Linear Response Theory and the Validity of the Semiclassical Approximation

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- Semiclassical backreaction equations are

$$G_{ab} + \Lambda g_{ab} = 8\pi \left(T_{ab}^{class} + \langle T_{ab}^{quant} \rangle \right)$$

- There are actually two semiclassical approximations
  - Loop expansion to $O(\hbar)$
    * Quantum fields can have only small effects on the background geometry
    * Graviton stress-energy tensor should be included for consistency.
  - Large N expansion to leading order
    * Quantum fields can have a large effect on the background geometry
    * Graviton stress-energy tensor is absent, coming in at next to leading order

- Validity of Semiclassical Approximation
  - Ford (1982)

    Semiclassical approximation is valid if

    $$\langle T_{ab}(x)T_{cd}(x')\rangle_{\text{con}} \ll \langle T_{ab}(x)\rangle\langle T_{cd}(x')\rangle$$

    $$\langle T_{ab}(x)T_{cd}(x')\rangle_{\text{con}} \equiv \langle T_{ab}(x)T_{cd}(x')\rangle - \langle T_{ab}(x)\rangle\langle T_{cd}(x')\rangle$$

  - Kuo - Ford (1993)

    $$\Delta_{abcd}(x,x') \equiv \left|\frac{\langle T_{ab}(x)T_{cd}(x')\rangle_{\text{con}}}{\langle T_{ab}(x)T_{cd}(x')\rangle}\right|$$

- Problems with the Kuo - Ford criterion
  - $\Delta_{abcd}(x, x')$ is not a tensor
  - Wu and Ford found state dependent divergences in $\Delta_{abcd}(x, x')$
  - Kuo and Ford computed
    $$\Delta(x) \equiv \left| \frac{\langle T_{00}(x) T_{00}(x) \rangle_{\text{con}}}{\langle T_{00}(x) T_{00}(x) \rangle} \right|$$
    for various states in flat space and found it is typically of $O(1)$ except for a coherent state
  - Phillips and Hu found its value in flat space depends on regularization scheme and how it is implimented

- One would guess that $\langle T_{ab}(x) T_{cd}(x') \rangle_{\text{con}}$ plays some role in the validity of the semiclassical approximation

- One place that it naturally occurs is in the linear response equations

- Semiclassical backreaction equations come from first variation of the effective action

$$
\begin{aligned}
\langle T_{ab}(x)\rangle &= -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{ab}}S^{(1)}_{\text{eff}}[g] \\
&= \frac{-i\hbar}{\sqrt{-g}}\text{Tr } G[g]\frac{\delta G^{-1}[g]}{\delta g^{ab}} \\
&\equiv \frac{-i\hbar}{\sqrt{-g}}\int d^4x'\sqrt{-g'}\int d^4x'' \\
&\quad \frac{\delta G^{-1}[g](x',x'')}{\delta g^{ab}(x)}\,G[g](x'',x') \\
&= -i\hbar\sqrt{-g}\,\mathcal{D}_{ab}\,G[g](x,x')|_{x'=x}
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{D}_{ab}\,G[g](x,x')|_{x'=x} = \Big[&\frac{1}{4}\left(2\nabla_a\nabla_b - \eta_{ab}\right) \\
&-\frac{1}{2}m^2\eta_{ab} + \xi\left(\eta_{ab} - \nabla_a\nabla_b\right)\Big]\,G[g](x,x) \\
+ \;& (-\delta_a^c\delta_b^d + \frac{1}{2}\eta_{ab}\eta^{cd})(\nabla'_c\nabla'_d)\,G[g](x,x')|_{x'=x}
\end{aligned}
$$

- To compute $\langle in|T_{ab}|in\rangle$ the CTP formalism is used

- In the CTP formalism the leading order effective action for the matter fields

$$S_{\text{eff}}^{(1)}[g] = \frac{i\hbar}{2}\,\text{Tr}\log G^{-1}[g]$$

$$G^{-1} \equiv -\Box + m^2 + \xi R$$

  is formally equivalent to the CTP effective action with

$$\mathbf{G}^{-1} \equiv c_{AB} G^{-1}[g^A]$$

$$c_{AB} = \text{diag}(1, -1)$$

- Varying with respect to g^1_{ab} gives $\langle in|T_{ab}|in\rangle$

- Linear response equation comes from the second variation of the effective action

  - First expand the effective action in the series and truncate at second order

$$S_{eff} = S_{eff}[g+h] + \int d^4x \frac{\delta S_{eff}[g]}{\delta g^A_{ab}(x)} h^A_{ab}(x)$$

$$+\frac{1}{2}\int d^4x \int d^4y \frac{\delta^2 S_{eff}[g]}{\delta g^A_{ab}(x)\delta g^B_{cd}(y)} h^A_{ab}(x) h^B_{cd}(y)$$

  - Choose a solution to the semiclassical equations. Then the second term is zero.

