Board of Governors of the Federal Reserve System

International Finance Discussion Papers

Number 716

December 2001

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VINTAGE CAPITAL GROWTH MODELS

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Abstract: Most growth models assume capital is homogeneous with regard to technology. This contradicts intuition and empirical evidence that the majority of technology is embodied in the capital stock. Berger (2001) showed that neoclassical vintage capital (embodied technology) and non-vintage capital (disembodied technology) models have different convergence rates, although identical steady state growth rates. Removing the neoclassical assumption that technological growth is exogenous, I examine two-sector, putty-putty, vintage capital models. Technological growth is tied to investment in the research sector. Savings rates and the allocation of labor differ between the vintage and non-vintage cases. It is shown for the first time that vintage and non-vintage versions of a model can have different steady state growth rates.

Keywords: productivity, technology,

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1. Introduction

Research on vintage capital models reached its zenith in the mid-1960's. Vintage capital research declined after the 1960's because of both technical difficulties inherent in the models and the conclusion of the papers that equilibrium growth rates are not affected by the choice of how technology is modeled. The classic papers of this era, Solow (1959), Phelps (1962), Phelps (1963), and Solow, Tobin, Weizsäcker, and Yarri (1966), examine neoclassical, non-optimization, growth models in which technological growth and the savings rate are exogenous parameters. Berger (2001) examines optimization versions of these classic papers, and showed that the vintage and non-vintage putty-putty models are characterized by substantially different transitional dynamics. The rate of σ-convergence, the rate at which per-capita output levels converge, and the rate of β-convergence, the rate at which the growth rate of output approaches the steady state growth rate, are significantly higher in putty-putty vintage capital models than their non-vintage counterparts. These differences are the result of the different optimal savings rates under the vintage and non-vintage regimes.

While the neoclassical framework illustrates potential differences arising from modeling technology as embodied (vintage capital) or disembodied (non-vintage capital), the assumption that the growth rate of technology is exogenous may be viewed as excessively restrictive. Improvements in technology often arise as the result of dedicated research. In this paper, I examine two-sector growth models in which there is a final goods sector and a research sector. The models are non-scale models in that the equilibrium growth rates of the economy are unaffected by the size or scale of the economy. The models are putty-putty, meaning that there is factor substitutability at both the time of installation of the capital and for all time thereafter.

Eicher and Turnovsky (1999, 2001) provide a general framework for examining non-scale balanced growth equilibria in two-sector models. I adapt their framework to the vintage capital models.

I examine two models that differ in the way technology is produced. The first model utilizes an AK production function in the research sector. In one-sector AK growth models, the equilibrium growth rate is the product of the savings rate and the exogenous parameter *A*, the constant marginal product of capital. AK models are therefore true endogenous growth models in which actors in the economy affect the long-run growth rates. The same type of result arises in the AK model of this paper. Different sectoral allocations of labor between the vintage and non-vintage capital cases lead to different equilibrium growth rates.

The second model is a vintage capital version of the general two-sector model used in Eicher and Turnovsky (1999, 2001). This model utilizes Cobb-Douglas production in the research sector. Because the equilibrium growth rates do not depend on choice variable, the growth rates are the same in the vintage and the non-vintage capital cases.[1] While the vintage and non-vintage versions have the same steady state growth rates, they do have different savings rates and sectoral allocations of labor. Therefore, the convergence rates in the two versions are likely to be different as they are in the neoclassical putty-putty model.

The paper is organized in the following manner. Section 2 discusses the common characteristics of the two models of the paper, and the modeling differences between the vintage and non-vintage cases. Section 3 is a discussion of the model with AK production in the research sector. Section 4 discusses the model with Cobb-Douglas production in the research

[1] . Jones (1995) referred to models in which the steady state growth rates are unaffected by agents' choices as "semi-endogenous" growth models.

sector. Section 5 contains concluding remarks. Appendix 1 examines the production maximization that occurs each period that leads to the formation of the aggregate capital stock and indirect production function.

