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Elliptic Lambert for Space Shuttle Onboard Software

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Lyndon B. Johnson Space Center
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SHUTTLE PROGRAM

ELLIPTIC LAMBERT FOR SPACE SHUTTLE ONBOARD SOFTWARE

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1.0 INTRODUCTION

This document is a detailed derivation of the equations for the Space Shuttle onboard software module to solve Lambert's rendezvous problem. The equations presented here pertain only to elliptical transfer orbits. These equations are developed from discussions and notes received from Stanley Shepperd of Draper Laboratory. The Lambert problem is defined as follows: Given an initial position vector ($\underline{r}_0$), a terminal position vector ($\underline{r}_1$), and a specified transfer time between $\underline{r}_0$ and $\underline{r}_1$, determine the required initial velocity vector $\underline{V}_0$. This subroutine does not include multirev capability.

2.0 SYMBOLS

- a semimajor axis
- r_0 initial position magnitude
- $\underline{r}_1$ terminal position magnitude
- $\underline{r}_0$ initial position vector
- r_1 final position vector
- C magnitude of chord connecting $\bar{r}_0$ and $\bar{r}_1$
- E eccentric anomaly
- b semiminor axis
- $\underline{t}$ time
- $\tilde{t}$ normalized time
- θ transfer angle
- f true anomaly
- e eccentricity
- p orbital parameter, semilatus rectum

3.0 DERIVATION OF EQUATIONS

Lambert's theorem (ref. 1) states that the time of flight is a function of three parameters: the semimajor axis, the sum of the two radii, and the chord of the transfer. The following development obtains expressions for a, $r_0 + r_1$, and C. The geometry is shown in figure 1.

Any point on an ellipse (X,Y) is related to its eccentric anomaly. For the initial and final points of the Lambert problem, the coordinate pairs are given by

