A NEW MATRIX DEVELOPMENT OF THE POTENTIAL AND ATTRACTION AT EXTERIOR POINTS AS A FUNCTION OF THE INERTIA TENSORS

TOMÁS SOLER
National Geodetic Survey, Charting and Geodetic Services, National Ocean Service, NOAA, Rockville, MD 20852, U.S.A.

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ABSTRACT. In this work a novel tensor-matrix notation is first introduced and later applied to develop a new general expression to compute the potential of a body at exterior points as a function of the full tensors of inertia. As a corollary the general matrix development of the gravitational attraction in function of the inertia tensors is also established. For clarity the first terms in both expansions are given explicitly in a simplified matrix form. Some classical particular cases still used in geophysical and geodetic literature are pointed out and discussed.

1. INTRODUCTION

It is well known that the gravitational potential V of a body at any exterior point P may be expressed as a spherical harmonic expansion. Much has been written about this subject, and among the representative basic references on the topic, one may mention (Hobson, 1931) and (Heiskanen and Moritz, 1967).

A general representation of the potential V as a function of the so-called inertial integrals and the partial derivatives of $1/\bar{r}$ ($\bar{r}$ being the radius vector of P) with respect to the Cartesian coordinates of P was given in (Thomson and Tait, 1912, I, p. 202) and (MacMillan, 1930, p. 329). Because of the practical limitations of the above formula, MacMillan (1930, p. 384) proceeds one step further, providing the expansion of the potential function with respect to the inertial integrals of the body and the powers of the Cartesian coordinates of the point in question.

In this paper a novel approach using a general matrix representation for the expansion of the exterior potential as a function of the full inertia tensors of rank r is presented.

This is followed by another general matrix development giving the Cartesian components of the gravitational attraction (force function) at the same point as a function of the inertia tensors.

A significant difference from the methods mentioned above is the possibility of writing the low order terms of the expansion in a conceptually simple matrix form, avoiding lengthy and cumbersome polynomial expansions of considerable complexity (see MacMillan, 1930, p. 87; Grafarend, 1980 or Doubochine, 1981).

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The matrix notation introduced here is especially useful when treating problems involving rotation of coordinate systems. A second important difference which contrasts with the spherical harmonic expansion is the fact that in this presentation the primary ingredients of the development are the full tensors of inertia, well defined physical quantities with fundamental dynamic properties in any body.

2. PRELIMINARY CONCEPTS AND NOTATION

A new notation is introduced in this work in order to simplify mathematical expressions. This direct notation is intended to be as clear and concise as feasible and is fully based on conventional matrix calculus, avoiding as much as possible the index convention commonly used in tensor treatises. In this way all the variables contained in the equations will be explicitly represented or implied in a straightforward manner.

Although matrix notation, in the author's opinion, is sometimes not as compact as formal tensor notation, it is more intuitive and easily comprehended by any reader with a basic knowledge of matrix algebra. As a by-product, the resulting matrix equations can be coded immediately in any computer language, taking advantage of matrix manipulation subroutines available in most computer facilities.

Unless otherwise stated, the following preliminary notations will be adopted throughout this paper.

2.1. Coordinate systems

Three-dimensional orthogonal Cartesian coordinates and those coordinate systems derived from them exist in what is known as an Euclidean three space represented by E^3. This is the space of our ordinary experience and the only one implied throughout this paper. However, it should be mentioned that there are spaces of both mathematical and physical interest which are neither three-dimensional nor Euclidean.

2.2. Vectors

In this study vectors of coordinates will be represented by column matrices denoted as

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}.$$

This column vector defines a point in the E^3 Euclidean space specified by the three real numbers x_i, $i = 1, 2, 3$, expressing the coordinates of the point along the Cartesian system.

To conform with matrix multiplication rules, sometimes the components of the same vector $\{x\}$ will be arranged in horizontal array

$$\{x\}^t = \{x_1 \quad x_2 \quad x_3\}$$

and termed a row vector. (The symbol t stands for transpose).

One-dimensional vectors are called scalars.

2.3. Matrices

In general a 3×3 real matrix will be denoted between two brackets [M]. Sometimes well-known types of matrices are written without the brackets. For example, this is the case of the rotation matrix R of the transformation between two Cartesian systems of coordinates, described in Section 6.

The following special types of matrix notations are introduced and subsequently used:

a) Skew-symmetric

To every vector $\{x\}$ it is possible to associate a skew-symmetric matrix denoted by

$$[\underline{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{2.1}$$

where clearly

$$[\underline{x}]^t = -[\underline{x}]. \tag{2.2}$$

b) Identity matrix

The 3×3 unit, or identity matrix, will always be denoted by [1]. In general a unit matrix of order n will be written $[1]_{n\times n}$.

c) Symmetric matrices

Such matrices will always be represented by the upper triangular elements only, and the letter s in the lower left of the matrix.

3. THE TENSOR-MATRIX CONCEPT

In principle, the algebra that has been devised for matrices may be used for tensors as well. In the E^3 Euclidean space all tensors are of dimension $d = 3$ and the distinction between covariant and contravariant components is non-existent. Nevertheless, tensors in E^3 may have different rank. It is well known that scalars are tensors of rank zero. Any 3×1 vector is a tensor of rank one, and 3×3 matrices are tensors of second rank. In particular, tensors of second rank, like their corresponding matrices, may be diagonal, symmetric, anti-symmetric, orthogonal, etc.

The total number of elements of a tensor of dimension d and rank r is d^r. In the particular case of tensors of inertia, only $\frac{1}{2}(r+1)(r+2)$ are distinct (i.e. not equal); consequently this is the maximum number of elements which must be computed in order to fully know any tensor of inertia of rank r.

At this point one should take note of the difference between tensor rank and matrix rank. Although it will not be used in this paper, the rank of a matrix [M] is the number of linearly independent columns (or rows) of [M]. Obviously for a 3×3 matrix $0 \leq \text{rank } [M] \leq 3$. Some modern texts of tensor calculus replace the more traditional word 'rank' (of a tensor) with 'order' but this terminology will not be followed here.

Three different tensor operations will be introduced having the following properties.

3.1. Tensor product

This is an operation between two tensors of rank r and s which results in a tensor of rank $r+s$. An example of this operation in matrix notation is the product of a vector and its transpose, namely $\{x\}\{x\}^t$. The term 'dyadic', infrequently employed today, occasionally is mentioned synonymously with the $\{x\}\{x\}^t$ tensor. The symbol $\otimes$ commonly used in mathematical literature to denote this operation will be adopted in the present discussion except in the case of the dyadic tensor.

3.2. Tensor 'inner' product

It is an operation between two tensors of rank r and s which gives as a result a tensor of rank $r+s-2$. The scalar product of two vectors is a particular example of this operation (e.g. $\{\bar{x}\}^t\{x\} = \text{scalar}$). No particular symbol for this operation will be used here. In subsequent sections it will be shown that it follows matrix multiplication rules with some restrictions.

3.3. Contraction of a tensor

It is an operation that, applied once to a tensor of rank r (for $r \geq 2$) gives as a result a tensor of rank $r-2$. The symbol $\mathscr{C}$ will be used to denote contraction (i.e., first order contraction); thus

$$\mathscr{C}[\text{tensor of rank } r] = \text{tensor of rank } r-2$$

As is well known, the contraction of a second rank tensor is equal to the trace of its matrix. It will be seen later that inertia tensors of rank greater than three may have contractions of several orders. Thus the contractions of order k will be denoted by $\overset{k}{\mathscr{C}}$. By definition the contraction of order zero is the identity contraction $\overset{o}{\mathscr{C}}[\ ] = [\ ]$.

The tensor inner product described above when mentioned in books is introduced as a contracted tensor product; nevertheless this practice will not be followed in this presentation.

Whereas 3.1, 3.2 and 3.3 do not completely define these operations, the remaining details will become clear from the examples considered below.

4. TENSORS OF INERTIA OF SECOND RANK

There is little agreement in physical and mathematical literature about what is referred to as tensors of inertia. In this work the nomenclature suggested by Hotine (1969, p. 165) is adopted, departing from the conventional terminology to which the readers may be accustomed. The following definitions and notations will be used throughout this paper.

4.1. Tensor of inertia of second rank (Inertia matrix of second rank)

The tensor of inertia of second rank of a body in a Cartesian coordinate system is defined by any of the forms given below

$$[I] = \int_M \{x\}\{x\}^t \, dm = \int_M [J] \, dm =$$

$$= \int_M \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ & x_2^2 & x_2x_3 \\ s & & x_3^2 \end{bmatrix} dm = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ & I_{22} & I_{23} \\ s & & I_{33} \end{bmatrix} \tag{4.1}$$

where the integration is to be extended over the total mass M of the body.

The diagonal elements I_{ii}, $i = 1, 2, 3$ are referred to as the moments of inertia with regard to the planes $x_i = 0$, $i = 1, 2, 3$ respectively, and I_{ij}, $i \neq j = 1, 2, 3$ as the products of inertia with respect to the planes $x_i = 0$ and $x_j = 0$, $i \neq j = 1, 2, 3$.

The symmetry of [I] follows immediately from the matrix equality

$$[\{x\}\{x\}^t]^t = \{x\}\{x\}^t. \tag{4.2}$$

Notice that the elements of the inertia matrix of second rank in (4.1) have two subindices. In general, elements in tensors or inertia matrices of rank r will have r subindices.

4.2. Associate tensor of inertia of second rank

The associate tensor of inertia of a body in a Cartesian coordinate system is defined by

$$[\mathbb{I}] = \int_M [\underline{x}][\underline{x}]^t \, dm = \int_M [\mathbb{J}] \, dm =$$

$$= \int_M \begin{bmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ & x_1^2 + x_3^2 & -x_2x_3 \\ s & & x_1^2 + x_2^2 \end{bmatrix} dm = \begin{bmatrix} A & -F & -E \\ & B & -D \\ s & & C \end{bmatrix} \tag{4.3}$$

where A, B, C are referred to as the moments of inertia of the body with respect to the x_1, x_2 and x_3 axes respectively, and $D = I_{23}$, $E = I_{13}$ and $F = I_{12}$ as defined above, or equivalently the products of inertia with respect to the axes x_2 and x_3, x_1 and x_3 and x_1 and x_2 respectively.

The moments and products of inertia are known as the six constants of the body with respect to a particular coordinate system, and clearly they are dependent on the choice of coordinate system.

The symmetry of the associate tensor of inertia of second rank is obvious from

$$[[\underline{x}][\underline{x}]^t]^t = [\underline{x}][\underline{x}]^t. \tag{4.4}$$

It may be proved that

$$[\underline{x}][\underline{x}]^t = (\{x\}^t\{x\})[1] - \{x\}\{x\}^t \tag{4.5}$$

giving a second representation of $[\mathbb{I}]$ as

$$[\mathbb{I}] = \int_M [(\{x\}^t\{x\})[1] - \{x\}\{x\}^t] \, dm. \tag{4.6}$$

By an orthogonal transformation (which preserves the rectangular Cartesian character of the coordinate axes), the matrix $[\mathbb{I}]$ may be reduced to the diagonal form

$$\mathrm{diag}[\mathbb{I}] = \begin{bmatrix} A_p & 0 & 0 \\ & B_p & 0 \\ s & & C_p \end{bmatrix}. \tag{4.7}$$

Then the values A_p, B_p and C_p are called the 'principal moments of inertia' and the axes of the transformed coordinate system are called the 'principal axes of inertia'. Notice that the origin is not changed by the orthogonal transformation.

The tensor of inertia referred to a coordinate system with origin at the center of mass (CM) will be called the 'central tensor of inertia'. If the central tensor is principal (i.e. diagonal) its associated coordinate axes are called 'central principal axes', or sometimes, 'axes of figure' of the body.

The coordinates of a point with respect to the axes of figure will be denoted by x_{op_i}, $i = 1, 2, 3$. Moments of inertia with respect to central principal axes will also be distinguished by subscripts op.

Observe that according to the above definitions the principal axes of inertia of a body are not necessarily central, a fact many times overlooked.