  - Vary with respect to h^1_{ab} and then set $h^1 = h^2 = h$ and $g^1 = g^2 = g$

- Linear response equation is equivalent to varying the semiclassical equations

$$\delta\left[\alpha\ {}^{(C)}H_{ab}+\beta\ {}^{(1)}H_{ab}+\frac{1}{8\pi G_N}\left(G_{ab}+\Lambda g_{ab}\right)\right]$$

$$=\delta\langle in|T_{ab}|in\rangle$$

$$=\frac{1}{4}M_{ab}{}^{cd}h_{cd}(x)$$

$$+\frac{1}{2}\int d^4x'\sqrt{-g(x')}\,\Pi^{(ret)cd}_{ab}(x,x')h_{cd}(x')$$

$$\Pi^{(ret)cd}_{ab}(x,x')=i\theta(t,t')\langle in|[T_{ab}(x),T^{cd}(x')]|in\rangle$$

- Our proposal is: Solutions to the semiclassical Einstein equations should be stable against linearized perturbations of the geometry. If not the semiclassical approximation breaks down.

- Advantages

  – Linear response equations naturally take quantum fluctuations of the stess-energy into account

  – Everything is computed completely within the semiclassical approximation - so none of the problems associated with quantizing gravity

  – Since the linear response equations are derived from the effective action

    * no new types of renormalization counterterms

    * no state dependent divergences

- Investigate semiclassical approximation for fields in Minkowski vacuum in flat space

- Previously instabilities were found by Horowitz (1980), Jordan (1987), Suen (1989), Flanagan and Wald (1996).

- An arbitrary metric perturbation, h_{ab}, around n dimensional flat space, η_{ab}, can be decomposed into scalar, vector, and tensor parts

$$h_{ab} = h^{\perp}_{ab} + \partial_a v^{\perp}_b + \partial_b v^{\perp}_a + \left(\partial_a \partial_b - \frac{1}{n}\eta_{ab}\right) w + \frac{\eta_{ab}}{n} h$$

where $h^{\perp}_{ab}$ is transverse and traceless with respect to the flat metric η_{ab}

$$\partial^a h^{\perp}_{ab} = 0 = \partial_b h^{\perp}_{ab} \,, \quad \text{and} \quad \eta^{ab} h^{\perp}_{ab} = 0$$

and $v^{\perp}_a$ is transverse

$$\partial^a v^{\perp}_a = 0$$

- Linear response equation around flat space background breaks into tensor and scalar parts

$$\left[\alpha - \frac{1}{16\pi G_N}\right] P_{ab}^{(T)\,cd} h_{cd}(x) + \left[6\beta + \frac{1}{8\pi G_N}\right] P_{ab}^{(S)\,cd} h_{cd}(x) = \frac{1}{4} M_{ab}{}^{cd} h_{cd}(x) - \frac{1}{2}\int d^4x' \, \Pi_{ab}^{(\mathrm{ret})\,cd}(x,x')\, h_{cd}(x')$$

- After substantial calculation we find for the tensor part the condition

$$k^2[2\alpha k^2 + \frac{1}{8\pi G_N} + (k^2)^2 \int_0^\infty \frac{ds}{s^3} \frac{\rho^{(T)}(s)}{(s + k^2 - i\epsilon\ \mathrm{sgn}(k^0))}] = 0$$

- For the scalar part

$$k^2[12\beta k^2 - \frac{1}{4\pi G_N} + (k^2)^2 \int_0^\infty \frac{ds}{s^3} \frac{\rho^{(S)}(s)}{(s + k^2 - i\epsilon\ \mathrm{sgn}(k^0))}] = 0$$

$$\rho^{(\mathrm{T})}(s) \geq 0$$

$$\rho^{(\mathrm{S})}(s) \geq 0$$

- $$k^2 = -(k^0)^2 + (\vec{k})^2 = 0$$

  is a solution which corresponds to the usual gravitational waves

- No other solutions occur for perturbations well below the Planck scale

- Thus there are no semiclassical instabilities in Minkowski space which are well below the Planck scale

Validity of the semiclassical approximation in de Sitter space

- To solve linear response equations you first need a solution to the semiclassical backreaction equations
  - Effective mass of a scalar field in de Sitter space is $m^2_{eff} \equiv m^2 + \xi R$
  - de Sitter space is an exact solution to the semiclassical equations if $m^2_{eff} > 0$ and field is in the Bunch-Davies state.
  - in a de Sitter space background the stress-energy tensor approaches a de Sitter invariant tensor if the field is in a homogeneous state and either $m^2_{eff} > 0$ or $m = \xi = 0$
  - stress-energy tensor diverges asymptotically in de Sitter space for all other values of m and ξ for homogeneous states

- To study stability a different approach is necessary
  - No spectral theorem
  - Time dependence of perturbations will not be in such a simple form

- Horowitz and Wald (1980) and Starobinsky (1981) have computed $\delta\langle T_{ab}\rangle$ for a conformally invariant field in a conformally flat spacetime

