A NEW BEAM THEORY
USING
FIRST-ORDER WARPING FUNCTIONS

C.A. IE and J.B. KOSMATKA

DEPARTMENT OF APPLIED MECHANICS AND ENGINEERING SCIENCE

UNIVERSITY OF CALIFORNIA, SAN DIEGO

LA JOLLA, CA 92093

BASIC IDEA

Due to a certain type of loading and geometrical boundary conditions, each beam will respond differently depending on its geometrical form of the cross section and its material definition. As an example, consider an isotropic rectangular beam under pure bending. Plane sections perpendicular to the longitudinal axis of the beam will remain plane and perpendicular to the deformed axis after deformation. However, due to the Poisson effect, particles in the planes will move relative to each other resulting in a form of anticlastic deformation. In other words, even in pure bending of an isotropic beam, each cross section will deform in the plane.

If the material of the beam above is replaced by a generally anisotropic material, then the cross sections will not only deform in the plane, but also out of plane. Hence, in general, both in-plane deformations and out-of-plane warping will exist and depend on the geometrical form and material definition of the cross sections and also on the loadings.

For the purpose of explanation, an analogy is made. The geometrical forms of the bodies of each individuals are unique. Hence, different sizes of clothes are needed. Finding the sizes of clothes for individuals is like determining the warping functions in beams.

A new beam theory using first-order warping functions is introduced. Numerical examples will be presented for an isotropic beam with rectangular cross section. The theory can be extended for composite beams. (Fig. 1.)
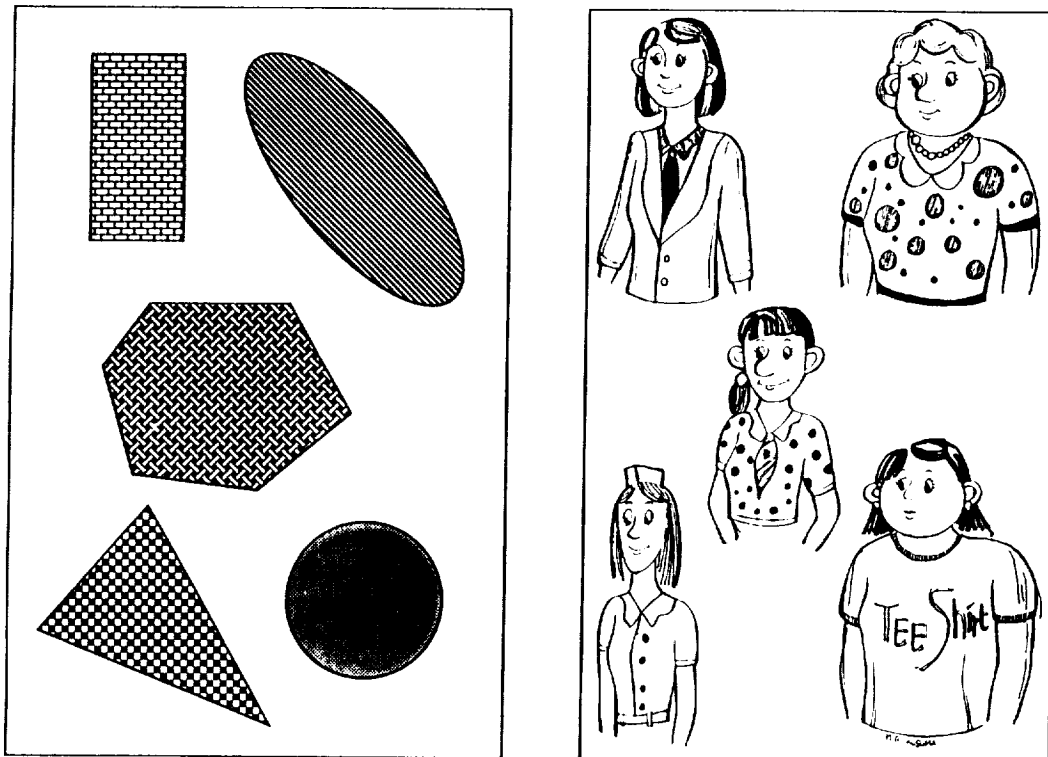

Figure 1.   Analogy between determining the (first-order) warping functions and the proper size of clothes for individuals.

CANTILEVER BEAM

Consider the case of an isotropic rectangular cantilever beam with a tip loading (P). For the purpose of **comparison** to the St. Venant elasticity solution, St. Venant boundary stresses shall be taken into account. These self-equilibrated boundary stresses are shown in the figure below. XYZ is the system coordinate ; L is the length ; H is the height, and B is the thickness of the beam. Comparison will be made with respect to the **plane stress** St. Venant elasticity solution. Hence, the comparison will be more valid as the thickness goes to zero.   (Fig. 2.)
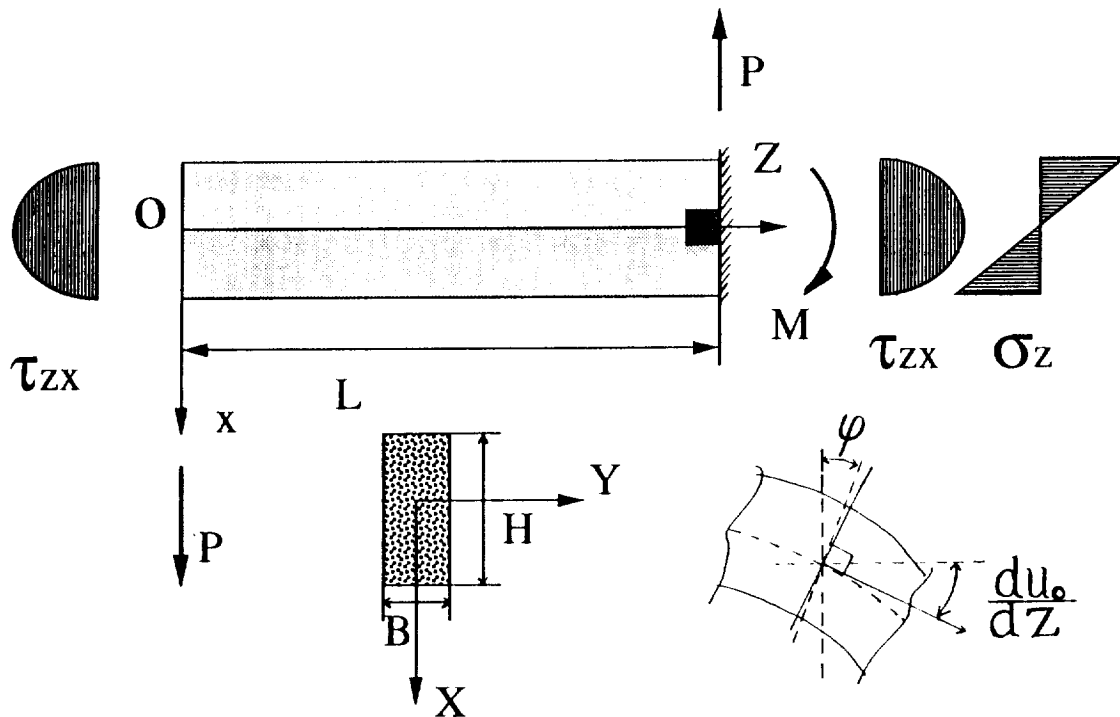

