IQI 04, Seminar 7

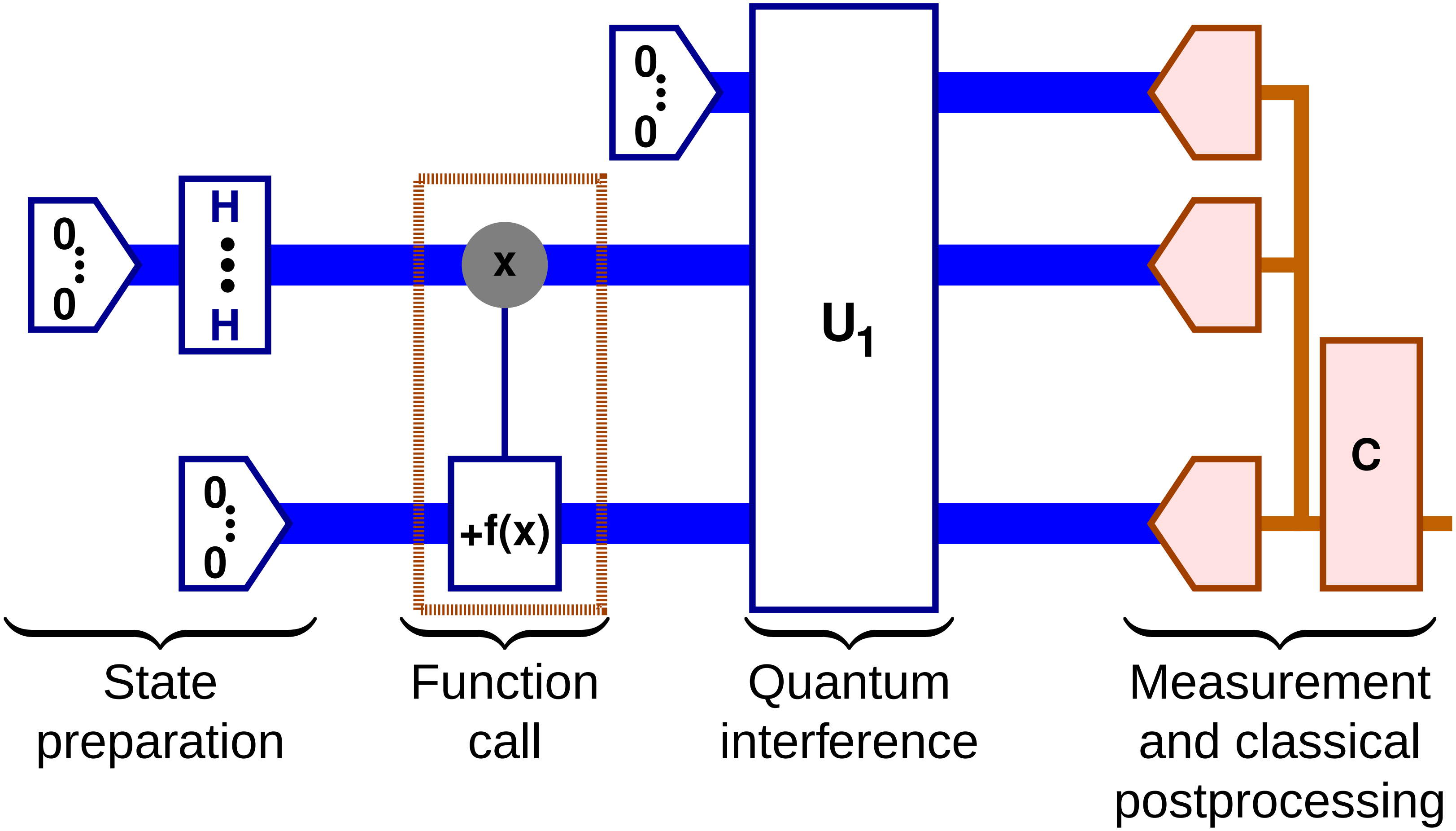
Produced with pdflatex and xfig

- Reversible classical computation.
- From irreversible to reversible computation.

E. “Manny” Knill: `knill@boulder.nist.gov`

Recognizing Patterns in Functions

- Algorithm structure for determining some **Property**(f).
  - One function call:



State preparation

Function call

Quantum interference

Measurement and classical postprocessing

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Recognizing Patterns in Functions

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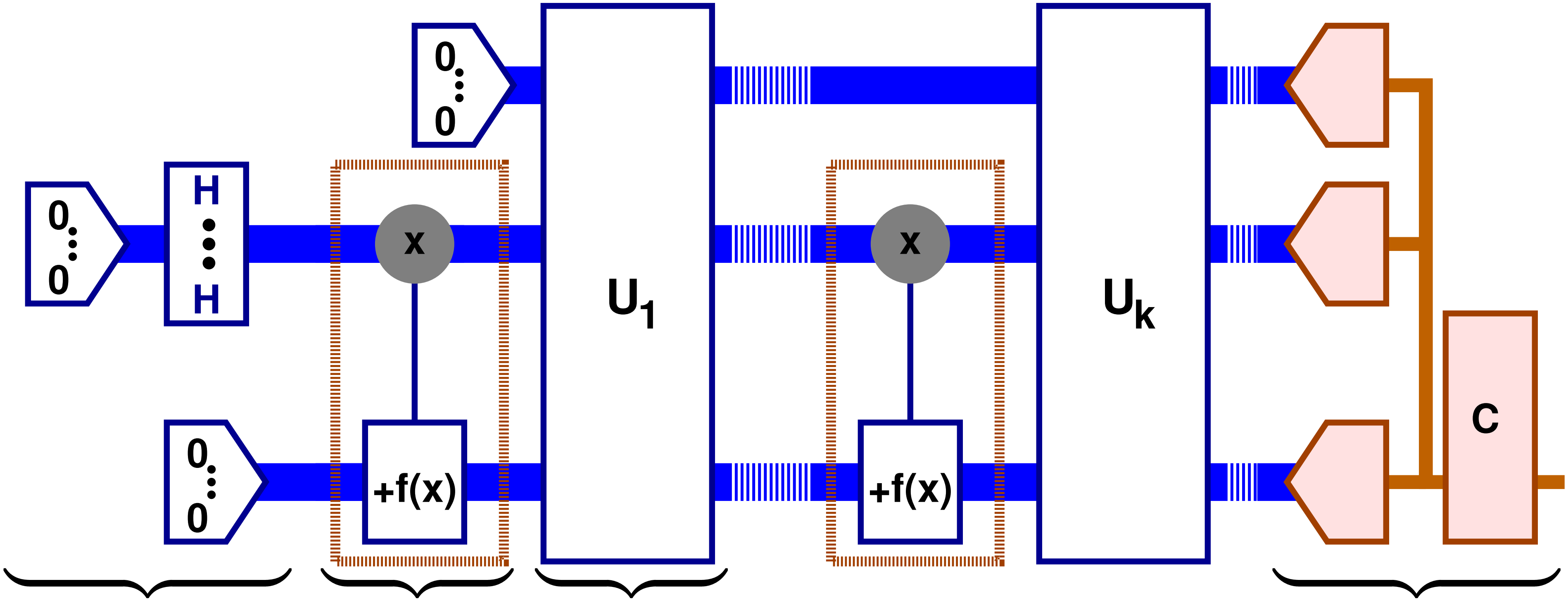
  - Multiple function calls:



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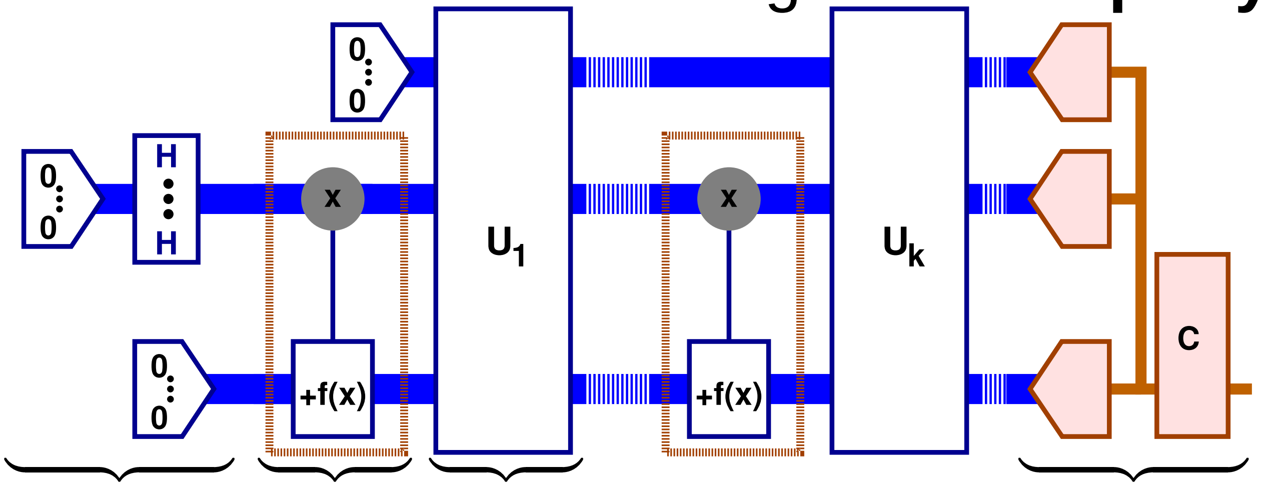

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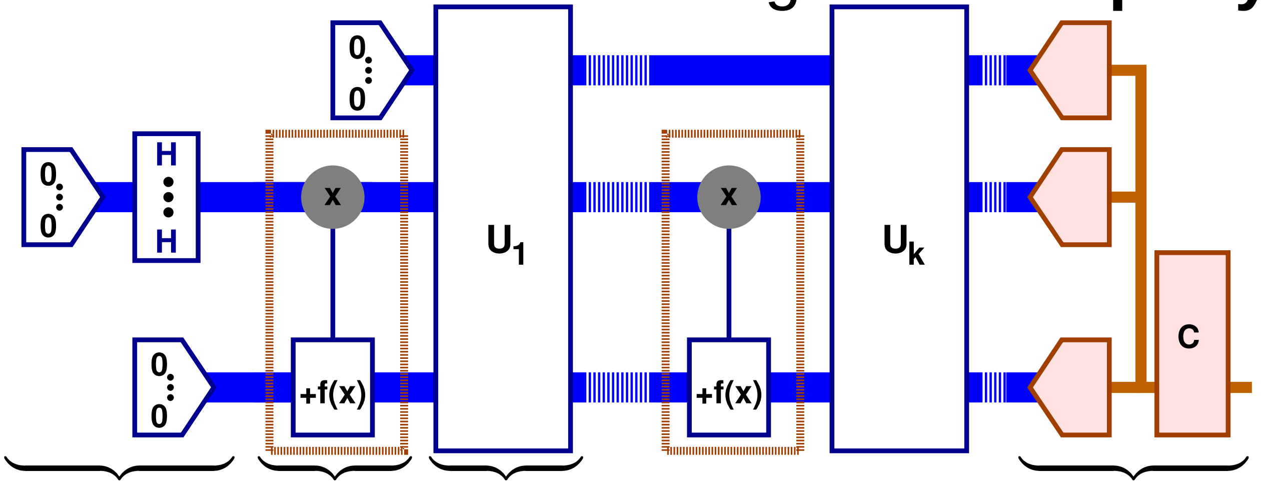

State preparation

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Quantum interference

Measurement and classical postprocessing

- A classical algorithm for $x \mapsto f(x)$ is known.
  - How to implement $\sum_x \alpha_x |xy\rangle \rightarrow \sum_x \alpha_x |x(y \oplus f(x))\rangle$?

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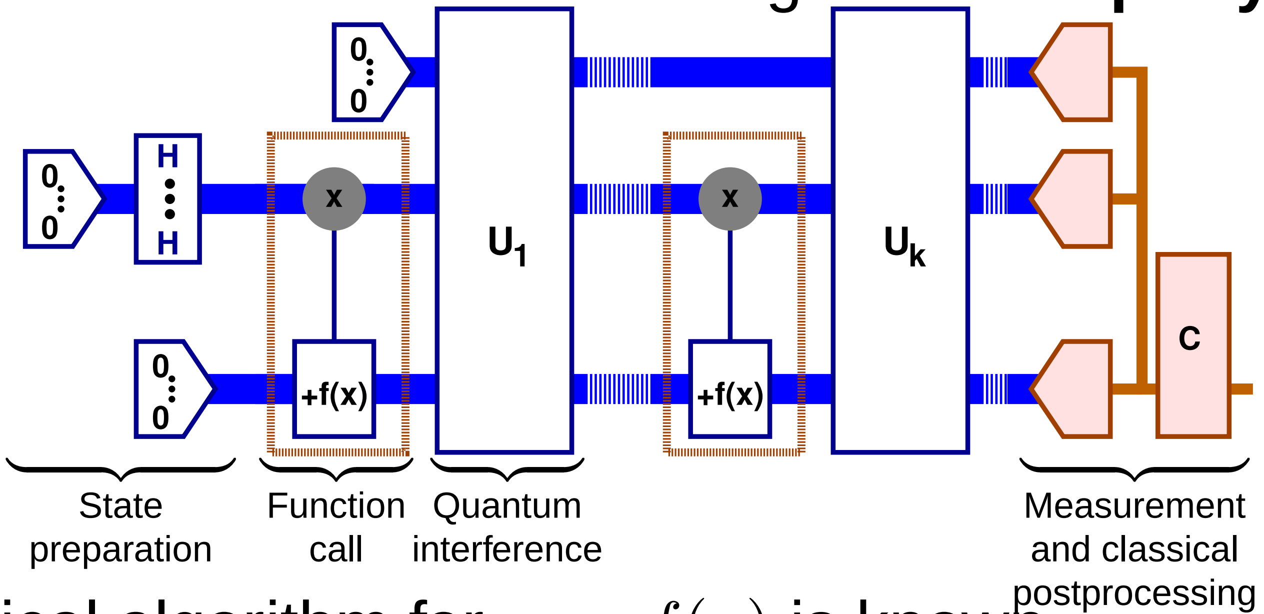

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- Solution:

  1. Algorithm → circuit$(|x|)$.

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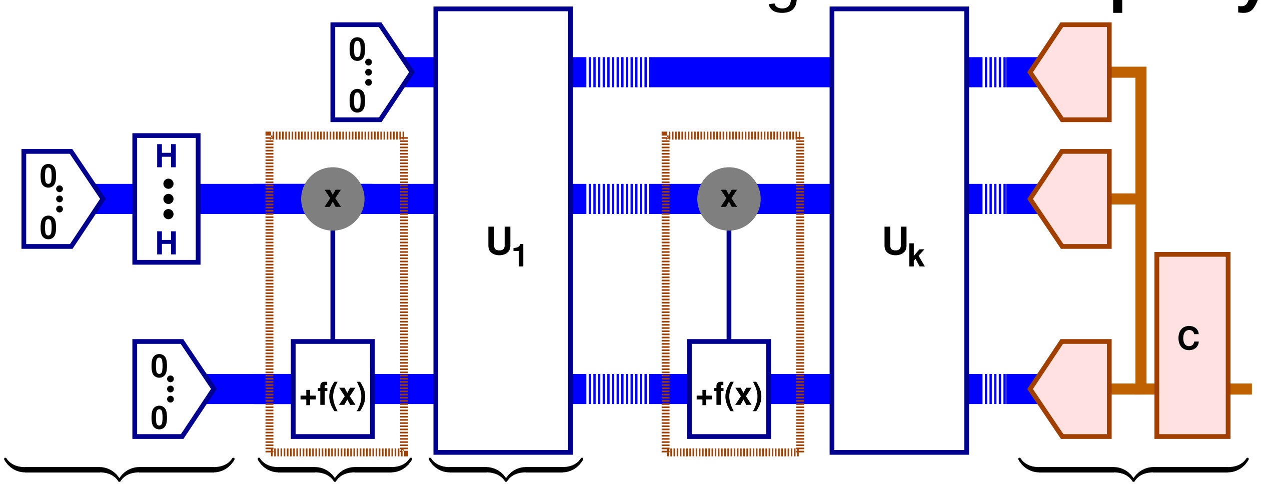

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1. Algorithm $\rightarrow$ circuit$(|x|)$.
2. Irreversible gates $\rightarrow$ reversible gates and memory.