- For other situations the calculations are more complicated

- Nonlocal part of the linear response equation is

$$\frac{i}{4}\int d^4y\sqrt{-g(y)}\theta(x^0,y^0)$$
$$\langle in|[T_{ab}(x),T^{cd}(y)]|in\rangle h_{cd}(y)$$

$$\langle T_{ab}(x)\rangle \;=\; \mathcal{D}_{ab}(x,x')\,\langle in|\phi(x)\phi(x')|in\rangle|_{x'=x}$$

- Green's function

$$\langle in|\phi(x)\phi(y)|in\rangle = -iG_>(x,y)$$
$$\langle in|\phi(y)\phi(x)|in\rangle = iG_>^*(x,y)$$

- Nonlocal part of the linear response equation becomes

$$\frac{-i}{4}\int d^4y\sqrt{-g(y)}\theta(x^0,y^0)$$
$$\lim_{x'\to x}\lim_{y'\to y}\mathcal{D}_{ab}(x,x')\mathcal{D}^{cd}(y,y')[G_>(x,y)G_>(x',y')$$
$$+G_>(x,y')G_>(x',y) - G_>^*(x,y)G_>^*(x',y')$$
$$-G_>^*(x,y')G_>^*(x',y)]h_{cd}(y)$$

- Compute the derivatives and then set $y' = y$ and $\vec{x}\,' = \vec{x}$ but leave $x^{0'} \neq x^0$ so that point splitting renormalization can be used

- Since the perturbed metric is $g_{ab} + h_{ab}$, expand the point splitting counterterms for the stress-energy tensor to linear order in h_{ab}

Conformally invariant field in a spatially flat Robertson-Walker spacetime

- Metric is $ds^2 = a^2(x^0)[-(dx^0)^2 + (d\vec{x})^2]$

- Green's function

$$
\begin{aligned}
G_>(x,y) &= -i\,\mathrm{sgn}(x^0 - y^0)\frac{\delta(\sigma_F(x,y))}{16\pi a(x^0)a(y^0)} \\
&+\frac{1}{8\pi^2 a(x^0)a(y^0)\sigma_F(x,y)} \\
\sigma_F(x,y) &\equiv \frac{1}{2}\left[-(x^0-y^0)^2 + |\vec{x}-\vec{y}|^2\right]
\end{aligned}
$$

- Consider a perturbation of the form

$$h_{ab}(x^0)\mathrm{e}^{i\vec{q}\cdot\vec{x}}$$

- Nonlocal part of the linear response equations is

$$\int \frac{d^4y}{64\pi^3} a^4(y^0) h_{cd}(y^0) \mathrm{e}^{i\vec{q}\cdot\vec{y}}\, \theta(x^0, y^0) \lim_{x'\to x} \lim_{y'\to y}$$
$$\mathcal{D}_{ab}(x,x')\mathcal{D}^{cd}(y,y') \left\{ \frac{1}{a(x^0)a(x^{0\prime})a(y^0)a(y'^0)} \right.$$
$$\left[\frac{\delta[\sigma_F(x,y)]}{\sigma_F(x',y')} + \frac{\delta[\sigma_F(x',y')]}{\sigma_F(x,y)} \right.$$
$$\left.\left. + \frac{\delta[\sigma_F(x,y')]}{\sigma_F(x',y)} + \frac{\delta[\sigma_F(x',y)]}{\sigma_F(x,y')} \right] \right\}$$

- Compute the derivatives and then set $y' = y$ and $\vec{x}\,' = \vec{x}$ but leave $x^{0'} \neq x^0$ so that point splitting renormalization can be used

- Integrate by parts to eliminate derivatives of $\delta(\sigma_F)$

- Integrating over $\delta(\sigma)$ results in $y^0 \to x^0 - |\vec{x} - \vec{y}|$ or $y^0 \to x^{0'} - |\vec{x} - \vec{y}|$

- Shift the integration variable from $\vec{y}$ to $\vec{z} = \vec{y} - \vec{x}$

- Subtract off point splitting counterterms and let $x^{0'} \to x^0$

- Find the linear response equation takes the form

$$\mathcal{G}_{ab}{}^{cd}[x^0, \vec{q}]h_{cd}(x^0) =$$

$$\lim_{x^{0'} \to x^0} \int_0^\infty dz \mathcal{F}_{ab}{}^{cd}[x^0, \vec{q}, z]h_{cd}(x^0 - z)$$

Conclusions

- A criterion for the validity of the semiclassical approximation is that solutions to the semiclassical backreaction equations should be stable to perturbations

- Quantum fluctuations in the stress-energy of the matter fields are naturally taken into account by the linear response equations

- In flat space in the Minkowski vacuum there are no unstable modes at scales well below the Planck scale so the semiclassical approximation is expected to be valid

- In a Robertson-Walker spacetime the linear response equations will reduce to a one-dimensional linear integrodifferential equation which can be renormalized using point splitting

- We plan to investigate the validity of the semiclassical approximation in de Sitter space