2. Common Model Characteristics

The models are central planner problems in which the planner maximizes the discounted utility resulting from a flow of consumption. These are continuous-time optimal control problems. The utility function used is a constant-elasticity of intertemporal-substitution utility function:

$$U(C) = \begin{cases} \dfrac{1}{1-\gamma}\left(C^{1-\gamma} - 1\right) & \text{for } \gamma \neq 1 \\ \ln C & \text{for } \gamma = 1 \end{cases} \tag{1}$$

The objective function of the central planner is:

$$\text{Maximize} \quad \int_{t=0}^{\infty} e^{-\rho t} U\left(C(t)\right) dt \qquad \rho > 0, \quad \gamma > 0 \tag{2}$$

At each instant, the central planner makes a consumption/savings choice, and a labor allocation choice. The choice variables are the level of consumption, *C*, and the fraction of labor devoted to the final goods sector, *m*. Given the labor, capital, and technology stocks at an instant in time, the labor allocation choice determines the output in the research and final goods sectors. The consumption choice determines the allocation of final goods between consumption and investment. Final goods not consumed are converted into capital at no cost. The model could be equivalently written with the savings rate, rather than the level of consumption, as a choice variable.

Depreciation is modeled in this paper as the standard "iceberg" depreciation for both the capital stock and the technology stock. In other words, exogenously given fractions of the stocks, δ_A and δ_K, wear out each period. In vintage capital models, the method of depreciation is less trivial than in non-vintage models. Because capital is heterogeneous in vintage capital models, it must be specified from which vintages the depreciation takes place. In this paper, each vintage loses the same fraction of its stock. This method translates into a constant fraction of the aggregate capital stock depreciating each period. The formulation of the aggregate capital stock is discussed below and in the appendix. Results are given in the vintage and non-vintage cases for δ_A and δ_K positive, and for $\delta_A = \delta_K = 0$. The no depreciation case is important because it highlights the fact that in the vintage capital versions of the models, older vintages are utilized less (i.e. less labor is assigned to older vintages) for purely economic reasons rather than any physical one.

The essence of vintage capital models is that the capital stock is heterogeneous. Generally the heterogeneity stems from capital embodying different levels of technology that makes newer vintages more productive. This increase in productivity can take many forms. For example, in fixed factor models such as Phelps (1963) and Solow, Tobin, Weizäcker, and Yarri (1966), the increase in productivity is often modeled as a lower labor requirement per unit of capital, rather than an increase in potential output per unit of capital. This form of improvement is called purely labor enhancing technological change. In this paper, there is a spectrum of vintages, and final goods production utilizing each vintage of capital is represented by a unique production function:

$$Y_v = A_v^{\sigma_A} N_v^{\sigma_N} K_v^{\sigma_K} \tag{3}$$

Y_v is the rate of output at an instant in time utilizing vintage *v*. A_v is the technology inherent in vintage *v*. σ_A, σ_N, and σ_K are the exogenous elasticities associated with the subscripted factor. N_v is the labor assigned for use with capital of vintage *v*. Unlike in the fixed factor case, the multiplicative nature of the Cobb-Douglas production function precludes differentiation between "labor" enhancing and "capital" enhancing technological change. Letting *m* be the fraction of labor assigned to the final goods sector, then $\int_{v=-V}^{t} N_v dv = mN$, where V is the number of vintages available at *t*=0.

In equation (4), K_v is the amount of capital of vintage *v* used. K_v at time *t* is equal to the investment corresponding to vintage *v* (or the exogenously given initial capital), discounted for the depreciation which has occurred since that vintage's installation:

$$K_v = I_v e^{-\delta_K (t-v)} \text{ for } 0 < v \le t \quad , \qquad K_v = \bar{K}_v e^{-\delta_K t} \text{ for } v \le 0 \tag{4}$$

The total amount of capital at time *t* is:

$$K(t) = \int_{v=0}^{t} I_v e^{-\delta_K (t-v)} dv + \int_{v=-V}^{0} \bar{K}_v e^{-\delta_K t} dv \tag{5}$$

where $\bar{K}_v$ is the given amount of each vintage at *t*=0. Taking the derivative of (5) with respect to time gives the standard capital accumulation equation:

$$\dot{K} = I - \int_{v=0}^{t} \delta_K I_v e^{-\delta_K (t-v)} dv - \int_{v=-V}^{0} \delta_K \bar{K}_v e^{-\delta_K t} dv = I - \delta_K K \tag{6}$$

Total output is $Y = \int_{v=-V}^{t} Y_v dv$. At each instant, output is maximized given the stocks of technology, capital, and labor assigned to the final goods sector. This maximization involves allocating labor to each vintage of capital (Appendix 1). The optimal allocation of labor resulting from this maximization enables the use of an indirect total production function *Y*, utilizing an aggregate capital stock, *Q*. Solow (1959) was the first paper to utilize such an aggregate stock. The indirect production function is:

$$Y = (mN)^{\sigma_N} Q^{1-\sigma_N} \tag{7}$$

where

$$Q = \int_{v=-V}^{t} A_v^{\frac{\sigma_A}{1-\sigma_N}} K_v^{\frac{\sigma_K}{1-\sigma_N}} dv \tag{8}$$

It is interesting to note that the indirect production function utilizing the aggregate capital stock exhibits constant returns to scale, regardless of the elasticities specified in the production function of each vintage. However, the elasticities of technology and capital appear in the new capital accumulation equation.

