rigid body matrix with 6 rows and g columns where each row represents the rigid body motion of all grid points about point MPFPNT. The default value for MPFPNT is grid point 0 that is interpreted in NASTRAN as the origin of the basic coordinate system. Another point about which rigid body motion is to be taken may be specified on a PARAM card in the Bulk Data deck. The format of this card is: PARAM, MPFPNT, Grid ID. Note that VECPLOT is commonly used in other DMAP routines and is therefore shown above apart from the rest of the modal mass DMAP.

2. VEC partitions USET into VRB to reduce the g-set into the f-set.
3. PARTN partitions RBGLOBAL (6 x g) into RBFSET (6 x f) based on VRB.
4. TRNSP transposes RBFSET into RBTF.
5. MPYAD multiplies KFF, the free-free stiffness matrix, by RBTF, the f x 6 rigid body matrix, to form KRB, an f x 6 matrix.
6. MPYAD multiplies RBFSET, the 6 x f rigid body matrix, by KRB, f x 6, to yield a 6 x 6 matrix, KRBF. This matrix is key to understanding if a structure is grounded. A "large" term in this array indicates grounding.
7. NORM normalizes KRB by the maximum value of the array to yield KRBFN.
8. MATPRN prints KRBF, the 6 x 6 matrix.
9. MATGPR prints KRBFN (f x 6) with associated grid point ID number. Only those values larger than 0.01 are printed. This matrix identifies the location of the ground.

***** NOTE *****

An equilibrium check of the F-set stiffness matrix has been described. It is highly recommended that equilibrium checks also be performed on the G-set (global DOFs), the N-set (those DOFs Not taken out by MPCs) and the T-set (equivalent to the A-set when dynamic reduction is not involved). An example of this DMAP package for MSC Version 64 is as shown:

```

ALTER 115
$$ G-SET CHECK
VECPLOT, , BGPDT, EQEXIN, CSTM, , /RBGL/V, Y, MPFPNT=0//4 $
TRNSP  RBGL/RBGLT $
MPYAD  KGG, RBGLT, /KPHIG/ $
MATGPR GPL, USET, SIL, KPHIG//G///1.-5 $
ALTER 121
$$ N-SET CHECK - CHECKS FOR PROBLEMS WITH MPCs
VEC    USET/VGM/G/M/COMP $
PARTN  RBGL, VGM, /, , RBNN, /1 $
TRNSP  RBNN/RBNNT $
MPYAD  KNN, RBNNT, /KPHIN/ $
MATGPR GPL, USET, SIL, KPHIN//N///1.-5 $
ALTER 126

```

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\$\$ F-SET CHECK

VEC USET/VGF/G/F/COMP \$
PARTN RBGL,VGF,/RBFF,,,/1 \$
TRNSP RBFF/RBFFT \$
MPYAD KFF,RBFFT,/KPHIF/ \$
DIAGONAL KFF/KFFD/SQUARE/-1. \$
MPYAD KFFD,KPHIF,/KPHIFN/ \$
MATGPR GPL,USET,SIL,KPHIFN//F///1.-5 \$

\$\$ A-SET (T-SET) CHECK

ALTER 174

VEC USET/VGT/G/T/COMP \$
PARTN RBGL,VGT,/RBTT,,,/1 \$
TRNSP RBTT/RBTTT \$
MPYAD KTT,RBTTT,/KPHIT/ \$
DIAGONAL KTT/KTTD/SQUARE/-1. \$
MPYAD KTTD,KPHIT,/KPHITN/ \$
MATGPR GPL,USET,SIL,KPHITN//T///1.-5 \$
ENDALTER

APPENDIX D

NASTRAN DMAP for Computing
Modal Participation Factors and Modal Weights
(This DMAP is for MSC Version 64).

```
SOL 3,0
ALTER 125
1  VECPLOT,, BGPDT,EQEXIN,CSTM,, /RBGL/V,Y,MPFPNT=0//4 $
   . . . . .
   $$$ COMPUTE MODAL PARTICIPATION AND MODAL WEIGHT
2  VEC          USET/VGT/G/T/COMP $
3  PARTN       RBGL,VGT,/RBASET,,,/1 $
4  MPYAD       RBASET,MAA,/RBMAA/ $
5  MPYAD       RBMAA,PHILA,/RBMPHI/ $
6  DIAGONAL    RBMPHI/MPHI2/WHOLE/2. $
7  ADD         MPHI2,/EFWGHT/(386.4,.0) $
8  MATGEN      ,/IDP/1/NEIGV $
9  DIAGONAL    IDP/CIDP/COLUMN/1. $
10 MPYAD       EFWGHT,CIDP,/WTSUM/ $
11 MATPRT      EFWGHT// $
12 MATPRT      WTSUM// $
   ENDALTER
```