Figure 2. Cantilever beam with its boundary conditions.

ASSUMED DISPLACEMENT FIELD

The u, v, w are displacement components parallel to the x, y, z coordinate axis (refer to figure 2.). u_0 is the transverse displacement of the axis of the beam parallel to the x coordinate axis. φ is the bending rotation of the beam axis and its positive sense is defined as in the direction of the positive y axis. M and Q are the bending moment and shear force, respectively. q is the distributed load. U and V are the in-plane deformation functions parallel to the x and y coordinate axis, respectively. W is the out-of-plane warping function parallel to z coordinate axis. E is the Young modulus, and I is the moment of inertia of the cross section.

Strains are computed from the first set of equations. From Hooke's law [$(\sigma) = [C](\epsilon)$], stresses can be calculated. Using the definition of moment, M can be solved in terms of E, I and φ. The final form of the displacement field is as shown in the second set of equations. The detail is as follows.

By definition of moment,

$$M(z) = \int_A x \, \sigma_z \, dA$$

Assuming that (in consistency with using only the first-order warping functions)

$Q'(z) = -q(z) = 0$, the moment will be expressible as being $M(z) = -E I \varphi'(z)$. By equilibrium $Q(z) = M'(z) = -E I \varphi''(z)$. Substituting M and Q into the first set of equations, the final form of the displacement field is obtained.

It is important to note that no assumption is being made except the assumed displacement field itself. (Fig. 3.)

$$u(x,y,z) = u_0(z) + M(z) \, \overline{U}(x,y) \qquad\qquad v(x,y,z) = M(z) \, \overline{V}(x,y)$$

$$w(x,y,z) = -x \, \varphi(z) + Q(z) \, \overline{W}(x,y)$$

where

$$\overline{U}(x,y) = -\frac{\upsilon}{2 E I} (x^2 - y^2) \qquad\qquad \overline{V}(x,y) = -\frac{\upsilon}{E I} x y$$

$$\overline{W}(x,y) = \overline{W}(x) = -\frac{2+\upsilon}{6 E I} x^3$$

Final Model

$$u(x,y,z) = u_0(z) + \varphi'(z) \, U(x,y) \qquad\qquad v(x,y,z) = \varphi'(z) \, V(x,y)$$

$$w(x,y,z) = -x \, \varphi(z) + \varphi''(z) \, W(x)$$

where now

$$U(x,y) = \frac{\upsilon}{2} (x^2 - y^2) \qquad\qquad V(x,y) = \upsilon \, x \, y$$

$$W(x) = \frac{2 + \upsilon}{6} x^3$$

Figure 3. Proposed model in a case of rectangular cross section.

LAYOUT OF THE NODAL POINTS

The finite-element model is developed using a layout of the nodal points as shown below. The layout is chosen such that all terms in the strain energy expression are taken into account. The minimum order of polynomials that is required for φ based upon the strain energy expression is three. Hence a four-node layout is required for φ. A five-node layout is selected for u_0 because, from the physical point of view, u_0 is one order higher. Other polynomials could also be selected (Fig. 4.)
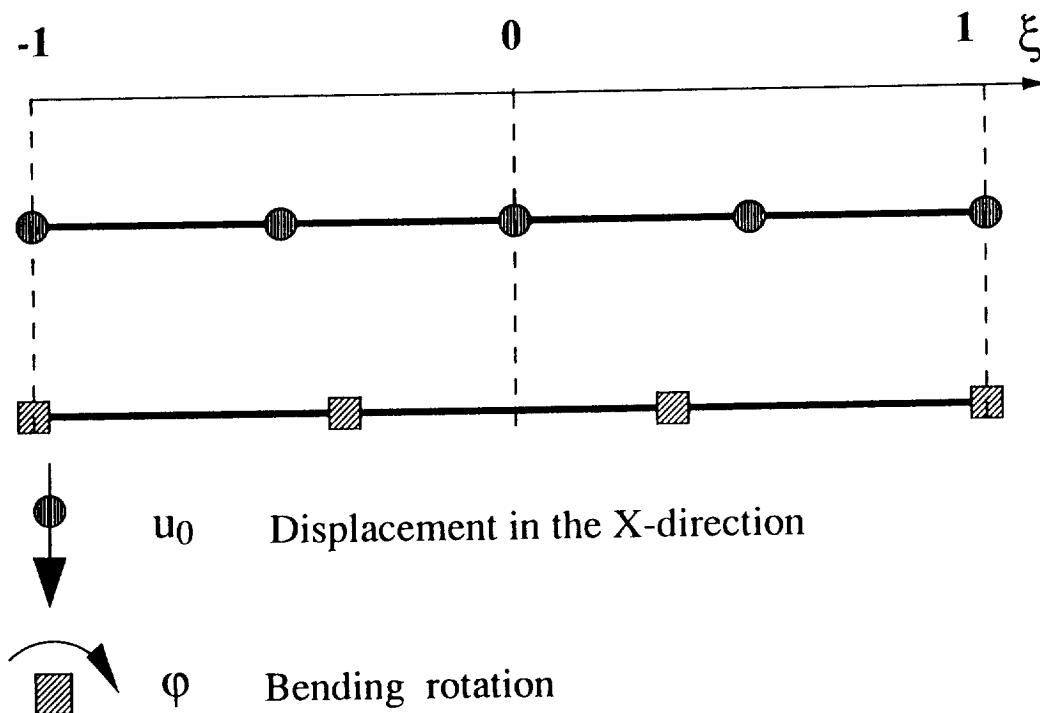

Figure 4. Layout of the nodal points in the finite-element model.