The first order contraction of [I] gives the moment of inertia with respect to the origin of the coordinate system

$$\mathscr{C}[I] = \mathrm{Tr}[I] = \int_M \{x\}^t\{x\} \, dm = \int_M (x_1^2 + x_2^2 + x_3^2) \, dm. \tag{4.8}$$

It is immediately proved from (4.1), (4.3) and (4.6) that the second rank tensors $[I]$ and $[\mathbb{I}]$ are related by the equations

$$\mathrm{Tr}[I] = \tfrac{1}{2}\mathrm{Tr}[\mathbb{I}] \tag{4.9}$$

$$[I] = \tfrac{1}{2}(\mathrm{Tr}[\mathbb{I}])[1] - [\mathbb{I}] \tag{4.10}$$

5. TENSORS OF INERTIA OF RANK HIGHER THAN TWO

5.1. Tensors of inertia of third rank

The tensor of inertia of third rank will be defined in matrix notation as

$$\{[I]\} = \int_M \{x\} \otimes [J]\, dm = \int_M \begin{Bmatrix} x_1[J] \\ x_2[J] \\ x_3[J] \end{Bmatrix} dm = \int_M \begin{Bmatrix} [J]_1 \\ [J]_2 \\ [J]_3 \end{Bmatrix} dm = \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} \tag{5.1}$$

where $[J]$ was defined previously in (4.1). Explicitly (5.1) may be written

$$\{[I]\} = \int_M \{[J]\}\, dm = \int_M \begin{Bmatrix} \begin{bmatrix} \underline{x_1^3} & \underline{x_1^2x_2} & \underline{x_1^2x_3} \\ & \underline{x_1x_2^2} & \underline{x_1x_2x_3} \\ s & & \underline{x_1x_3^2} \end{bmatrix} \\ \begin{bmatrix} x_1^2x_2 & x_1x_2^2 & x_1x_2x_3 \\ & \underline{x_2^3} & \underline{x_2^2x_3} \\ s & & \underline{x_2x_3^2} \end{bmatrix} \\ \begin{bmatrix} x_1^2x_3 & x_1x_2x_3 & x_1x_3^2 \\ & x_2^2x_3 & x_2x_3^2 \\ s & & \underline{x_3^3} \end{bmatrix} \end{Bmatrix} dm =$$

$$= \begin{Bmatrix} \begin{bmatrix} \underline{I_{111}} & \underline{I_{112}} & \underline{I_{113}} \\ & \underline{I_{122}} & \underline{I_{123}} \\ s & & \underline{I_{133}} \end{bmatrix} \\ \begin{bmatrix} I_{211} & I_{212} & I_{213} \\ & \underline{I_{222}} & \underline{I_{223}} \\ s & & \underline{I_{233}} \end{bmatrix} \\ \begin{bmatrix} I_{311} & I_{312} & I_{313} \\ & I_{322} & I_{323} \\ s & & \underline{I_{333}} \end{bmatrix} \end{Bmatrix} = \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} \tag{5.2}$$

Therefore, in E^3 tensors of inertia of third rank can be considered 3×1 'vectors', the 'components' of which are three matrices which will be called 'inertia matrices of third rank'. Notice that the rank of an inertia matrix in this context is the rank of the tensor which it is a part of, i.e. the number of subindices in every element or the degree of the products $x_i x_j x_k$, rather than conventional matrix rank. The total number of elements of a three-dimensional tensor of rank three will be 27, but only 10 of them are distinct. (They are underlined in Equation (5.2).)

One may define 3×3 inertia matrices of any rank as a function of the basic matrix [J]

$$[I]_{ijk\ldots p} = \int_M x_i x_j x_k \ldots x_p [J] \, dm =$$

$$= \int_M x_i x_j x_k \ldots x_p \begin{bmatrix} x_1^2 & x_1 x_2 & x_1 x_3 \\ & x_2^2 & x_2 x_3 \\ s & & x_3^2 \end{bmatrix} dm \tag{5.3}$$

where the subindices i, j, k...p are positive integers equal to 1, 2 or 3. If n is the total number of subindices, then the rank of the inertia matrix is n + 2.

Incidentally, note that any permutation of the n subindices i, j, k...p does not alter the result, assuring the invariance of inertia matrices of any rank with respect to the same subindices, i.e.

$$[I]_{ijk} = [I]_{ikj} = [I]_{jki} = [I]_{jik} = [I]_{kij} = [I]_{kji}. \tag{5.4}$$

Also notice that in general the elements of an inertia matrix of rank r are the so-called 'inertial integrals' which are in the form

$$\int_M x_1^q x_2^s x_3^t \, dm \quad \text{for} \quad q + s + t = r \tag{5.5}$$

5.2. <u>Inertia tensors of fourth rank</u>

Fourth rank inertia tensors are defined as follows:

$$[[I]] = \int_M \{x\} \otimes \{[J]\} \, dm =$$

$$= \int_M \begin{bmatrix} [J]_{11} & [J]_{12} & [J]_{13} \\ & [J]_{22} & [J]_{23} \\ s & & [J]_{33} \end{bmatrix} dm = \int_M [[J]] \, dm \tag{5.6}$$

or explicitly

$$[[I]] = \int_M \begin{bmatrix} \begin{bmatrix} \underline{x_1^4} & \underline{x_1^3 x_2} & \underline{x_1^3 x_3} \\ & \underline{x_1^2 x_2^2} & \underline{x_1^2 x_2 x_3} \\ s & & \underline{x_1^2 x_3^2} \end{bmatrix} & \begin{bmatrix} x_1^3 x_2 & x_1^2 x_2^2 & x_1^2 x_2 x_3 \\ & \underline{x_1 x_2^3} & \underline{x_1 x_2^2 x_3} \\ s & & \underline{x_1 x_2 x_3^2} \end{bmatrix} & \begin{bmatrix} x_1^3 x_3 & x_1^2 x_2 x_3 & x_1^2 x_3^2 \\ & x_1 x_2^2 x_3 & x_1 x_2 x_3^2 \\ s & & \underline{x_1 x_3^3} \end{bmatrix} \\ & \begin{bmatrix} x_1^2 x_2^2 & x_1 x_2^3 & x_1 x_2^2 x_3 \\ & \underline{x_2^4} & \underline{x_2^3 x_3} \\ s & & \underline{x_2^2 x_3^2} \end{bmatrix} & \begin{bmatrix} x_1^2 x_2 x_3 & x_1 x_2^2 x_3 & x_1 x_2 x_3^2 \\ & x_2^3 x_3 & x_2^2 x_3^2 \\ s & & \underline{x_2 x_3^3} \end{bmatrix} \\ & & \begin{bmatrix} x_1^2 x_3^2 & x_1 x_2 x_3^2 & x_1 x_3^3 \\ & x_2^2 x_3^2 & x_2 x_3^3 \\ s & & \underline{x_3^4} \end{bmatrix} \\ s & & \end{bmatrix} dm =$$

$$= \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix} \tag{5.7}$$

Hence the inertia tensor of fourth rank is formed by a matrix of 3×3 submatrices which are denoted as 'inertia matrices of fourth rank'.

Therefore the inertia tensor of fourth rank has $3^4 = 81$ elements, of which only $\frac{1}{2}(4+1)(4+2) = 15$ are distinct. One choice of these distinct elements is underlined in Equation (5.7) following a pattern previously used for third rank inertia tensors and easy to generalize. Consequently fourth rank tensors of inertia are composed of 3×3 symmetric matrices, therefore the tensor 'components' are inertia matrices of fourth rank. The number of distinct inertia matrices in a tensor of rank r is $\frac{1}{2}r(r-1)$. Thus, for a tensor of $r = 4$ one has only 6 distinct inertia matrices of fourth rank.

5.3. Tensors of inertia of higher rank

Following the same logic outlined in the previous sections, the tensor of inertia of fifth rank can be written in any of the forms given below

$$\{[[I]]\} = \int_M \{x\} \otimes [[J]]\, dm = \int_M \begin{Bmatrix} x_1 [[J]] \\ x_2 [[J]] \\ x_3 [[J]] \end{Bmatrix} dm$$

or, showing all inertia matrices explicitly,

$$\{[[I]]\} = \int_M \left\{ \begin{matrix} x_1 \begin{bmatrix} [J]_{11} & [J]_{12} & [J]_{13} \\ & [J]_{22} & [J]_{23} \\ s & & [J]_{33} \end{bmatrix} \\ x_2 \begin{bmatrix} [J]_{11} & [J]_{12} & [J]_{13} \\ & [J]_{22} & [J]_{23} \\ s & & [J]_{33} \end{bmatrix} \\ x_3 \begin{bmatrix} [J]_{11} & [J]_{12} & [J]_{13} \\ & [J]_{22} & [J]_{23} \\ s & & [J]_{33} \end{bmatrix} \end{matrix} \right\} dm = \left\{ \begin{matrix} \begin{bmatrix} [I]_{111} & [I]_{121} & [I]_{131} \\ & [I]_{221} & [I]_{231} \\ s & & [I]_{331} \end{bmatrix} \\ \begin{bmatrix} [I]_{112} & [I]_{122} & [I]_{132} \\ & [I]_{222} & [I]_{232} \\ s & & [I]_{332} \end{bmatrix} \\ \begin{bmatrix} [I]_{113} & [I]_{123} & [I]_{133} \\ & [I]_{223} & [I]_{233} \\ s & & [I]_{333} \end{bmatrix} \end{matrix} \right\} =$$

$$= \left\{ \begin{matrix} \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_1 \\ \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_2 \\ \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_3 \end{matrix} \right\} = \left\{ \begin{matrix} [[I]]_1 \\ [[I]]_2 \\ [[I]]_3 \end{matrix} \right\} \qquad (5.8)$$

The three 'components' of the tensor of inertia of fifth rank $[[I]]_i$, $i = 1, 2, 3$ are called <u>clusters</u>. Because our space is a E^3 Euclidean space, every inertia tensor of odd rank r will be a vector of three clusters, so named because they are formed by clusters of submatrices, the primary elements of which are inertia matrices of rank r.

For example, explicitly the element $[I]_{123}$ is known to be the following inertia matrix of rank five.

$$[I]_{123} = \int_M \begin{bmatrix} x_1^3 x_2 x_3 & x_1^2 x_2^2 x_3 & x_1^2 x_2 x_3^2 \\ & x_1 x_2^3 x_3 & x_1 x_2^2 x_3^2 \\ s & & x_1 x_2 x_3^3 \end{bmatrix} dm =$$

$$= \begin{bmatrix} I_{11123} & I_{11223} & I_{11233} \\ & I_{12223} & I_{12233} \\ s & & I_{12333} \end{bmatrix} \qquad (5.9)$$

Thus in general every tensor of odd rank $r \geq 3$ will be a vector of three clusters and may be denoted in short by

$$\{[\overset{1}{\ldots}[I]\overset{m}{\ldots}]\} = \begin{Bmatrix} [\overset{1}{\ldots}[I]\overset{m}{\ldots}]_1 \\ [\ldots[I]\ldots]_2 \\ [\ldots[I]\ldots]_3 \end{Bmatrix} \equiv \begin{Bmatrix} \square_1 \\ \square_2 \\ \square_3 \end{Bmatrix} \tag{5.10}$$

where $\square_i$ are clusters, and

$$m = \text{greater integer less or equal to } r/2 \equiv \langle r/2 \rangle \tag{5.11}$$

and $r = 2m + 1$. In general the symbol $\langle\nu\rangle$ will be used to denote the largest integer $\leqq \nu$.

Similarly, tensors of inertia of even rank will be 3×3 matrices of clusters represented symbolically by

$$[[\overset{1}{\ldots}[I]\overset{m}{\ldots}]] = \begin{bmatrix} [\overset{1}{\ldots}[I]\overset{m-1}{\ldots}]_{11} & [\overset{1}{\ldots}[I]\overset{m-1}{\ldots}]_{12} & [\overset{1}{\ldots}[I]\overset{m-1}{\ldots}]_{13} \\ & [\ldots[I]\ldots]_{22} & [\ldots[I]\ldots]_{23} \\ s & & [\ldots[I]\ldots]_{33} \end{bmatrix} \tag{5.12}$$

m as defined above and rank (even) $= 2m$.

Equation (5.12) may also be expressed in compact cluster form as

$$[[\overset{1}{\ldots}[I]\overset{m}{\ldots}]] \equiv \begin{bmatrix} \square_{11} & \square_{12} & \square_{13} \\ & \square_{22} & \square_{23} \\ s & & \square_{33} \end{bmatrix}$$

Thus, in a simplified notation the tensor of inertia of six rank can be written

$$[[[I]]] = \begin{bmatrix} [[I]]_{11} & [[I]]_{12} & [[I]]_{13} \\ & [[I]]_{22} & [[I]]_{23} \\ s & & [[I]]_{33} \end{bmatrix} \tag{5.13}$$

where the clusters $[[I]]_{ij}$ may be obtained from the previously defined matrices in (5.8), namely

$$[[I]]_{ij} = x_i[[I]]_j$$

Each element of the cluster $[[I]]_{ij}$ is an inertia matrix of rank six $[I]_{ijkl}$.