Because technology appears in (8), it is necessary to discuss depreciation of technology before deriving the aggregate capital accumulation equation. Unlike the capital stock, which is heterogeneous, there is a single technology stock for the economy. This technology stock can also be thought of as a stock of knowledge. The level that the technology stock attains at each point in time (corresponding to a vintage) is designated by A_v. In this paper, technology depreciation only appears in the technology accumulation equation, even though the technology stock appears in the definition of the aggregate capital stock. The rationale is that in order to keep the stock of knowledge constant, there must be some minimal investment in research in order to maintain skills. This rationale does not imply, however, that the knowledge or technology embodied by an existing unit of capital somehow depreciates within that capital. The actual physical depreciation of capital is accounted for by δ_K, and therefore it is not necessary to take into account any additional loss of aggregate capital that occurs due to technology decreasing in the previously obtained stock. Using (4) and taking the time derivative of (8):

$$\dot{Q} = A^{\frac{\sigma_A}{1-\sigma_N}} I^{\frac{\sigma_K}{1-\sigma_N}} - \left(\frac{\sigma_K}{1-\sigma_N}\right)\delta_K Q \tag{9}$$

$\dot{Q}$ is the instantaneous change in the aggregate capital stock with respect to time. A is the level of technology represented in the newest capital. If all capital utilizes the latest technology, then from (3), the indirect total production function is simply:

$$Y = A^{\sigma_A}(mN)^{\sigma_N} K^{\sigma_K} \tag{10}$$

and (6), the standard accumulation equation, is used.

3. Model #1: AK Research Sector

In the standard neoclassical growth model, technological growth is exogenous and increasing subject to a constant growth rate. The most natural extension of this phenomenon to a two-sector model is the adoption of "AK" technology in the research sector:

$$\dot{A} = (1-m)\overline{g}A - \delta_A A \tag{11}$$

where $\overline{g}$ is an exogenous parameter, and δ_A is the depreciation rate of technology. The notation is a little confusing because the "K" in "AK" refers to the stock, which in the research sector of this paper is designated by "A." The "A" in "AK" refers to the productivity of the stock. In this paper, that term is the product $(1-m)\overline{g}$. This is a natural extension of the neoclassical model because if *m*, the sectoral allocation of labor, is exogenous as opposed to being a choice variable, then the model reduces to the neoclassical model with an exogenous growth rate of technology.

Dividing both sides of (11) by A:

$$g_A = \frac{\dot{A}}{A} = (1-m)\bar{g} - \delta_A \tag{12}$$

From (12), it can be seen that the labor allocation choice determines the growth rate of technology, and that, ignoring depreciation, $\bar{g}$ is the rate of technological growth if all labor is applied to research $(m = 0)$.

There are intuitive reasons why this is an appropriate form for the research sector. While the central planner may choose the growth rate of technology, the choice is bounded. This has intuitive appeal in that one can imagine that the growth of knowledge may be bounded for a given time period. Basic research often takes time in order to be applied as shown in Adams (1990). Also, new research builds on previous research. The requirement that many discoveries must be found sequentially combined with empirically found gestation periods, limits the rate at which new ideas can be discovered. $\bar{g}$ can be used to represent this upper bound.

For tractability and to coincide with the previous literature, this section of the paper will focus on the case in which final goods production exhibits constant returns to scale in labor and capital $(\sigma_N + \sigma_K = 1)$, and exhibits constant returns in technology $(\sigma_A = 1)$. These are the parameter values used in Phelps (1962) and Berger (2001).