$$X_0 = a \cos E_o \qquad Y_0 = b \sin E_0$$

$$X_1 = a \cos E_1 \qquad Y_1 = b \sin E_1$$

so that

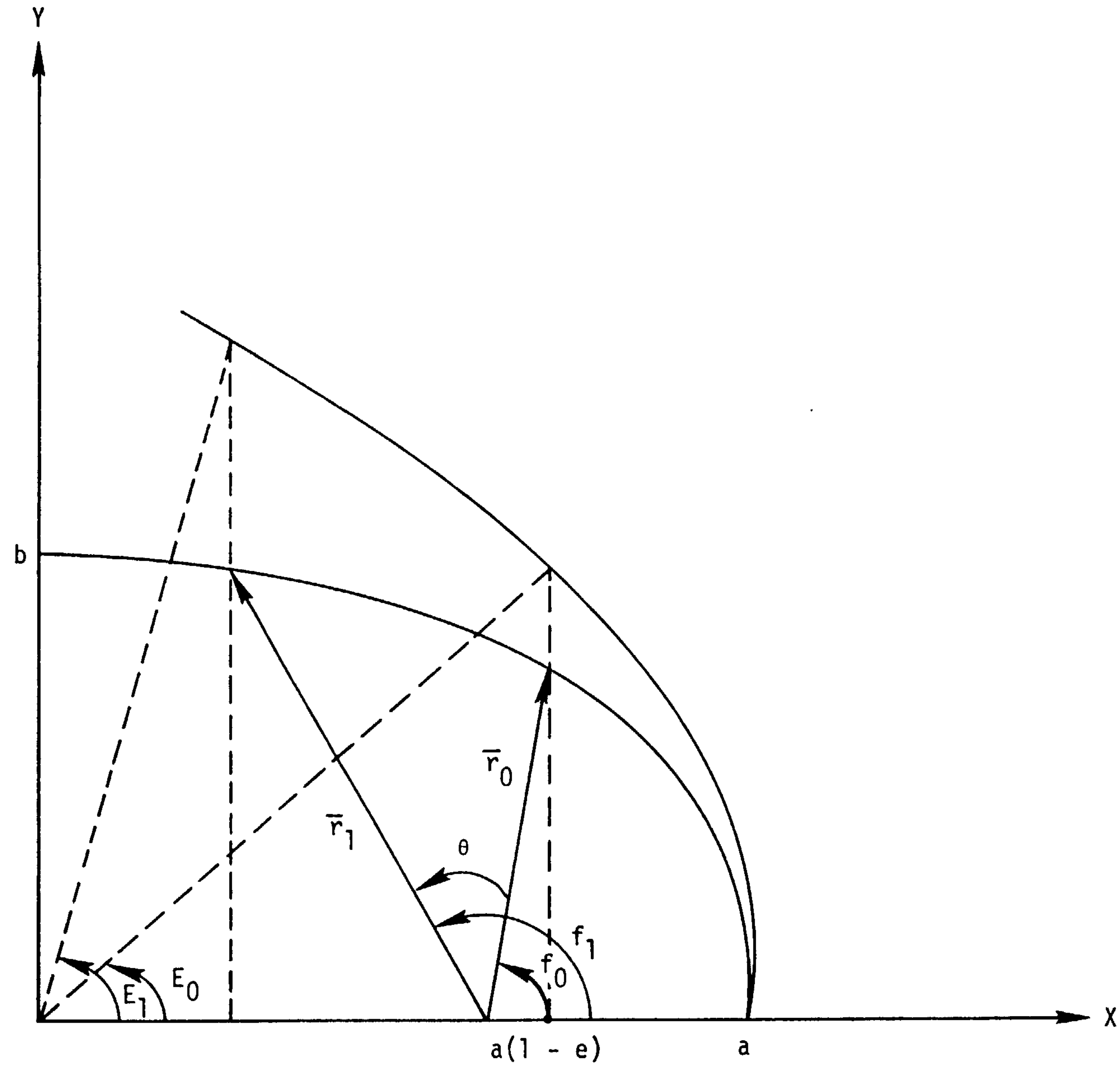

Figure 1.- Geometry of ellipse.

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

The chord between $\bar{r}_0$ and $\bar{r}_1$, is given by

$$c^2 = (X_1 - X_0)^2 + (Y_2 - Y_0)^2$$

$$= a^2 (\cos E_1 - \cos E_0)^2 + b^2 (\sin E_1 - \sin E_0)^2$$

$$= 4a^2 \sin^2 \left(\frac{E_1 + E_0}{2}\right) \sin^2 \left(\frac{E_1 - E_0}{2}\right)$$

$$+ 4b^2 \cos^2 \left(\frac{E_1 + E_0}{2}\right) \sin^2 \left(\frac{E_1 - E_0}{2}\right)$$

$$= 4 \sin^2 \left(\frac{E_1 - E_0}{2}\right) \left[a^2 \sin^2 \left(\frac{E_1 + E_0}{2}\right) + a^2 (1 - e^2) \cos^2 \left(\frac{E_1 + E_0}{2}\right)\right]$$

and

$$\left(\frac{c}{2}\right)^2 = a^2 \sin^2 \left(\frac{E_1 - E_0}{2}\right) \left[\sin^2 \left(\frac{E_1 + E_0}{2}\right) + (1 - e^2) \cos^2 \left(\frac{E_1 + E_0}{2}\right)\right]$$

$$= a^2 \sin^2 \left(\frac{E_1 - E_0}{2}\right) \left[1 - e^2 \cos^2 \left(\frac{E_1 + E_0}{2}\right)\right] \quad (1)$$

The radius of an orbit for the initial and final points of the Lambert problem can be expressed in terms of the eccentric anomaly as follows:

$$r_0 = a(1 - e \cos E_0)$$

$$r_1 = a(1 - e \cos E_1)$$

Adding these two equations yields

$$r_0 + r_1 = a\left[2 - e\,(\cos E_1 + \cos E_0)\right]$$

$$= a\left[2 - 2e\cos\left(\frac{E_1 + E_0}{2}\right)\cos\left(\frac{E_1 - E_0}{2}\right)\right]$$

and

$$\frac{r_0 + r_1}{2} = a\left(1 - e\cos\frac{E_1 + E_0}{2}\cos\frac{E_1 - E_0}{2}\right) \qquad (2)$$

Kepler's equation (ref. 1) may be written for the initial and final points of the Lambert problem as follows:

$$\sqrt{\frac{\mu}{a^3}}\; t_0 = E_0 - e\sin E_0$$

$$\sqrt{\frac{\mu}{a^3}}\; t_1 = E_1 - e\sin E_1$$

where time (t) is the time from pericenter. Subtraction of these equations yields

$$\sqrt{\frac{\mu}{a^3}}\,(t_1 - t_0) = (E_1 - E_0) - e\,(\sin E_1 - \sin E_0)$$

$$= (E_1 - E_0) - 2\,e\cos\left(\frac{E_1 + E_0}{2}\right)\sin\left(\frac{E_1 - E_0}{2}\right),$$

or

$$\sqrt{\frac{\mu}{a^3}}\;\frac{t_1 - t_0}{2} = \frac{E_1 - E_0}{2} - e\cos\left(\frac{E_1 + E_0}{2}\right)\sin\left(\frac{E_1 - E_0}{2}\right) \qquad (3)$$

Equations (1), (2), and (3) yield three equations in the following three unknowns:

$$a, \quad \frac{E_1 - E_0}{2}, \quad e \cos \frac{E_1 + E_0}{2}$$

In order to solve these three equations, the following change of variable is made:

$$\frac{\alpha - \beta}{2} \equiv \frac{E_1 - E_0}{2} \tag{4}$$

$$\cos \frac{\alpha + \beta}{2} \equiv e \cos \frac{E_1 + E_0}{2} \tag{5}$$

The angles α and β are defined to satisfy the following limits:

$$0 \leq \alpha - \beta < 2\pi$$

$$0 \leq \alpha + \beta < 2\pi$$

These definitions and constraints imply that

$$0 \leq \alpha < 2\pi$$

$$-\pi \leq \beta < \pi$$

This is shown in figure 2.