<u>Remark</u>. Multiplication of 'vectors' and 'matrices' consisting of these 'components' (i.e. clusters for high rank inertia tensors) follows the ordinary multiplication of matrices except 'component' by 'component' rather

than element by element. The following examples explicitly illustrate the convention applied, replacing the standard notation using $\otimes$.

a) The expression $[\{x\}^t \otimes [1]]\{[I]\}$ will be abbreviated to

$$\{x\}^t\{[I]\} = \{x\}^t \begin{Bmatrix} [I]_1 \\ \hline [I]_2 \\ \hline [I]_3 \end{Bmatrix} = x_1[I]_1 + x_2[I]_2 + x_3[I]_3 \tag{5.14}$$

b) Similarly $[\{x\}^t \otimes [1]][[I]][\{x\} \otimes [1]]$ will be shortened to

$$\{x\}^t[[I]]\{x\} = \{x\}^t \left[\begin{array}{c:c:c} [I]_{11} & [I]_{12} & [I]_{13} \\ \hdashline & [I]_{22} & [I]_{23} \\ \hdashline s & & [I]_{33} \end{array}\right]\{x\}$$

$$= x_1^2[I]_{11} + x_2^2[I]_{22} + x_3^2[I]_{33} +$$

$$+ 2x_1x_2[I]_{12} + 2x_1x_3[I]_{13} + 2x_2x_3[I]_{23} \tag{5.15}$$

c) and $[[A] \otimes [1]]\{[I]\}$ by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} = \begin{Bmatrix} a_{11}[I]_1 + a_{12}[I]_2 + a_{13}[I]_3 \\ a_{21}[I]_1 + a_{22}[I]_2 + a_{23}[I]_3 \\ a_{31}[I]_1 + a_{32}[I]_2 + a_{33}[I]_3 \end{Bmatrix} \tag{5.16}$$

5.4. <u>The associate tensor of inertia of third rank</u>

Although not defined or given explicitly in books on mechanics, one can also extend the concept of higher tensors of inertia to the <u>associate</u> tensor of inertia. For example, the associate tensor of inertia of third rank can be written

$$\{[\mathbb{I}]\} = \int_M \{x\} \otimes [\mathbb{J}]\, dm = \int_M \begin{Bmatrix} \begin{bmatrix} x_1x_2^2 + x_1x_3^2 & -x_1^2x_2 & -x_1^2x_3 \\ & x_1^3 + x_1x_3^2 & -x_1x_2x_3 \\ s & & x_1^3 + x_1x_2^2 \end{bmatrix} \\ \begin{bmatrix} x_2^3 + x_2x_3^2 & -x_1x_2^2 & -x_1x_2x_3 \\ & x_1^2x_2 + x_2x_3^2 & -x_2^2x_3 \\ s & & x_1^2x_2 + x_2^3 \end{bmatrix} \\ \begin{bmatrix} x_2^2x_3 + x_3^3 & -x_1x_2x_3 & -x_1x_3^2 \\ & x_1^2x_3 + x_3^3 & -x_2x_3^2 \\ s & & x_1^2x_3 + x_2^2x_3 \end{bmatrix} \end{Bmatrix} dm =$$

$$= \left\{ \begin{matrix} \begin{bmatrix} A_1 & -F_1 & -E_1 \\ & B_1 & -D_1 \\ s & & C_1 \end{bmatrix} \\ \begin{bmatrix} A_2 & -F_2 & -E_2 \\ & B_2 & -D_2 \\ s & & C_2 \end{bmatrix} \\ \begin{bmatrix} A_3 & -F_3 & -E_3 \\ & B_3 & -D_3 \\ s & & C_3 \end{bmatrix} \end{matrix} \right\} = \left\{ \begin{matrix} [\mathbb{I}]_1 \\ [\mathbb{I}]_2 \\ [\mathbb{I}]_3 \end{matrix} \right\}. \tag{5.17}$$

To emphasize the difference between the tensor and associate tensor of inertia of third rank, notice that although $D_1 = E_2 = F_3$, the diagonal elements of the matrices are always different from the non-diagonal elements; therefore we have 15 distinct elements in the associate tensor of inertia, as opposed to 10 in the tensor of inertia of third rank. The extension to higher rank associate tensor of inertia, if required, become fairly obvious.

5.5. Contraction of tensors of rank higher than two

Contractions of higher rank tensors are defined as follows:

a) The contraction of tensors of even rank is equal to the sum of the three clusters forming the diagonal of the tensor, that is

$$\mathscr{C}[\overset{m}{[}\ldots\overset{1}{[I]}\ldots]] = \sum_{i=1}^{3} \overset{m-1}{[}\ldots\overset{1}{[I]}\ldots]_{ii} = \overset{m-1}{[}\ldots\overset{1}{[I]}\ldots]. \tag{5.18}$$

b) The contraction of tensors of odd rank is equal to the contraction of each of the clusters, namely

$$\mathscr{C}\{\overset{m}{[}\ldots\overset{1}{[I]}\ldots]\} = \left\{ \begin{matrix} \mathscr{C}\overset{m}{[}\ldots\overset{1}{[I]}\ldots]_1 \\ \mathscr{C}[\ldots[I]\ldots]_2 \\ \mathscr{C}[\ldots[I]\ldots]_3 \end{matrix} \right\} = \left\{ \begin{matrix} \overset{m-1}{[}\ldots\overset{1}{[I]}\ldots]_1 \\ [\ldots[I]\ldots]_2 \\ [\ldots[I]\ldots]_3 \end{matrix} \right\}. \tag{5.19}$$

For example, the contraction of tensors of third rank may be written

$$\mathscr{C}\{[I]\} = \left\{ \begin{matrix} \mathscr{C}[I]_1 \\ \mathscr{C}[I]_2 \\ \mathscr{C}[I]_3 \end{matrix} \right\} = \left\{ \begin{matrix} \mathrm{Tr}[I]_1 \\ \mathrm{Tr}[I]_2 \\ \mathrm{Tr}[I]_3 \end{matrix} \right\} = \left\{ \begin{matrix} I_{111} + I_{122} + I_{133} \\ I_{211} + I_{222} + I_{233} \\ I_{311} + I_{322} + I_{333} \end{matrix} \right\} \tag{5.20}$$

where in general the contraction of any 3×3 inertia matrix is its trace

$$\mathscr{C}[I]_{ijk\ldots p} = \mathrm{Tr}[I]_{ijk\ldots p}. \tag{5.21}$$

With the above definitions, equations similar to (4.9) and (4.10) can be written involving the tensor and associate tensor of inertia of third rank

$$\mathscr{C}\{[I]\} = \tfrac{1}{2}\mathscr{C}\{[\mathbb{I}]\} \tag{5.22}$$

and

$$\{[I]\} = \tfrac{1}{2}[[1]]\mathscr{C}\{[\mathbb{I}]\} - \{[\mathbb{I}]\} \tag{5.23}$$

where $[[1]]$ is the unit tensor of fourth rank (i.e. $\underset{9\times 9}{[1]}$).

In general one may take the contraction of a tensor of rank r, k times, where $k = 0, 1, 2, \ldots, m$, and the maximum contraction m given by $m = \langle r/2\rangle$ where $\langle r/2\rangle$ was defined above in (5.11).

Thus, neglecting the contraction of order zero which is the identity contraction, a tensor of inertia of third rank may take only one contraction while fourth rank tensors of inertia may take a maximum of two. Multiple contractions will be denoted by

$$\mathscr{C},\ \overset{2}{\mathscr{C}} = \mathscr{C}\mathscr{C},\ \overset{3}{\mathscr{C}} = \mathscr{C}\mathscr{C}\mathscr{C},\ \ldots,\ \overset{m}{\mathscr{C}} = \mathscr{C}\mathscr{C}\mathscr{C}\overset{m}{\ldots}\mathscr{C}. \tag{5.24}$$

The maximum contraction $\overset{m}{\mathscr{C}}$ when applied to a tensor of even rank will result in a scalar, while when applied to a tensor of odd rank, it will result in a 3×1 vector.

Therefore

$$\overset{m}{\mathscr{C}}[\text{Tensor of rank } r] = \begin{cases}\text{Scalar for } r \text{ even}\\ \text{Vector for } r \text{ odd}\end{cases}. \tag{5.25}$$

For example, $[[[I]]]$ being a tensor of sixth rank, may have a maximum of $m = 3$ contractions. Thus one can write for the first order contraction,

$$\mathscr{C}[[[I]]] = [[I]]_{11} + [[I]]_{22} + [[I]]_{33} = \sum_{i=1}^{3} [[I]]_{ii}. \tag{5.26}$$

The contraction of second order will be

$$\begin{aligned}\overset{2}{\mathscr{C}}[[[I]]] &= \mathscr{C}\mathscr{C}[[[I]]] = \mathscr{C}[[[I]]_{11} + [[I]]_{22} + [[I]]_{33}] =\\ &= [I]_{1111} + [I]_{2222} + [I]_{3333} + 2[I]_{1122} +\\ &\quad + 2[I]_{1133} + 2[I]_{2233}\end{aligned} \tag{5.27}$$

and finally for the maximum (third order) contraction

$$\begin{aligned}\overset{3}{\mathscr{C}}[[[I]]] &= \mathscr{C}\overset{2}{\mathscr{C}}[[[I]]] = \mathrm{Tr}[I]_{1111} + \mathrm{Tr}[I]_{2222} + \mathrm{Tr}[I]_{3333} +\\ &\quad + 2\mathrm{Tr}[I]_{1122} + 2\mathrm{Tr}[I]_{1133} + 2\mathrm{Tr}[I]_{2233}.\end{aligned} \tag{5.28}$$

Consequently the contraction of order k ($k \leq m$) of a tensor of rank r will result in a tensor of rank $r - 2k$.

$$\overset{k}{\mathscr{C}}[\text{Tensor of rank } r] = \text{Tensor of rank } r - 2k. \tag{5.29}$$

6. EFFECT OF ROTATIONS ON THE TENSORS OF INERTIA AND THEIR CONTRACTIONS

Assume that one knows the rotation R between two coordinate systems with a common origin. The transformation of coordinate between the two systems due to the rotation is given by the following equation:

$$\{\tilde{x}\} = R\{x\} \tag{6.1}$$

where the 3×3 matrix of direction cosines R may be parameterized by, e.g. Eulerian or Cardanian angles (Magnus, 1971, p. 32).

It is well known that tensors of inertia of second rank (3×3 matrices in E^3) transform according to the equation

$$[\tilde{I}] = R[I]R^t \tag{6.2}$$

and that their contractions (traces) are invariant under rotation, namely

$$\mathscr{C}[\tilde{I}] = \mathscr{C}[I] \quad \text{or} \quad \mathrm{Tr}[\tilde{I}] = \mathrm{Tr}[I]. \tag{6.3}$$

This invariance of the contractions under rotation in general is lost for higher rank tensors. It may be shown (see Appendix A) that the transformation involving R between higher rank inertia tensors will follow the equations presented below.

Third rank tensors.

Given the transformation $\{\tilde{x}\} = R\{x\}$ and denoting by $\{[\tilde{I}]\}$ and $\{[I]\}$ the inertia tensors of third rank in the transformed and original system respectively, then

$$\{[\tilde{I}]\} = \begin{Bmatrix} [\tilde{I}]_1 \\ [\tilde{I}]_2 \\ [\tilde{I}]_3 \end{Bmatrix} = R \begin{Bmatrix} R[I]_1R^t \\ R[I]_2R^t \\ R[I]_3R^t \end{Bmatrix} \tag{6.4}$$

Notice that

$$[\tilde{I}]_i \neq R[I]_iR^t, \quad i = 1, 2, 3 \tag{6.5}$$

implying that in general the clusters are not tensors. In this case, clearly, inertia matrices of third rank are not tensors.

The only possible contraction of third rank inertia tensors transforms according to the equation

$$\mathscr{C}\{[\tilde{I}]\} = R \begin{Bmatrix} \mathscr{C}[I]_1 \\ \mathscr{C}[I]_2 \\ \mathscr{C}[I]_3 \end{Bmatrix} = R \begin{Bmatrix} \mathrm{Tr}[I]_1 \\ \mathrm{Tr}[I]_2 \\ \mathrm{Tr}[I]_3 \end{Bmatrix}. \tag{6.6}$$

Similarly fourth rank tensors will transform under rotation according to the equation

$$[[\tilde{I}]] = R \begin{bmatrix} R[I]_{11}R^t & R[I]_{12}R^t & R[I]_{13}R^t \\ & R[I]_{22}R^t & R[I]_{23}R^t \\ s & & R[I]_{33}R^t \end{bmatrix} R^t \tag{6.7}$$

and taking contractions up to $m = 2$

$$\mathscr{C}[[\tilde{I}]] = R[[I]_{11} + [I]_{22} + [I]_{33}]R^t = R\mathscr{C}[[I]]R^t \tag{6.8}$$

$$\overset{2}{\mathscr{C}}[[\tilde{I}]] = \mathscr{C}\mathscr{C}[[\tilde{I}]] = \overset{2}{\mathscr{C}}[[I]]. \tag{6.9}$$

Therefore

$$\mathscr{C}R\{[I]\} = R\mathscr{C}\{[I]\} = \mathscr{C}\{[\tilde{I}]\} \tag{6.10}$$

and

$$\overset{2}{\mathscr{C}}[[I]] = \overset{2}{\mathscr{C}}[[\tilde{I}]]. \tag{6.11}$$

For better understanding, the following example shows the effect of a rotation R on a fifth rank tensor, the corresponding formulation for higher rank tensors being evident.

$$\{[[\tilde{I}]]\} = R \left\{ \begin{array}{c} R \begin{bmatrix} R[I]_{11}R^t & R[I]_{12}R^t & R[I]_{13}R^t \\ & R[I]_{22}R^t & R[I]_{23}R^t \\ s & & R[I]_{33}R^t \end{bmatrix}_1 R^t \\ \hdashline R \begin{bmatrix} R[I]_{11}R^t & R[I]_{12}R^t & R[I]_{13}R^t \\ & R[I]_{22}R^t & R[I]_{23}R^t \\ s & & R[I]_{33}R^t \end{bmatrix}_2 R^t \\ \hdashline R \begin{bmatrix} R[I]_{11}R^t & R[I]_{12}R^t & R[I]_{13}R^t \\ & R[I]_{22}R^t & R[I]_{23}R^t \\ s & & R[I]_{33}R^t \end{bmatrix}_3 R^t \end{array} \right\} \tag{6.12}$$

and hence

$$\mathscr{C}\{[[\tilde{I}]]\} = R\mathscr{C}\begin{Bmatrix} \square_1 \\ \square_2 \\ \square_3 \end{Bmatrix} = R\begin{Bmatrix} R[[I]_{111} + [I]_{122} + [I]_{133}]R^t \\ R[[I]_{211} + [I]_{222} + [I]_{233}]R^t \\ R[[I]_{311} + [I]_{322} + [I]_{333}]R^t \end{Bmatrix}. \tag{6.13}$$

But a fifth rank tensor may have a maximum of two contractions; therefore,

$$\overset{2}{\mathscr{C}}\{[[\tilde{I}]]\} = \mathscr{C}\mathscr{C}\{[[\tilde{I}]]\} = R\begin{Bmatrix} Tr[I]_{111} + Tr[I]_{122} + Tr[I]_{133} \\ Tr[I]_{211} + Tr[I]_{222} + Tr[I]_{233} \\ Tr[I]_{311} + Tr[I]_{322} + Tr[I]_{333} \end{Bmatrix}. \tag{6.14}$$

which is the product of a 3×3 matrix by a column vector, thus resulting in a column vector or a tensor of rank one.