FINITE-ELEMENT MODEL

Using the principle of stationary potential energy a finite-element model is obtained. The term $\{\delta M\}$ in the equation is due to the fact that, instead of applied concentrated resultant forces, 'applied' St. Venant distributed stresses (through the cross section) are to be considered. If applied concentrated resultant forces do exist in the reality, then this term will vanish. The term $\{\delta m\}$ is present due to the fact that the distributed forces are applied on the upper surface of the beam. These additional terms exist because a beam theory that accounts for in-plane deformations and out-of-plane warping is used. Had an Euler-Bernoulli beam theory been used (or likewise Timoshenko beam theory), all these terms will vanish no matter how the loads are applied. All other terms are the usual terms that result when developing a finite-element model based upon an Euler- Bernoulli theory. For example

$$\{f\} = \int q [N]^T dz$$

where q is the distributed load and [N] are the shape functions. (Fig. 5.)

$$u_o(z) = \sum_{j=1}^{5} u_{oj} \, \phi_j(z)$$

$$\varphi(z) = \sum_{j=1}^{4} \varphi_j \, \theta_j(z)$$

u_{oj} , φ_j are Lagrangian type shape functions.

Equilibrium equations

$$\begin{bmatrix} [K^{11}]_{5 \times 5} & [K^{12}]_{5 \times 4} \\ [K^{21}]_{4 \times 5} & [K^{22}]_{4 \times 4} \end{bmatrix} \begin{pmatrix} \{u_o\}_{5 \times 1} \\ \{\varphi\}_{4 \times 1} \end{pmatrix} = \begin{pmatrix} \{P\}_{5 \times 1} \\ \{M\}_{4 \times 1} \end{pmatrix} + \begin{pmatrix} \{0\}_{5 \times 1} \\ \{\delta M\}_{4 \times 1} \end{pmatrix} + \begin{pmatrix} \{f\}_{5 \times 1} \\ \{\delta m\}_{4 \times 1} \end{pmatrix}$$

Figure 5. Finite-element model.

TIP DISPLACEMENT OF A CANTILEVER BEAM

A semi-logarithmic plot of aspect ratio versus nondimensionalized tip displacement is shown below where the tip displacement is nondimensionalized by dividing by the Euler-Bernoulli solution for a given load and geometry. Aspect ratio is defined as the length divided by the height of the beam. The Poisson ratio is taken equal to 0.25. All theories are in agreement for slender beams. For this type of loading, the elasticity solution, the proposed theory, and the Timoshenko using k equal to 2/3 are in perfect agreement. Using a k factor equal to 0.8475 [1] in the Timoshenko theory results in a stiffer beam (compare to using k equal to 2/3).

Solutions were calculated for extremely long slender beams (L/H = 100) to insure that the current beam element converged to the Euler-Bernoulli solution and did not "shear-lock". All calculations were performed assuming that B/L is equal to 1/8000. This selection was made to insure that the current model can be directly compared to the plane stress elasticity solutions. (Fig. 6.)
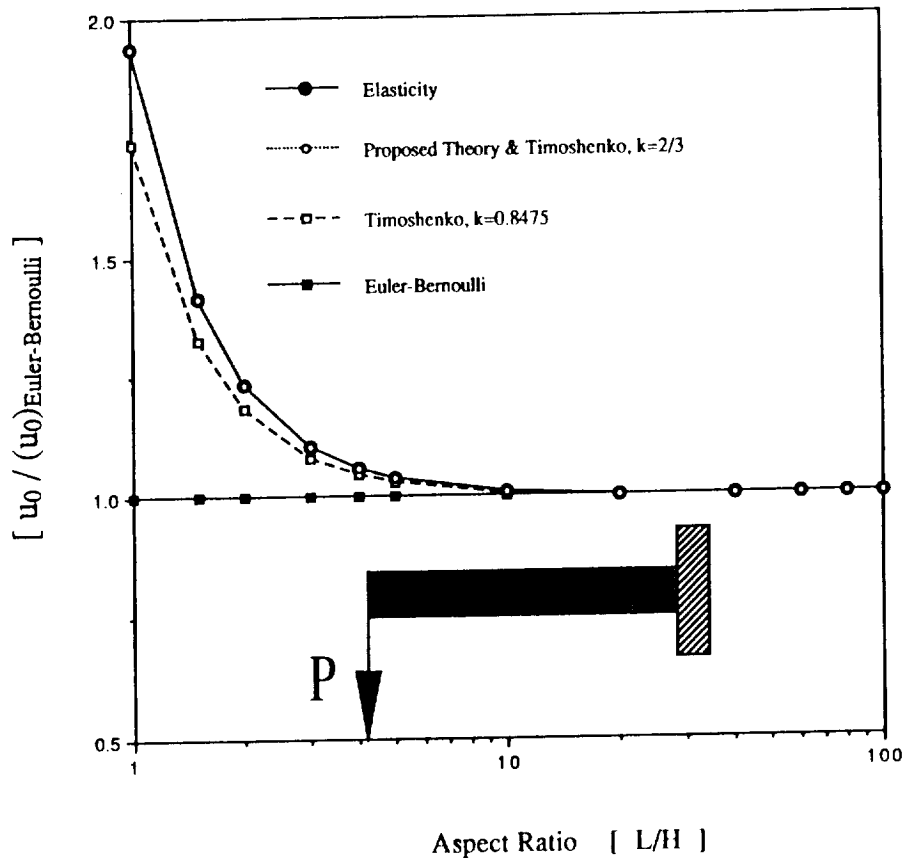

Figure 6. Tip displacements of a tip-loaded cantilever beam.