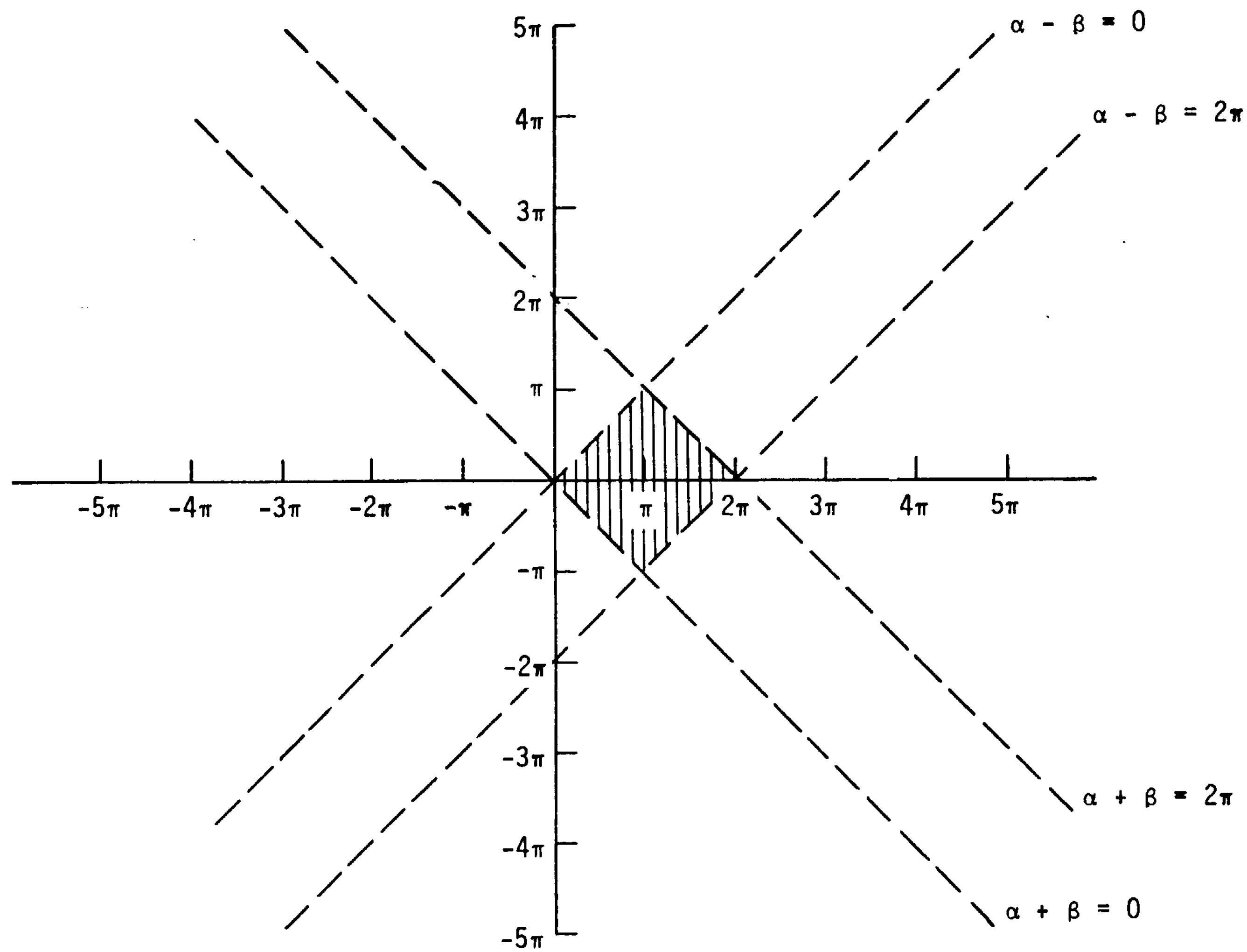

Figure 2.- α, β constraints.