In general it may be proved that

$$\overset{m}{\mathscr{C}}\{[\ldots[\overset{1}{\tilde{I}}]\ldots\overset{m}{]}\} = R\overset{m}{\mathscr{C}}\{[\ldots[\overset{1}{I}]\ldots\overset{m}{]}\} \tag{6.15}$$

and

$$\overset{m}{\mathscr{C}}[\ldots[\overset{1}{\tilde{I}}]\ldots\overset{m}{]} = \overset{m}{\mathscr{C}}[\ldots[\overset{1}{I}]\ldots\overset{m}{]}. \tag{6.16}$$

Thus the invariance of the contraction under rotation is conserved only when the maximum possible contraction m (see Equation 5.25) is applied to even rank inertia tensors. This was expected because the maximum order contraction of even rank tensors are scalars.

In the following sections the basic operations among inertia tensors outlined above will be applied to express the potential of a body at an exterior point as a function of the inertia tensors.

7. THE EXPANSION OF THE POTENTIAL OF A FINITE BODY AS A FUNCTION OF THE TENSORS OF INERTIA

Consider a rigid body of arbitrary shape and mass M, and let Ox denote the origin of the coordinate system x_i, i = 1, 2, 3 fixed in the body. Let CM be

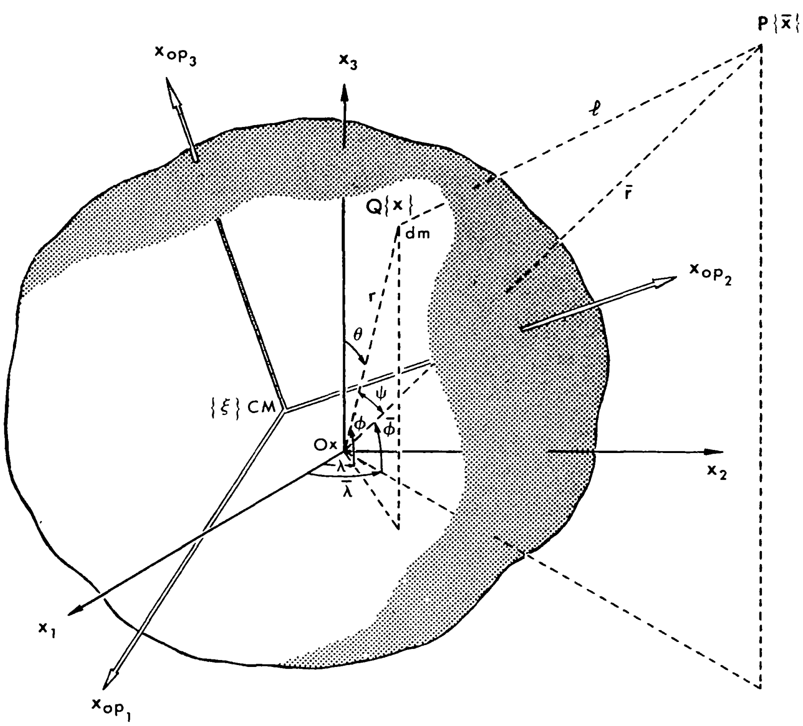

Fig. 1. Coordinate systems notation and configuration.

the body's center of mass and x_{opi}, $i = 1, 2, 3$ its central principal axes. Denote by $\{\xi\}$ the column vector of the coordinates of the CM in the x_i system.

Let $\{\bar{x}\}$ represent the column vector of the coordinates of a point P exterior to the body at a distance $\bar{r}$ from Ox. That is (see Fig. 1)

$$\bar{r}^2 = \{\bar{x}\}^t\{\bar{x}\}. \tag{7.1}$$

Similarly, let $\{x\}$ be the coordinates of a mass-point Q interior to the body at a distance r from Ox and with element of mass dm.

The gravitational potential of any body at an exterior point $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ can be expressed by the well known equation

$$V = G \int_M \frac{dm}{\ell} \tag{7.2}$$

where G is the gravitational constant and the integration is to be extended over the complete mass M of the body. The distance ℓ between P and Q is given by

$$\ell^2 = \{\bar{x} - x\}^t\{\bar{x} - x\} \tag{7.3}$$

where

$$\{x\} = \begin{Bmatrix} r \cos \phi \cos \lambda \\ r \cos \phi \sin \lambda \\ r \sin \phi \end{Bmatrix} = \begin{Bmatrix} r \sin \theta \cos \lambda \\ r \sin \theta \sin \lambda \\ r \cos \theta \end{Bmatrix}; \quad \theta = \frac{\pi}{2} - \phi \tag{7.4}$$

with similar expressions holding for the barred variable defining the position of point P.

Thus Equation (7.3) may be written

$$\ell^2 = \{\bar{x}\}^t\{\bar{x}\} + \{x\}^t\{x\} - 2\{\bar{x}\}^t\{x\} = \bar{r}^2 + r^2 - 2\bar{r}rt \tag{7.5}$$

where, in this case

$$t = \cos \psi = \frac{1}{\bar{r}r} \{\bar{x}\}^t\{x\} = \cos \bar{\theta} \cos \theta + \sin \bar{\theta} \sin \theta \cos(\bar{\lambda} - \lambda). \tag{7.6}$$

When $r < \bar{r}$ one can write (Heiskanen and Moritz, 1967, p. 33)

$$\frac{1}{\ell} = \sum_{n=0}^{\infty} \frac{r^n}{\bar{r}^{n+1}} P_n(\cos \psi) \tag{7.7}$$

where $P_n(\cos \psi)$ are the Legendre polynomials of degree n.

It is well known that the series in (7.7) converges absolutely and uniformly in a certain domain for arbitrary values of the angle ψ.

Substituting (7.7) into (7.2) and integrating term by term

$$V(\bar{r},\ \bar{\theta},\ \bar{\lambda}) = G \sum_{n=0}^{\infty} \frac{1}{\bar{r}^{n+1}} \int_M r^n P_n(\cos\ \psi)\ dm \tag{7.8}$$

where by the addition theorem (Hobson, 1931, p. 143)

$$P_n(\cos\ \psi) = P_n(\cos\ \bar{\theta}) P_n(\cos\ \theta) + \\ + 2 \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} P_{nm}(\cos\ \theta) P_{nm}(\cos\ \bar{\theta}) \cos m(\bar{\lambda} - \lambda). \tag{7.9}$$

Therefore (7.8) can be written in the form

$$V(\bar{r},\ \bar{\theta},\ \bar{\lambda}) = G \sum_{n=0}^{\infty} \frac{1}{\bar{r}^{n+1}} Y_n(\bar{\theta},\ \bar{\lambda}) \tag{7.10}$$

and the surface spherical harmonic of degree n is given by

$$Y_n(\bar{\theta},\ \bar{\lambda}) = \int_M r^n P_n(\cos\ \psi)\ dm. \tag{7.11}$$

It is known that any homogeneous harmonic polynomial of degree n, $H_n(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)$, when expressed as a function of spherical coordinates $(\bar{r},\ \bar{\theta},\ \bar{\lambda})$ may be rearranged as a product of two functions, only one of which is dependent on $\bar{r}$ and the other dependent on $(\bar{\theta},\ \bar{\lambda})$

$$H_n(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3) = \bar{r}^n Y_n(\bar{\theta},\ \bar{\lambda}) \tag{7.12}$$

where in the notation of (Heiskanen and Moritz, 1967, p. 29)

$$Y_n(\bar{\theta},\ \bar{\lambda}) = \sum_{m=0}^{n} [a_{nm} R_{nm}(\bar{\theta},\ \bar{\lambda}) + b_{nm} S_{nm}(\bar{\theta},\ \bar{\lambda})] = \\ = \sum_{m=0}^{n} [a_{nm} \cos m\bar{\lambda} + b_{nm} \sin m\bar{\lambda}] P_{nm}(\cos\ \bar{\theta}). \tag{7.13}$$

For clarity, it must be emphasized that the above does not mean that all homogeneous polynomials are harmonic functions; only some of them are (see Hotine, 1969, p. 162).

From (7.12) therefore

$$Y_n(\bar{\theta},\ \bar{\lambda}) = \frac{H_n(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)}{\bar{r}^n} \tag{7.14}$$

and substituting Equation (7.14) in (7.10) finally it is possible to write the potential V as a function of the Cartesian coordinates of P, namely

$$V(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3) = G \sum_{n=0}^{\infty} \frac{H_n(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3)}{\bar{r}^{2n+1}}. \tag{7.15}$$

As mentioned above, the series on the right-hand side of (7.15) converges absolutely and uniformly for $\bar{r} > a$ and for any a such that all points of the body satisfy the inequality $r \leq a$.

The polynomial $H_n(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ may be obtained from (7.12) and (7.11)

$$H_n(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \bar{r}^n \int_M r^n P_n(\cos \psi)\ dm. \tag{7.16}$$

Using Equation (1-62) in (Heiskanen and Moritz, 1967, p. 24), namely

$$P_{nm}(t) = 2^{-n}(1 - t^2)^{m/2} \sum_{k=0}^{<(n-m)/2>} (-1)^k \times$$

$$\times \frac{(2n - 2k)!}{k!(n - k)!(n - m - 2k)!} t^{n-m-2k} \tag{7.17}$$

where

$<(n - m)/2>$ = greater integer less or equal to $(n - m)/2$

and the particular case $m = 0$, one gets

$$P_n(t) = 2^{-n} \sum_{k=0}^{<n/2>} (-1)^k \frac{(2n - 2k)!}{k!(n - k)!(n - 2k)!} t^{n-2k}. \tag{7.18}$$

Recalling that now

$$t = \cos \psi = \frac{\{\bar{x}\}^t\{x\}}{\bar{r}r} \tag{7.19}$$

one can write (7.18) in the form

$$P_n(\cos \psi) = \sum_{k=0}^{<n/2>} T_{nk}(\{\bar{x}\}^t\{x\})^{n-2k}(\bar{r}r)^{2k-n} \tag{7.20}$$

where

$$T_{nk} = (-1)^k \frac{(2n - 2k)!}{2^n k!(n - k)!(n - 2k)!} \tag{7.21}$$

and finally, substituting in (7.16), one arrives at

$$H_n(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \sum_{k=0}^{<n/2>} T_{nk} \int_M (\{\bar{x}\}^t\{x\})^{n-2k}(\bar{r}r)^{2k}\ dm. \tag{7.22}$$

The potential $V(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ may be obtained from (7.15) as

$$V(\bar{x}_1, \bar{x}_2, \bar{x}_3) = G \sum_{n=0}^{\infty} \frac{1}{\bar{r}^{2n+1}} \sum_{k=0}^{<n/2>} T_{nk} \int_M (\{\bar{x}\}^t\{x\})^{n-2k}(\bar{r}r)^{2k}\ dm \tag{7.23}$$

and using only matrix notation inside the integral

$$V(\bar{x}_1, \bar{x}_2, \bar{x}_3) = G \sum_{n=0}^{\infty} \frac{1}{\bar{r}^{2n+1}} \sum_{k=0}^{<n/2>} T_{nk} \bar{r}^{2k} \int_M (\{\bar{x}\}^t\{x\})^{n-2k} (\{x\}^t\{x\})^k \, dm. \tag{7.24}$$

The above equation can finally be written in the simplified form

$$V(\bar{x}_1, \bar{x}_2, \bar{x}_3) = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} \frac{1}{\bar{r}^{2(n-k)+1}} T_{nk} \mathscr{I}_{nk} \tag{7.25}$$

where T_{nk} was given in (7.21) and

$$\mathscr{I}_{nk} = \int_M (\{\bar{x}\}^t\{x\})^{n-2k} (\{x\}^t\{x\})^k \cdot dm. \tag{7.26}$$

In the following sections it will be proved that $\mathscr{I}_{nk}$ is a function of the tensors of inertia of rank n and its contractions $\overset{k}{\mathscr{C}}$, where $k = 0, 1 \ldots <n/2>$.