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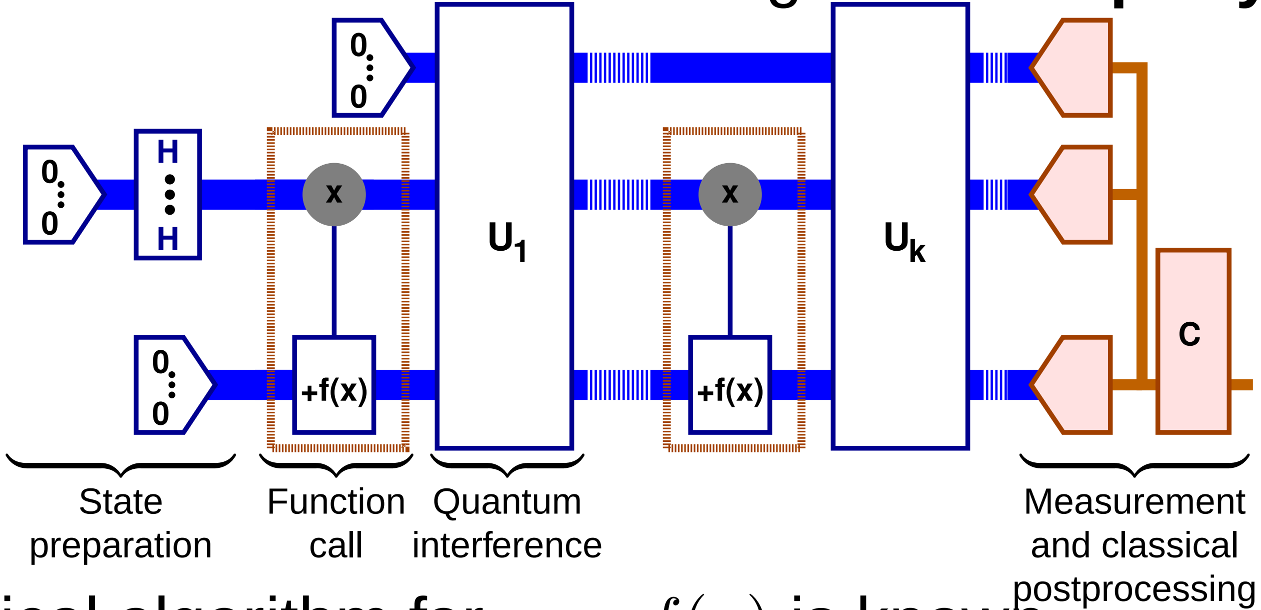

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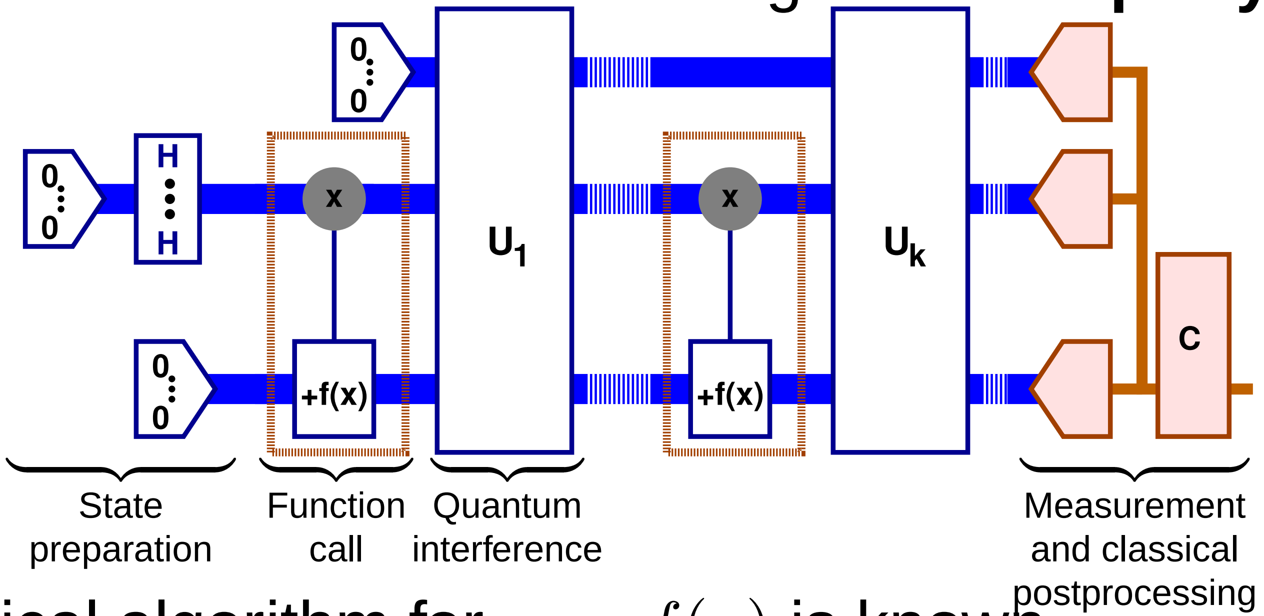

- A classical algorithm for $x \mapsto f(x)$ is known.
  - How to implement $\sum_x \alpha_x|xy\rangle \to \sum_x \alpha_x|x(y \oplus f(x))\rangle$?
- Solution:

1. Algorithm $\to$ circuit$(|x|)$.
2. Irreversible gates $\to$ reversible gates and memory.
3. Reversibly erase memory.
4. Bits $\to$ qubits. Reversible gates $\to$ unitary gates.

Example: Number Comparison

- Algorithm for comparing two binary numbers.

COMP(x, y)
Input: Zero-filled n-bit numbers $x = x_{n-1} \dots x_0$ and $y = y_{n-1} \dots y_0$.
Output: 0 if $x < y$ and 1 if $x \geq y$.

```
$k \leftarrow n - 1$
$c \leftarrow -1$
while $k \geq 0 \ \& \ c < 0$
    if $x_k < y_k$
        $c \leftarrow 0$
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k	:						
c	:						
x	:	0	1	0	1	0	1
y	:	0	1	0	0	1	1

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k : 5

c :

x	:	0	1	0	1	0	1
y	:	0	1	0	0	1	1

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```

$$\begin{array}{cc|c|ccccc}
k & : & 5 \\
c & : & -1 \\
\cline{3-3}
x & : & 0 & 1 & 0 & 1 & 0 & 1 \\
y & : & 0 & 1 & 0 & 0 & 1 & 1 \\
\cline{3-3}
\end{array}$$

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    **end**
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    **return** $c$
```

$$\begin{array}{ccl} k & : & 5 \\ c & : & -1 \end{array}$$

$$\begin{array}{cccccccc} x & : & \boxed{0} & 1 & 0 & 1 & 0 & 1 \\ y & : & \boxed{0} & 1 & 0 & 0 & 1 & 1 \end{array}$$

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$$\begin{array}{lcl} k & : & 4 \\ c & : & -1 \end{array}$$

$$\begin{array}{cc|c|ccccc} \cline{3-3} x & : & 0 & 1 & 0 & 1 & 0 & 1 \\ y & : & 0 & 1 & 0 & 0 & 1 & 1 \\ \cline{3-3} \end{array}$$

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\begin{array}{ccc}
k & : & 4 \\
c & : & -1
\end{array}
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x & : & 0 & 1 & 0 & 1 & 0 & 1 \\
y & : & 0 & 1 & 0 & 0 & 1 & 1
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$$k : 3$$

$$c : -1$$

$$\begin{array}{ccc|c|cccc} x & : & 0 & 1 & 0 & 1 & 0 & 1 \\ y & : & 0 & 1 & 0 & 0 & 1 & 1 \end{array}$$

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```

$$\begin{array}{ccc}
k & : & 2 \\
c & : & -1
\end{array}$$

$$\begin{array}{cccc|c|ccc}
x & : & 0 & 1 & 0 & 1 & 0 & 1 \\
y & : & 0 & 1 & 0 & 0 & 1 & 1
\end{array}$$

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- Algorithm for comparing two binary numbers.

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$k \ : \ 2$

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$$\begin{array}{ccl} k & : & 2 \\ c & : & -1 \end{array}$$

$$\begin{array}{cccccccc} x & : & 0 & 1 & 0 & \multirow{2}{*}{$\boxed{\begin{array}{c} 1 \\ 0 \end{array}}$} & 0 & 1 \\ y & : & 0 & 1 & 0 & & 1 & 1 \end{array}$$

Example: Number Comparison

- Algorithm for comparing two binary numbers.

```
COMP($x, y$)
Input: Zero-filled $n$-bit numbers $x = x_{n-1} \ldots x_0$ and $y = y_{n-1} \ldots y_0$.
Output:  $0$ if $x < y$ and $1$ if $x \geq y$.

    $k \leftarrow n - 1$
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$$\begin{array}{ccl} k & : & 2 \\ c & : & -1 \end{array}$$

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$$\begin{array}{ccc} k & : & 2 \\ c & : & 1 \end{array}$$

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```

k : 1

c : 1

x	:	0	1	0	$\boxed{1}$	0	1
y	:	0	1	0	$\boxed{0}$	1	1

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            $c \leftarrow 0$
        **else if** $x_k > y_k$
            $c \leftarrow 1$
        **end**
        $k \leftarrow k - 1$
    **end**
⇒   **if** $c < 0$ **then** $c \leftarrow 1$
    **return** $c$
```

$$\begin{array}{lcl} k & : & 1 \\ c & : & 1 \end{array}$$

$$\begin{array}{lcccc|c|cc} x & : & 0 & 1 & 0 & 1 & 0 & 1 \\ y & : & 0 & 1 & 0 & 0 & 1 & 1 \end{array}$$

Example: Number Comparison

- Algorithm for comparing two binary numbers.

COMP(x, y)
Input: Zero-filled n-bit numbers $x = x_{n-1} \ldots x_0$ and $y = y_{n-1} \ldots y_0$.
Output: 0 if $x < y$ and 1 if $x \geq y$.

```
    $k \leftarrow n - 1$
    $c \leftarrow -1$
    while $k \geq 0$ & $c < 0$
        if $x_k < y_k$
            $c \leftarrow 0$
        else if $x_k > y_k$
            $c \leftarrow 1$
        end
        $k \leftarrow k - 1$
    end
    if $c < 0$ then $c \leftarrow 1$
$\Rightarrow$ return $c$
```

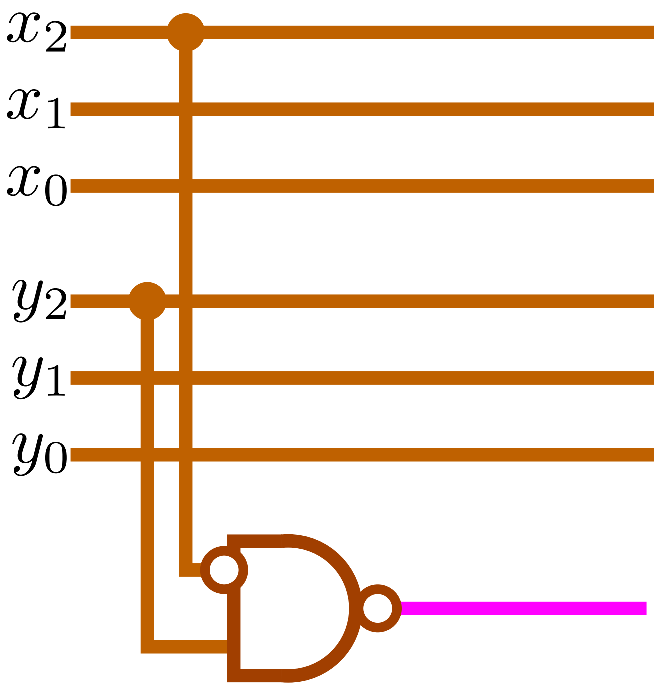
k : 1

c : 1

x	:	0	1	0	$\boxed{1}$	0	1
y	:	0	1	0	$\boxed{0}$	1	1

Classical Comparison Circuit

- 3 bit comparison circuit for “$\textbf{if } x < y \textbf{ then } 0 \textbf{ else } 1$”.



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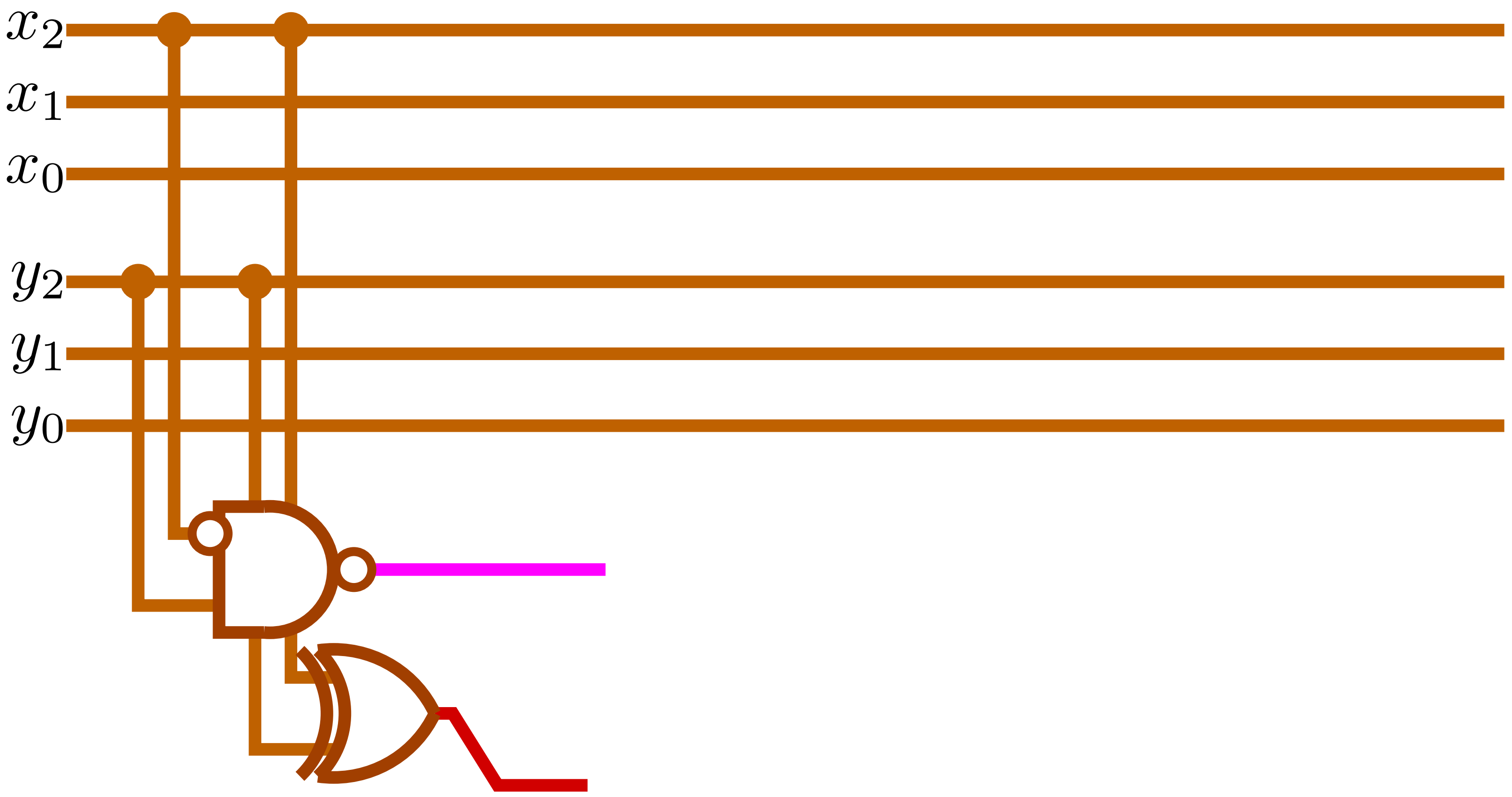

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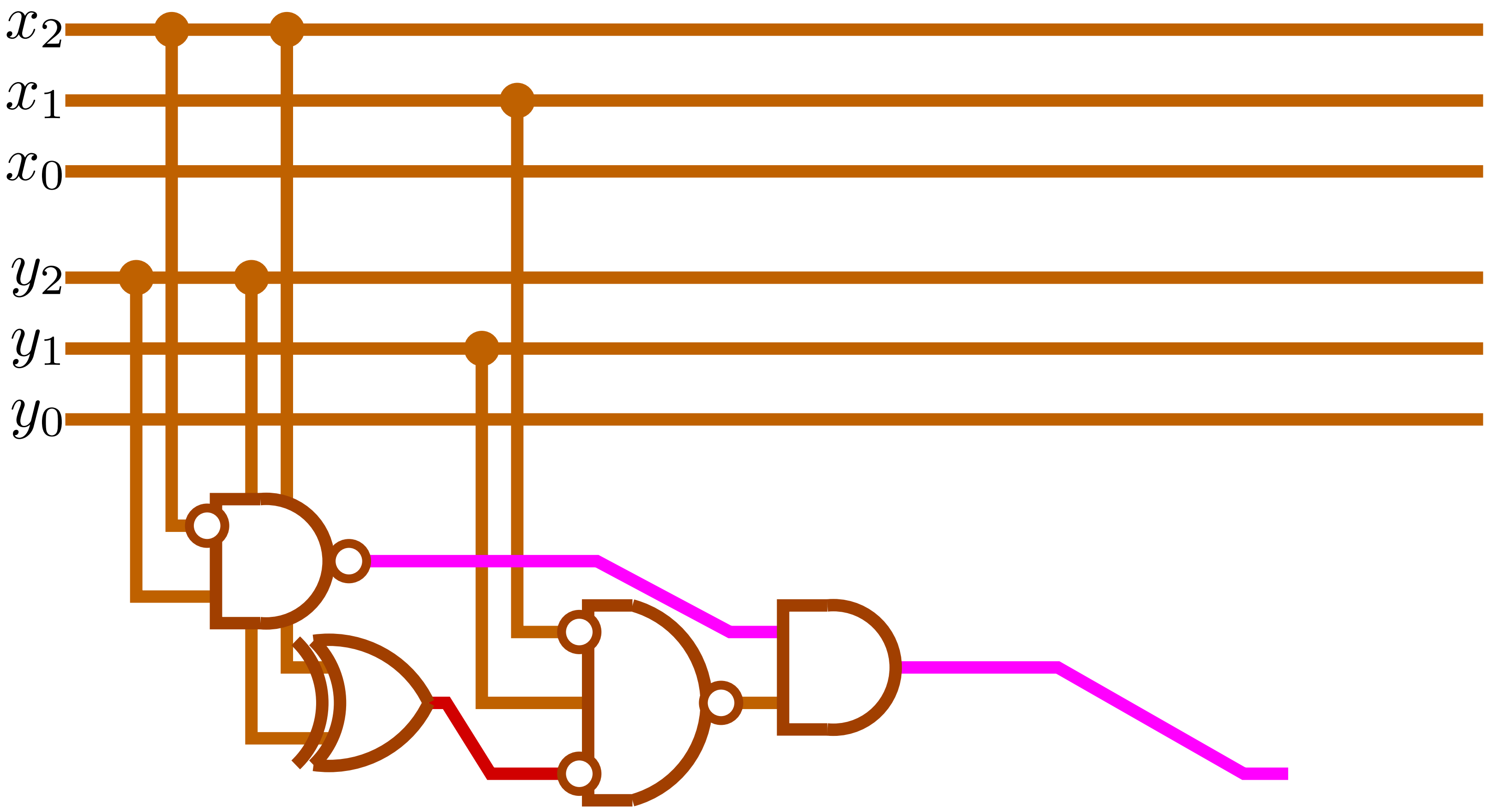