Substituting equations (4) and (5) for equations (1), (2), and (3) yields the following results:

$$C/_{2a} = \sin\left(\frac{\alpha - \beta}{2}\right)\sin\left(\frac{\alpha + \beta}{2}\right) \tag{6}$$

$$\frac{r_0 + r_1}{2a} = 1 - \cos\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right) \tag{7}$$

$$\sqrt{\frac{\mu}{a^3}}\;\frac{t_1 - t_0}{2} = \frac{\alpha - \beta}{2} - \sin\left(\frac{\alpha - \beta}{2}\right)\cos\left(\frac{\alpha + \beta}{2}\right) \tag{8}$$

or equivalently,

$$C/_a = \cos\beta - \cos\alpha \tag{9}$$

$$\frac{r_0 + r_1}{a} = 2 - (\cos\beta + \cos\alpha) \tag{10}$$

$$\sqrt{\frac{\mu}{a^3}}\;(t_1 - t_0) = (\alpha - \beta) - (\sin\alpha - \sin\beta) \tag{11}$$

The three unknowns are now a, α, and β.

From equations (9) and (10), the equations for α and β can be separated to obtain

$$\cos\alpha = 1 - \frac{r_0 + r_1 + C}{2a} \tag{12}$$

and

$$\cos\beta = 1 - \frac{r_0 + r_1 + C}{2a} \tag{13}$$

The semi-perimeter of the transfer angle S is given by

$$S = \frac{r_0 + r_1 + C}{2}$$

Equations (11) and (13) can now be written as

$$1 - \cos \alpha = {}^{S}/_{a}$$

and

$$1 - \cos \beta = S - {}^{C}/_{a}$$

or, since $\sin \frac{X}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$

$$\sin^2 \frac{\alpha}{2} = \frac{S}{2a} \qquad (14)$$

and

$$\sin^2 \frac{\beta}{2} = \frac{S - C}{2a} \qquad (15)$$

Equations (14) and (15) can be combined to yield

$$\sin \frac{\beta}{2} = \pm \sqrt{\frac{S - C}{Sa}} \quad \sin \left(\frac{\alpha}{2}\right)$$

Define

$$\lambda = \pm \sqrt{\frac{S - C}{S}}$$

so that

$$\sin\frac{\beta}{2} = \lambda \sin\frac{\alpha}{2}$$

From simple trigonometric relations

$$\pm\sqrt{\frac{s - c}{s}} = \frac{\sqrt{r_0 + r_1}}{s} \cos\frac{\theta}{2}$$

From this relationship, λ is a constant for a particular problem in the range of $-1 \leq \lambda \leq 1$. Positive values occur for transfers less than 180° and negative values for transfers greater than 180°.

The independent variable u is chosen to be

$$u = \cos\frac{\alpha}{2} \tag{16}$$

The other sines and cosines follow from the following sequence of operations:

$$\sin\frac{\alpha}{2} = \sqrt{1 - u^2} \quad , \quad \sin\frac{\alpha}{2} \text{ is always positive}$$

since $0 \leq \alpha < 2\pi$

$$\sin\frac{\beta}{2} = \lambda \sin\frac{\alpha}{2} \quad , \quad \text{sign of } \sin\frac{\beta}{2} \text{ determined by } \lambda$$

$$\cos\frac{\beta}{2} = \sqrt{1 - \sin^2\frac{\beta}{2}} \quad , \quad \text{sign of } \cos\frac{\beta}{2} \text{ must be positive}$$

since $-\pi \leq \beta < \pi$

The above relations allow evaluation of Kepler's equation to evaluate t corresponding to the current value of u. A Newton-Raphson iteration is performed to determine the value of u to yield the desired value t. Since

$$0 \leq \alpha \leq 2\pi$$

and $u = \cos \frac{\alpha}{2}$, the value of u is between

$$-1 < u < 1$$

4.0 DETERMINATION OF INITIAL VELOCITY EQUATION

From basic two-body motion, the equation for the required velocity vector $\bar{V}_0$ is given by

$$\bar{V}_0 = Vr_0 \, \bar{i}r_0 + Vh_0 \, (\bar{i}_n \times \bar{i}r_0),$$

where

$\bar{i}r_0$ = a unit vector along $\bar{r}_0$

$\bar{i}_n$ = a unit vector along the angular momentum vector

Vr_0 = the component of velocity along $\bar{i}r_0$

Vh_0 = the component of velocity along $\bar{i}_n \times \bar{i}r_0$

$$Vr_0 = \sqrt{\frac{-\mu\rho}{r_0^{\,2}} + \frac{2\mu}{r_0} - \frac{\mu}{a}} = \sqrt{\mu} \quad \sqrt{-\rho + 2r_0 - \frac{r_0^{\,2}}{a}}$$

$$\rho = \frac{V_0^{\,2} r_0^{\,2}}{\mu} - \sigma_0^2$$

$$Vr_0 = \frac{\sqrt{\mu}}{r_0} \, \sigma_0$$

$$Vh_0 \quad \frac{\sqrt{\mu}}{r_0} \sqrt{\rho} \quad ,$$

where ρ is the semilatus rectum and

$$\sigma_0 = \frac{\bar{r}_0 \cdot \bar{V}_0}{\sqrt{\mu}}$$

Therefore,

$$\bar{V}_0 = \frac{\sqrt{\mu}}{r_0} \left(\sigma_0 \, \underline{i}_{r0} + \sqrt{\rho} \, (\bar{i}_n \, X \bar{i}r_0)\right)$$