Non-Vintage Case

In the non-vintage case, the central planner's problem can be summarized by the following four equations: the objective function, the final goods production function, the technology accumulation equation, and the capital accumulation equation respectively:

$$\frac{1}{1-\gamma}\int_{t=0}^{\infty} C^{1-\gamma}e^{-\rho t}dt \qquad \rho > 0, \qquad \gamma > 0 \tag{13}$$

$$Y = A(mN)^{1-\sigma_K} K^{\sigma_K} \qquad 0<\text{m}<1, \quad 0<\sigma_K < 1 \tag{14}$$

$$\dot{A} = (1-m)\overline{g}A - \delta_A A \qquad 0 < \overline{g} < 1, \qquad \delta_A \geq 0 \tag{15}$$

$$\dot{K} = Y - C - \delta_K K = A(mN)^{1-\sigma_K} K^{\sigma_K} - C - \delta_K K \qquad \delta_K \geq 0 \tag{16}$$

The optimality and transversality conditions resulting from the central planner problem can be summarized by the following equations and (14)-(16):

$$C^{-\gamma} = \Lambda \tag{17}$$

$$\frac{\Lambda}{\Phi} = \frac{\overline{g}}{(1-\sigma_K)m^{-\sigma_K} N^{1-\sigma_K} K^{\sigma_K}} \tag{18}$$

$$(1-m)\overline{g} - \delta_A + m^{1-\sigma_K} N^{1-\sigma_K} K^{\sigma_K} \frac{\Lambda}{\Phi} = \rho - \frac{\dot{\Phi}}{\Phi} \tag{19}$$

$$\sigma_K A m^{1-\sigma_K} N^{1-\sigma_K} K^{\sigma_K - 1} - \delta_K = \rho - \frac{\dot{\Lambda}}{\Lambda} \tag{20}$$

$$\lim_{t\to\infty} e^{-\rho t}\Phi A = \lim_{t\to\infty} e^{-\rho t}\Lambda K = 0 \tag{21}$$

Φ and Λ are the shadow prices of technology and capital respectively. (17) equates the marginal utility of consumption to the shadow value of capital. This makes intuitive sense since a unit of consumption can be traded one-to-one for a unit of capital. Because laobr is homogeneous, (18) equates the marginal value of fractional labor in the technology sector with that in the final goods sector. (19) and (20) equate the returns to consumption with the returns to technology and capital, respectively.

I define a balanced growth path be one in which i) all stocks and flows grow at a constant rate, ii) *m* is constant, and iii) the savings rate, *s*, is constant. From these conditions and the production functions, the balanced growth rates of the primal variables can be determined. Repeating equation (12):

$$g_A = \frac{\dot{A}}{A} = (1-m)\overline{g} - \delta_A \tag{22}$$

From (16), $\dot{K} = sY - \delta_K K$, so $g_K = \frac{\dot{K}}{K} = s\frac{Y}{K} - \delta_K$. In the steady state, *s*, g_K, and δ_K are constant so $g_Y = g_K$. Dividing (16) by *K*, and taking the time derivative:

$$g_K = g_Y = g_C = \frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{1}{1-\sigma_K} g_A + n = \frac{(1-m)}{1-\sigma_K}\overline{g} - \frac{\delta_A}{1-\sigma_K} + n \tag{23}$$

(23) is very similar to the growth rate in the neoclassical one-sector model. The difference is only that g_A would be replaced by the exogenous growth rate of technology.

The growth rates of the shadow prices can be found by using (22) and (23) in combination with the optimality conditions. Taking the time derivative of (17):

$$g_\Lambda = \frac{\dot{\Lambda}}{\Lambda} = -\gamma g_Y = -\frac{\gamma(1-m)}{1-\sigma_K}\overline{g} + \frac{\gamma\delta_A}{1-\sigma_K} - \gamma n \tag{24}$$

Taking the time derivative of (18):

$$g_\Phi = \frac{\dot{\Phi}}{\Phi} = (1-\sigma_K)n + \sigma_K g_K + g_\Lambda = \frac{(\sigma_K-\gamma)(1-m)}{(1-\sigma_K)}\overline{g} - \frac{(\sigma_K-\gamma)\delta_A}{(1-\sigma_K)} + (1-\gamma)n \tag{25}$$