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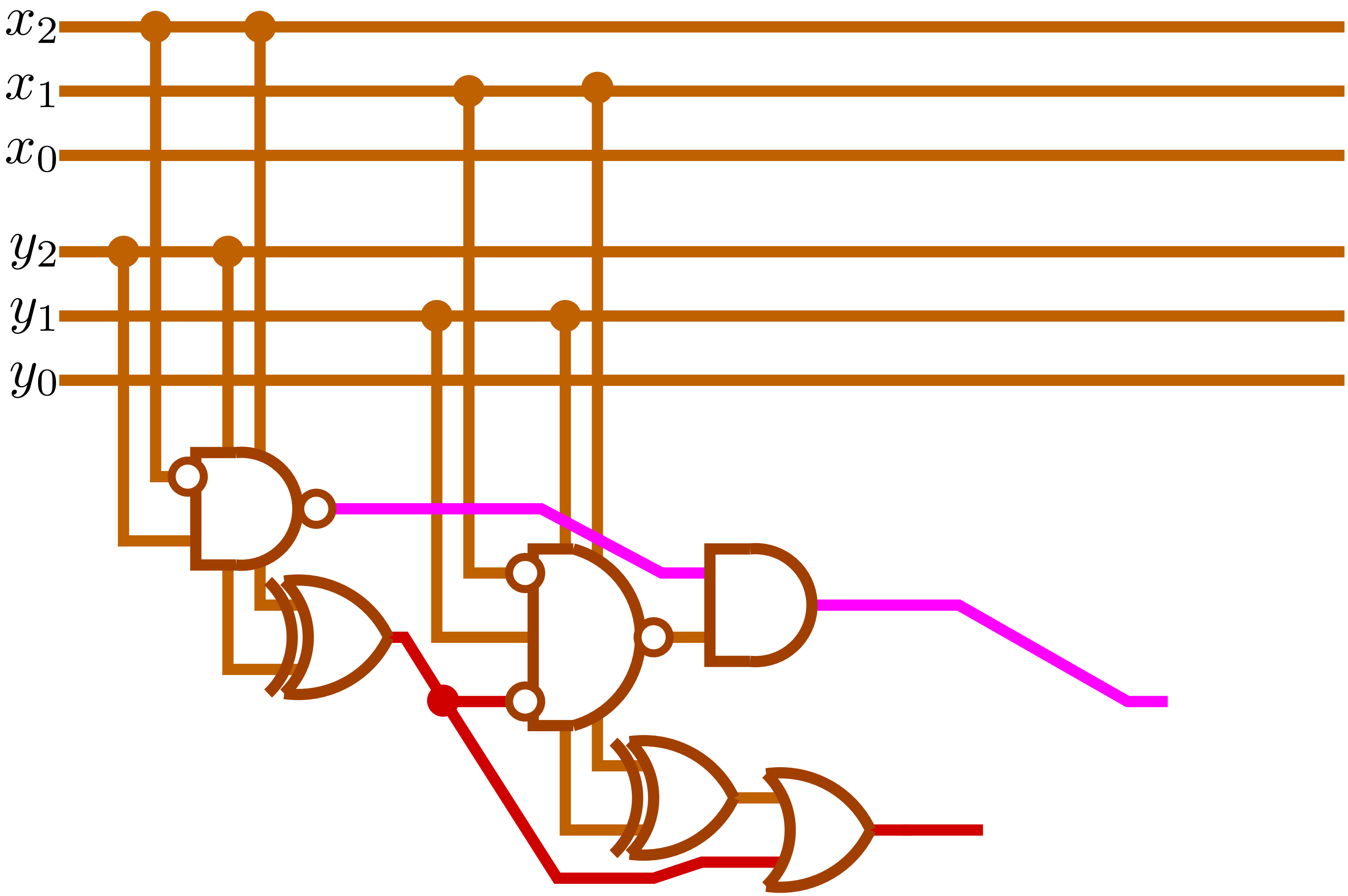

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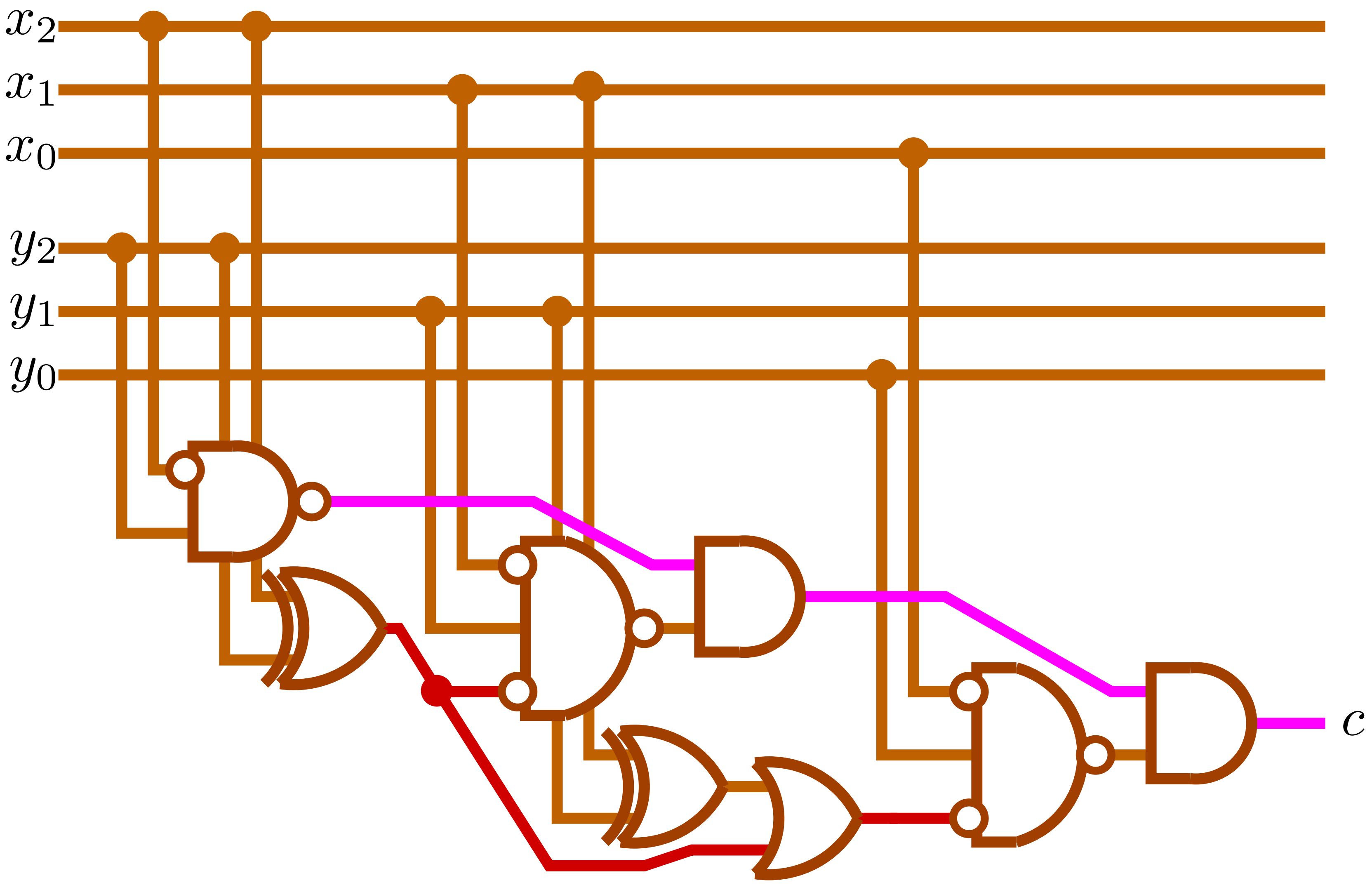
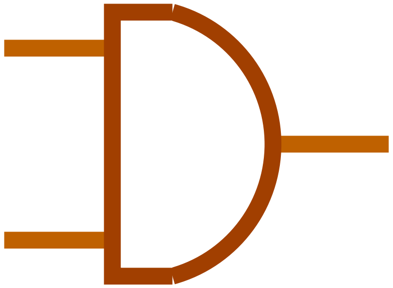


Converting to Reversible Logic

- Reversifying the and gate.



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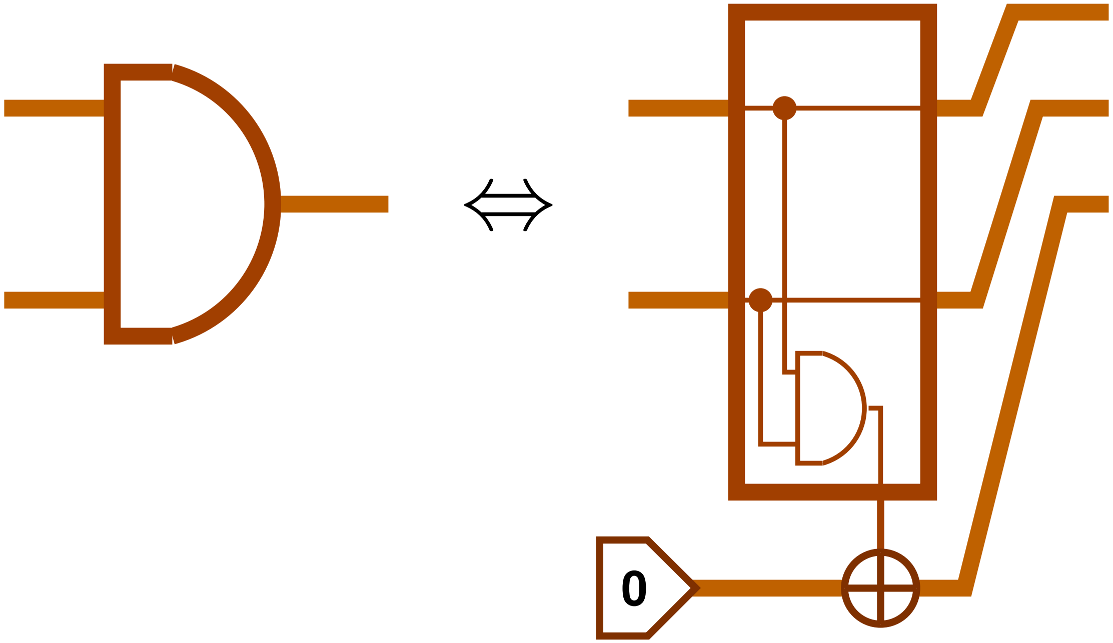

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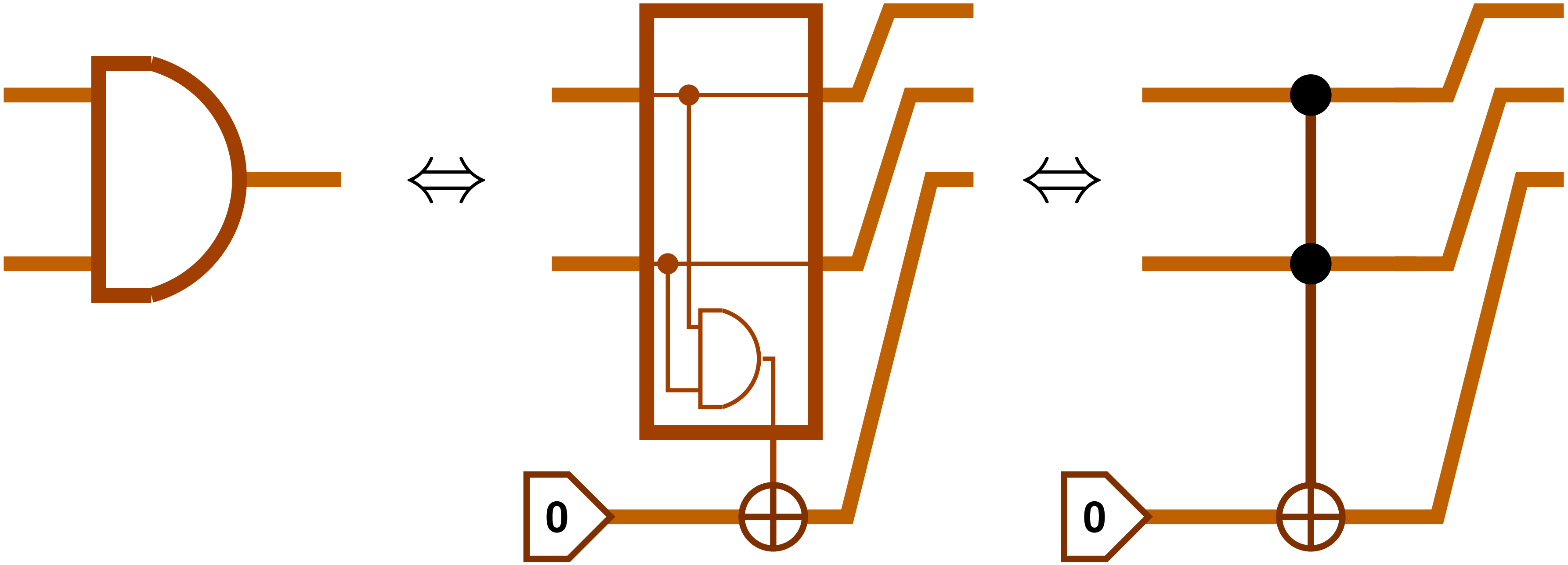

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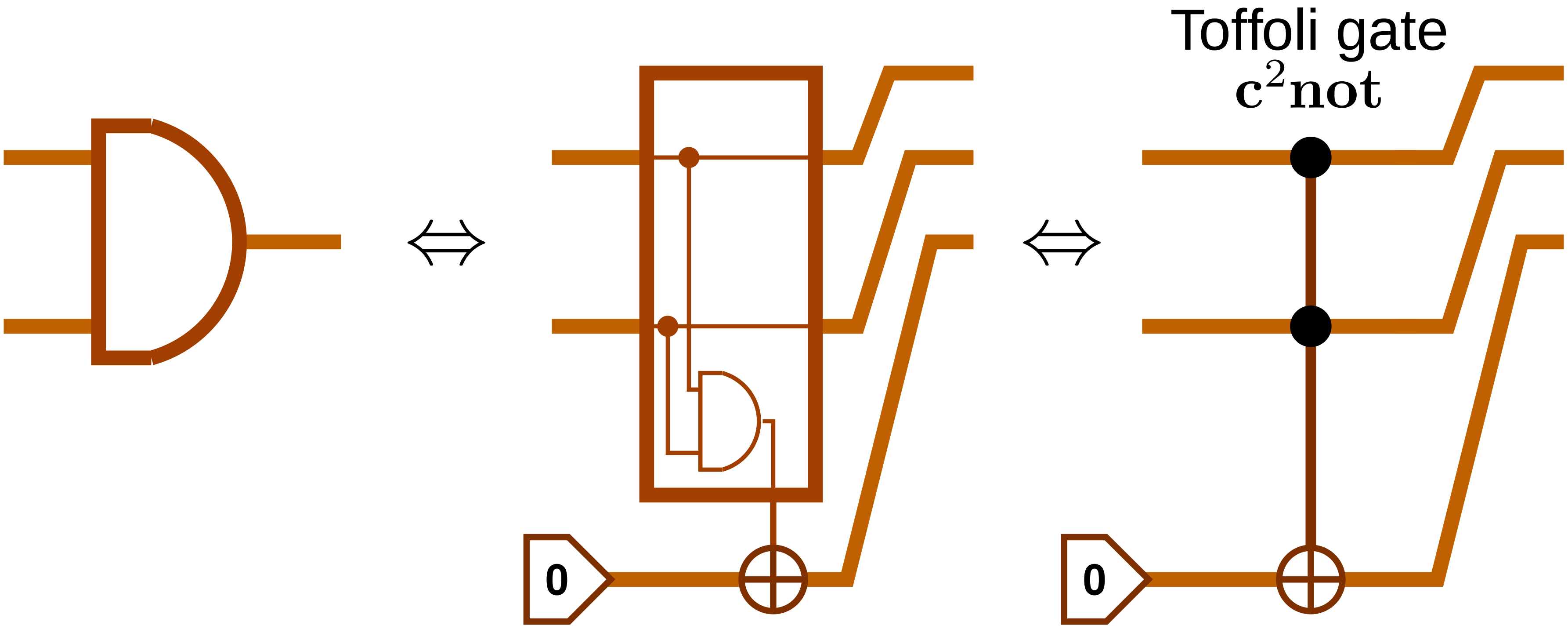
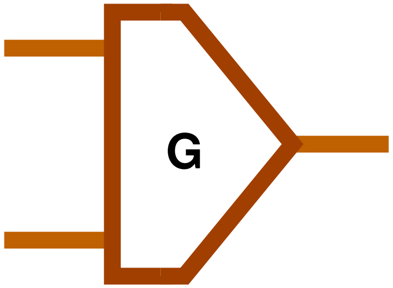


Converting to Reversible Logic

- Reversifying a general gate.



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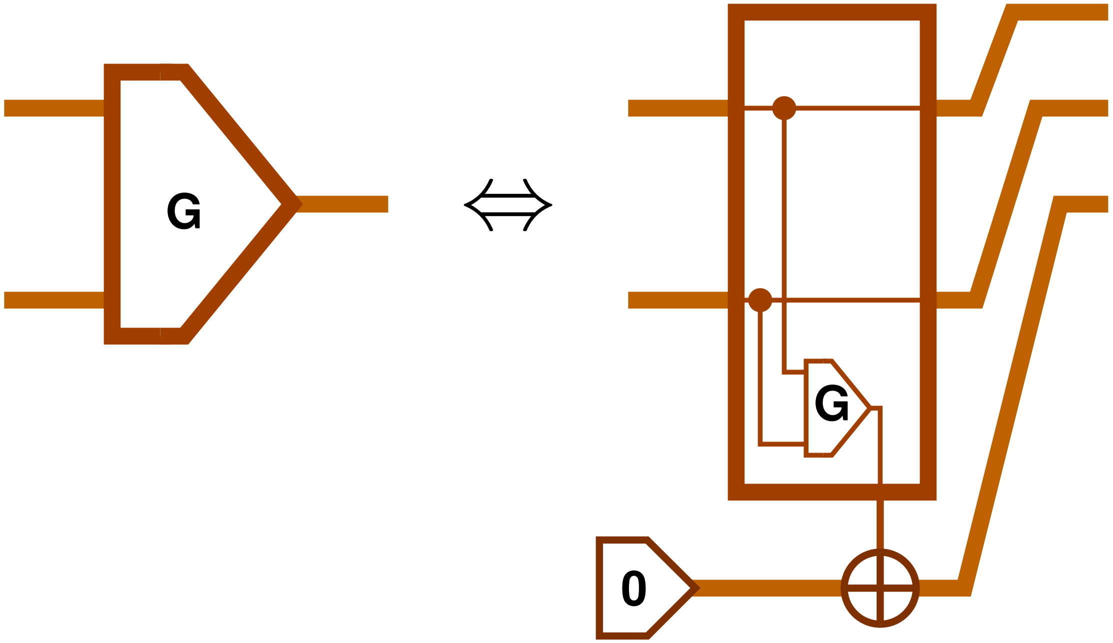