NORMAL STRESSES AT THE ROOT OF A CANTILEVER BEAM

Consider a case of the cantilever beam with an aspect ratio equal to three. The abscissa is the nondimensionalized normal stresses with respect to the Euler-Bernoulli normal stress at the top of the surface, i.e., $\overline{\sigma}_z$. The ordinate is the nondimensionalized X-coordinate where the top and bottom surfaces of the beam are defined as -1 and 1, respectively. As can be seen, all theories are in perfect agreement. (Fig. 7.)
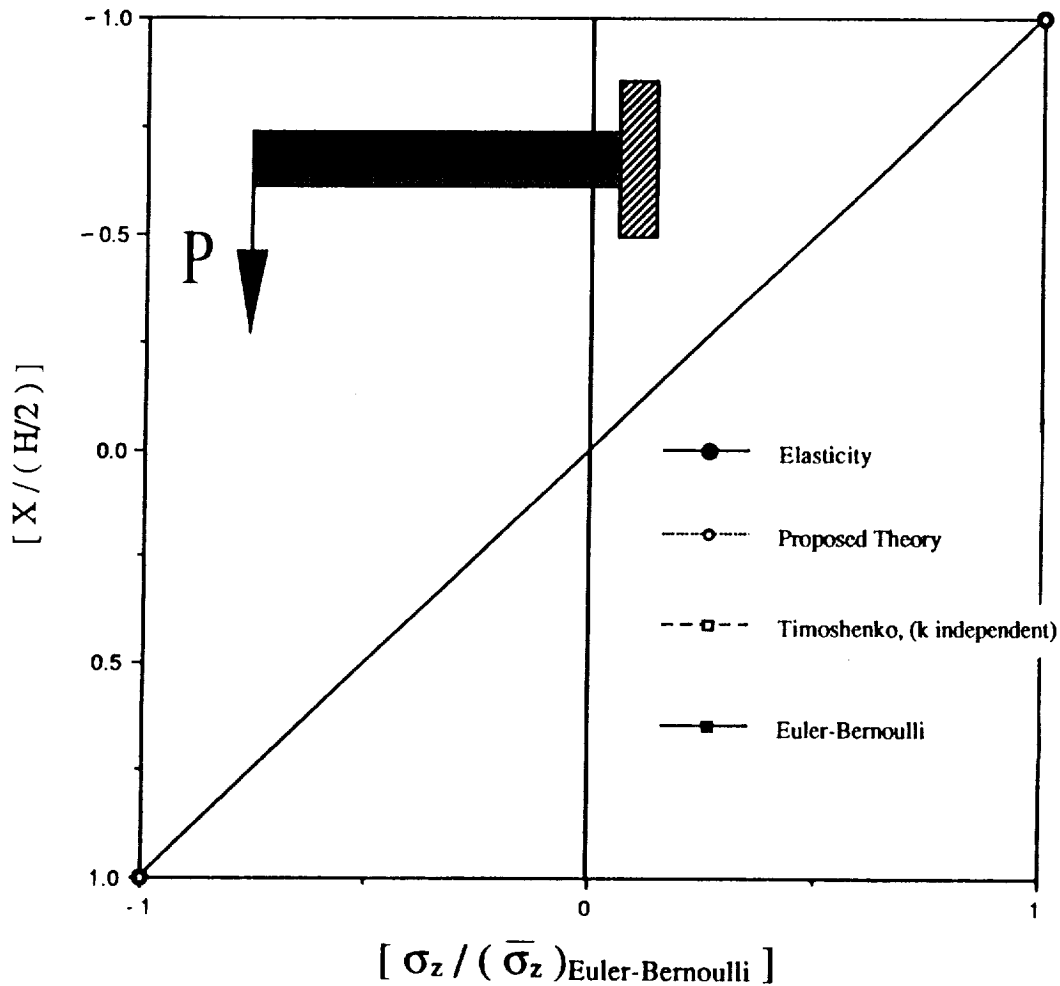

Figure 7. Normal stress at the root of a tip-loaded cantilever beam.

TRANSVERSE SHEAR STRESSES
AT THE ROOT OF A CANTILEVER BEAM

Nondimensionalized shear stresses with respect to the elasticity shear stress at x equal to zero are made, i.e., nondimensionalized with respect to $\bar{\tau}_{zx}$. The ordinate is the nondimensionalized X-coordinate. The results from the proposed theory are in perfect agreement with the elasticity solutions. If the Timoshenko theory is applied, constant shear stress distribution is obtained. In fact, their values are equal to the average shear stress, i.e., P/A where A is the area of the cross section and P is the applied concentrated load. In this case, they do not depend on the value of the shear correction factor (k). Hence, the shear correction factor will influence the stiffness and shear strains of the beam, but not the shear stresses. Since the shear stresses are independent of k, then the shear strains must vary proportionally to the inverse of k. For an Euler-Bernoulli beam, a contradiction exists. If shear stresses are computed from the shear strains, then their values will vanish. On the other hand, from the equilibrium point of view, shear stresses cannot be zero. Using the principle of equilibrium, shear stresses can be obtained. (Fig. 8.)
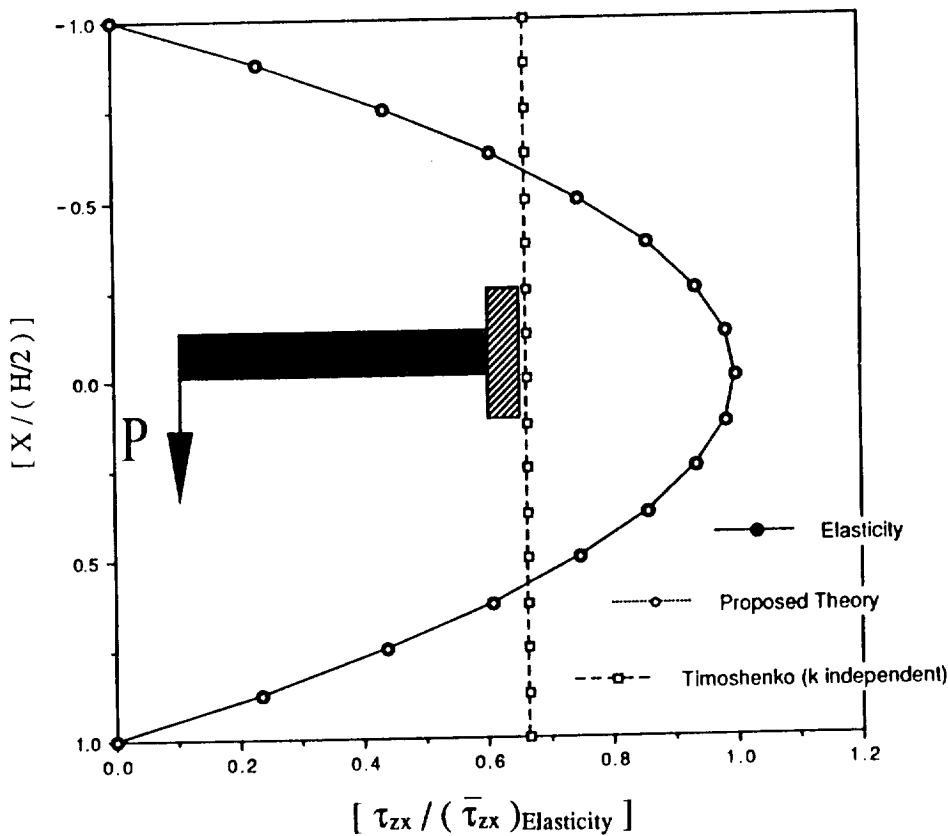