It can be shown that:

$$g_\Lambda - g_\Phi = g_A - g_Y \tag{26}$$

Using the above growth rates, it is possible to write the optimality conditions in terms of stationary variables. Let $\beta_Y = g_Y / n$, $\beta_A = g_A / n$, $\beta_\Phi = g_\Phi / n$, and $\beta_\Lambda = g_\Lambda / n$. Then on the equilibrium growth path the following are stationary variables: $c = C / N^{\beta_Y}$, $a = A / N^{\beta_A}$, $k = K / N^{\beta_Y}$, $\phi = \Phi / N^{\beta_\Phi}$, and $\lambda = \Lambda / N^{\beta_\Lambda}$. Substituting the stationary variables into the optimality conditions, final goods production, and the accumulation equations:

$$c^{-\gamma} = \lambda \tag{27}$$

$$\frac{\lambda}{\phi} = \frac{\bar{g} m^{\sigma_K}}{(1-\sigma_K) k^{\sigma_K}} \tag{28}$$

$$(1-m)\bar{g} - \delta_A + m^{1-\sigma_K} k^{\sigma_K} \frac{\lambda}{\phi} = \rho - g_\Phi \tag{29}$$

$$\sigma_K a m^{1-\sigma_K} k^{\sigma_K - 1} - \delta_K = \rho - g_\Lambda \tag{30}$$

$$y = a m^{1-\sigma_K} k^{\sigma_K} \tag{31}$$

$$g_A = (1-m)\bar{g} - \delta_A \tag{32}$$

$$g_K = \frac{y}{k} - \frac{c}{k} - \delta_K \tag{33}$$

Substituting (28) into (29), I obtain an equation in which *m* is the only choice variable. Solving for *m*:

$$m = \frac{(1-\sigma_K)(\rho + \gamma n - n) + (\bar{g} - \delta_A)(\gamma - 1)}{\bar{g}\gamma} \tag{34}$$

Analyzing (34) provides insight into what parameter values will provide an interior solution ($0<m<1$). The results are shown in (35) and (36):

$$\bar{g} > (1-\sigma_K)(\rho + \gamma n - n) - (\gamma - 1)\delta_A \quad \Rightarrow \quad m < 1 \tag{35}$$

$$\left.\begin{array}{ll}\gamma<1, & \bar{g}<\dfrac{(1-\sigma_K)(\rho+\gamma n-n)-\delta_A(\gamma-1)}{1-\gamma} \\ \gamma=1 & \\ \gamma>1, & \bar{g}>\dfrac{(1-\sigma_K)(\rho+\gamma n-n)-\delta_A(\gamma-1)}{1-\gamma}\end{array}\right\}\Rightarrow \quad m>0 \tag{36}$$

(35) is a statement that the addition to growth of sacrificing labor for research must be sufficiently high if there is to be sustained sacrifice in the long run. Given $\gamma \geq 1$, (36) will hold for any $\bar{g} > \delta_A$. If $\gamma < 1$, there is always a range of $\bar{g}$ for which (36) holds, but the size of the range goes to zero as $\gamma \to 0$.

Table 1 gives the benchmark parameter values for this model, while Tables 2 and 3 provide the values of *m* for a range of γ and $\bar{g}$. Table 2 contains the results using the benchmark parameters and Table 3 contains the results for when there is no depreciation. The benchmark parameters are substantially the same as those used in the literature. The production elasticities are the same as those in Berger (2001), while the depreciation rates are equal to those used in Eicher and Turnovsky (1999, 2001). As would be expected, *m* is decreasing in $\bar{g}$ and increasing in γ.

$$\frac{\partial m}{\partial \bar{g}} = -\frac{1}{(\bar{g}\gamma)^2} < 0 \tag{37}$$

$$\frac{\partial m}{\partial \gamma} = \frac{\left[(1-\sigma_K)n + (\bar{g}-\delta_A)\right](\gamma-1)\bar{g}}{(\bar{g}\gamma)^2} \geq 0 \qquad \text{if} \quad \gamma \geq 1, \ \ \bar{g} > \delta_A \tag{38}$$

(37) and (38) are expected because increasing $\bar{g}$ increases the return to allocating labor to the research sector, and increasing γ increases the relative return to earlier consumption. Tables 6 and 7 show the growth rate of output corresponding to the levels of *m*. All of the tables

in this paper corresponding to growth rates and savings rates show non-zero values only when the corresponding values of *m* are between 0 and 1.