8. COMPUTATION OF THE LOWER TERMS IN THE SERIES

8.1. Zero order term

In this particular case $n = 0$, implying $k = 0$.

Therefore

$$V_0(\bar{x}_1, \bar{x}_2, \bar{x}_3) = G \frac{1}{\bar{r}} \int_M dm. \tag{8.1}$$

Knowing that

$$\int_M dm = M \tag{8.2}$$

which may be considered the tensor of inertia of rank zero, one gets the well known result

$$V_0(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{GM}{\bar{r}}. \tag{8.3}$$

8.2. First order term

Assuming $n = 1$ and consequently $k = 0$, and substituting these values in (7.24),

$$V_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{G}{\bar{r}^3} \int_M (\{\bar{x}\}^t\{x\}) \, dm = \frac{G}{\bar{r}^3} \{\bar{x}\}^t \int_M \{x\} \, dm \tag{8.4}$$

but

$$\int_M \{x\} \, dm = M\{\xi\} = M \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} \tag{8.5}$$

where $\{\xi\}$ was defined previously as the coordinates of the center of mass of the body with respect to the x_i, $i = 1, 2, 3$ coordinate system. Equation (8.5) may be considered the tensor of inertia of rank one.

Replacing (8.5) in (8.4) one has

$$V_1(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{GM}{\bar{r}^3}\{\bar{x}\}^t\{\xi\} \tag{8.6}$$

8.3. <u>Second order term</u>

Assuming $n = 2$, then $k = 0,1$ and substituting into (7.24) one can write

$$V_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{3G}{2\bar{r}^5}\int_M (\{\bar{x}\}^t\{x\})^2\, dm - \frac{G}{2\bar{r}^3}\int_M (\{x\}^t\{x\})\, dm \tag{8.7}$$

but

$$(\{\bar{x}\}^t\{x\})^2 = \{\bar{x}\}^t\{x\}\{x\}^t\{\bar{x}\} = \{\bar{x}\}^t[J]\{\bar{x}\} \tag{8.8}$$

and

$$\{x\}^t\{x\} = \mathscr{C}[J] = \mathrm{Tr}[J] \tag{8.9}$$

which after integration becomes as a function of the tensor of inertia of second rank

$$V_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{3G}{2\bar{r}^5}\{\bar{x}\}^t[I]\{\bar{x}\} - \frac{G}{2\bar{r}^3}\mathrm{Tr}[I]. \tag{8.10}$$

The above equation may be written as a function of the associate tensor of inertia $[\mathcal{I}]$. Recalling Equations (4.9) and (4.10) it follows that

$$\begin{aligned} V_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) = &- \frac{3G}{2\bar{r}^5}\{\bar{x}\}^t[\mathcal{I}]\{\bar{x}\} + \\ &+ \frac{3G}{4\bar{r}^5}\mathrm{Tr}[\mathcal{I}]\{\bar{x}\}^t\{\bar{x}\} - \frac{G}{4\bar{r}^3}\mathrm{Tr}[\mathcal{I}] \end{aligned} \tag{8.11}$$

or

$$V_2(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{G}{2\bar{r}^3}\mathrm{Tr}[\mathcal{I}] - \frac{3G}{2\bar{r}^5}\{\bar{x}\}^t[\mathcal{I}]\{\bar{x}\}. \tag{8.12}$$

It is remarkable that V_2 as a function of $[I]$ is equal to the negative value of V_2 as a function of $[\mathcal{I}]$. Nevertheless this property cannot be generalized for any n, as will be shown later when determining V_3.

It can be proved immediately that Equation (8.12) when referred to the central principal axis, reduces to the second order term of the potential expansion, as given in (Thomson and Tait, 1912, II, p. 87 or Heiskanen and Moritz, 1967, p. 63).

Another particular form of Equation (8.12) included in the development of the potential is attributed to MacCullagh and consequently called 'MacCullagh's formula'. See (Jeffreys, 1970, p. 176 or Hotine, 1969, p. 167).

Noticing that

$$\{\bar{x}\}^t[\mathbb{I}]\{\bar{x}\} = \bar{r}^2\{\alpha\}^t[\mathbb{I}]\{\alpha\} = \bar{r}^2 I_\rho \tag{8.13}$$

where $\{\alpha\}$ is the column vector of direction cosines of line OxP, namely (see Fig. 1),

$$\{\alpha\} = \begin{Bmatrix} \sin\bar{\theta}\cos\bar{\lambda} \\ \sin\bar{\theta}\sin\bar{\lambda} \\ \cos\bar{\theta} \end{Bmatrix} \tag{8.14}$$

and I_ρ is the moment of inertia of the body with respect to the line OxP, another way to write (8.12) is

$$V_2 = \frac{G}{2\bar{r}^3}(A + B + C) - \frac{3G}{2\bar{r}^3} I_\rho = \frac{G}{2\bar{r}^3}(A + B + C - 3I_\rho). \tag{8.15}$$

Notice the generality of the above equation where A, B and C are not the central principal moments of inertia, but the moments of inertia (not even principal) with respect to a given coordinate system. This term is more general than the corresponding one included in the equation originally credited to MacCullagh and published as a lecture account by one of his students (Allman, 1855) 'with the view of securing to Prof. MacCullagh the merit of whatever is original in the investigation or its results'.

MacCullagh's formula refers to the central principal axes of the body; consequently Equation (8.15) takes the simplified form

$$V_{2op} = \frac{G}{2\bar{r}^3}(A_{op} + B_{op} + C_{op} - 3I_{o\rho}) \tag{8.16}$$

where now

$$I_{o\rho} = \{\alpha\}^t \begin{bmatrix} A_{op} & 0 & 0 \\ & B_{op} & 0 \\ s & & C_{op} \end{bmatrix} \{\alpha\} = \alpha_1^2 A_{op} + \alpha_2^2 B_{op} + \alpha_3^2 C_{op}. \tag{8.17}$$

Because the axes of figure now constitute the basic reference system, $\{\xi\} = 0$, thus $V_1 = 0$, and finally the potential, neglecting terms in the expansion higher than the second, is written

$$V_{op} = \frac{GM}{\bar{r}} + \frac{G}{2\bar{r}^3}(A_{op} + B_{op} + C_{op} - 3I_{o\rho}). \tag{8.18}$$

The above equation is quoted and used in geophysical literature when studying the contributions to the associate tensor of inertia due to tidal, rotational or any other type of deformation. For example, one can consult (Munk and MacDonald, 1960, p. 25 or Israel and Ben-Menahem, 1975).

It is easy to prove that Equations (8.10), (8.12) or (8.15) correspond exactly to the second degree term in the expansion of the potential in spherical harmonics.

8.4. Third order term

Assuming $n = 3$ and thus $k = 0,1$ and after substitution into (7.24)

$$V_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{5G}{2\bar{r}^7} \int_M (\{\bar{x}\}^t\{x\})^3 \, dm - \\ - \frac{3G}{2\bar{r}^5} \int_M (\{\bar{x}\}^t\{x\})(\{x\}^t\{x\}) \, dm. \tag{8.19}$$

After some matrix manipulation and simplification (see Appendix B), the above equality can be written

$$V_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{5G}{2\bar{r}^7} \{\bar{x}\}^t[\bar{x}_1[I]_1 + \bar{x}_2[I]_2 + \bar{x}_3[I]_3]\{\bar{x}\} - \\ - \frac{3G}{2\bar{r}^5} \{\bar{x}\}^t \begin{Bmatrix} Tr[I]_1 \\ Tr[I]_2 \\ Tr[I]_3 \end{Bmatrix} \tag{8.20}$$

where the term between brackets may be expressed as the 'inner tensor product' of $\{\bar{x}\}$ by the third rank inertia tensor, so that

$$\{\bar{x}\}^t\{[I]\} = \{\bar{x}\}^t \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} = \bar{x}_1[I]_1 + \bar{x}_2[I]_2 + \bar{x}_3[I]_3. \tag{8.21}$$

Therefore

$$V_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \frac{5G}{2\bar{r}^7} \{\bar{x}\}^t[\{\bar{x}\}^t\{[I]\}]\{\bar{x}\} - \frac{3G}{2\bar{r}^5} \{\bar{x}\}^t\mathscr{C}\{[I]\}. \tag{8.22}$$

Equation (8.20) as a function of the associate tensor of inertia takes the form

$$V_3(\bar{x}_1, \bar{x}_2, \bar{x}_3) = -\frac{5G}{2\bar{r}^7} \{\bar{x}\}^t[\bar{x}_1[\mathbb{I}]_1 + \bar{x}_2[\mathbb{I}]_2 + \bar{x}_3[\mathbb{I}]_3]\{\bar{x}\} + \\ + \frac{G}{2\bar{r}^5} \{\bar{x}\}^t \begin{Bmatrix} Tr[\mathbb{I}]_1 \\ Tr[\mathbb{I}]_2 \\ Tr[\mathbb{I}]_3 \end{Bmatrix}. \tag{8.23}$$

Hence

$$V_3(\bar{x}_1,\ \bar{x}_2,\ \bar{x}_3) = -\frac{5G}{2\bar{r}^7}\{\bar{x}\}^t[\{\bar{x}\}^t\{[\mathbb{I}]\}]\{\bar{x}\} + \frac{G}{2\bar{r}^5}\{\bar{x}\}^t\mathscr{C}\{[\mathbb{I}]\}. \tag{8.24}$$

Notice that the coefficient of the term containing $\mathscr{C}\{[\mathbb{I}]\}$ above is opposite in sign to the one containing $\mathscr{C}\{[I]\}$ in (8.22), but smaller by a factor of three. Thus

$$V_3 \text{ (as a function of } [I]) \neq -V_3 \text{ (as a function of } [\mathbb{I}]).$$

9. HIGHER ORDER TERMS

Computing only $\mathscr{I}_{nk}$ in (7.26) and therefore bypassing the calculation of the coefficients of the expansion, easily obtainable from (7.25) for any value of n, it may be proved that the higher order terms take the following forms.

9.1. Fourth order term

In this case $n = 4$ and k takes the values $k = 0,\ 1,\ 2$.

If $k = 0$, then

$$\mathscr{I}_{4,0} = \{\bar{x}\}^t\ [\{\bar{x}\}^t \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix} \{\bar{x}\}]\ \{\bar{x}\} =$$

$$= \{\bar{x}\}^t[\{\bar{x}\}^t[[I]]\{\bar{x}\}]\{\bar{x}\}. \tag{9.1}$$

For the value $k = 1$, one has

$$\mathscr{I}_{4,1} = \{\bar{x}\}^t\ [\mathscr{C} \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}]\ \{\bar{x}\} = \{\bar{x}\}^t\mathscr{C}[[I]]\{\bar{x}\} \tag{9.2}$$

and finally, for $k = 2$

$$\mathscr{I}_{4,2} = \overset{2}{\mathscr{C}} \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix} = \overset{2}{\mathscr{C}}[[I]]. \tag{9.3}$$

9.2. Fifth order term

Assuming now $n = 5$, and therefore $k = 0,\ 1,\ 2$, one can write the following; if $k = 0$

$$\mathscr{I}_{5,0} = \{\bar{x}\}^t\ [\{\bar{x}\}^t\ [\{\bar{x}\}^t \left\{\begin{matrix} \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_1 \\ \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_2 \\ \begin{bmatrix} [I]_{11} & [I]_{12} & [I]_{13} \\ & [I]_{22} & [I]_{23} \\ s & & [I]_{33} \end{bmatrix}_3 \end{matrix}\right\}]\ \{\bar{x}\}]\ \{\bar{x}\} =$$

$$= \{\bar{x}\}^t\ [\{\bar{x}\}^t\ [\{\bar{x}\}^t \left\{\begin{matrix} [[I]]_1 \\ [[I]]_2 \\ [[I]]_3 \end{matrix}\right\}]\ \{\bar{x}\}]\ \{\bar{x}\} =$$

$$= \{\bar{x}\}^t[\{\bar{x}\}^t[\{\bar{x}\}^t\{[[I]]\}]\{\bar{x}\}]\{\bar{x}\}. \tag{9.4}$$

For $k = 1$

$$\mathscr{I}_{5,1} = \{\bar{x}\}^t\ [\{\bar{x}\}^t\mathscr{C}\left\{\begin{matrix} [[I]]_1 \\ [[I]]_2 \\ [[I]]_3 \end{matrix}\right\}]\ \{\bar{x}\} = \{\bar{x}\}^t[\{\bar{x}\}^t\mathscr{C}\{[[I]]\}]\{\bar{x}\}. \tag{9.5}$$

For $k = 2$

$$\mathscr{I}_{5,2} = \{\bar{x}\}^t\overset{2}{\mathscr{C}}\left\{\begin{matrix} [[I]]_1 \\ [[I]]_2 \\ [[I]]_3 \end{matrix}\right\} = \{\bar{x}\}^t\overset{2}{\mathscr{C}}\{[[I]]\}. \tag{9.6}$$

9.3. Sixth order term

To provide more familiarization with the particular notation for matrix expansion introduced in this paper, $\mathscr{I}_{nk}$ values for the sixth order term will be given explicitly.