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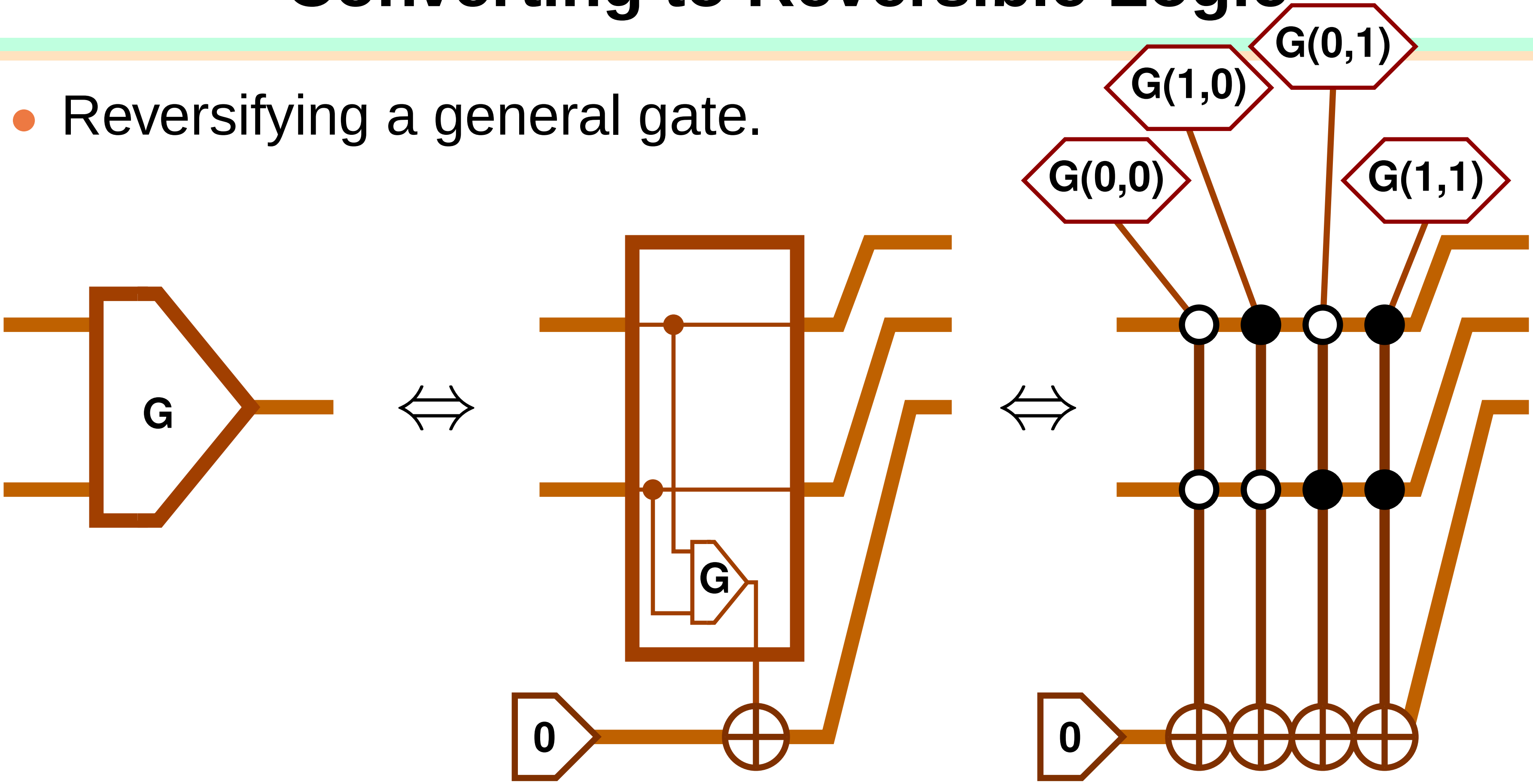

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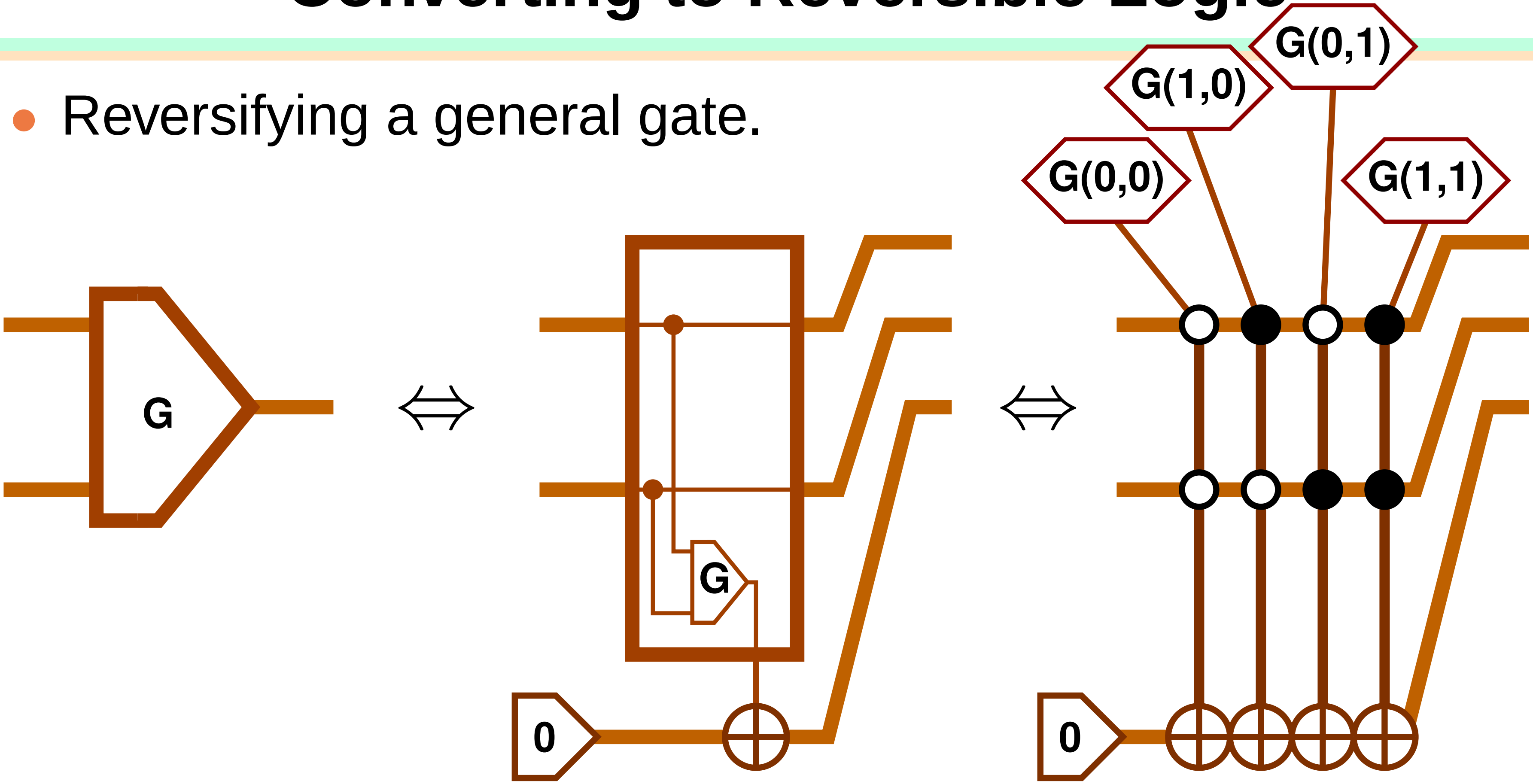

- . . . then optimize the conditional gates.

Converting to Reversible Logic

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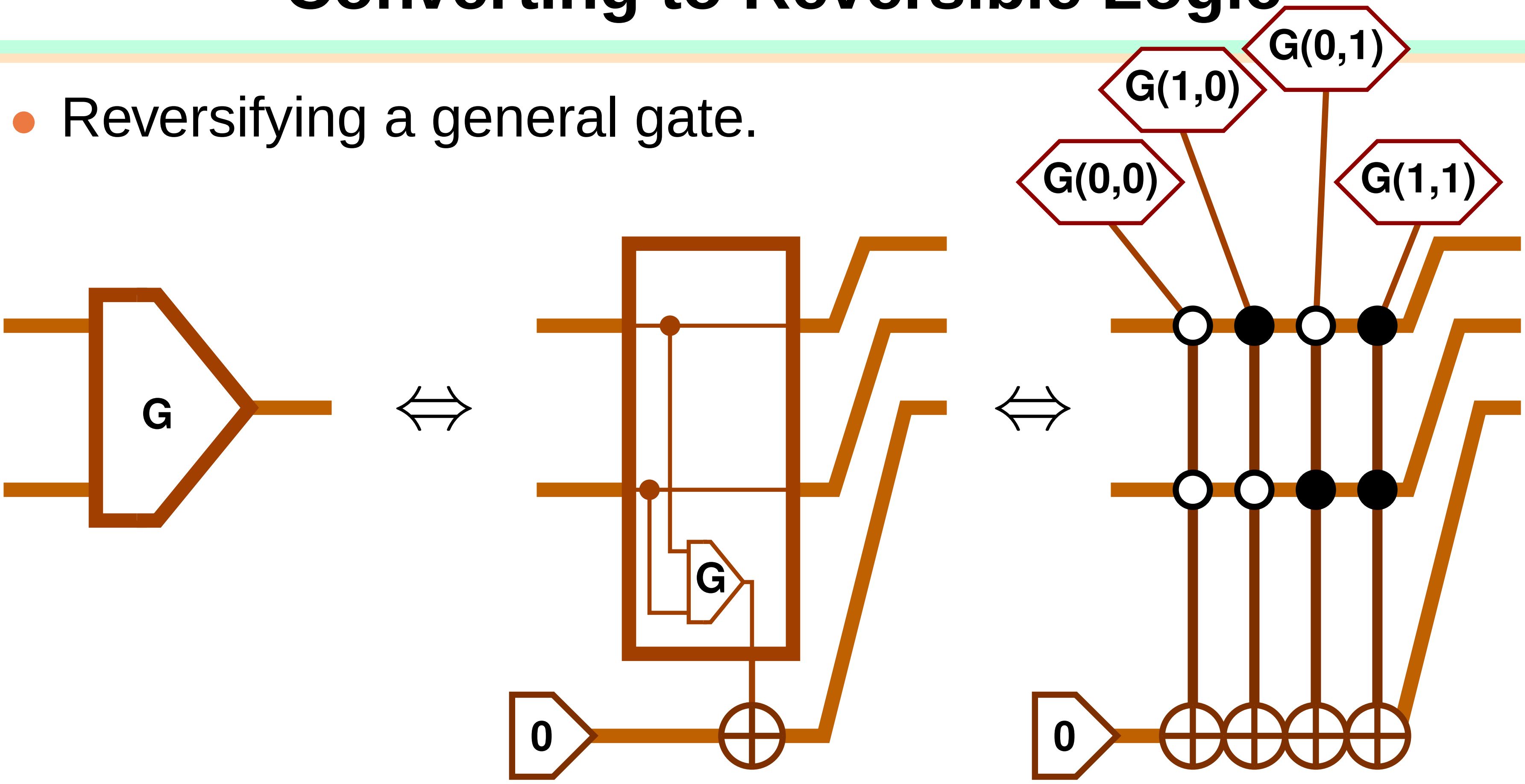
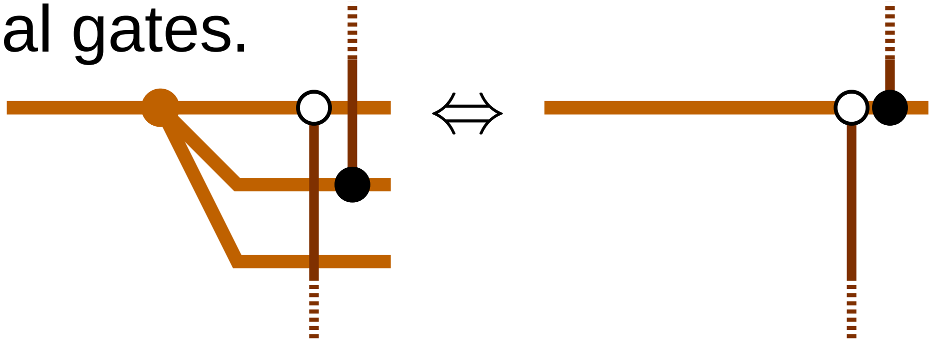
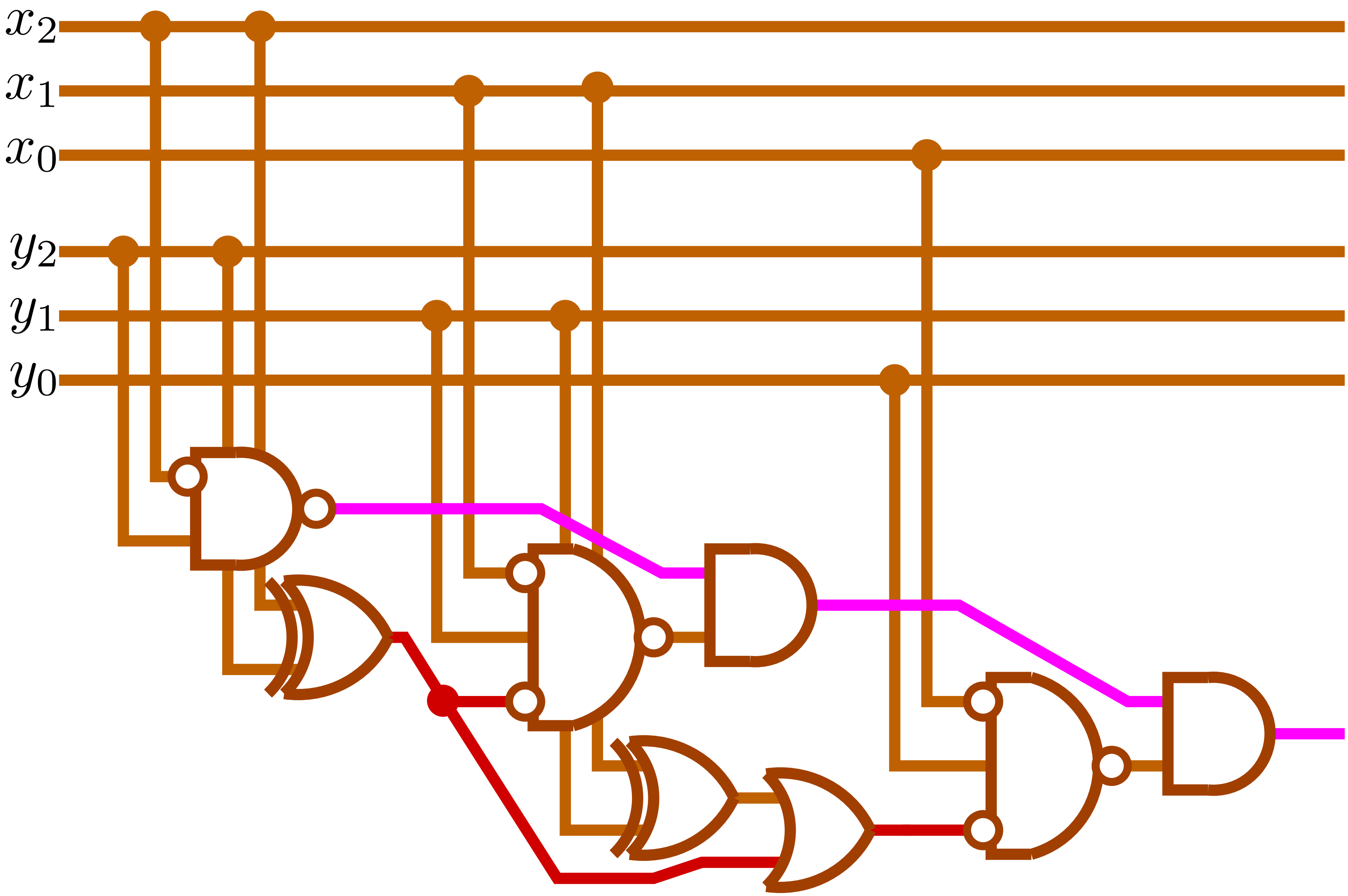


- . . . then optimize the conditional gates.
- Remove redundant fanout.



Reversifying the Comparison Circuit



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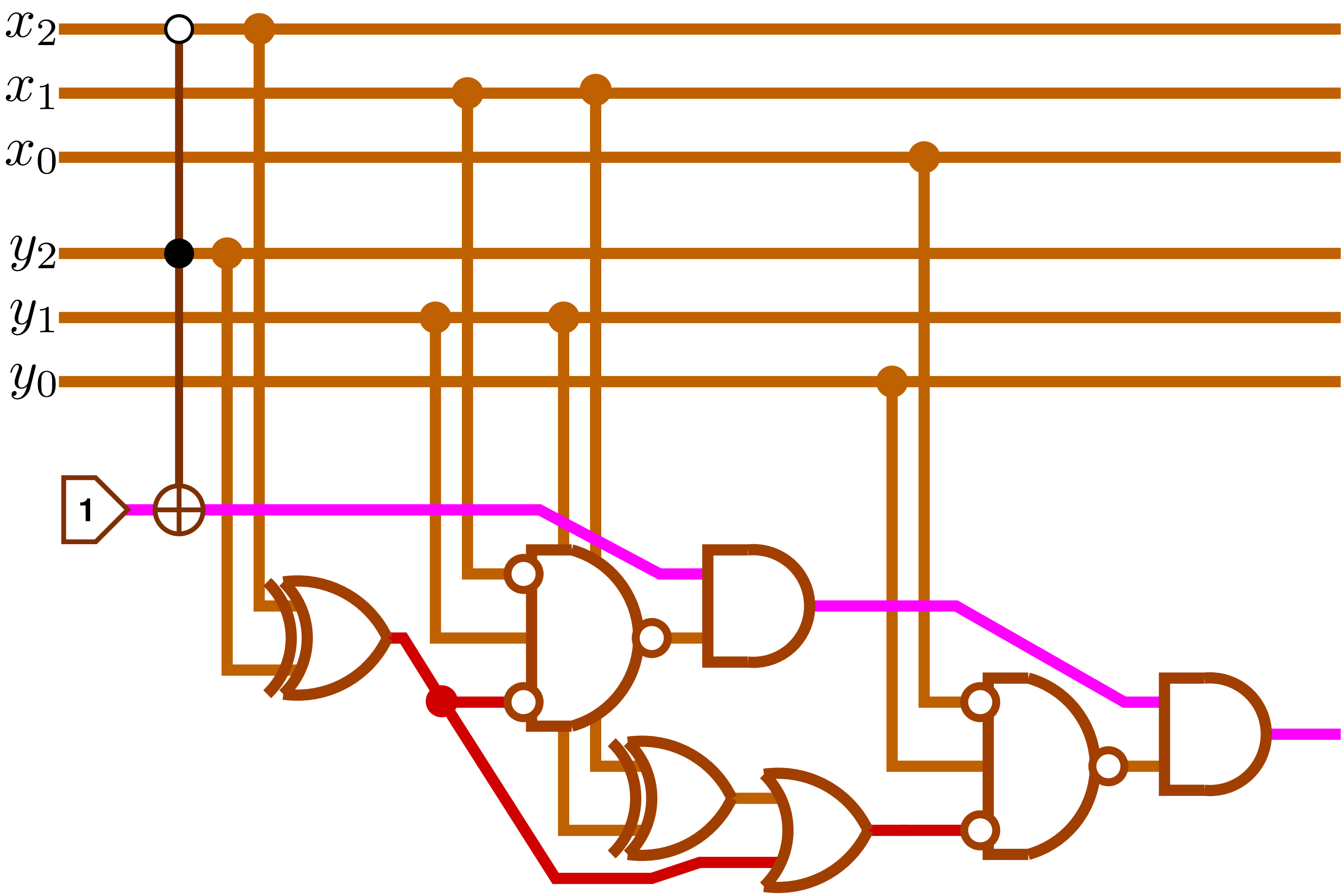