From reference 3:

$$r_0 \cos\left(\frac{E_1 - E_0}{2}\right) + \sigma_0 \sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right) = \lambda S$$

and

$$\sigma_0 = \frac{\lambda S - r_0 \cos\left(\frac{E_1 - E_0}{2}\right)}{\sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right)}$$

Also from reference 3,

$$\sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right) = \frac{\sqrt{r_0 r_1}}{\sqrt{\rho}} \sin\frac{\Theta}{2} ,$$

and

$$\sqrt{\rho} = \frac{\sqrt{r_0 r_1} \sin\frac{\Theta}{2}}{\sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right)}$$

Therefore,

$$\bar{V}_0 = \frac{\sqrt{\mu}}{r_0} \left(\frac{\lambda S - r_0 \cos\left(\frac{E1 - E_0}{2}\right)}{\sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right)} \bar{i}_{r0} + \frac{\sqrt{r_0 r_1} \sin \frac{\theta}{2}}{\sqrt{\alpha} \sin\left(\frac{E_1 - E_0}{2}\right)} (\bar{i}_n \times \bar{i}r_0) \right)$$

5.0 CODED EQUATIONS

This section presents and explains the equations for this routine as coded and tested. The routine is coded in FORTRAN and is named LCONVR. The coded equations presented represent a bench program version of the onboard Lambert targeting routine specification as documented in reference 2.

The input and output arguments for the routine are as follows:

Input list: RO, R1, NUNIT, T
Output list: VO, ALARM

where

RO = the initial position vector

R1 = the final position vector

NUNIT = a unit vector in the direction of the angular momentum vector

T = the desired transfer time

VO = the required initial velocity

ALARM = a flag to signal possible problems with input data or iteration procedures

For descriptive purposes, the code has been blocked into nine sections. Each section is listed and explained as follows:

a. Code section 1

```
      ALARM=0
      ROMAG=VECMG(RO)
      R1MAG=VECMG(R1)
      DO 1 I=1,3
      TEMP(I)=R1(I)-RO(I)
    1 CONTINUE
      RPARA=0.5*(ROMAG+R1MAG+VECMG(TEMP))
      VPARA=SQRT(2.0*GM/RPARA)
      CALL CROSS(RO,R1,TEMP)
      Z=ROMAG*R1MAG-DOT(RO,R1)
      IF(     Z .GT.EPTRAN*ROMAG*R1MAG) GO TO 2
      ALARM=2
      RETURN
```

Comment section 1 - This segment of the code calculates some program constants and does some initial testing. The ALARM flag is set to zero. The value of ALARM will be changed to a positive integer if a test is failed

$$ROMAG = |\overline{RO}|$$

$$R1MAG = |\overline{R1}|$$

RPARA is equal to the semiperimeter of the transfer triangle and is used as a normalizing factor. VPARA is equal to the parabolic velocity at perigee for VPARA.

The parameter Z is calculated as

$$Z = |\overline{RO}||\overline{R1}| - \overline{RO} \cdot \overline{R1} = |\overline{RO}||\overline{R1}| \; (1 - \cos\theta).$$

If the transfer angle is close to 0° or 360°, some of the equations break down numerically. For example,

$$\sqrt{\frac{\mu}{a^3}} \; (t_1 - t_0) = (\alpha - \beta) - (\sin\alpha - \sin\beta)$$

A test is made at this point. If the transfer angle θ is close to 0° or 360°, ALARM is set to 2 and an exit is made.

b. Code section 2

```
    2 CONTINUE
      U=DOT(TEMP,NUNIT)/Z
      VH=SQRT(ROMAG*R1MAG/(1.0+U*U))
      N=0
      UMAX=1.0-0.5*EPSDU
      UMIN=-1.0
```

Comment section 2 - The cotangent of $\theta/2$ is calculated as

$$Z = (\overline{RO} \times \overline{R1}) \cdot \overline{NUNIT}/Z$$

$\overline{NUNIT}$ is a unit vector in the direction of the angular momentum.
The parameter VH is calculated as

$$VH = \sqrt{|\overline{RO}||\overline{R1}| \;/\; (1 + Z^2)}$$

The initial value of the counter is set on the number of Newton-Raphson iterations:

$$N = 0$$

The upper and lower limits on u is set, and u can take on the values between

$$-1 < 0 < +1$$

However, if the orbit is near parabolic, u will be close to 1 (u = 1 if the orbit is parabolic), and some of the equations will numerically break down. For this reason, the lower limit on u is set to -1 where

$$UMIN = -1$$

and the upper limit on u is set to equal

$$UMAX = 1 - \frac{DU}{2}$$

DU is a calculated number. Figure 3 explains the development of DU.