For $n = 6$ and $k = 0, 1, 2, 3$, if $k = 0$

$$\mathscr{I}_{6,0} = \{\bar{x}\}^t\ [\{\bar{x}\}^t\ [\{\bar{x}\}^t \begin{bmatrix} [[I]]_{11} & [[I]]_{12} & [[I]]_{13} \\ & [[I]]_{22} & [[I]]_{23} \\ s & & [[I]]_{33} \end{bmatrix} \{\bar{x}\}]\ \{\bar{x}\}]\ \{\bar{x}\} =$$

$$= \{\bar{x}\}^t[\{\bar{x}\}^t[\{\bar{x}\}^t[[[I]]]\{\bar{x}\}]\{\bar{x}\}]\{\bar{x}\}. \tag{9.7}$$

For $k = 1$

$$\mathscr{I}_{6,1} = \{\bar{x}\}^t[\{\bar{x}\}^t[\mathscr{C}[[[I]]]]\{\bar{x}\}]\{\bar{x}\}. \tag{9.8}$$

For k = 2

$$\mathscr{I}_{6,2} = \{\bar{x}\}^t \overset{2}{\mathscr{C}}[[[I]]]\{\bar{x}\} \tag{9.9}$$

and finally, for k = 3

$$\mathscr{I}_{6,3} = \overset{3}{\mathscr{C}}[[[I]]]. \tag{9.10}$$

10. GENERAL EQUATION

In conclusion, a general equation for $\mathscr{I}_{n,k}$ using the notation and operations involving inertia tensors defined in this paper can be written as follows:

$$\mathscr{I}_{nk} = \int_M (\{\bar{x}\}^t\{x\})^{n-2k}(\{x\}^t\{x\})^k \, dm =$$

$$= \{\bar{x}\}^t\{\bar{x}\}^t \overset{<\frac{n+1}{2}>-k}{\cdots\cdots\cdots} \{\bar{x}\}^t \overset{k}{\mathscr{C}}\begin{bmatrix}\text{Inertia tensor}\\ \text{of rank n}\end{bmatrix}\{\bar{x}\}\{\bar{x}\}\overset{<\frac{n}{2}>-k}{\cdots\cdots}\{\bar{x}\} \tag{10.1}$$

where the symbol $<\nu>$ was previously defined following Equation (5.11).

In order to facilitate understanding of the new notation, the general Equation (10.1) will be given explicitly below for even and odd rank inertia tensors when $n \geqq 2$.

n is even

$$\mathscr{I}_{nk} = \{\bar{x}\}^t \Big[\overset{m-k}{\cdots\cdots} \Big[\{\bar{x}\}^t \; \overset{k}{\mathscr{C}}[[\ldots\overset{1}{[}I]\ldots\overset{m}{]}]] \; \{\bar{x}\}\Big] \overset{m-k}{\cdots\cdots}\Big] \{\bar{x}\} \tag{10.2}$$

n is odd

$$\mathscr{I}_{nk} = \{\bar{x}\}^t \Big[\overset{m-k}{\cdots\cdots} \Big[\{\bar{x}\}^t \; \{\bar{x}\}^t\overset{k}{\mathscr{C}}\{[\ldots\overset{1}{[}I]\ldots\overset{m}{]}\} \; \{\bar{x}\}\Big] \overset{m-k}{\cdots\cdots}\Big] \{\bar{x}\} \tag{10.3}$$

where $m = <n/2>$ and $k = 0, 1, 2 \ldots m$.

Finally, when the expression (10.1) is substituted in Equation (7.25) and the rules defined here for manipulating inertia tensors are followed, the potential at an exterior point $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in matrix notation is obtained

$$V(\bar{x}_1, \bar{x}_2, \bar{x}_3) = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} \times$$

$$\times \left(\{\bar{x}\}^t \overset{<\frac{n+1}{2}>-k}{\cdots\cdots\cdots} \{\bar{x}\}^t \overset{k}{\mathscr{C}}\begin{bmatrix}\text{Inertia}\\ \text{tensor of}\\ \text{rank n}\end{bmatrix} \{\bar{x}\}\overset{<\frac{n}{2}>-k}{\cdots\cdots}\{\bar{x}\}\right). \tag{10.4}$$

11. INVARIANCE OF THE POTENTIAL WITH RESPECT TO ROTATION OF THE COORDINATE SYSTEM

The invariance of the gravitational potential V with respect to a rotation of the coordinate system is immediately obtained from Equation (7.25) using simple matrix notation. (The body remains fixed in this rotation).

Recalling (6.1) in general one can write the transformation between the original coordinates $\{x\}$ and the rotated ones $\{\tilde{x}\}$ as

$$\{\tilde{x}\} = R\{x\}.$$

Therefore

$$\{\tilde{x}\}^t = \{x\}^t R^t$$

and similarly

$$\{\tilde{\bar{x}}\}^t = \{\bar{x}\}^t R^t.$$

Hence

$$\{\tilde{\bar{x}}\}^t\{\tilde{x}\} = \{\bar{x}\}^t R^t R\{x\} = \{\bar{x}\}^t\{x\}$$

and similarly

$$\{\tilde{x}\}^t\{\tilde{x}\} = \{x\}^t\{x\}.$$

Therefore the expression of $\mathscr{I}_{nk}$ in Equation (7.26) is invariant with respect to a rotation of the coordinate system, namely

$$\int_M (\{\tilde{\bar{x}}\}^t\{\tilde{x}\})^{n-2k}(\{\tilde{x}\}^t\{\tilde{x}\})^k\, dm = \int_M (\{\bar{x}\}^t\{x\})^{n-2k}(\{x\}^t\{x\})^k\, dm \quad (11.1)$$

or

$$\tilde{\mathscr{I}}_{nk} = \mathscr{I}_{nk}. \quad (11.2)$$

Because the radius vector of point P will not change after a rotation of the coordinate system, that is, $\bar{r} = \tilde{\bar{r}}$, the very well known property that the potential V is invariant with respect to a rotation of the coordinate system is established.

$$V(\tilde{\bar{x}}_1, \tilde{\bar{x}}_2, \tilde{\bar{x}}_3) = V(\bar{x}_1, \bar{x}_2, \bar{x}_3). \quad (11.3)$$

This result emphasizes the well known fact that tensors of rank zero are always invariant under the change of coordinates.

12. COMPUTATION OF THE GRAVITATIONAL ATTRACTION

The components of the gravitational attraction f at P along the x_i, $i = 1, 2, 3$ coordinate axes can be expressed by

$$\{f\} = \left\{\frac{\partial}{\partial \bar{x}}\right\} V \quad (12.1)$$

or explicitly,

$$f_1 = \frac{\partial V}{\partial \bar{x}_1}, \quad f_2 = \frac{\partial V}{\partial \bar{x}_2}, \quad f_3 = \frac{\partial V}{\partial \bar{x}_3}. \tag{12.2}$$

Therefore, taking the partial derivatives of the potential given by Equation (7.25) with respect to the vector $\{\bar{x}\}$ one obtains

$$\begin{aligned}\{f\} = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} T_{nk}(2k - 2n - 1)\bar{r}^{2(k-n)-3} \times \\ \times \int_M (\{\bar{x}\}^t\{x\})^{n-2k}(\{x\}^t\{x\})^k \, dm\{\bar{x}\} + \\ + G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} T_{nk}\bar{r}^{2(k-n)-1} \times \\ \times \int_M (n - 2k)(\{\bar{x}\}^t\{x\})^{n-2k-1}(\{x\}^t\{x\})^k\{x\} \, dm.\end{aligned}$$

The above expression can be written

$$\begin{aligned}\{f\} = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} T_{nk}\bar{r}^{2(k-n)-1} \times \\ \times \{(2k - 2n - 1)\bar{r}^{-2}\mathscr{I}_{nk}\{\bar{x}\} + (n - 2k)\{\mathscr{K}\}_{nk}\}\end{aligned} \tag{12.3}$$

where $\mathscr{I}_{nk}$ was given previously by Equation (10.1), and

$$\{\mathscr{K}\}_{nk} = \int_M (\{\bar{x}\}^t\{x\})^{n-2k-1}(\{x\}^t\{x\})^k\{x\} \, dm. \tag{12.4}$$

Equation (12.3) may be rewritten in the final form

$$\{f\} = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} \left\{\frac{2(k - n) - 1}{\bar{r}^2} \{\mathscr{I}\}_{nk} + (n - 2k)\{\mathscr{K}\}_{nk}\right\} \tag{12.5}$$

where

$$\{\mathscr{I}\}_{nk} = \mathscr{I}_{nk}\{\bar{x}\} = \left(\{\bar{x}\}^t \overset{<\frac{n+1}{2}>-k}{\cdots\cdots\cdots} \{\bar{x}\}^t \overset{k}{\mathscr{C}} \begin{bmatrix}\text{Inertia} \\ \text{tensor of} \\ \text{rank } n\end{bmatrix} \{\bar{x}\} \overset{<\frac{n}{2}>-k}{\cdots\cdots} \{\bar{x}\}\right) \{\bar{x}\}. \tag{12.6}$$

It can be proved following a procedure similar to the one used in the previous sections that

$$\{\mathscr{K}\}_{nk} = \{\bar{x}\}^t \overset{<\frac{n-1}{2}>-k}{\cdots\cdots\cdots} \{\bar{x}\}^t \overset{k}{\mathscr{C}} \begin{bmatrix}\text{Inertia} \\ \text{tensor of} \\ \text{rank } n\end{bmatrix} \{\bar{x}\} \overset{<\frac{n}{2}>-k}{\cdots\cdots} \{\bar{x}\}. \tag{12.7}$$

Notice that for $n = 2k$ the coefficient multiplying $\{\mathscr{K}\}_{nk}$ in Equation (12.3) is equal to zero; thus Equation (12.4) needs only to be defined for values of $n \neq 2k$.

13. EXPLICIT FORM OF THE FIRST TERMS OF THE EXPANSION OF THE GRAVITATIONAL ATTRACTION

13.1. Zero order term

In this case $n = 0$ and $k = 0$; therefore, taking into account (12.3)

$$\{f\}_0 = - \frac{GM}{\bar{r}^3} \{\bar{x}\} \tag{13.1}$$

which is the well known value for the components of the gravitational attraction at $P(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ at a distance of $\bar{r}$ from a point mass M located at the origin of the x_i, $i = 1, 2, 3$ coordinate system.

13.2. First order term

Substituting the values of $n = 1$ and $k = 0$ in (12.5), one gets

$$\{f\}_1 = - \frac{3GM}{\bar{r}^5}(\{\bar{x}\}^t\{\xi\})\{\bar{x}\} + \frac{GM}{\bar{r}^3} \{\xi\}. \tag{13.2}$$

The above equation, as expected, depends on the coordinates of the center of mass of the body. Observe that as in the case of the potential, the components of the first order term of the gravitational attraction expansion will be zero if one selects as reference a central coordinate system.

13.3. Second order term

For $n = 2$, k takes the values $k = 0, 1$. Thus, using the general Equation (12.5) one gets

$$\{f\}_2 = - \frac{15G}{2\bar{r}^7} (\{\bar{x}\}^t[I]\{\bar{x}\})\{\bar{x}\} + \frac{3G}{\bar{r}^5} [I]\{\bar{x}\} + \frac{3G}{2\bar{r}^5} \mathscr{C}[I]\{\bar{x}\}. \tag{13.3}$$

It is possible to express the above equation as a function of the associate tensor of inertia [ ]. Substituting (4.9) and (4.10) in (13.3), after some matrix manipulation and simplification, one gets as expected the same expression, but with a change of sign and with [I] replaced by $[\mathbb{I}]$.

$$\{f\}_2 = \frac{15G}{2\bar{r}^7} (\{\bar{x}\}^t[\mathbb{I}]\{\bar{x}\})\{\bar{x}\} - \frac{3G}{\bar{r}^5} [\mathbb{I}]\{\bar{x}\} - \frac{3G}{2\bar{r}^5} \mathscr{C}[\mathbb{I}]\{\bar{x}\}. \tag{13.4}$$

Recalling (8.13) one can write the above equation

$$\{f\}_2 = - \frac{3G}{2\bar{r}^5} (\mathscr{C}[\mathbb{I}] - 5I_\rho)\{\bar{x}\} - \frac{3G}{\bar{r}^5} [\mathbb{I}]\{\bar{x}\} \tag{13.5}$$

or

$$\{f\}_2 = - \frac{3G}{2\bar{r}^5} (A + B + C - 5I_\rho)\{\bar{x}\} - \frac{3G}{\bar{r}^5} [\mathbb{I}]\{\bar{x}\}. \tag{13.6}$$

Finally, if the selected coordinate system is a central principal system, Equation (13.6) reduces to

$$\{f\}_{2_{op}} = -\frac{3G}{2\bar{r}^5}(A_{op} + B_{op} + C_{op} - 5I_{o_\rho})\{\bar{x}\} - \frac{3G}{\bar{r}^5}\operatorname{diag}[\mathbb{I}_0]\{\bar{x}\} \quad (13.7)$$

where the value of I_{o_ρ} was given in (8.17) and $\operatorname{diag}[\mathbb{I}_o]$ is a diagonal matrix the elements of which are the three central principal moments of inertia, i.e. A_{op}, B_{op}, C_{op}.