NOTES:

1. VECPLOT generates RBGL, the global rigid body matrix with 6 rows and g columns where each row represents the rigid body motion of all grid points about point MPFPNT. The default value for MPFPNT is grid point 0 that is interpreted in NASTRAN as the origin of the basic coordinate system. Another point about which rigid body motion is to be taken may be specified on a PARAM card in the Bulk Data deck. The format of this card is: PARAM, MPFPNT, Grid ID. Note that VECPLOT is commonly used in other DMAP routines and is therefore shown apart from the rest of the modal mass DMAP.
2. VEC partitions USET into VGT to reduce the g-set into the t-set. For practical purposes the t-set is equivalent to the a-set.
3. PARTN partitions RBGL (6 x g) into RBASET (6 x a) based on VGT.
4. MPYAD multiplies the rigid body a-set matrix, RBASET, by the a-set mass matrix, MAA (a x a), to produce RBMAA (6 x a).

5. MPYAD multiplies RBMAA (6 x a) by elastic modes, PHILA (a x modes) to produce RBMPHI, 6 x modes, the matrix of Modal Participation Factors.
6. DIAGONAL squares all the terms in RMBPHI to produce MPHI2.
7. ADD scales MPHI2 by 386.4 to produce EFWGHT, the matrix of modal weights (6 x modes).
8. MATGEN produces IDP, an n x n identity matrix, where n is NEIGV, the number of modes specified on the EIGR card in Bulk Data.
9. DIAGONAL strips the diagonal from IDP (n x n) to produce CIDP (n x 1).
10. MPYAD multiplies EFWGHT (6 x n) by CIDP (n x 1) to produce WTSUM (6 x 1), the sum of the modal weights for all modes in each direction.
11. MATPRT prints EFWGHT.
12. MATPRT prints WTSUM.

APPENDIX E

NASTRAN DMAP for Generating an LTM
(This DMAP is for MSC Version 64).

```
SOL 3,0
ALTER 416,417
$$ This DMAP outputs data blocks OEF2 and OEF4 from
$$ which a post-processor (such as the Fortran program
$$ LTMGEN) can generate an LTM.
$$
$$ The DMAP from APPENDIX B for generating a Craig-
$$ Bampton model should be inserted here since data
$$ blocks PHIL, MP and PHIRB which were generated there
$$ are also of required here.
$$
$$ LTM DMAP
$$
$$ ACCELERATION DATA RECOVERY
$$
1  MPYAD      MLL,DM,MLR/MIF/ $
2  ADD       MIF,/AMIF/(-1.,0.) $
3  MPYAD      MLL,PHIL,/LFOR//-1 $
4  MERGE     AMIF,,LFOR,,MP,/LFORA/1 $
5  FBS       LLL,,LFORA/LDISPA/1/1/ $
6  UMERGE    USET,LDISPA,/ADISP/A/L/R $
7  SDR1      USET,,ADISP,,,GO,GM,,,,/PHIZ2,,1/REIG $
8  OUTPUT4   PHIZ2,,,,//-1/24 $
9  SDR2      CASECC,CSTM,MPT,DIT,EQEXIN,SIL,ETT,,BGPDT,LAMA,,
        PHIZ2,EST,/OPG2,OQG2,OPHIZ2,OES2,OEF2,PUGV2/REIG
10 OFP       OEF2// $
    $$      OUTPUT TABLES LISTED BELOW
    $$
    $$      OQG2 - SPC FORCES DUE TO ACCELERATIONS
    $$      OPHIZ - DISPLACEMENTS DUE TO ACCELERATIONS
    $$      OEF2 - ELEMENT FORCES DUE TO ACCELERATIONS
    $$
    $$      REDUNDANT FORCES
    $$
11 SDR1      USET,,PHIRB,,,GO,GM,,KFS,,/PHIZ4,,/1/REIG $
12 SDR2      CASECC,CSTM,MPT,DIT,EQEXIN,SIL,ETT,,BGPDT,LAMA,,
        PHIZ4,EST,/OPG4,OQG4,OPHIZ4,OES4,OEF4,PUGV4/REIG $
13 OFP       OEF4// $
    $$
    $$      OUTPUT TABLES LISTED BELOW
    $$
    $$      OEF4 - ELEMENT FORCES DUE TO I/F DISPLACEMENTS
    $$      OQG4 - SPC FORCES DUE TO I/F DISPLACEMENTS
    $$      OPHIZ4 - DISPLACEMENTS DUE TO I/F DISPLACEMENTS
    ENDALTER
```