Figure 8. Transverse shear stresses at the root of a cantilever beam.

TABLE OF COMPARISON FOR THE STRESS COMPONENTS
OF A CANTILEVER BEAM

Results from the theory of elasticity for the stresses are available. Due to the computational round-off errors, the coefficients ε_i will exist in the proposed theory. These terms become smaller as the value of B (the thickness) goes to zero. If truncation errors could be eliminated, the ε_i terms will go to zero as B goes to zero. This is due to the fact that the out-of-plane warping function $W(x,y)$ was taken from the plane stress solution, which is then only a function of x, i.e., $W = W(x)$. This is done mainly for the purpose of comparison and simplicity. In order for the current comparison to be valid, the thickness of the beam should be taken very small (B --> 0). As one can see, apart from the round-off errors, the proposed theory is in perfect agreement with the elasticity solution for the whole body (plane). (Fig. 9.)

$$\sigma_z = \varepsilon_1 \; x \; z^2 + \underline{c_1 \; x \; z} + \varepsilon_2 \; x^3 + \varepsilon_3 \; x$$

$$\tau_{zx} = \varepsilon_4 \; z^3 + \varepsilon_5 \; z^2 + \varepsilon_6 \; x^2 \; z + \varepsilon_7 \; z + \underline{c_2 \; x^2} + \underline{c_3}$$

$$\sigma_x = \sigma_y = \varepsilon_8 \; x^3 \qquad\qquad \tau_{yz} = \tau_{xy} = 0$$

	Proposed Theory Elasticity	Timoshenko Elasticity	Euler-Bernoulli Elasticity
c_1	1.00	1.00	1.00
c_2	1.00	0.00	0.00
c_3	1.00	0.67	0.00
$\varepsilon_i \to 0$ as B (the thickness) goes to 0			

Figure 9. Comparison of the stress components relative to
the elasticity solutions.

SIMPLY SUPPORTED BEAM

Since the model is developed for tip loadings, it may have slight deficiencies for analyzing beams with higher-order distributed loads. In this example, the effects of the higher-order warping functions will be shown. Although it is possible to extend the model incorporating some degree of higher-order warping functions, for beam-type structures it may not be necessary. Higher-order warping functions play an important role for beams with small aspect ratios (closer to solid structures). But, as the aspect ratio gets smaller, the St. Venant solution becomes trivial in the practical sense (in reality). Another way of defining beams is as follows. A structure (structural element) can be considered as being a beam if the higher-order warping functions play insignificant roles. (Fig. 10.)
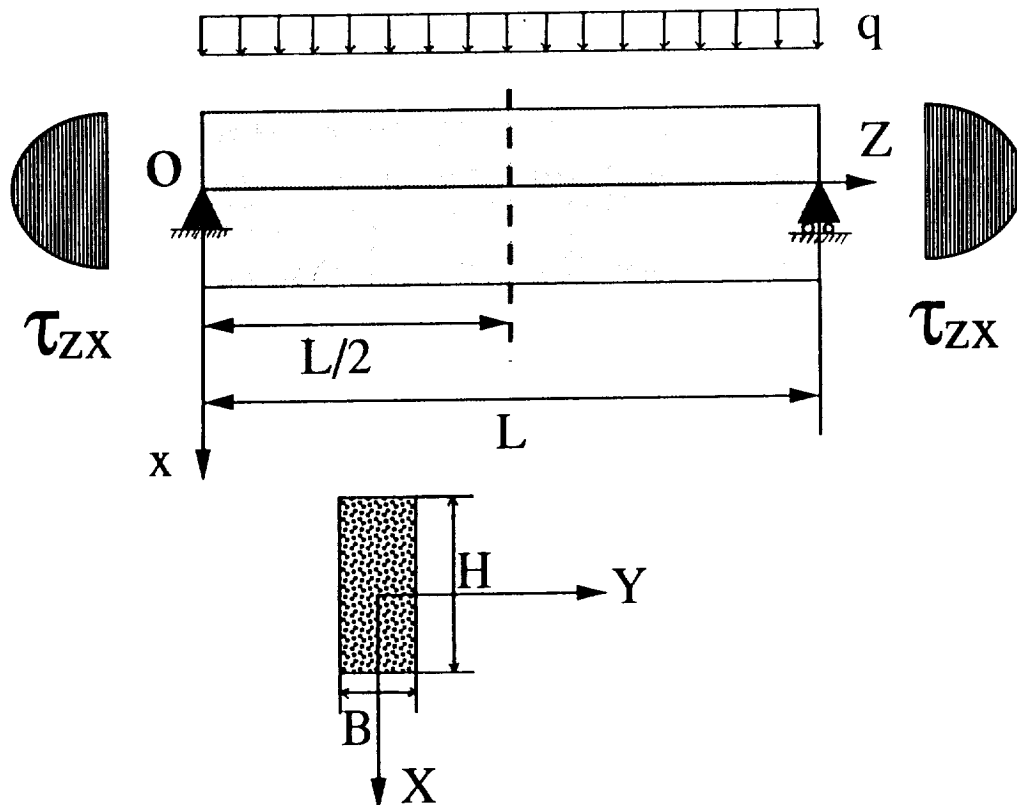

Figure 10. Simply supported beam with its boundary conditions.