After substituting $\{\bar{x}\} = \bar{r}\{\alpha\}$ with $\{\alpha\}$ as defined previously in (8.14), Equation (13.7) reduces in matrix notation to the form attributed to MacCullagh (Allman, 1855).

$$\{f\}_{2_{op}} = -\frac{3G}{2\bar{r}^4}(A_{op} + B_{op} + C_{op} - 5I_{o_\rho})\{\alpha\} - \frac{3G}{\bar{r}^4}\operatorname{diag}[\mathbb{I}_0]\{\alpha\}. \quad (13.8)$$

Finally, a new form of Equation (13.4) can be obtained adding to and subtracting from it the value

$$\frac{G}{\bar{r}^5}\mathscr{C}[\mathbb{I}]\{\bar{x}\}. \quad (13.9)$$

After simple matrix manipulation and simplification and the substitution $\mathscr{C}[\mathbb{I}] = \operatorname{Tr}[\mathbb{I}]$, one arrives at

$$\{f\}_2 = -\frac{5G}{2\bar{r}^7}(\{\bar{x}\}^t[\operatorname{Tr}[\mathbb{I}][1] - 3[\mathbb{I}]]\{\bar{x}\})\{\bar{x}\} +$$
$$+ \frac{G}{\bar{r}^5}[\operatorname{Tr}[\mathbb{I}][1] - 3[\mathbb{I}]]\{\bar{x}\}. \quad (13.10)$$

The above general expression when referred to the central principal axes x_{op_i}, $i = 1, 2, 3$ reduces to the formula given by Thomson and Tait (1912, II, p. 87), namely

$$\{f\}_{2_{op}} = -\frac{5G}{2\bar{r}^7}\Big((B_{op} + C_{op} - 2A_{op})\bar{x}_1^2 + (A_{op} + C_{op} - 2B_{op})\bar{x}_2^2 +$$
$$+ (A_{op} + B_{op} - 2C_{op})\bar{x}_3^2\Big)\{\bar{x}\} + \frac{G}{\bar{r}^5}\begin{Bmatrix}(B_{op} + C_{op} - 2A_{op})\bar{x}_1 \\ (A_{op} + C_{op} - 2B_{op})\bar{x}_2 \\ (A_{op} + B_{op} - 2C_{op})\bar{x}_3\end{Bmatrix}. \quad (13.11)$$

14. HIGHER ORDER TERMS

Equations (12.6) and (12.7) will now be used to show explicitly the value of the quantities $\{\mathscr{J}\}_{nk}$ and $\{\mathscr{K}\}_{nk}$ for the third, fourth and fifth order contributions to $\{f\}$.

14.1. Third order term

In this case $n = 3$ and $k = 0, 1$. For $n = 3$ and $k = 0$, one gets

$$\{\mathscr{I}\}_{3,0} = \mathscr{I}_{3,0}\{\bar{x}\} = (\{\bar{x}\}^t[\{\bar{x}\}^t\{[I]\}]\{\bar{x}\})\{\bar{x}\}, \tag{14.1}$$

$$\{\mathscr{K}\}_{3,0} = [\{\bar{x}\}^t\{[I]\}]\{\bar{x}\}. \tag{14.2}$$

For $n = 3$ and $k = 1$,

$$\{\mathscr{I}\}_{3,1} = \mathscr{I}_{3,1}\{\bar{x}\} = (\{\bar{x}\}^t\mathscr{C}\{[I]\})\{\bar{x}\}, \tag{14.3}$$

$$\{\mathscr{K}\}_{3,1} = \mathscr{C}\{[I]\}. \tag{14.4}$$

14.2 Fourth order term

Now $n = 4$ and $k = 0, 1, 2$. For $n = 4$ and $k = 0$,

$$\{\mathscr{I}\}_{4,0} = \mathscr{I}_{4,0}\{\bar{x}\} = (\{\bar{x}\}^t[\{\bar{x}\}^t[[I]]\{\bar{x}\}]\{\bar{x}\})\{\bar{x}\}, \tag{14.5}$$

$$\{\mathscr{K}\}_{4,0} = [\{\bar{x}\}^t[[I]]\{\bar{x}\}]\{\bar{x}\}. \tag{14.6}$$

For $n = 4$, $k = 1$,

$$\{\mathscr{I}\}_{4,1} = \mathscr{I}_{4,1}\{\bar{x}\} = (\{\bar{x}\}^t\mathscr{C}[[I]]\{\bar{x}\})\{\bar{x}\}, \tag{14.7}$$

$$\{\mathscr{K}\}_{4,1} = \mathscr{C}[[I]]\{\bar{x}\}. \tag{14.8}$$

For $n = 4$, $k = 2$,

$$\{\mathscr{I}\}_{4,2} = \mathscr{I}_{4,2}\{\bar{x}\} = \overset{2}{\mathscr{C}}[[I]]\{\bar{x}\}. \tag{14.9}$$

Observe that in this case the coefficient $n - 2k$, multiplying $\{\mathscr{K}\}_{4,2}$ in Equation (12.5) is equal to zero; therefore the value of $\{\mathscr{K}\}_{4,2}$ is not needed.

14.3. Fifth order term

Now for $n = 5$, k will take the values $k = 0, 1, 2$. Thus for $n = 5$ and $k = 0$, one has,

$$\{\mathscr{I}\}_{5,0} = \mathscr{I}_{5,0}\{\bar{x}\} = (\{\bar{x}\}^t[\{\bar{x}\}^t[\{\bar{x}\}^t\{[[I]]\}]\{\bar{x}\}]\{\bar{x}\})\{\bar{x}\}, \tag{14.10}$$

$$\{\mathscr{K}\}_{5,0} = [\{\bar{x}\}^t[\{\bar{x}\}^t\{[[I]]\}]\{\bar{x}\}]\{\bar{x}\}. \tag{14.11}$$

For $n = 5$, $k = 1$,

$$\{\mathscr{I}\}_{5,1} = \mathscr{I}_{5,1}\{\bar{x}\} = (\{\bar{x}\}^t[\{\bar{x}\}^t\mathscr{C}\{[[I]]\}]\{\bar{x}\})\{\bar{x}\}, \tag{14.12}$$

$$\{\mathscr{K}\}_{5,1} = [\{\bar{x}\}^t\mathscr{C}\{[[I]]\}]\{\bar{x}\}. \tag{14.13}$$

Finally, for $n = 5$ and $k = 2$,

$$\{\mathscr{J}\}_{5,\,2} = (\{\bar{x}\}^t \overset{2}{\mathscr{C}}\{[[I]]\})\{\bar{x}\}, \tag{14.14}$$

$$\{\mathscr{K}\}_{5,\,2} = \overset{2}{\mathscr{C}}\{[[I]]\}. \tag{14.15}$$

15. SIMPLIFIED EQUATIONS

A further simplification of Equation (12.5) can be established by introducing the 'notational device' described below.

For values of $n = 2k$, Equation (12.7) takes the form

$$\{\mathscr{K}\}_{n=2k,\,k} = (\{\bar{x}\}^t)^{-1} \overset{k}{\mathscr{C}} \begin{bmatrix} \text{Inertia} \\ \text{tensor of} \\ \text{rank } n \end{bmatrix}. \tag{15.1}$$

In the above equation, the symbol $(\{\bar{x}\}^t)^{-1}$ although literally lacking proper meaning (at least in a Cayleyian matrix algebra sense) is to be understood as the 'inverse' value of $\{\bar{x}\}^t$, i.e.

$$(\{\bar{x}\}^t)^{-1}\{\bar{x}\}^t\{\bar{x}\} = \{\bar{x}\} \tag{15.2}$$

or

$$\{\bar{x}\}^t(\{\bar{x}\}^t)^{-1}\{\bar{x}\} = \{\bar{x}\}. \tag{15.3}$$

Assuming the above 'operational artifice', the 'inverse' value of $\{\bar{x}\}$ can be introduced similarly, and the following general relationships between the vectors $\{\mathscr{J}\}_{nk}$, $\{\mathscr{K}\}_{nk}$ and the scalar $\mathscr{I}_{nk}$ may be written.

From Equations (12.6) and (12.7), for any value of n it immediately follows that

$$\{\mathscr{J}\}_{nk} = \mathscr{I}_{nk}\{\bar{x}\} = (\{\bar{x}\}^t\{\mathscr{K}\}_{nk})\{\bar{x}\}. \tag{15.4}$$

Consequently

$$\mathscr{I}_{nk} = \{\bar{x}\}^t\{\mathscr{K}\}_{nk}. \tag{15.5}$$

The above equalities are easily seen in explicit form by comparing the values given in Sections 9 and 14 for the lower order terms.

As a result of the above, it is possible to write a final equation for the attraction and potential at any exterior point P of a body which depends only on the quantity $\{\mathscr{K}\}_{nk}$.

The components of the attraction along any x_i, $i = 1, 2, 3$ coordinate system at point $P\{\bar{x}\}$ can be written

$$\{f\} = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} \times$$

$$\times \left\{\frac{2(k-n)-1}{\bar{r}^2} (\{\bar{x}\}^t\{\mathscr{K}\}_{nk})\{\bar{x}\} + (n - 2k)\{\mathscr{K}\}_{nk}\right\} \tag{15.6}$$

and the potential at the same point

$$V = G \sum_{n=0}^{\infty} \sum_{k=0}^{<n/2>} \frac{T_{nk}}{\bar{r}^{2(n-k)+1}} (\{\bar{x}\}^t \{\mathscr{K}\}_{nk}), \tag{15.7}$$

where T_{nk} was given in Equation (7.21), $\{\mathscr{K}\}_{nk}$ in (12.7) and $<n/2>$, as defined previously, is the largest integer $\leq n/2$.

Notice that in order to know $\{\bar{f}\}$ or V at any point exterior to the body with coordinates $\{\bar{x}\}$, one needs a set of inertia tensors (up to rank n) for the body in question, referred to the basic coordinate system x_i, $i = 1, 2, 3$. A value of the vector $\{\mathscr{K}\}_{nk}$ must be computed from the inertia tensors at each point $\{\bar{x}\}$ using Equation (12.7). Substitution of this value in (15.7) or (15.6) will provide the potential or the components of the gravitational attraction along the x_i coordinate system at P.

16. EFFECT ON {f} OF ROTATION OF COORDINATE SYSTEMS

It is clear that a transformation of coordinates under the rotation R is given by

$$\{\tilde{x}\} = R\{x\} \tag{16.1}$$

should result in

$$\{\tilde{f}\} = R\{f\}. \tag{16.2}$$

That this is the case can easily be proved using the matrix notation of this work. Clearly,

$$\{\tilde{\mathscr{K}}\}_{nk} = R\{\mathscr{K}\}_{nk} \tag{16.3}$$

and

$$(\{\tilde{\bar{x}}\}^t \{\tilde{\mathscr{K}}\}_{nk})\{\tilde{\bar{x}}\} = (\{\bar{x}\}^t \{\mathscr{K}\}_{nk}) R\{\bar{x}\} = R(\{\bar{x}\}^t \{\mathscr{K}\}_{nk})\{\bar{x}\}. \tag{16.4}$$

Substituting (16.3) and (16.4) in (15.6), one finally obtains, as expected, (16.2). Employing the same reasoning, it is obvious from Equation (15.7) that the potential is invariant under rotation, a fact already proved.

17. CONCLUSIONS

In this work a new methodology has been formulated for the calculation of the gravitational potential and attraction of a body at any exterior point. The final expressions are original in the sense that they depend exclusively on two major parameters: the Cartesian coordinates of the point and a set of inertia tensors of the body. The theory is developed in a general matrix form introduced through a novel notation which uses basic matrix algebra operations. Some applications of the general equations are obtained and discussed, arriving at simple formulas for the first terms of the expansion, which are very easy to visualize and comprehend without the necessity of writing laborious and lengthy polynomial expressions.

The author has more practical problems in mind. For example, the calculation of the potential or its gravitational force created by a disturbing field such as the earth's crust is the most immediate and obvious one. In this particular instance any spherical region of the upper crust can be modeled by a finite number of elements (or blocks) using a discrete density function depending only on elevation and based on known geophysical hypotheses (e.g. Airy-Heiskanen isostatic compensation). In this way crustal influence on the attracting force at satellite altitude can be computed. It should be noted that this procedure is equivalent to obtaining the disturbing effect originated by the mass irregularities of the crust at satellite heights. Naturally, if the resolution of the data is high (e.g. $1° \times 1°$ terrain elevations or oceanic depths of equiangular blocks are available), very few terms in the expansion will be required. The number will be dependent primarily on the altitude of the satellite. Recall that if one uses spherical harmonics a resolution of $1° \times 1°$ will require approximately an $180° \times 180°$ earth model, which is always expensive to implement.