Using the solution for *m*, (34), one can solve for the equilibrium growth rates of the primal and dual variables. Substituting (31) into (30), I solve for $\frac{y}{k}$. Substituting $\frac{y}{k}$ into (33), yields $\frac{c}{k}$. $\frac{y}{k}$ and $\frac{c}{k}$ give the savings rate:

$$s = \frac{\left[(1-\sigma_K)(\rho - n - \gamma\delta_K) - \overline{g} + \delta_A\right]\sigma_K}{\left[(1-\sigma_K)(-n-\delta_K) - \overline{g} + \delta_A\right]\gamma} \tag{39}$$

From (39) it can be shown that:

$$\begin{aligned}
\gamma &< \frac{\delta_K + \rho}{\delta_K} & \Rightarrow \quad & \frac{\partial s}{\partial \overline{g}} > 0 \\
\gamma &= \frac{\delta_K + \rho}{\delta_K} & \Rightarrow \quad & \frac{\partial s}{\partial \overline{g}} = 0 \\
\gamma &> \frac{\delta_K + \rho}{\delta_K} & \Rightarrow \quad & \frac{\partial s}{\partial \overline{g}} < 0
\end{aligned} \tag{40}$$

If $\delta_K = 0$:

$$\frac{\partial s}{\partial \overline{g}} = \frac{\rho\sigma_K\gamma(1-\sigma_K)}{\left[(1-\sigma_K)(-n-\delta_K) - \overline{g} + \delta_A\right]^2\gamma^2} > 0 \tag{41}$$

Tables 4 and 5 provide the values of the savings rate for ranges of γ and $\overline{g}$ under depreciation and no depreciation regimes. Also from (39),

$$\operatorname{sgn}\left(\frac{\partial s}{\partial \gamma}\right) = \operatorname{sgn}\left(\left[(1-\sigma_K)(-n-\delta_K) - \overline{g} + \delta_A\right]\left[-(1-\sigma_K)(\rho - n) + \overline{g} - \delta_A\right]\right) \tag{42}$$

For reasonable parameter values, the $\overline{g}$ terms will dominate the other terms, giving $\frac{\partial s}{\partial \gamma} < 0$.

Vintage Case

In the vintage case, equations (43)-(46), the objective function, the final goods production function, the capital accumulation equation, and the technology accumulation equation, summarize the central planner's problem. The differences between the vintage and non-vintage cases appear in equations (44) and (46), the production function and capital accumulation equations respectively. Writing the problem using aggregate capital, Q, the technology stock no longer appears in the production function and now appears in the capital accumulation equation.

$$\frac{1}{1-\gamma}\int_0^\infty C^{1-\gamma}e^{-\rho t}dt \qquad \rho>0; \qquad \gamma>0 \tag{43}$$

$$Y=(mN)^{1-\sigma_K}Q^{\sigma_K} \qquad 0<\text{m}<1, \quad 0<\sigma_K<1 \tag{44}$$

$$\dot{A}=(1-m)\bar{g}A-\delta_A A \qquad 0<\bar{g}<1, \qquad \delta_A\geq 0 \tag{45}$$

$$\dot{Q}=A^{1/\sigma_K}(Y-C)-\delta_K Q=A^{1/\sigma_K}\left((mN)^{1-\sigma_K}Q^{\sigma_K}-C\right)-\delta_K Q \tag{46}$$

The optimality and transversality conditions are summarized by the following equations, where Φ and Λ are the shadow prices of technology and capital respectively, along with (44)-(46). The interpretations of the equations are the same as in the non-vintage case.

$$C^{-\gamma}=A^{1/\sigma_K}\Lambda \tag{47}$$

$$\frac{\Lambda}{\Phi}=\frac{\bar{g}}{(1-\alpha)A^{\frac{1-\sigma_K}{\sigma_K}}m^{-\sigma_K}N^{1-\sigma_K}Q^{\sigma_K}} \tag{48}$$

$$(1-m)\bar{g}-\delta_A+\frac{1}{\sigma_K}A^{\frac{1-\sigma_K}{\sigma_K}}(m^{1-\sigma_K}N^{1-\sigma_K}Q^{\sigma_K}-C)\frac{\Lambda}{\Phi}=\rho-\frac{\dot{\Phi}}{\Phi} \tag{49}$$

$$\sigma_K A^{1/\sigma_K}m^{1-\sigma_K}N^{1-\sigma_K}Q^{\sigma_K-1}-\delta_K=\rho-\frac{\dot{\Lambda}}{\Lambda} \tag{50}$$

$$\lim_{t\to\infty} e^{-\rho t}\Phi A = \lim_{t\to\infty} e^{-\rho t}\Lambda Q = 0 \tag{51}$$

Using the definition of a balanced growth path from the non-vintage case, it is possible to solve for the equilibrium growth rates of the primal and dual variables using the production functions and optimality conditions. From (45):

$$g_A = \frac{\dot{A}}{A} = (1-m)\overline{g} - \delta_A \tag{52}$$

This is the same growth rate equation as in the non-vintage case. However, it will be shown that the equilibrium value of *m* will not generally be the same in the two cases. Thus, the equilibrium growth rate of technology will be different, and therefore all of the growth rates will differ between the two cases.