MID-LENGTH DISPLACEMENT OF A SIMPLY SUPPORTED BEAM

It is very interesting to note that, although the Timoshenko theory for k equal to 2/3 gives the exact results (for the displacements) in the case of a cantilever beam, it gives the most flexible structure in the case of simply supported beams with constant distributed load. The proposed theory still gives very accurate results for the displacements. As can be seen, the effect of higher-order warping functions in the displacements, in this typical case, is very insignificant. (Fig. 11.) The effects in the stresses can be seen in the next figure.
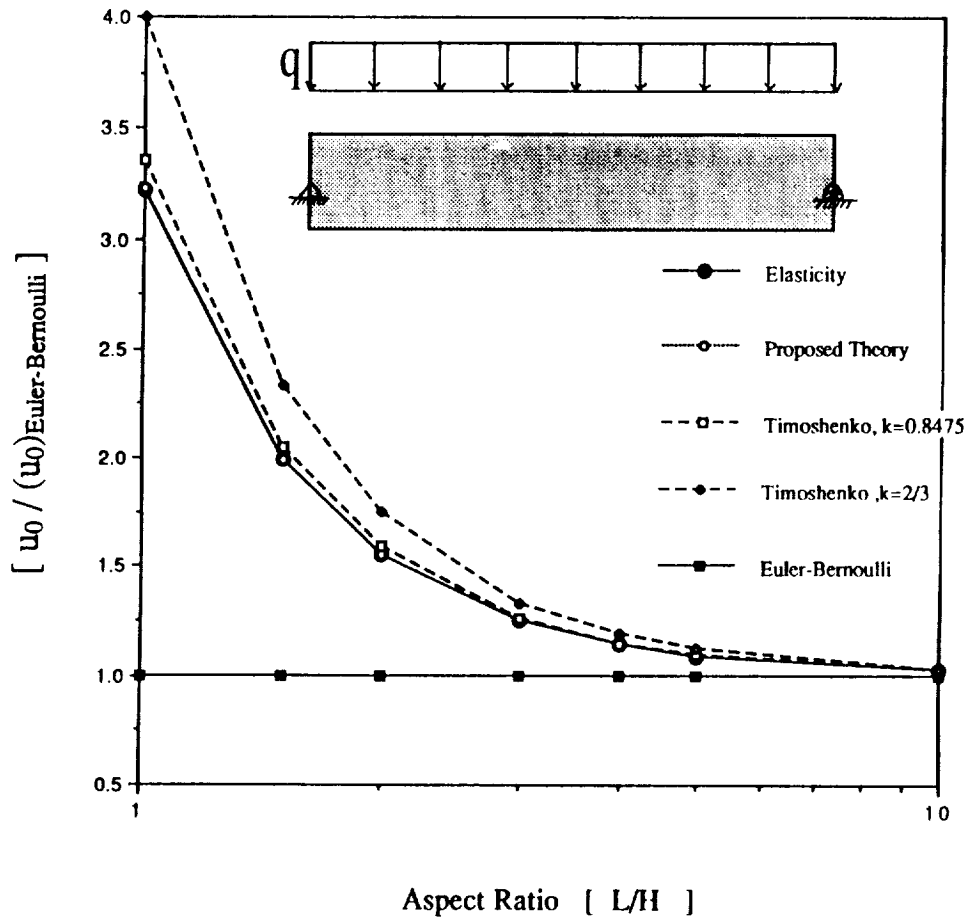

Figure 11. Displacements at mid-length of a simply supported beam.

96

NORMAL STRESSES AT MID-LENGTH OF
A SIMPLY SUPPORTED BEAM

Consider the case of a simply supported beam with a uniformly distributed load where the aspect ratio (L/B) is equal to three. Nondimensionalized normal stresses are with respect to the Euler-Bernoulli stress on the bottom surface of the beam. It is important to note that the elasticity solution results in a cubic normal stress distribution as opposed to linear as suggested by the Euler-Bernoulli or Timoshenko beam theories. (Fig. 12.) In the next figure, a closer look at the lower portion of the cross section is presented.
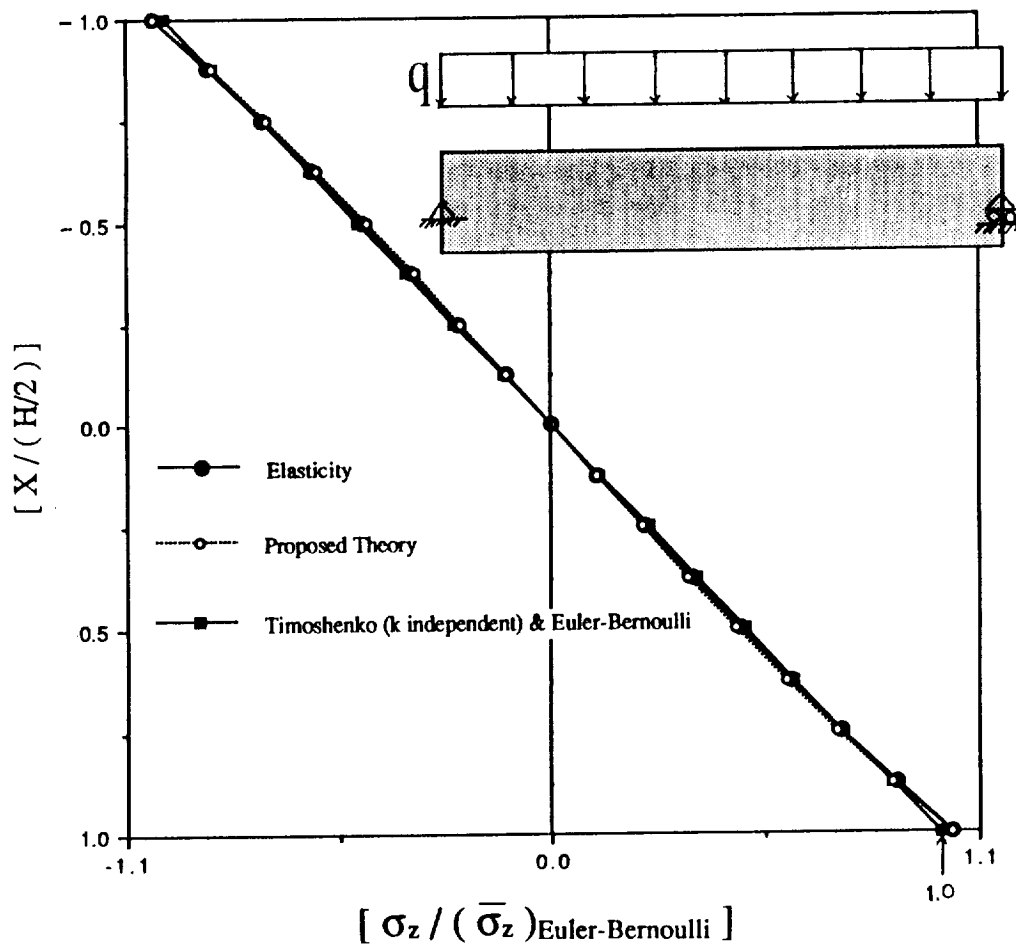