The method outlined here proposes an alternative to other currently used methods: for example, surface density layer, point masses, etc. The surface density layer approach is equivalent to computing the tensors of inertia of the elements, assuming they have varying density and a differentially small height. The point mass approach may be considered a particular case of the expansion described in this paper, when only the zero order term is considered. In areas where the existing elevation data bases are very detailed (e.g. the U.S. where one point of elevation exists for every 30 seconds of latitude and longitude), an accurate computation of the disturbing effects of the modeled crust would be feasible. The author in a previous unrelated investigation already computed the second rank inertia tensors for $1° \times 1°$ earth crustal blocks of 50 km depth, using a density distribution based on the isostatic compensation theory (Soler, 1977, or Soler and Mueller, 1978). The extension of this numerical integration to higher rank inertia tensors does not present insurmountable difficulties, although it must be stressed that now the set of inertia tensors of each element should be computed with respect to a local coordinate system in the block.

Nevertheless, the intention of the material set forth here was limited to introducing the theory without putting any emphasis on practical examples related specifically to the fields of geodesy or geophysics. Utilization of the equations discussed above is easy to envision. Thus it is left to the reader to apply them to problems related to the gravitational potential and attraction of a body, and in particular the earth.

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APPENDIX A

A. Transformation under rotation of tensors of inertia of third and fourth rank

Assume an orthogonal transformation between two Cartesian coordinate systems, where the matrix of direction cosines R is defined in general by

$$R = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}.$$

It is well known that tensors of first and second rank between the respective systems transform according to the equations

$$\begin{aligned} \{\tilde{x}\} &= R\{x\}, \\ [\tilde{I}] &= R[I]R^t, \\ \mathscr{C}[\tilde{I}] &= \mathscr{C}[I]. \end{aligned}$$

If the original and transformed inertia tensors of third rank are denoted by

$$\{[I]\} = \begin{Bmatrix} [I]_1 \\ [I]_2 \\ [I]_3 \end{Bmatrix} \quad \text{and} \quad \{[\tilde{I}]\} = \begin{Bmatrix} [\tilde{I}]_1 \\ [\tilde{I}]_2 \\ [\tilde{I}]_3 \end{Bmatrix}$$

they will transform under R as follows; from the basic definitions, the inertia matrix $[\tilde{I}]_1$ will take the form

$$\begin{aligned} [\tilde{I}]_1 &= \int_M \tilde{x}_1\{\tilde{x}\}\{\tilde{x}\}^t \, dm = \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3) R[J]R^t \, dm = \\ &= \alpha_1 \int_M x_1 R[J]R^t \, dm + \alpha_2 \int_M x_2 R[J]R^t \, dm + \alpha_3 \int_M x_3 R[J]R^t \, dm = \\ &= \alpha_1 R \int_M x_1 [J] \, dm \, R^t + \alpha_2 R \int_M x_2 [J] \, dm \, R^t + \alpha_3 R \int_M x_3 [J] \, dm \, R^t = \\ &= \alpha_1 R[I]_1 R^t + \alpha_2 R[I]_2 R^t + \alpha_3 R[I]_3 R^t. \end{aligned}$$

The inertia matrices $[\tilde{I}]_i$, $i = 2, 3$ can be obtained similarly:

$$[\tilde{I}]_2 = \beta_1 R[I]_1 R^t + \beta_2 R[I]_2 R^t + \beta_3 R[I]_3 R^t,$$

$$[\tilde{I}]_3 = \gamma_1 R[I]_1 R^t + \gamma_2 R[I]_2 R^t + \gamma_3 R[I]_3 R^t.$$

Therefore, finally

$$\{[\tilde{I}]\} = \begin{Bmatrix} [\tilde{I}]_1 \\ [\tilde{I}]_2 \\ [\tilde{I}]_3 \end{Bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \begin{Bmatrix} R[I]_1 R^t \\ R[I]_2 R^t \\ R[I]_3 R^t \end{Bmatrix} = R \begin{Bmatrix} R[I]_1 R^t \\ R[I]_2 R^t \\ R[I]_3 R^t \end{Bmatrix}.$$

A.1 Transformation of contractions of third rank tensors of inertia under a rotation

As above, denoting the transformed and the original tensors of inertia of third rank by $\{[\tilde{I}]\}$ and $\{[I]\}$ respectively, and using the basic definitions, one can write

$$\mathscr{C}[\tilde{I}]_1 = \int_M \tilde{x}_1 \{\tilde{x}\}^t \{\tilde{x}\}\, dm = \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\{x\}^t R^t R\{x\}\, dm =$$

$$= \alpha_1 \int_M x_1 \{x\}^t \{x\}\, dm + \alpha_2 \int_M x_2 \{x\}^t \{x\}\, dm + \alpha_3 \int_M x_3 \{x\}^t \{x\}\, dm =$$

$$= \alpha_1 \mathscr{C}[I]_1 + \alpha_2 \mathscr{C}[I]_2 + \alpha_3 \mathscr{C}[I]_3.$$

Similarly,

$$\mathscr{C}[\tilde{I}]_2 = \beta_1 \mathscr{C}[I]_1 + \beta_2 \mathscr{C}[I]_2 + \beta_3 \mathscr{C}[I]_3,$$

$$\mathscr{C}[\tilde{I}]_3 = \gamma_1 \mathscr{C}[I]_1 + \gamma_2 \mathscr{C}[I]_2 + \gamma_3 \mathscr{C}[I]_3.$$

Thus

$$\mathscr{C}\{[\tilde{I}]\} = \begin{Bmatrix} \mathscr{C}[\tilde{I}]_1 \\ \mathscr{C}[\tilde{I}]_2 \\ \mathscr{C}[\tilde{I}]_3 \end{Bmatrix} = R \begin{Bmatrix} \mathscr{C}[I]_1 \\ \mathscr{C}[I]_2 \\ \mathscr{C}[I]_3 \end{Bmatrix}.$$

A.2 Transformation under rotation of fourth rank inertia tensors

If $[[\tilde{I}]]$ and $[[I]]$ are the transformed and original tensors of inertia of fourth rank respectively, it is possible to write

$$[\tilde{I}]_{11} = \int_M \tilde{x}_1 [\tilde{J}]_1\, dm = \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\tilde{x}_1 \{\tilde{x}\}\{\tilde{x}\}^t\, dm =$$

$$= \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)^2 R\{x\}\{x\}^t R^t\, dm =$$

$$= \int_M (\alpha_1^2 x_1^2 + \alpha_2^2 x_2^2 + \ldots + 2\alpha_1\alpha_3 x_1 x_3 + 2\alpha_2\alpha_3 x_2 x_3) R[J]R^t\, dm =$$

$$= \int_M (\alpha_1^2 R x_1^2 [J] R^t + \ldots + 2\alpha_2\alpha_3 R x_2 x_3 [J] R^t)\, dm =$$

$$= \alpha_1^2 R[I]_{11}R^t + \alpha_2^2 R[I]_{22}R^t + \alpha_3^2 R[I]_{33}R^t +$$
$$+ 2\alpha_1\alpha_2 R[I]_{12}R^t + 2\alpha_1\alpha_3 R[I]_{13}R^t + 2\alpha_2\alpha_3 R[I]_{23}R^t.$$

A non-diagonal cluster of the tensor $[[\tilde{I}]]$, for example $[\tilde{I}]_{12}$, may be obtained similarly,

$$[\tilde{I}]_{12} = \int_M \tilde{x}_1[\tilde{J}]_2 \, dm = \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)\tilde{x}_2\{\tilde{x}\}\{\tilde{x}\}^t \, dm =$$
$$= \int_M (\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3)(\beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3) R[J]R^t \, dm =$$
$$= \alpha_1\beta_1 R[I]_{11}R^t + \alpha_2\beta_1 R[I]_{12}R^t + \alpha_3\beta_1 R[I]_{13}R^t +$$
$$+ \alpha_1\beta_2 R[I]_{12}R^t + \alpha_2\beta_2 R[I]_{22}R^t + \alpha_3\beta_2 R[I]_{23}R^t +$$
$$+ \alpha_1\beta_3 R[I]_{13}R^t + \alpha_2\beta_3 R[I]_{23}R^t + \alpha_3\beta_3 R[I]_{33}R^t.$$

Therefore, finally, with the simplified notation used in this work,

$$[[\tilde{I}]] = R \begin{bmatrix} R[I]_{11}R^t & R[I]_{12}R^t & R[I]_{13}R^t \\ & R[I]_{22}R^t & R[I]_{23}R^t \\ s & & R[I]_{33}R^t \end{bmatrix} R^t.$$

The same procedure can be extended, if required, to higher rank tensors.

APPENDIX B

B. Powers of scalar products in matrix notation

In this appendix some of the matrix operations used to derive the equations in Sections 8 and 9 are given explicitly.

Recalling the value of $(\{\bar{x}\}^t\{x\})^2$ from Equation (8.8), one can write

$$(\{\bar{x}\}^t\{x\})^3 = (\{\bar{x}\}^t\{x\})^2(\{\bar{x}\}^t\{x\}) = \{\bar{x}\}^t[J]\{\bar{x}\}\{x\}^t\{\bar{x}\},$$

but it can be proved that

$$\{\bar{x}\}\{x\}^t = [\underline{x}][\underline{\bar{x}}] + \mathrm{Tr}[\{x\}\{\bar{x}\}^t][1].$$

Therefore, substituting the above and knowing that $[\underline{\bar{x}}]\{\bar{x}\} = \{0\}$, one gets

$$(\{\bar{x}\}^t\{x\})^3 = \{\bar{x}\}^t[J](\mathrm{Tr}[\{x\}\{\bar{x}\}^t])\{\bar{x}\} =$$
$$= \{\bar{x}\}^t[(\{\bar{x}\}^t\{x\})[J]]\{\bar{x}\} =$$
$$= \{\bar{x}\}^t[\bar{x}_1 x_1[J] + \bar{x}_2 x_2[J] + \bar{x}_3 x_3[J]]\{\bar{x}\} =$$

$$= \{\bar{x}\}^t \left[\{\bar{x}\}^t \begin{Bmatrix} x_1[J] \\ x_2[J] \\ x_3[J] \end{Bmatrix} \right] \{\bar{x}\} = \{\bar{x}\}^t \left[\{\bar{x}\}^t \begin{Bmatrix} [J]_1 \\ [J]_2 \\ [J]_3 \end{Bmatrix} \right] \{\bar{x}\}.$$

Thus

$$(\{\bar{x}\}^t\{x\})^3 = \{\bar{x}\}^t[\{\bar{x}\}^t\{[J]\}]\{\bar{x}\}.$$

Using similar reasoning,

$$(\{\bar{x}\}^t\{x\})^4 = (\{\bar{x}\}^t\{x\})^3(\{\bar{x}\}^t\{x\}) =$$

$$= \{\bar{x}\}^t[(\{\bar{x}\}^t\{x\})[J]]\{\bar{x}\}(\{\bar{x}\}^t\{x\}) =$$

$$= \{\bar{x}\}^t[(\{\bar{x}\}^t\{x\})^2[J]]\{\bar{x}\} =$$

$$= \{\bar{x}\}^t[\bar{x}_1^2[J]_{11} + \bar{x}_2^2[J]_{22} + \dots + 2\bar{x}_2\bar{x}_3[J]_{23}]\{\bar{x}\} =$$

$$= \{\bar{x}\}^t[\{\bar{x}\}^t[[J]]\{\bar{x}\}]\{\bar{x}\} \quad \text{etc.}$$

It is easy to see that

$$(\{\bar{x}\}^t\{x\})\mathscr{C}[J] = \{\bar{x}\}^t\mathscr{C}[J]\{x\} = \{\bar{x}\}^t\mathscr{C}\{[J]\} = \{\bar{x}\}^t \begin{Bmatrix} \mathrm{Tr}[J]_1 \\ \mathrm{Tr}[J]_2 \\ \mathrm{Tr}[J]_3 \end{Bmatrix}$$

and

$$(\{\bar{x}\}^t\{x\})^2\mathscr{C}[J] = (\{\bar{x}\}^t\{x\})(\{\bar{x}\}^t\{x\})\mathscr{C}[J] = \{\bar{x}\}^t[J]\{\bar{x}\}\mathscr{C}[J] =$$

$$= \{\bar{x}\}^t\mathscr{C}[J][J]\{\bar{x}\} = \{\bar{x}\}^t\mathscr{C}[[J]]\{\bar{x}\}.$$

Similarly,

$$(\{\bar{x}\}^t\{x\})^3\mathscr{C}[J] = \{\bar{x}\}^t[\{\bar{x}\}^t\mathscr{C}\{[[J]]\}]\{\bar{x}\} \quad \text{etc.}$$

Finally, it follows immediately

$$(\{x\}^t\{x\})^2 = \{x\}^t[J]\{x\} = \overset{2}{\mathscr{C}}[[J]],$$

$$(\{x\}^t\{x\})^3 = \{x\}^t[J]^2\{x\} = \overset{3}{\mathscr{C}}[[[J]]],$$

$$(\{x\}^t\{x\})^4 = \{x\}^t[J]^3\{x\} = \overset{4}{\mathscr{C}}[[[[J]]]], \quad \text{etc.}$$

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