The normalized time t versus u is shown in figure 3. From reference 3, t can be written as

$$\tilde{t} = 4 \lambda D + D^3 Q$$

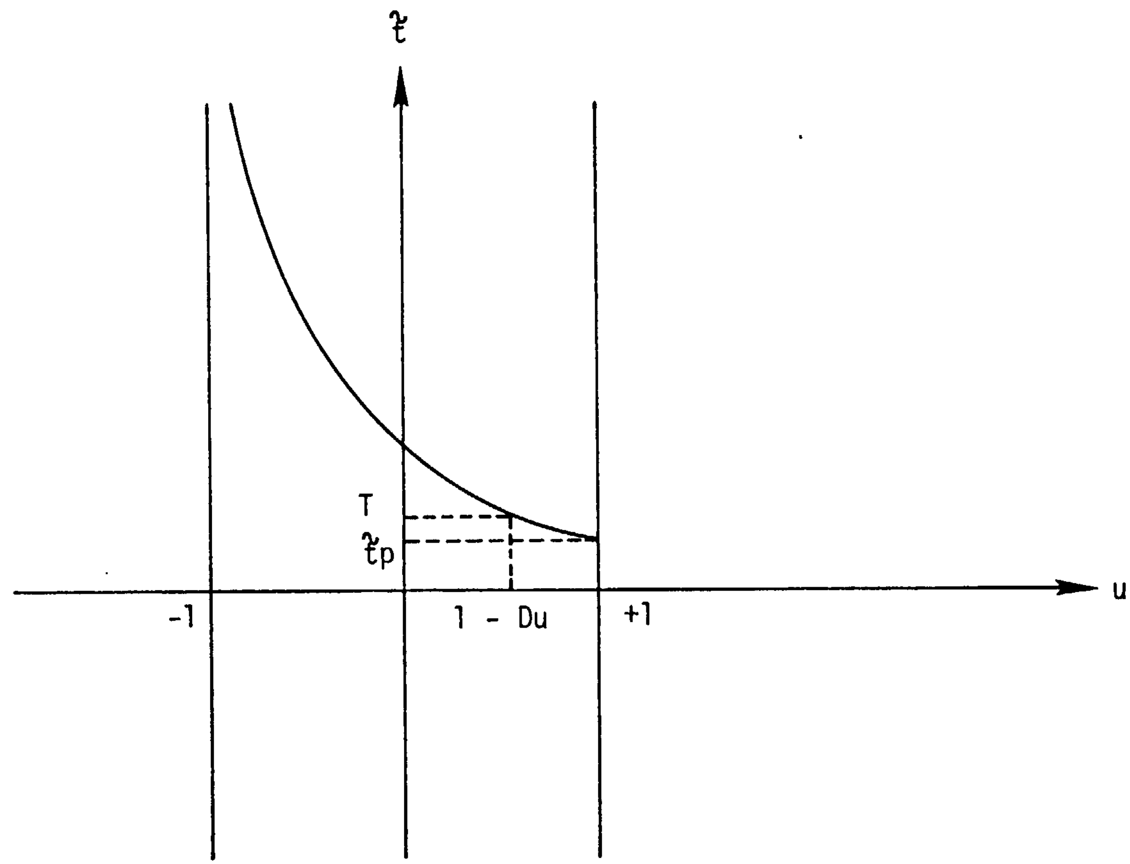

Figure 3.- $\tilde{t}$ versus u.

where

$$\tilde{t} = \sqrt{\frac{8\mu}{S^3}} \quad \frac{(t_1 - t_0)}{2}$$

$$D = y - \lambda u$$

$$Y = \sqrt{1 - \lambda^2 (1 - u^2)}$$

$$Q = \frac{4}{3} F(1, 3, \frac{5}{2}, q) \text{ (ref. 3)} = f (w (u))$$

$$W = \sqrt{\frac{1 + E}{2}}$$

$$E = \lambda + uD$$

The purpose of the following development is to calculate the slope of $\tilde{t}_{u = 1}$ and estimate the value of t when $u = 1 - DU$. When

$$U = 1$$

$$Y = 1$$

$$D = 1 - \lambda$$

$$E = 1$$

$$W = 1$$

$$Q = \frac{4}{3}$$

and

$$\tilde{t}p = 4\lambda (1 - \lambda) + (1 - \lambda)3 \frac{4}{3}$$

$$= \frac{4}{3} (1 - \lambda^3)$$

The slope of $\tilde{t}$ at $u = 1$ is given by

$$\frac{d\tilde{t}}{du} = \left[4\lambda + 3D^2Q\right] \frac{dD}{du} + D^3 \frac{dQ}{du}$$