Figure 12. Normal stresses at mid-length of a simply supported beam ..

NORMAL STRESSES AT MID-LENGTH FOR
THE LOWER PORTION OF THE CROSS SECTION

As can be seen, the elasticity solution gives higher normal stresses at the top and bottom of the beam compared to the Euler-Bernoulli or Timoshenko beam. For this typical numerical example (with an aspect ratio equal to three), the elasticity normal stress at the bottom (or the top) is 3 percent higher, and the proposed theory gives 2.7 percent higher. This effect is due to the presence of distributed load or, in other words, due to the presence of higher-order warping functions. As the aspect ratio gets smaller, its effect will be more significant. (Fig. 13.)
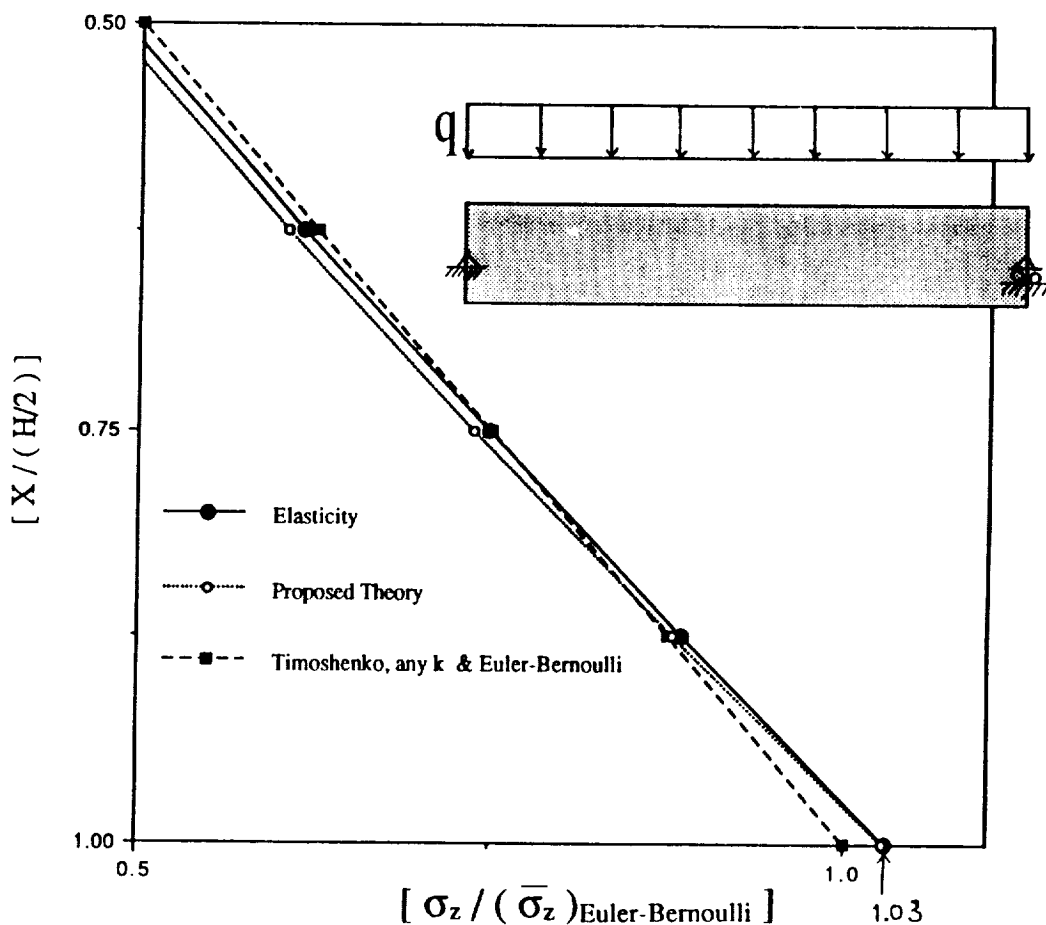

Figure 13. Detail of figure 12 for lower portion of the cross section.