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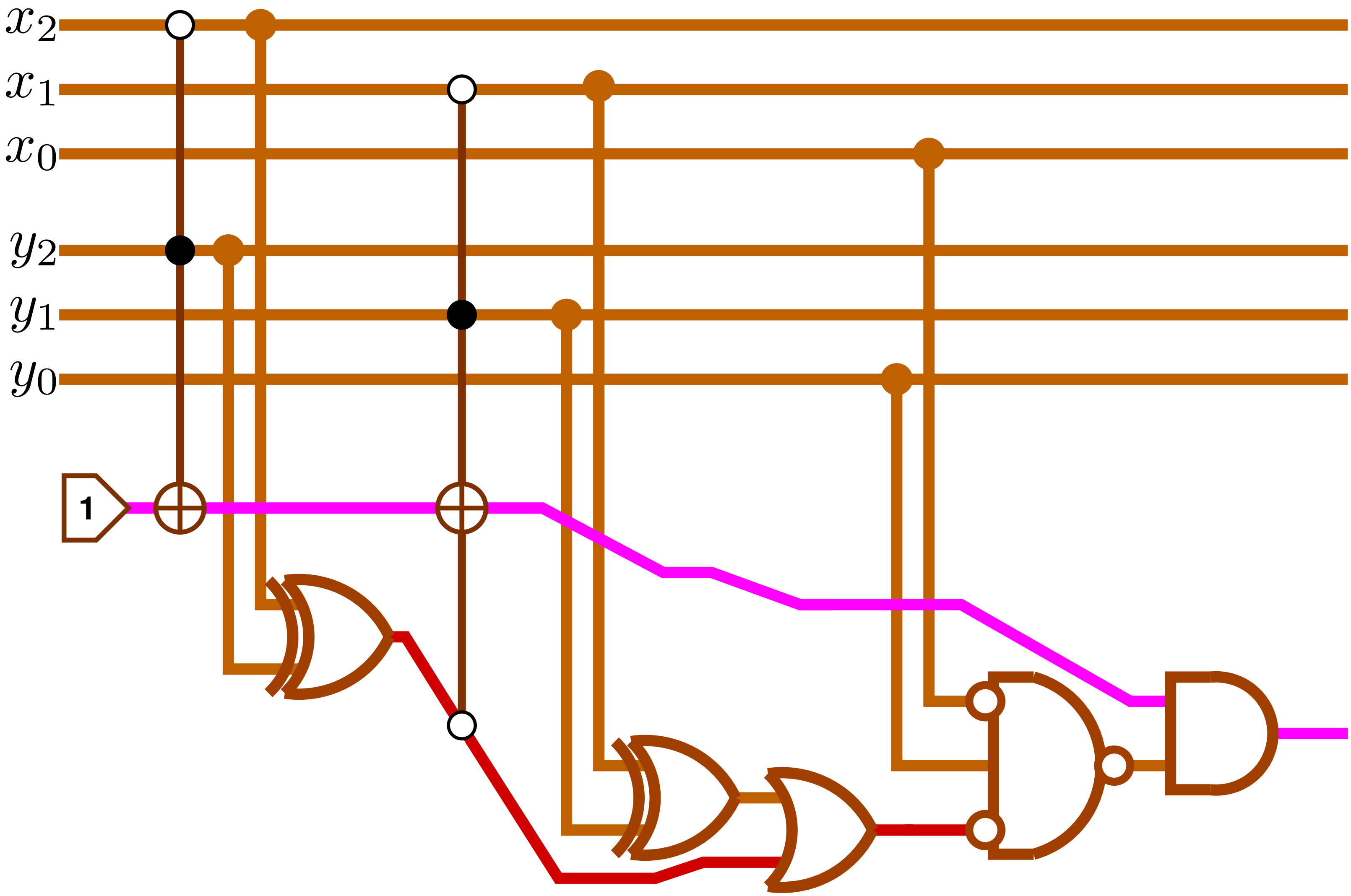

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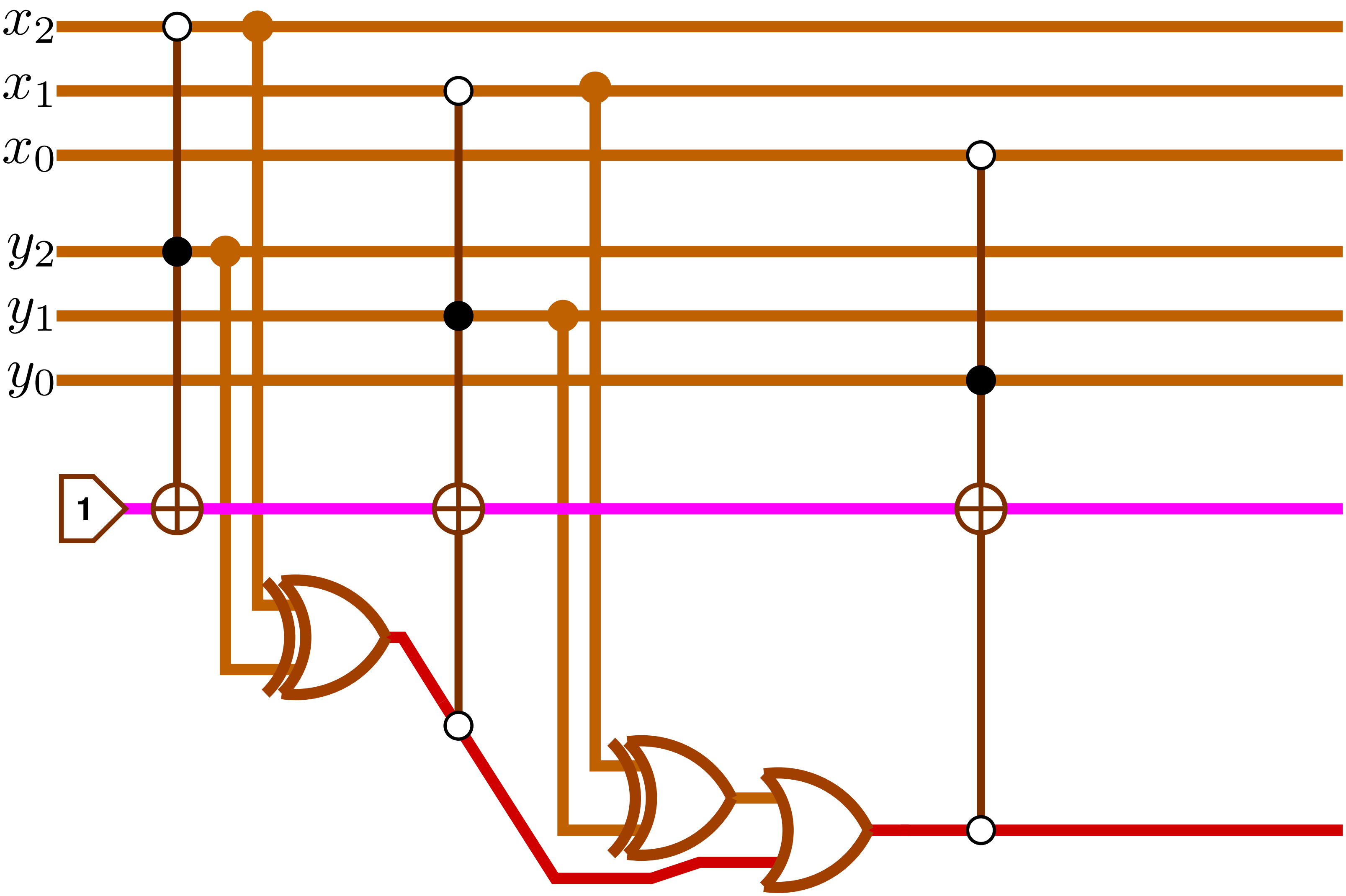

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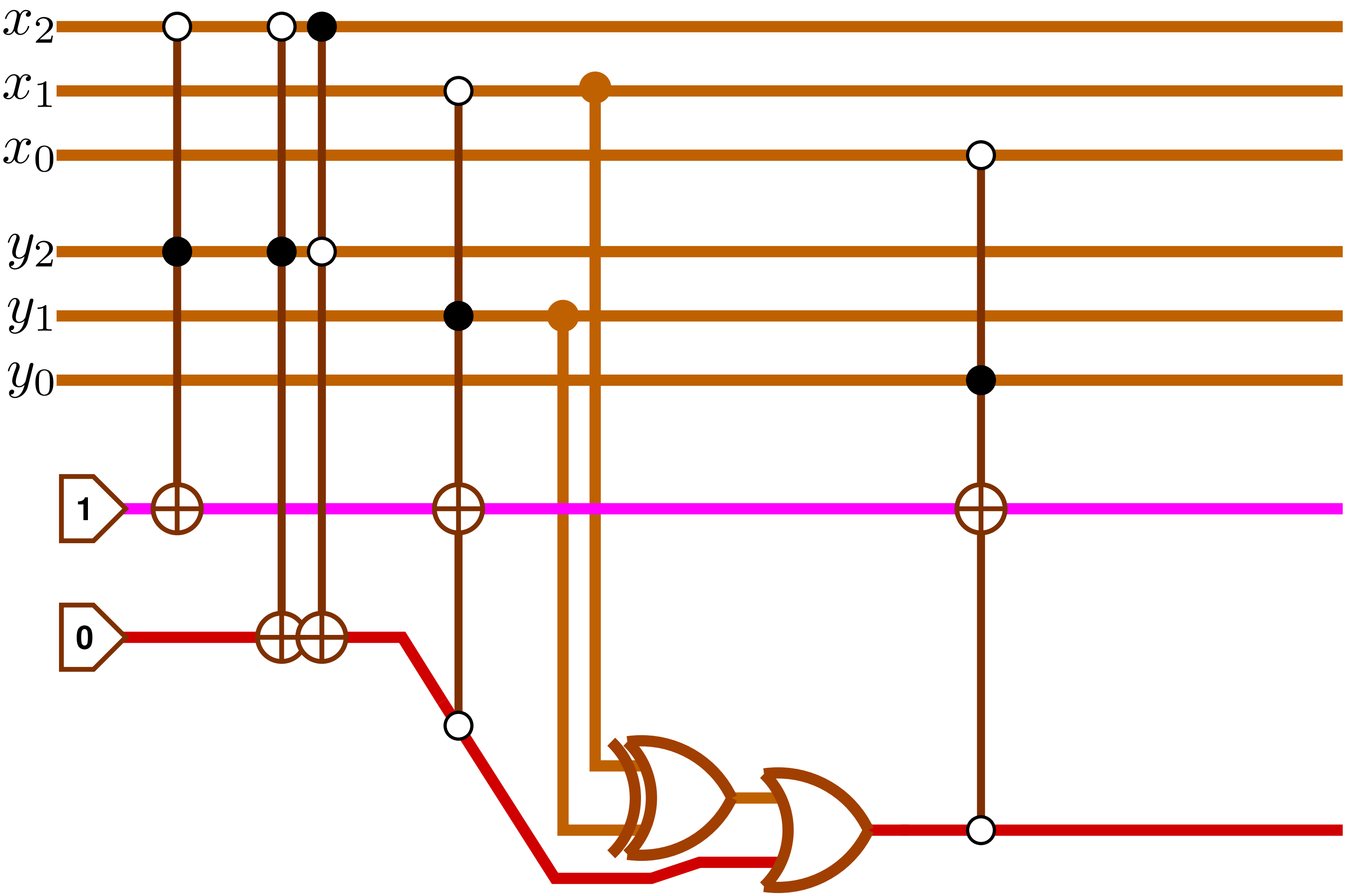

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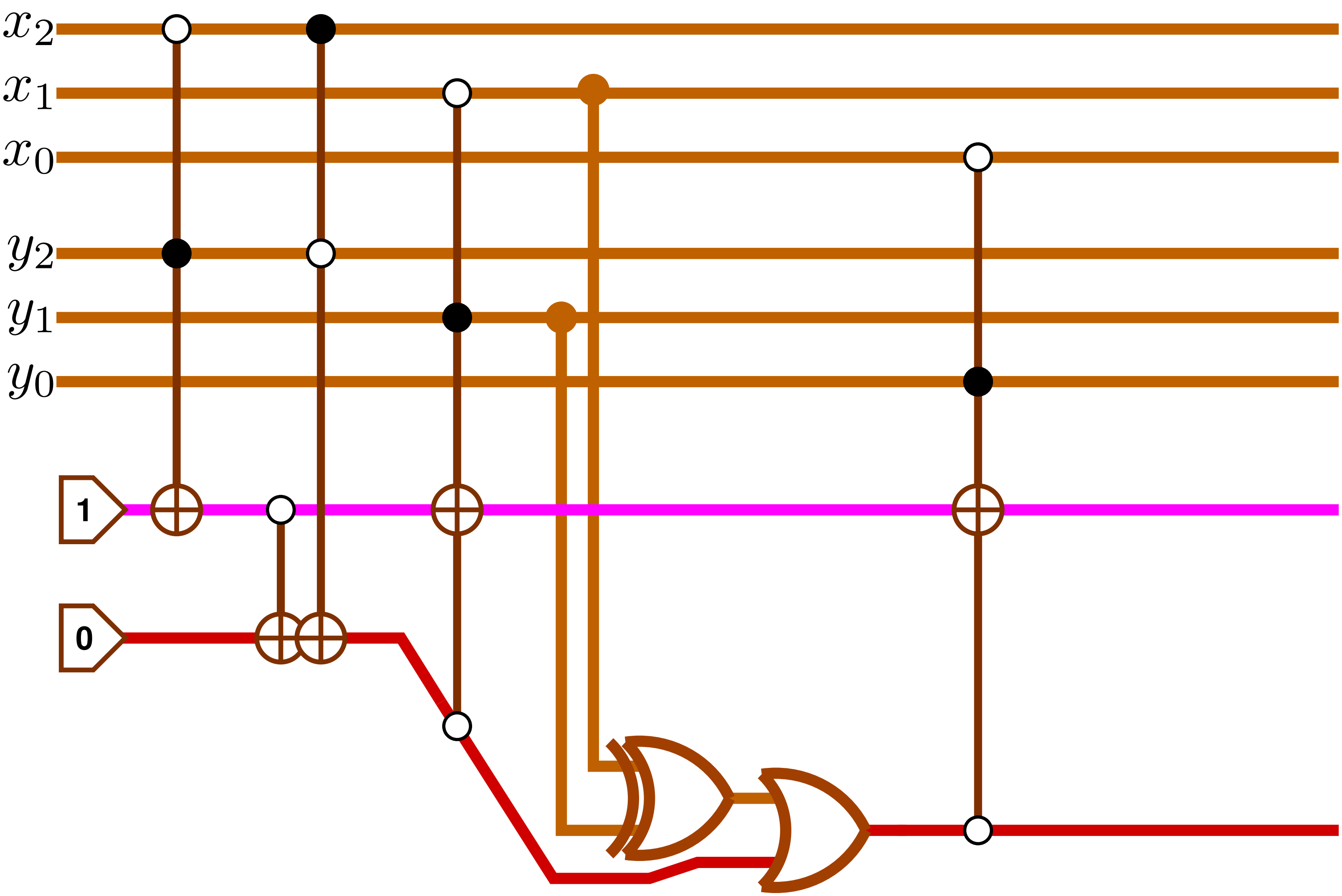

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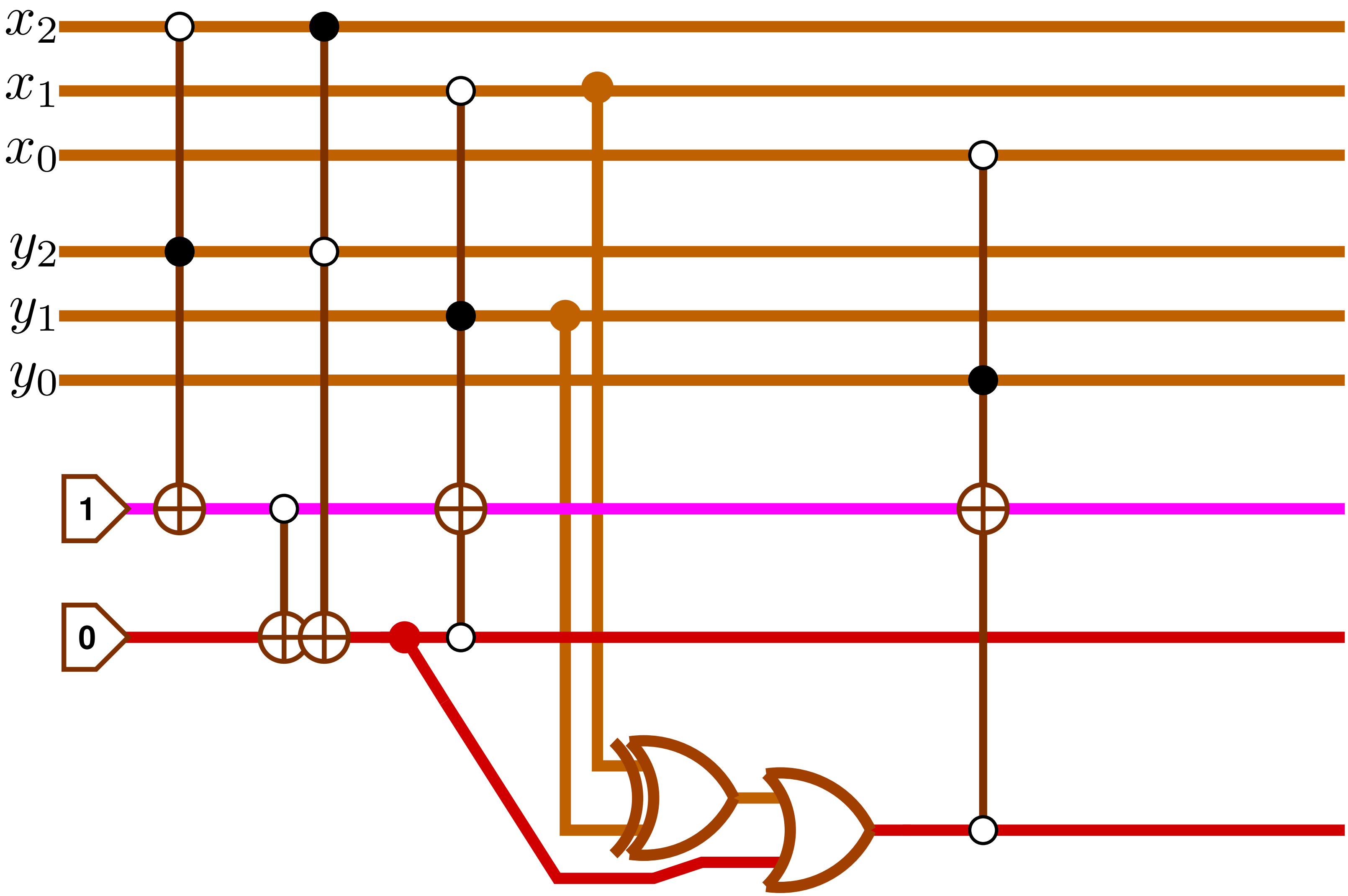