$$\frac{dy}{du} = \frac{\lambda^2 u}{y} = \lambda^2$$

$$\frac{dD}{du} = \frac{-\lambda D}{y} = -\lambda\ (1 - \lambda)$$

$$\frac{dE}{du} = \frac{D^2}{y} = (1 - \lambda)^2$$

$$\frac{dw}{du} = \frac{D^2}{4wy} = \frac{(1 - \lambda)^2}{4}$$

$$\frac{dQ}{du} = \frac{dQ}{dw}\ \frac{dw}{du}$$

$$\frac{dw}{du} = \frac{(y - \lambda u)^2}{4wy} = \frac{(1 - \lambda)^2}{4}$$

$$\frac{dQ}{dw} = \frac{5\ (w + 1)(w + 2) + 2F}{5w^2(w + 1)^3} - \frac{3Q}{w} = -\frac{16}{5}$$

$$\frac{dQ}{du} = \frac{(1 - \lambda)^2}{4} \left(-\frac{16}{5}\right) = -\frac{4}{5}\ (1 - \lambda)^2$$

$$\frac{dtp}{du} = -\lambda \left[4\lambda + 3(1 - \lambda)^2 \frac{4}{3}\right] (-\lambda)\ (1 - \lambda) - (1 - \lambda)^3 \frac{4}{5}\ (1 - \lambda)^2$$

$$= -\frac{4}{5}\ (1 - \lambda^5)$$

Now

$$T = \tilde{t}p + \Delta\tilde{t}p = \tilde{t}p - DU\,\frac{d\tilde{t}p}{du} = \frac{4}{3}(1-\lambda)^3 + \frac{4}{5}(1-\lambda^5)\,DU$$

However, only half of this value is taken to be conservative and

$$T = \frac{2}{3}(1-\lambda)^3 + \frac{2}{5}(1-\lambda^5)\,DU$$

c. Code section 3

```
LAMBDA=VH/RPARA*U
IF(ABS(LAMBDA).LT.1.0E-09) LAMBDA=0.0
U=LAMBDA/SQRT(1.0+LAMBDA*LAMBDA)
TILDES=VPARA/RPARA*T
```

Comment section 3 - The constant parameter LAMBDA is calculated as

$$LAMBDA = \left(\frac{VH}{VPARA}\right) u$$

The first guess for the independent variable u is calculated. This first guess assumes a circular orbit and is calculated as

$$u^2 = \cos^2\frac{\alpha}{2} = 1 - \sin^2\frac{\alpha}{2}$$

$$= 1 - \frac{S}{2}a$$

For a circular orbit,

$$2a = r_0 + r_1$$

However,

$$r_0 + r_1 = 2\,S - C$$

$$= S\left(1 + \frac{S - C}{S}\right)$$

Since $\lambda = \pm \sqrt{\dfrac{S - C}{S}}$,

$$r_0 + r_1 = S\,(1 + \lambda^2).$$

Therefore,

$$u = \sqrt{1 - \frac{S}{S\,(1 + \lambda^2)}} = \lambda \quad \sqrt{1 + \lambda^2}$$

The normalized transfer time is calculated as

$$\text{TILDES} = \frac{\text{VPARA}}{\text{RPARA}} \cdot \ T$$

A problem may occur in λ if the transfer angle is near 180°. Since λ contains the cross product of $\bar{R}_0$, $\bar{R}_1$, λ may be near zero, and λ^2 may cause an underflow. The code must consider this problem.

d. Code section 4

```
TOTHE5=0.0
IF(ABS(LAMBDA).GT.1.0E-07) TOTHE5=LAMBDA**5
TMIN=2.0/3.0*(1.0-LAMBDA**3)+0.4*(1.0-TOTHE5)*EPSDU
IF(TILDES.GT.TMIN) GO TO 3
ALARM=1
RETURN
```

Comment section 4 - A transfer time is determined which is slightly greater than the parabolic transfer time. The parabolic transfer time is given by $u = 1$, or

$$\tilde{t}_{u\,=\,1} = \frac{2}{3}\,(1 - \text{LAMBDA}^3)$$

The correction for the DU shift away from u = 1 is also considered, so that

$$T\_MIN = \frac{2}{3}(1 - LAMBDA^3) + \frac{2}{5}(1 - LAMBDA^5)\ DU$$

The value of LAMBDA must be tested to prevent an underflow in LAMBDA5 if the transfer angle is near 180°. The normalized value of the transfer time is tested against the value of T_MIN. If the normalized time is less than T_MIN, the desired transfer is parabolic, or close to parabolic. The ALARM flag is set to 3, and an exit is made.