TRANSVERSE SHEAR STRESSES AT LEFT-END
OF A SIMPLY SUPPORTED BEAM

The same nondimensionalization as is found in figure 8 is made. The proposed theory results in perfect agreement with the elasticity solution. Again, the Timonshenko beam gives a constant shear stress distribution which is equal to R/A where R is the reaction force at the left end. (Fig. 14.)
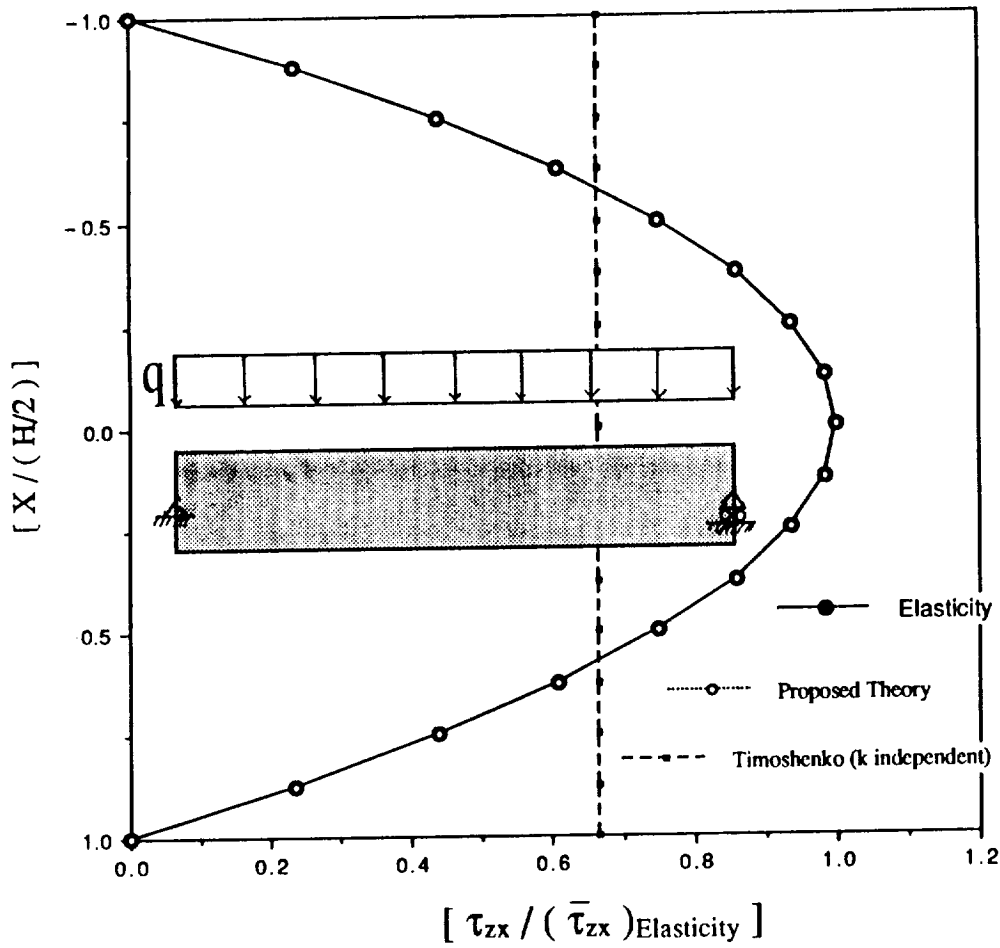

Figure 14. Shear stresses at the left-end of a simply supported beam.

TABLE OF COMPARISONS FOR THE STRESS COMPONENTS
OF A SIMPLY SUPPORTED BEAM

All underlined terms exist in the plane stress elasticity solution. The terms with the coefficients h_1 and h_2 are the nonclassical terms and become important only for beams with very small aspect ratios. As a matter of fact, these two terms, found in the expression of σ_z, are self-equilibrated in the section planes. Obviously both the Timoshenko and Euler-Bernoulli beam theories cannot capture these higher-order terms. The proposed theory is still able to capture these two terms. Although it is not accurate inside the body of the beam, it gives an accurate result at the top and bottom of the beam, which are, in fact, the most important points (for normal stresses) for practical purposes.

Due to the presence of the distributed load, the stress component σ_x will not vanish. The proposed theory gives a meaningless result in the sense that it does not satisfy the boundary conditions at the top and bottom of the beam. This is not surprising because the theory accounts for only first-order warping functions. Although it is possible to extend the theory incorporating higher-order warping functions, it may not be necessary for practical purposes. It is important to note that for this typical beam structure, the most important stress components are σ_z and τ_{zx}. (Fig. 15.)

$$\sigma_z = \underline{c_1 \; x \; z^2} + \underline{c_2 \; x \; z} + \underline{h_1 \; x^3} + \underline{h_2 \; x}$$

$$\tau_{zx} = \varepsilon_1 \; z^3 + \varepsilon_2 \; z^2 + \underline{c_3 \; x^2 \; z} + \underline{c_4 \; z} + \underline{c_5 \; x^2} + \underline{c_6}$$

$$\sigma_x = \underline{c_7 \; x^3} + \underline{c_8 \; x} + \underline{c_9} \qquad\qquad \sigma_y = d_1 \; x^3 \qquad\qquad \tau_{yz} = \tau_{xy} = 0$$

	Proposed Theory Elasticity	Timoshenko Elasticity	Euler-Bernoulli Elasticity
c_1	1.00	1.00	1.00
c_2	1.00	1.00	1.00
h_1	1.35	0.00	0.00
h_2	1.63	0.00	0.00
c_3	1.00	0.00	0.00
c_4	1.00	0.67	0.00
c_5	1.00	0.00	0.00
c_6	1.00	0.67	0.00
In the proposed theory, $\sigma_x = \sigma_y = d_1 \; x^3$; $d_1 \rightarrow 10^{-4}$.			

Figure 15. Comparison of the stress compoments relative to the elasticity solutions of a simply supported beam.

FUTURE STUDIES

Future studies will include vibration analysis, methods to determine the warping functions for a typical beam cross section, geometrical nonlinearity, and an extension for composite beams. Basically the proposed theory will be extended for general cross sections and material definition (composite beams) such that they can be applied for any special case. (Fig. 16.)

FUTURE STUDIES :

    □□□>   VIBRATION ANALYSIS

    □□□>   METHOD TO DETERMINE WARPING FUNCTIONS

    □□□>   GEOMETRIC NONLINEARITY

    □□□>   COMPOSITE BEAMS

Figure 16. Future studies.

REFERENCES

1. Cowper, G. R. : The Shear Coefficient in Timoshenko's Beam Theory.   ASME Journal of Applied Mechanics, June 1966, pp. 335-340.