Euler Equation - Wave Equation Connection

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Wave Equation

1. The One-dimensional (1-D) Wave Equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

   with a the wave speed.

2. Is a Good Representative Equation For the Euler Equations

3. First Part of The Course We Will Use the 1-D Wave Equation to Derive and Analyze Various Aspects of Accuracy, Stability and Efficiency

4. What Motivates This Model Equation?

One Dimensional Euler Equations

1. The Euler Equations are

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0 \tag{2}$$

$$Q = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e + p) \end{bmatrix} \tag{3}$$

Equation of state

$$p = (\gamma - 1)\left(e - \frac{1}{2}\rho(u^2)\right) \tag{4}$$

where γ is the ratio of specific heats, generally taken as 1.4.

Quasi-Linear Form

1. First We Re-Write the Euler Equations, Eq. 2, in Change Rule Form (Quasi-Linear)
2. Let $\frac{\partial E}{\partial x} = \left(\frac{\partial E}{\partial Q}\right)\frac{\partial Q}{\partial x}$, where $\frac{\partial E}{\partial Q}$ needs to be defined since E and Q are vectors.
3. The term $\frac{\partial E}{\partial Q}$ is A Tensor, Actually a Matrix Defined As the Jacobian of the Flux Vector E with respect to Q.
4. Eq.2 can be rewritten as (A defined below)

$$\frac{\partial Q}{\partial t} + A\frac{\partial Q}{\partial x} = 0 \tag{5}$$

Generalized Forms

1. Redefine Q and E In terms of Independent Variables q_1, q_2, q_3 as

$$Q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}$$

$$E = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} q_2 \\ \frac{q_2^2}{q_1} + (\gamma - 1)\left(q_3 - \frac{1}{2}\frac{q_2^2}{q_1}\right) \\ \frac{q_2}{q_1}\left(q_3 + (\gamma - 1)\left(q_3 - \frac{1}{2}\frac{q_2^2}{q_1}\right)\right) \end{bmatrix}$$

Jacobian Derivation

1. The Definition of the Jacobian $A = \frac{\partial E}{\partial Q}$,

$$A = \begin{bmatrix} \frac{\partial e_1}{\partial q_1} & \frac{\partial e_1}{\partial q_2} & \frac{\partial e_1}{\partial q_3} \\ \frac{\partial e_2}{\partial q_1} & \frac{\partial e_2}{\partial q_2} & \frac{\partial e_2}{\partial q_3} \\ \frac{\partial e_3}{\partial q_1} & \frac{\partial e_3}{\partial q_2} & \frac{\partial e_3}{\partial q_3} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma-1 \\ -\frac{\gamma e u}{\rho} + (\gamma-1)u^3 & \frac{\gamma e}{\rho} - \frac{3(\gamma-1)u^2}{2} & \gamma u \end{bmatrix}$$

Linear Diagonalized Form of Euler Equations

1. Freeze the Jacobian Matrix A at a Reference State A_0
2. This Can Be Justified By Small Perturbation Theory, Asymptotic Analysis, etc.
3. We now have

$$\frac{\partial Q}{\partial t} + A_0 \frac{\partial Q}{\partial x} = 0 \tag{6}$$

4. The matrix A (and the corresponding A_0) has a complete set of eigenvectors and eigenvalues.

Eigensystem of A

1. Let $A = X \Lambda X^{-1}$ and conversely $\Lambda = X^{-1} A X$
   (a) X is the $3x3$ eigenvector matrix of A
   (b) Λ is the diagonal eigenvalue matrix with elements, $\lambda_1, \lambda_2 \lambda_3$.
   (c) For the Euler Equations, $\lambda_1 = u, \lambda_2 = u + c$, and $\lambda_3 = u - c$ with $c = \sqrt{\frac{\gamma p}{\rho}}$ the speed of sound.

Diagonalization of Euler Equations

1. Using the Eigen-System of A_0 we can transform Eq.6 to

$$X_0^{-1}\left[\frac{\partial Q}{\partial t} + A_0 X_0 X_0^{-1}\frac{\partial Q}{\partial x}\right] = 0$$

$$\frac{\partial\left[X_0^{-1}Q\right]}{\partial t} + \left[X_0^{-1}A_0X_0\right]\frac{\partial\left[X_0^{-1}Q\right]}{\partial x} = 0$$

$$\frac{\partial W}{\partial t} + \Lambda_0\frac{\partial W}{\partial x} = 0 \tag{7}$$

with $W = X_0^{-1}Q$

Characteristic Form of Euler Equations

1. The Equations in Characteristic Form are uncoupled

$$\frac{\partial w_i}{\partial t} + \lambda_{0\,i}\frac{\partial w_i}{\partial x} = 0 \qquad (8)$$

   for $i = 1, 2, 3$

2. So for each i we have the wave equation, Eq.1, where $u = w_i$ and $a = \lambda_{0\,i}$

3. Therefore, any process, analysis, stability, etc results applied to the wave equation holds for each characteristic equation of w_i

Model Equation Justification

1. To Complete the process
   (a) Transform back to physical variables $Q = X_0 W$
   (b) X_0 is a constant matrix (it is made up of elements at the frozen state and therefore not a function of x, t)
   (c) The resulting Q is just linear combinations of the w_i and any results applied to w_i also apply to q_i.
   (d) For example, if any of the w_i are divergent (unstable, going to infinity, inaccurate, etc), the q_i behave consistent with the w_i

CONCLUSIONS

1. The wave equation Eq:1:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{9}$$

   is an appropriate model equation for the Euler Equations

2. PS: GET USE TO SEEING IT FOR THE NEXT FEW WEEKS!!