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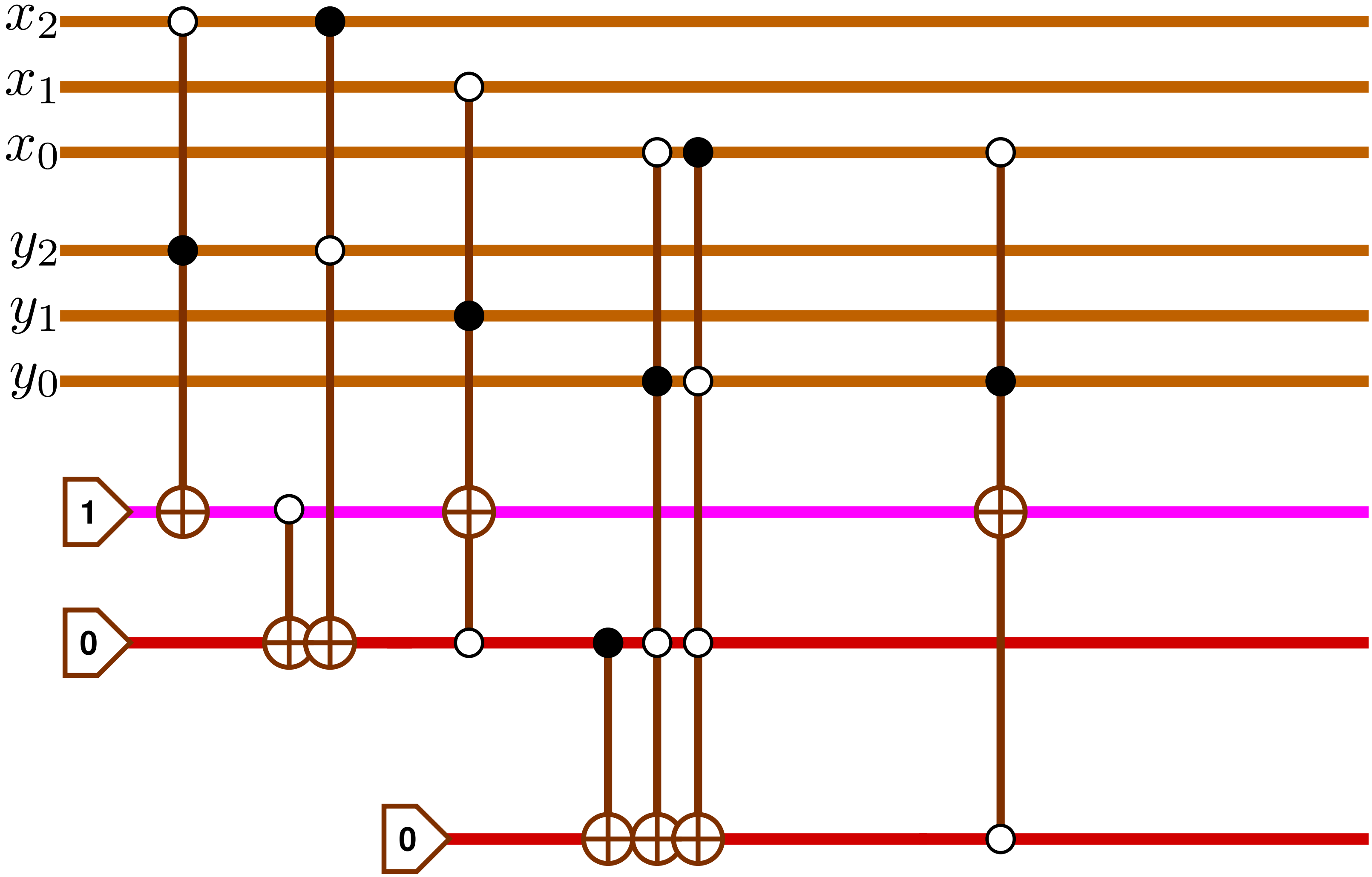
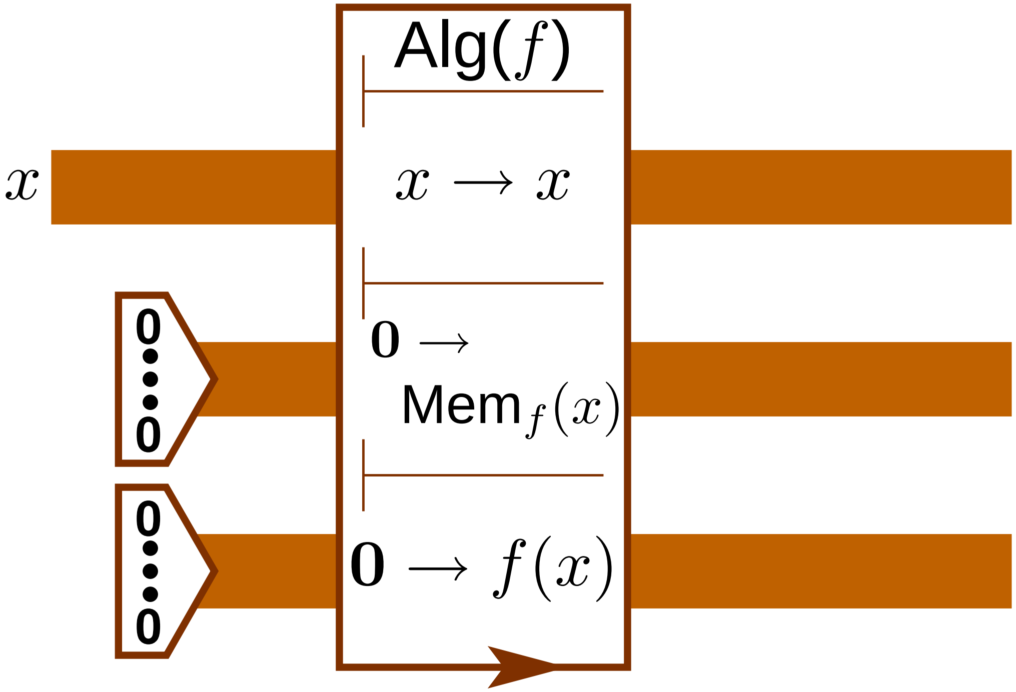


Erasing Memory

- Erasing memory by copying output and reversing.



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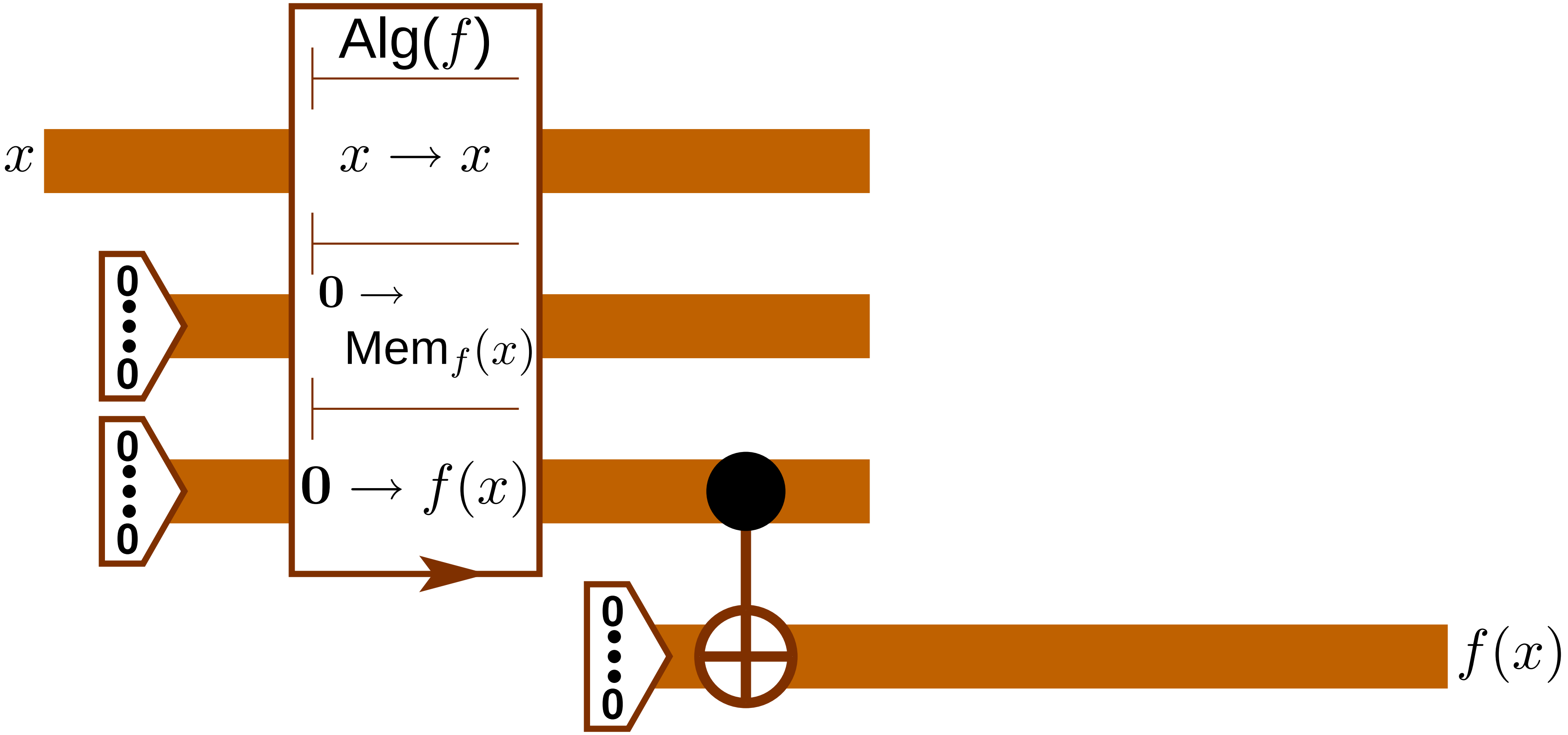
Alg(f)
x
$x \rightarrow x$
0
⋮
0
$\mathbf{0} \rightarrow$ Mem$_f(x)$
0
⋮
0
$\mathbf{0} \rightarrow f(x)$
0
⋮
0
$f(x)$

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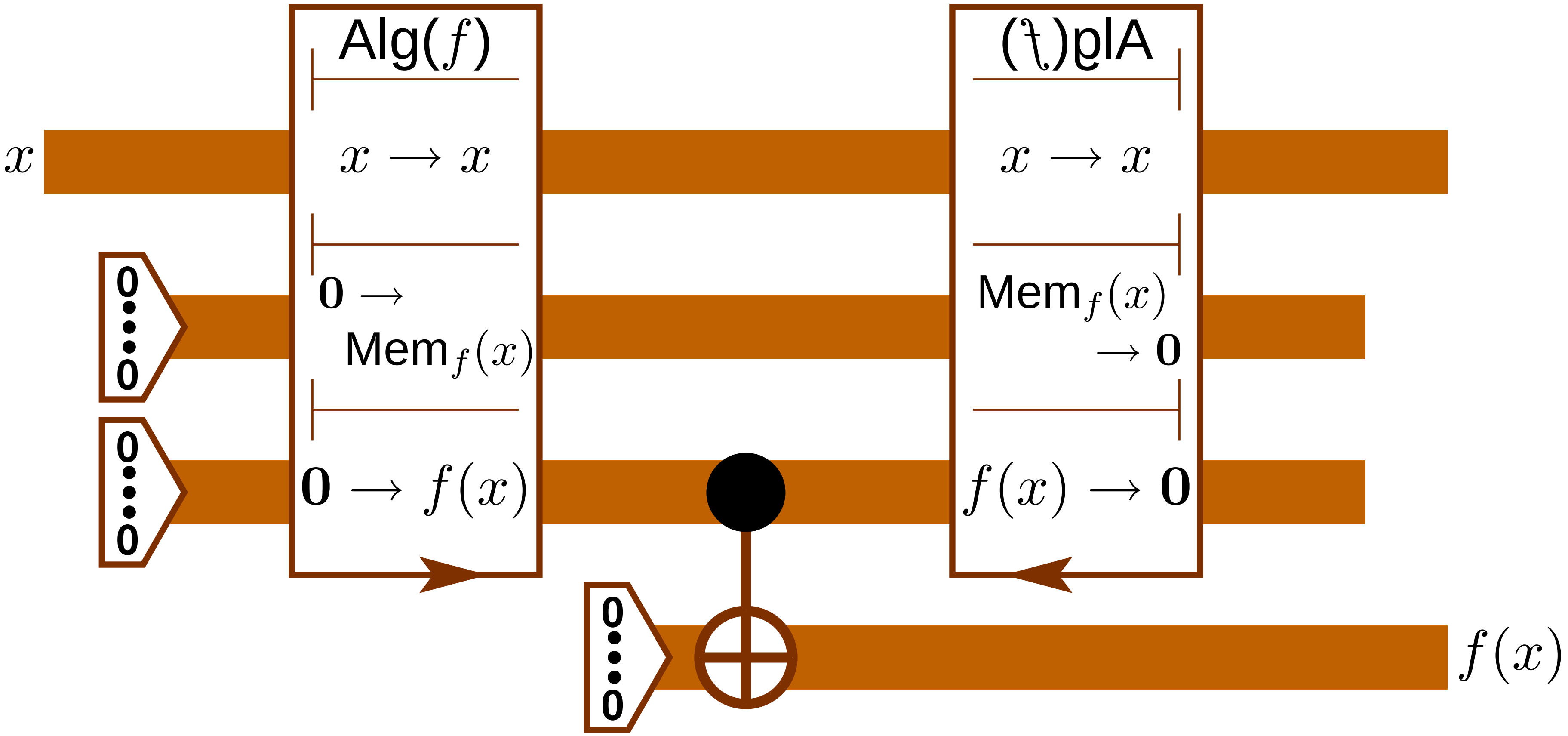

x

Alg(f)

$x \to x$

$\mathbf{0} \to$ Mem$_f(x)$

$\mathbf{0} \to f(x)$

Alg(f)

$x \to x$

Mem$_f(x) \to \mathbf{0}$

$f(x) \to \mathbf{0}$

$f(x)$

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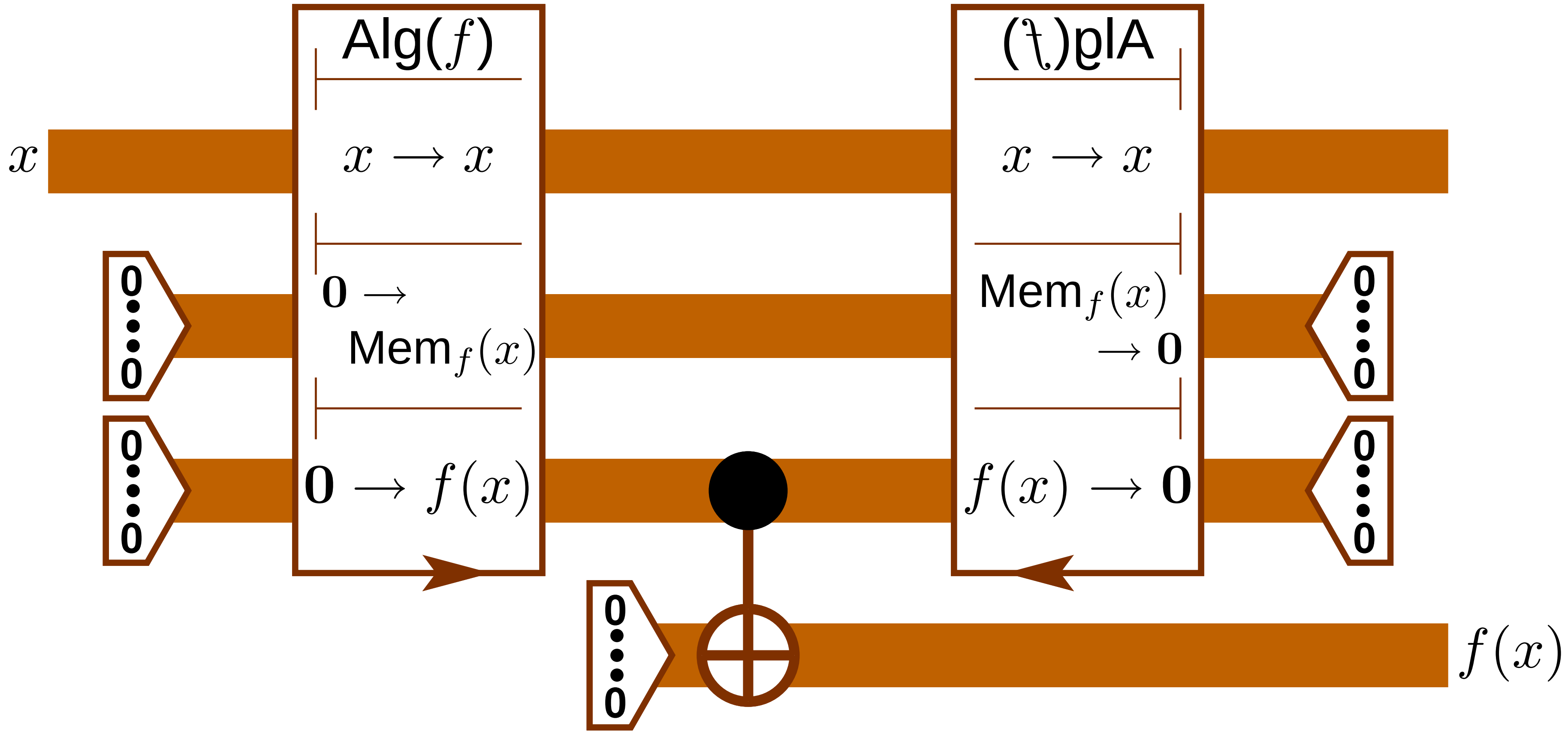