10/06/00
Craig Bampton_2.doc

E.1

NOTES:

1. MPYAD generates data block MIF that equals $[M_{LR} + M_{LL} \phi_R]$ in Eq. 2.5.4c. The terms in MIF are related to InterFace Mass. Its size is LxR.
2. ADD generates data block AMIF, the negative of MIF.
3. MPYAD generates data block LFOR that equals $[M_{LL} \phi_L]$ in Eq. 2.4.3. Its size is Lxm. The terms in LFOR are related to the L-set forces.
4. MERGE generates data block LFORA, the matrix $[AMIF \quad LFOR]$. Its size is Lx(R+m).
5. FBS (Matrix Forward/Backward Substitution) generates data block LDISPA that equals $\left\{ \begin{matrix} -K_{LL}^{-1} (M_{LR} + M_{LL} \phi_R) \\ -K_{LL}^{-1} M_{LL} \phi_L \end{matrix} \right\}$ which are two of the sub-matrices in Eq. 3.4.4. The terms in LDISPA are related to the L-set displacements.
6. UMERGE takes the 1x2 matrix LDISPA and generates the 2x2 matrix ADISP by placing zeros in row 1. The terms in ADISP are related to the A-set displacements.
7. SDR1 is a module that generates data block PHIZ2, which is the displacement matrix ADISP blown-up to G-set size.
8. OUTPUT4 writes the matrix PHIZ2 onto Fortran unit 24.
9. SDR2 is a module which outputs several data blocks (G-set size) related to forces, stresses and displacements that result from unit modal accelerations.
10. OFP outputs data block OEF2, the element forces due to modal accelerations.
11. SDR1 is a module that generates data block PHIZ4, which is the support vector matrix PHIRB blown-up to G-set size. It contains the rigid body constraint modes.
12. SDR2 is a module which outputs several data blocks (G-set size) related to forces, stresses and displacements that result from redundant interface forces.
13. OFP outputs data block OEF4, the element forces due to the redundant interface forces.

Note that the following symbolic relationships apply to this DMAP sequence.

$$[U_A] = \begin{bmatrix} 0 & 0 & I \\ -K_{LL}^{-1} [M_{LR} + M_{LL} \phi_R] & -K_{LL}^{-1} M_{LL} \phi_L & \phi_R \end{bmatrix} \begin{bmatrix} U_R \\ q_m \\ U_R \end{bmatrix}$$

where

$$\begin{aligned} & \begin{bmatrix} 0 & 0 & I \\ -K_{LL}^{-1} [M_{LR} + M_{LL} \phi_R] & -K_{LL}^{-1} M_{LL} \phi_L & \phi_R \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & I \\ -K_{LL}^{-1} (AMIF) & -K_{LL}^{-1} (LFOR) & DM \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ -K_{LL}^{-1} (LFORA) & DM \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ LDISPA & DM \end{bmatrix} \\ &= [ADISP \quad PHIRB] \end{aligned}$$

Note also for this DMAP sequence the G-set displacement matrix, $[U_G]$, may be expressed symbolically as

$$[U_G] = [PHIZ2 \quad PHIZ4] \begin{bmatrix} U_R \\ q_m \\ U_R \end{bmatrix}$$

Finally note that the data blocks OEF2 and OEF4 constitute the LTM and are written into the NASTRAN output data file.

Typically, a post-processor (such as the Fortran program LTMGEN) reads this file and re-formats it appropriately.