e. Code section 5

```
    3 CONTINUE
      N=N+1
      CALL CALA
      IF(ABS(TILERR).LT.EPSILN*TILDES) GO TO 4
```

Comment section 5 - The counter on the number of Newton-Raphson iterations is incremented by calling the subroutine CALA. This subroutine calculates the transfer time corresponding to the current value of u. If the transfer time obtained from Kepler's equation yields the desired transfer time, within tolerances, the program calculates the desired initial velocity.

f. Code section 6

```
      CALL CALB
      IF(N.LT.NMAX) GO TO 3
      ALARM=5
```

Comment section 6 - If the transfer time obtained from Kepler's equation does not yield a satisfactory answer, a call is made to the Newton Raphson iterator to obtain a better value of u.

g. Code section 7

```
    4 CONTINUE
      VH=VH/ROMAG
      VR=RPARA/ROMAG*LAMBDA-G
      COEF=VPARA/(Y-LAMBDA*U)
      COEF=COEF/ROMAG
      CALL CROSS(NUNIT,RO,TEMP)
      DO 5 I=1,3
      VO(I)=COEF*(VR*RO(I)+VH*TEMP(I))
    5 CONTINUE
      RETURN
```

Comment section 7 - The code section 7 equations are the terms to calculate the required initial velocity previously discussed. The equations are slightly modified to remove the normalizing factor.

h. Code section 8

```
SUBROUTINE CALA
W=SQRT(1.0-U*U)
X=LAMBDA*W
Y=SQRT(1.0-X*X)
F=W*Y-U*X
G=U*Y+W*X
TILDA=(    ARTAN( F, G)-(U*W-X*Y))/(W*W*W)
STILDA=(3.0*U*TILDA-2.0*(1.0-(U/Y)*LAMBDA**3))/(W*W)
TILERR=TILDES-TILDA
RETURN
```

Comment section 8 - The following sequence of calculations leads to the determination of the normalized transfer time corresponding to the current value of u.

$$W = \sin\frac{\alpha}{2} = \sqrt{1 - U^2}$$

$$X = \sin\frac{\beta}{2} = \lambda W$$

$$Y = \cos\frac{\beta}{2} = \sqrt{1 - X^2}$$

$$F = \sin\frac{1}{2}(\alpha - \beta) = WY - UX$$

$$G = \cos\frac{1}{2}(\alpha - \beta) = UY + WX$$

$$\tilde{t} = \frac{1}{W^3}\left[\tan^{-1}\left(\frac{F}{G}\right) - (UW - XY)\right]$$

Since $0 \leq \alpha - \beta < 2\pi$,

$$0 \leq \tan^{-1}\left(\frac{F}{G}\right) < \pi$$

Also, since $t = \frac{1}{W^3}\left[\tan^{-1}\left(\frac{F}{G}\right) - (UW - XY)\right]$,

$$\frac{d\tilde{t}}{du} = \frac{-3}{W^4} \frac{dW}{dW} \left[\tan^{-1} \left(\frac{F}{G} \right) - (UW - XY) \right]$$

$$+ \frac{1}{W^3} \frac{d}{du} \left[\tan^{-1} \left(\frac{F}{G} \right) - (UW - XY) \right]$$

With a significant amount of manipulating and rearranging,

$$\frac{d\tilde{t}}{du} = \frac{1}{W^2} \left[3\ at - 2 \left(1 - \frac{X^2}{W^2} \frac{\lambda \mu}{Y} \right) \right]$$

The error in the desired transfer time and the transfer time obtained from Kepler's equation is given by TILE RR.

i. Code section 9

```
00234              SUBROUTINE CALB
00237              USTEP=TILERR/STILDA
00240              IF(USTEP.LE.0.0) GO TO 100
00242              UMIN=U
00243              U=U+USTEP
00244              IF(U.GT.UMAX) U=0.5*(UMIN+UMAX)
00246              RETURN
00247          100 CONTINUE
00250              UMAX=U
00251              U=U+USTEP
00252              IF(U.LT.UMIN) U=0.5*(UMIN+UMAX)
00254              RETURN
00255              END
```

Comment section 9 - This code is just the Newton-Raphson step to find the change in u for the next step.

6.0 REFERENCES

1. Battin, R. H.: Astronautical Guidance. McGraw-Hill Book Company, Inc., 1964.

2. OFT Level C FSSR. GN&C-PARTA, Guidance Onorbit I (Rev. 1) SD 76-SH-0003A, Mar. 1, 1978.

3. Uzzell, B. R.: JSC IN 77-FM-3, Jan. 1977, Derivation and Documentation of the New Battin Conic Subroutines.