Alg(f)
x
$x \rightarrow x$
$\mathbf{0} \rightarrow$ Mem$_f(x)$
$\mathbf{0} \rightarrow f(x)$
0 • • • • 0
0 • • • • 0
0 • • • • 0
(f)glA
$x \rightarrow x$
Mem$_f(x)$ $\rightarrow \mathbf{0}$
$f(x) \rightarrow \mathbf{0}$
0 • • • • 0
0 • • • • 0
$f(x)$

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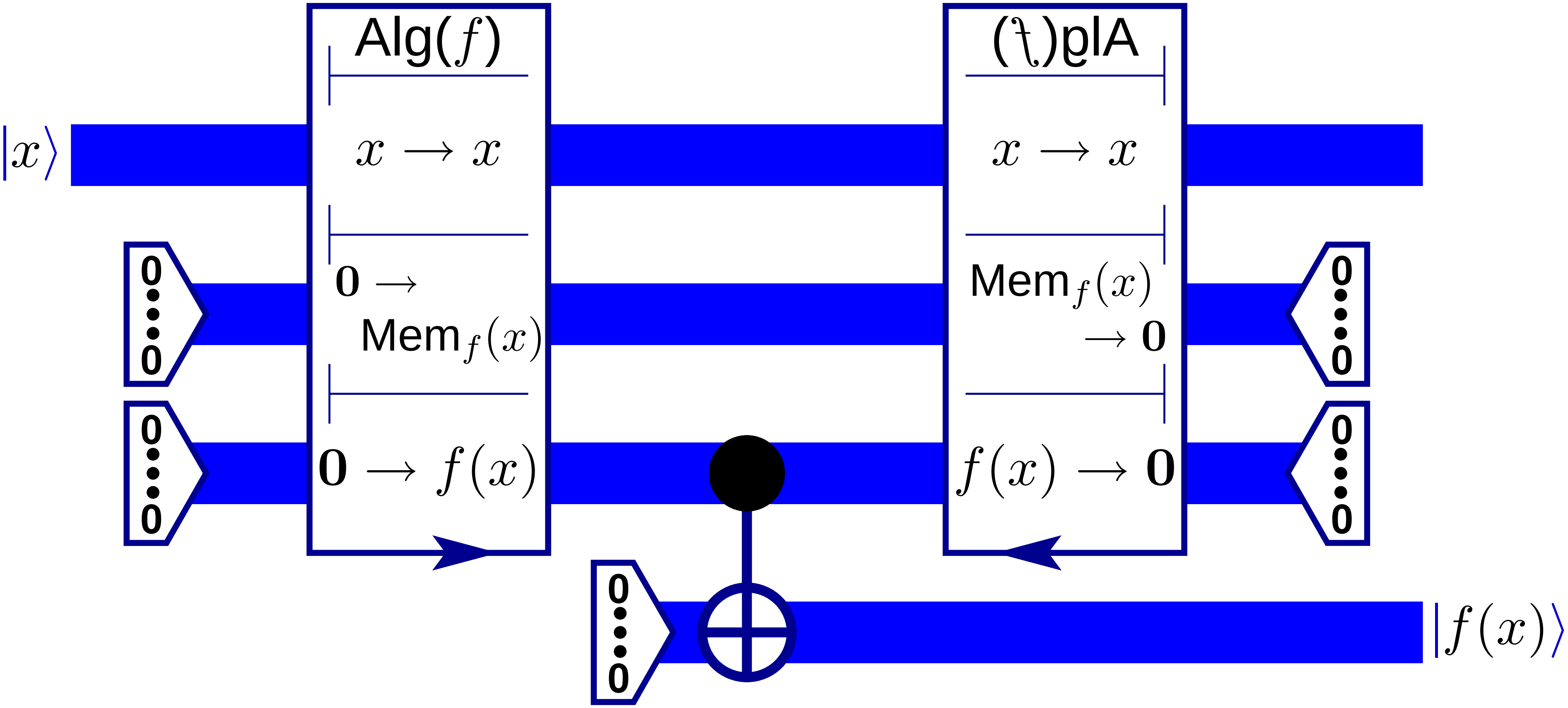

Alg(f)
(f)glA
$|x\rangle$
$x \rightarrow x$
$x \rightarrow x$
$\mathbf{0} \rightarrow$ Mem$_f(x)$
Mem$_f(x)$ $\rightarrow \mathbf{0}$
$\mathbf{0} \rightarrow f(x)$
$f(x) \rightarrow \mathbf{0}$
0 ⋮ 0
0 ⋮ 0
0 ⋮ 0
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0 ⋮ 0
$|f(x)\rangle$

- Classical reversible gates → quantum gates.

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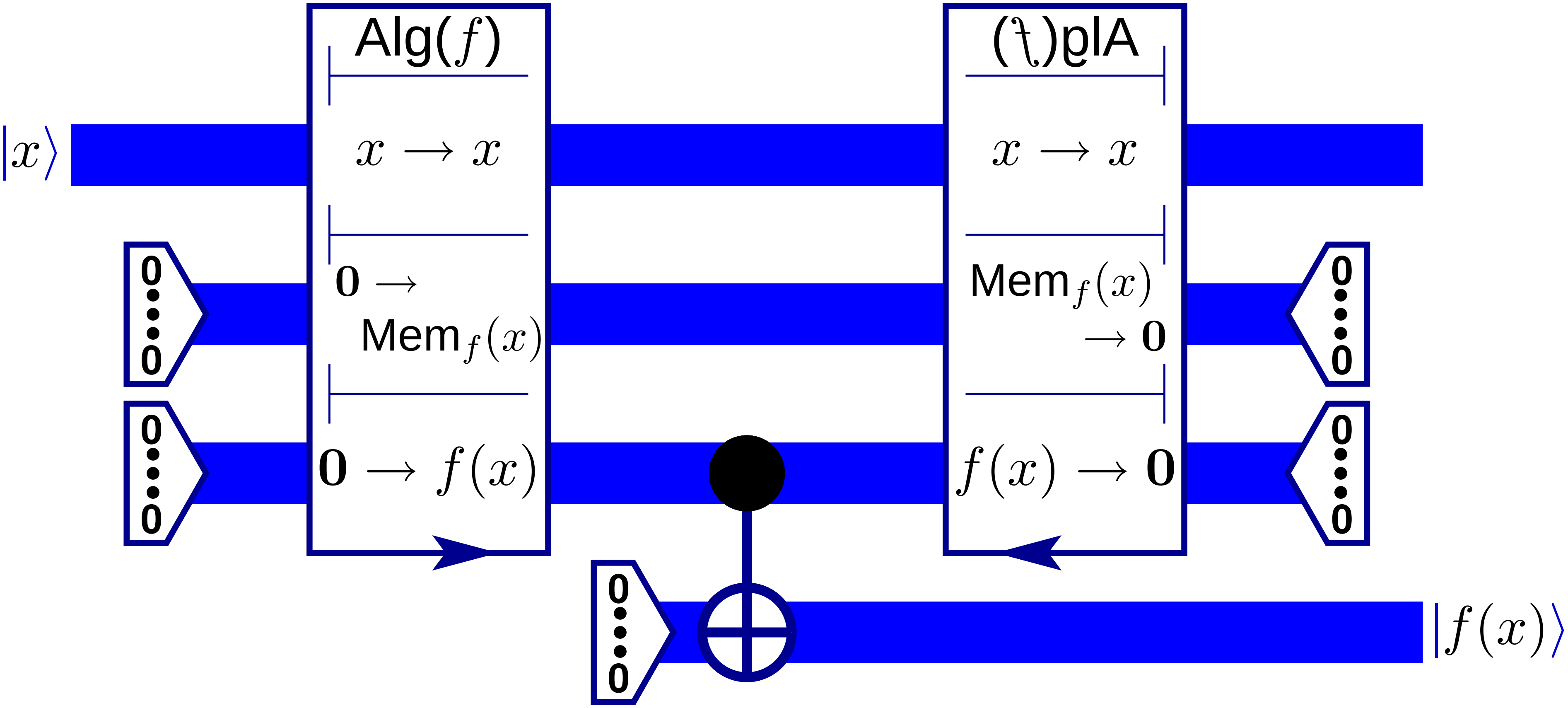

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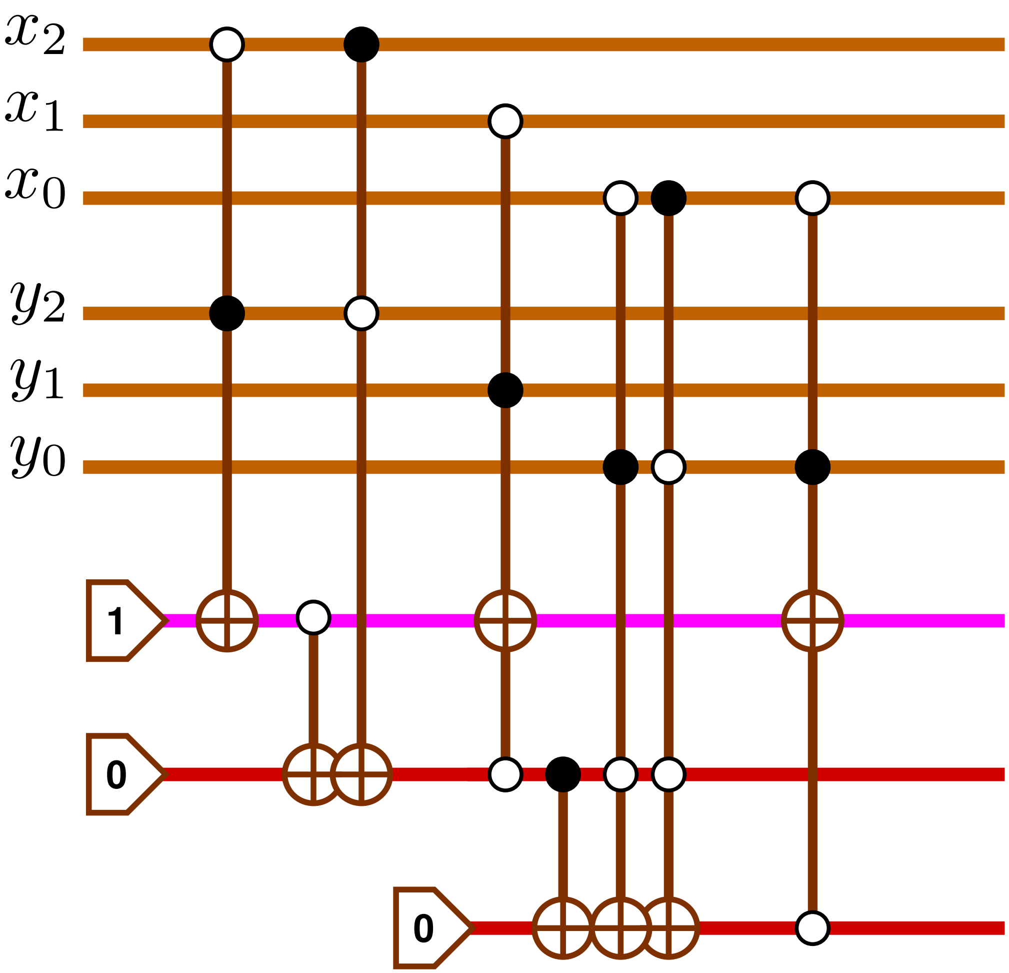
$$\sum_x \alpha_x |x\rangle_{\mathsf{I}} \rightarrow \sum_x \alpha_x |x\rangle_{\mathsf{I}} |f(x)\rangle_{\mathsf{O}}$$

Coherent Comparison



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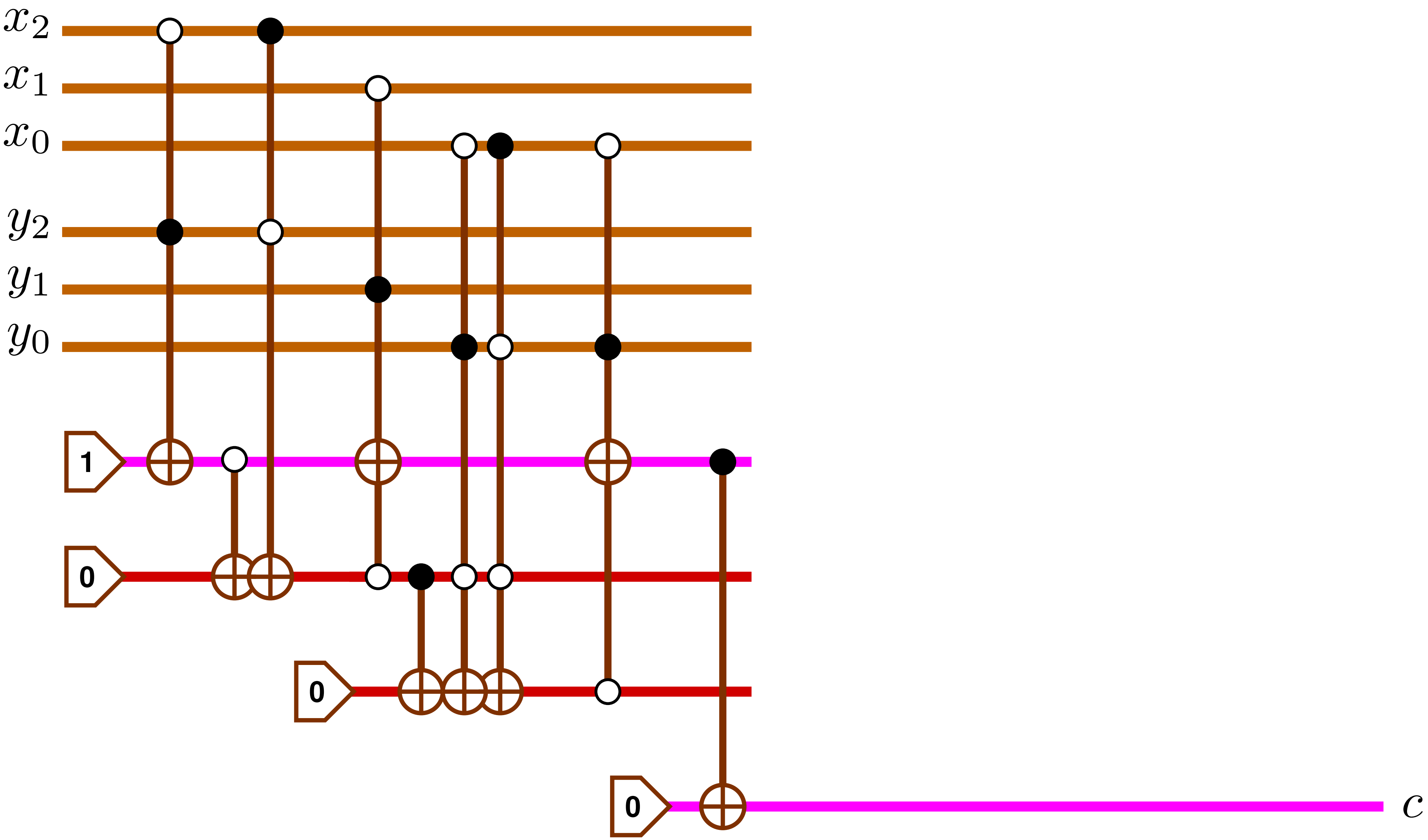

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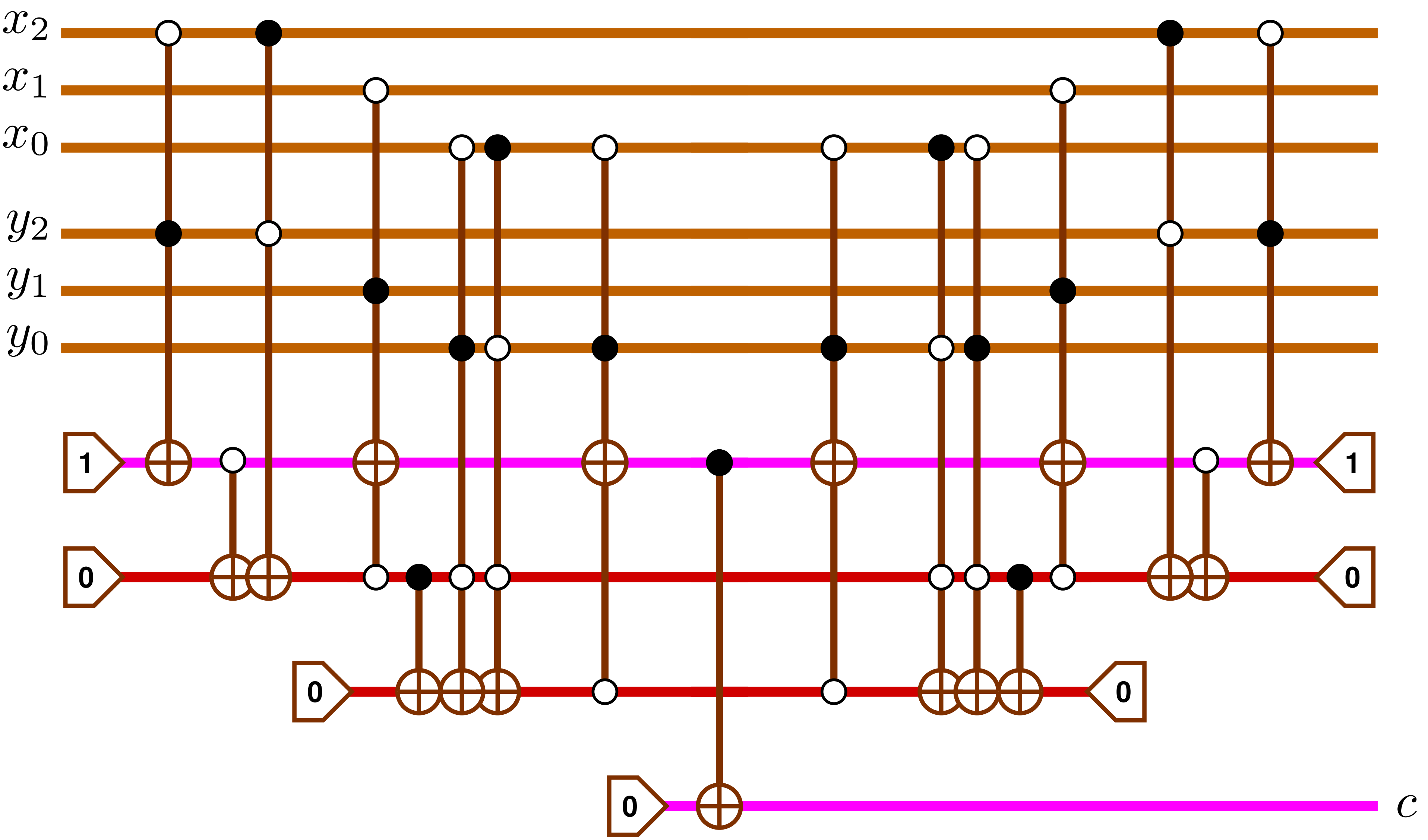

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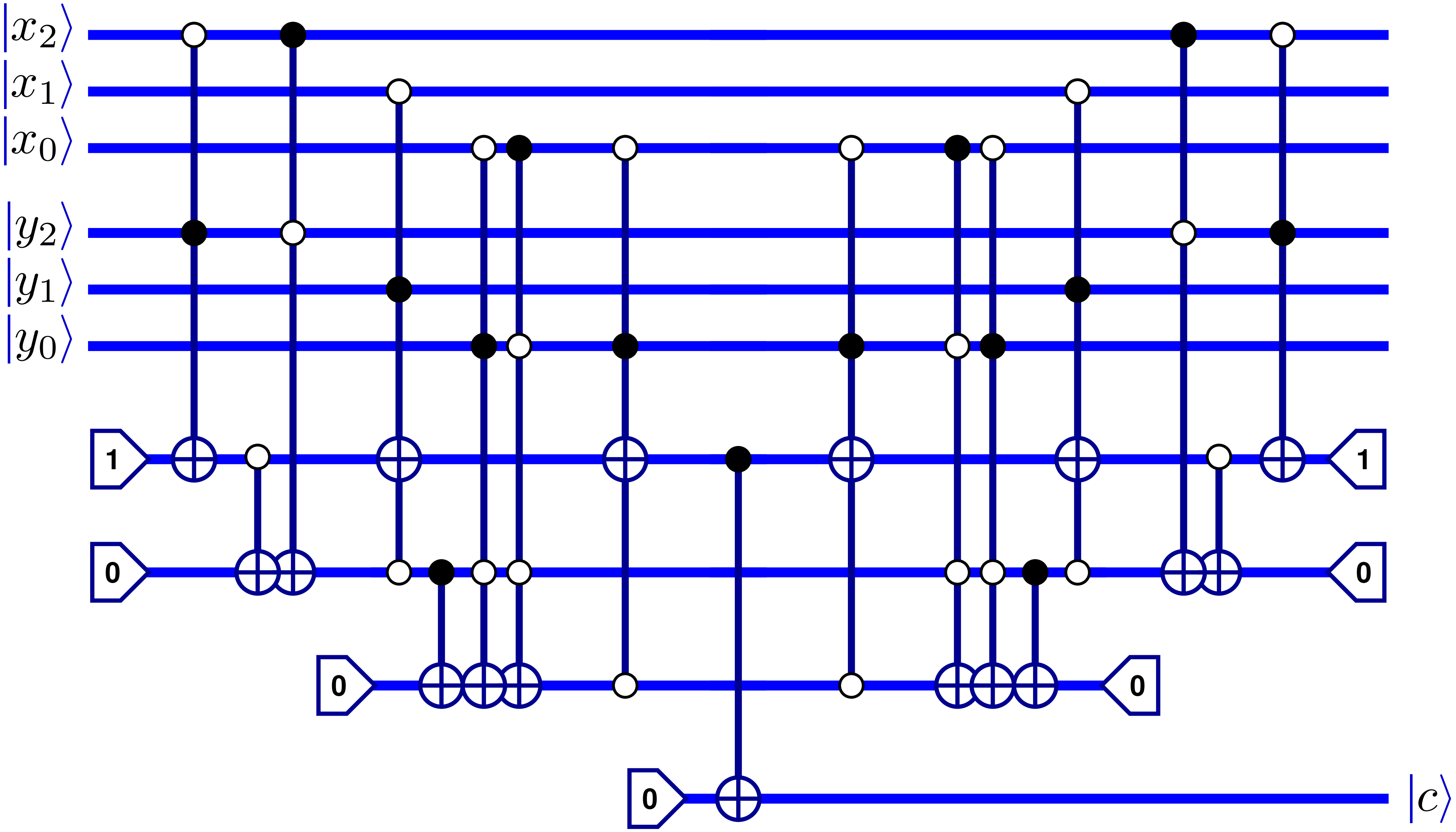

Invertible Functions

- When can one coherently implement $\sum_x \alpha_x|x\rangle \to \sum_x \alpha_x|f(x)\rangle$?

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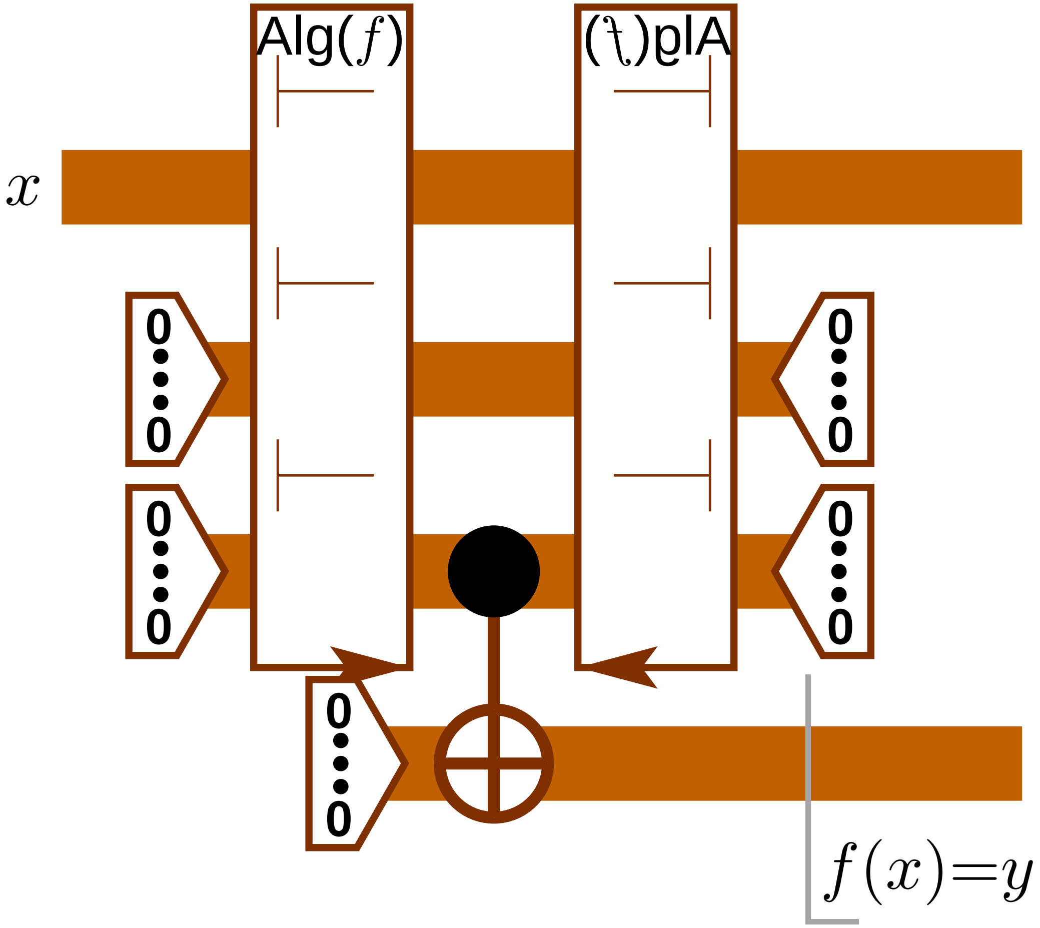

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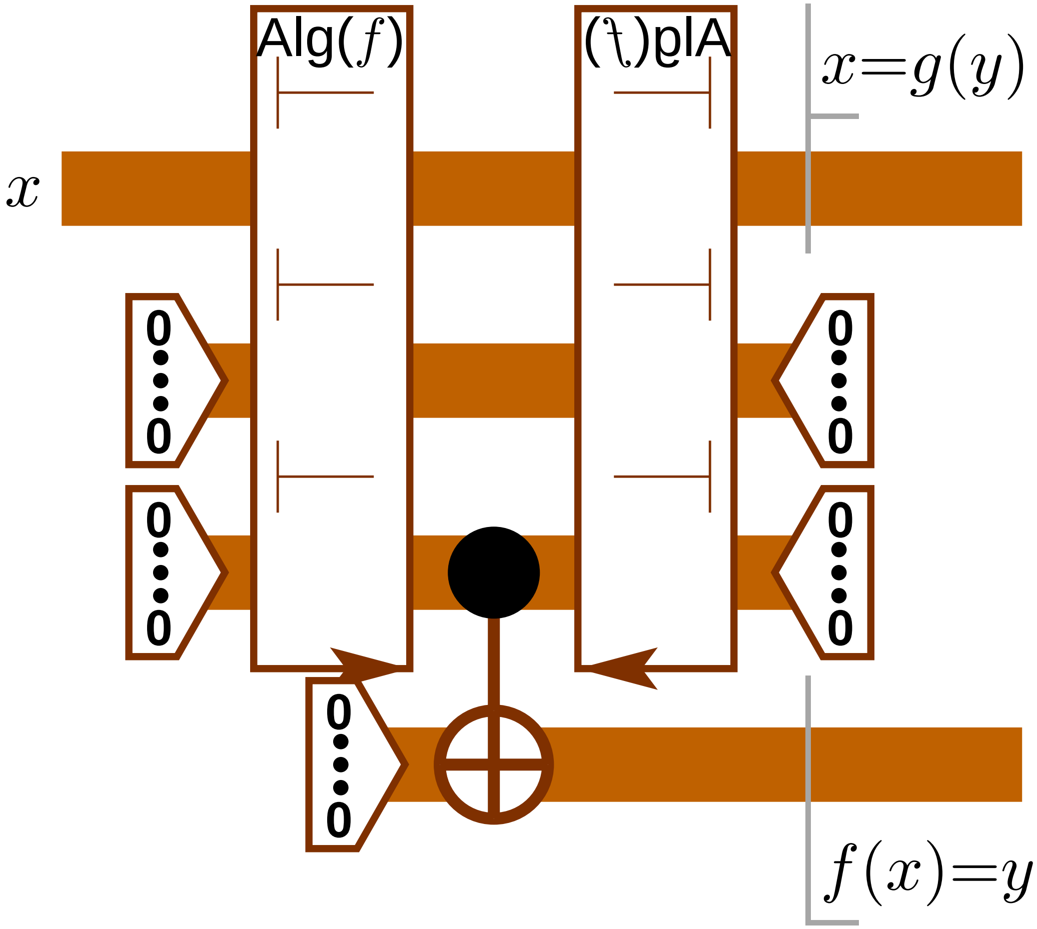

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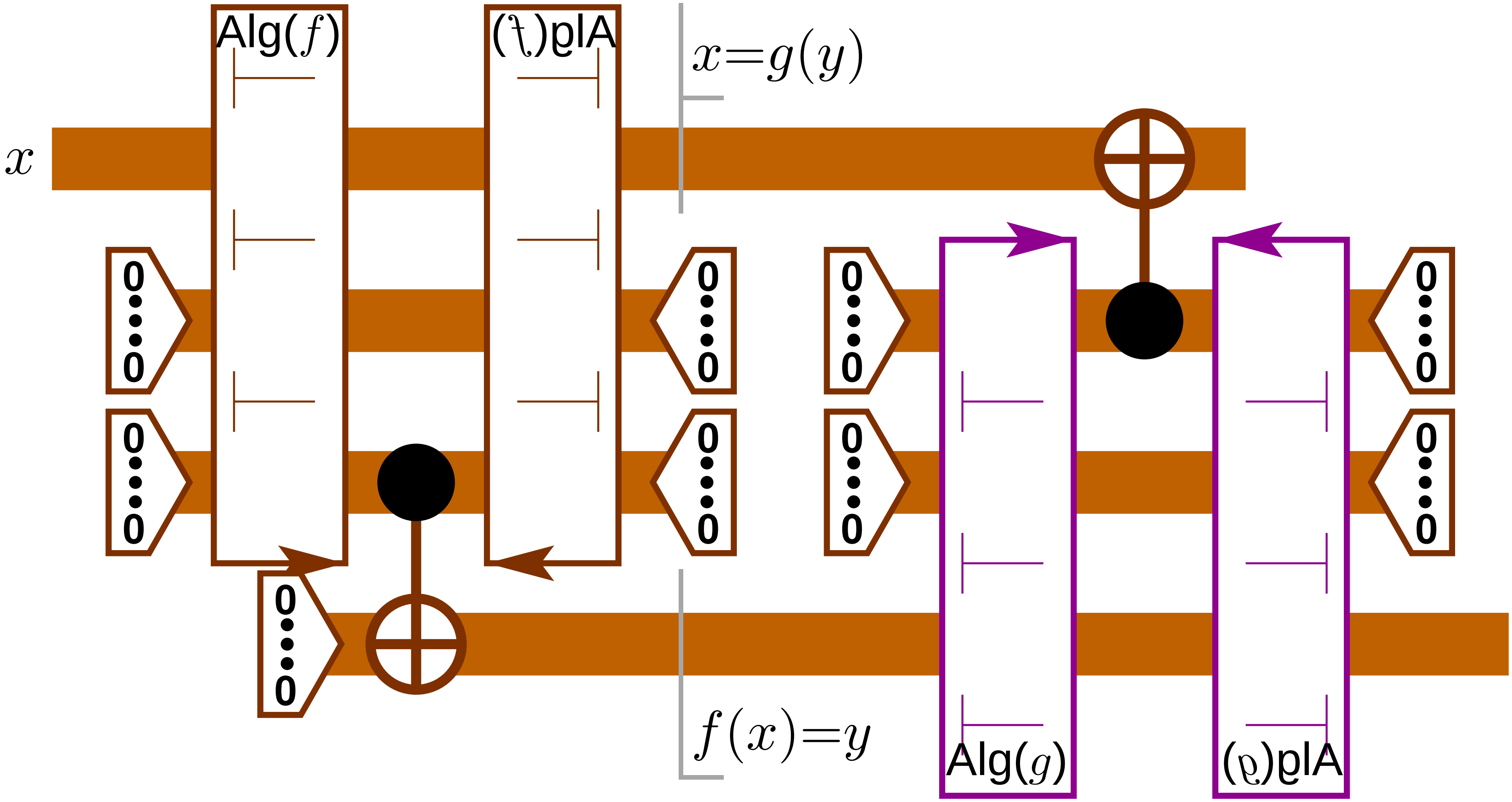

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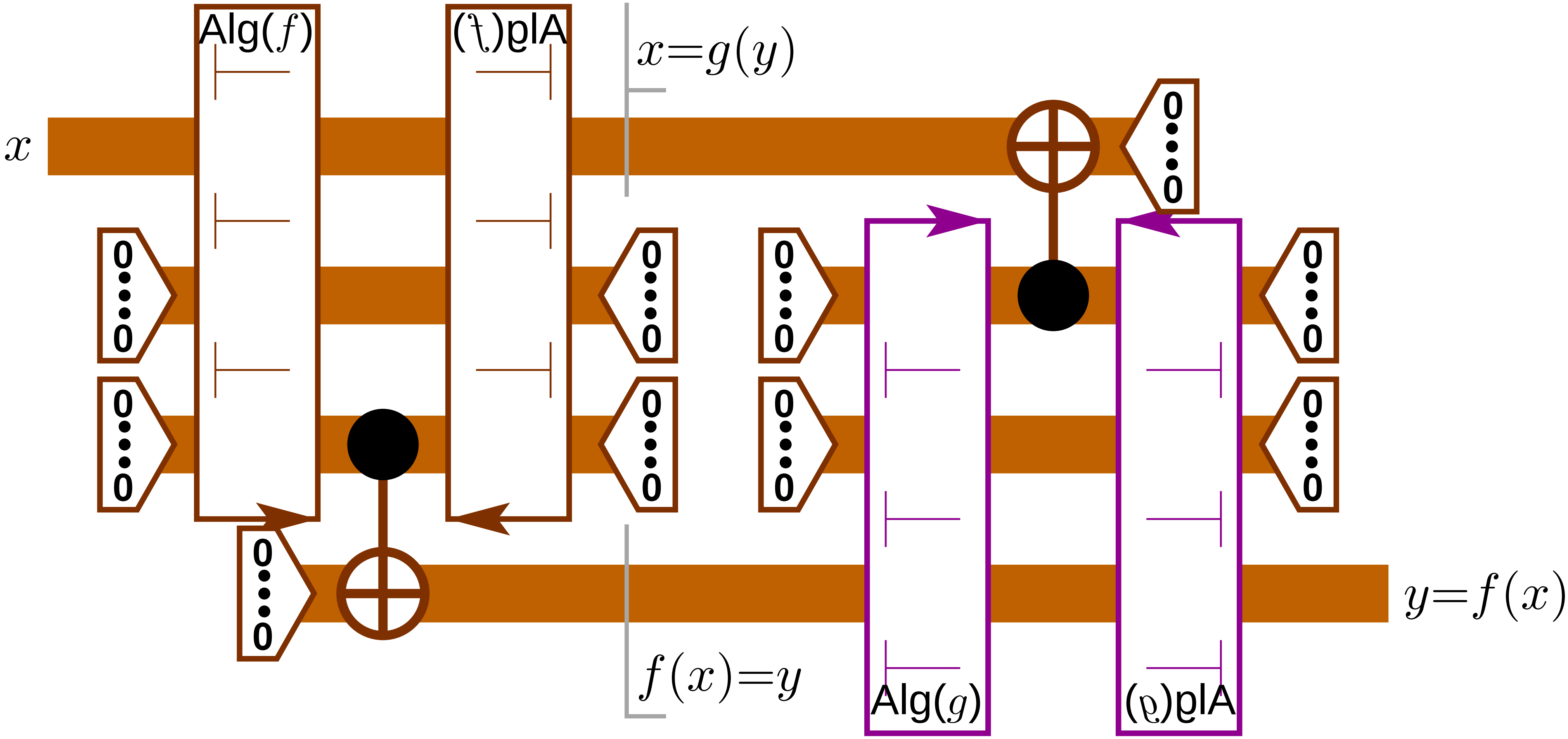

Contents

References

[1] C. H. Bennett, G. Brassard, S. Breidbart, and S. Wiesner. Quantum cryptography, or unforgeable subway tokens. In *Advances in Cryptology: Proceedings of Crypto'82*, pages 267–275. Plenum Press, 1982.

[2] C. H. Bennett. Time/space trade-offs for reversible computation. *SIAM J. Comput.*, 18:766–776, 1989.