From (46), $\dot{Q} = sA^{1/\sigma_K}Y - \delta_K Q$, so $g_Q = \frac{\dot{Q}}{Q} = sA^{1/\sigma_K} m^{1-\sigma_K} N^{1-\sigma_K} Q^{\sigma_K - 1} - \delta_K$. Taking the time derivative of g_Q, and then solving for g_Q:

$$g_Q = \frac{1}{\sigma_K(1-\sigma_K)} g_A + n = \frac{1-m}{\sigma_K(1-\sigma_K)}\overline{g} - \frac{\delta_A}{\sigma_K(1-\sigma_K)} + n \tag{53}$$

Given the same labor allocation, the aggregate capital stock grows at a faster rate than the capital stock in the non-vintage case. However, by the assumption of a constant savings rate in the definition of equilibrium, the non-aggregated units of capital must grow at the same rate as output in both cases. The growth rate of output is shown in (54) to have the same functional form as in the non-vintage case. Taking the time derivative of (44) and then dividing by Y:

$$g_Y = g_C = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \sigma_K g_Q + (1-\sigma_K)n = \frac{1-m}{1-\sigma_K}\overline{g} - \frac{1}{1-\sigma_K}\delta_A + n \tag{54}$$

As in the non-vintage case, it can be shown that the difference in the growth rates of the shadow values of the two sectors is equal to the difference in the growth rates of the sectors themselves.

One can solve for the stationary variables using the same methodology as in the non-vintage case. This leads to equation (55) which is solely a function of *m* (g_A, g_Q, g_Φ, and g_Λ being functions of *m*).

$$\frac{(g_Q+\delta_K)m\overline{g}}{(1-\sigma_K)(\rho+\delta_K-g_\Lambda)}=\rho-g_\Phi-g_A \tag{55}$$

(55) is a quadratic equation in m, which can be written in the form $d_1 m^2+d_2 m+d_3=0$, where:

$$d_1=-\frac{(\gamma^2\sigma_K+\gamma-2\gamma\sigma_K+\sigma_K)\overline{g}^2}{\sigma_K(1-\sigma_K)} \tag{56}$$

There will be a unique solution to m if and only if d_1, d_2, and d_3 form a perfect square or $d_1=0$. It can be shown however that there is no combination of positive σ_K and γ such that $d_1=0$. Therefore, there will generally be two solutions to m, but depending on parameter values, there may be no solutions such that $0<m<1$. Tables 8 and 9 give the values of m over a range of γ and $\overline{g}$ values, given the benchmark parameter values and the no depreciation case. Zeros in tables 8 and 9 represent occurrences in which the solution to *m* is an imaginary number. Only one solution for *m* is given for each $(\gamma,\overline{g})$ combination. The other solution is not included because it is very much a knife's edge solution. For each γ, there is only a range of $\overline{g}$ of .02 for which $0<m<1$.

Using the solution to m, one can solve for the equilibrium growth rates of the primal and dual variables. Tables 10 and 11 give the savings rates corresponding to the values of m. From

Table 10, it can be seen that $\frac{\partial s}{\partial \bar{g}} > 0$ for values of $\gamma < 5$. Because of the complicated nature of the solution for m, it is not possible to get closed form conditions for the sign of $\frac{\partial s}{\partial \bar{g}}$ and $\frac{\partial s}{\partial \gamma}$ as in the non-vintage case.

The optimal sectoral allocation of labor is different between the vintage and non-vintage cases. Thus, the equilibrium growth rates will be different between the two models, as shown in tables 12 and 13. This is the first model that shows different equilibrium growth rates for vintage and non-vintage cases. The growth rate of the non-vintage case is higher (i.e. m is lower) than in the vintage case for parameter values in which m is an interior solution for both cases. Intuitively this is because the marginal returns from increasing technology are higher in the non-vintage case. In disembodied models, technology improves the entire production process. For example, technological growth of 3% in the non-vintage case leads to output increasing by 3% if there is no increase in factors. In the embodied technology case on the other hand, an increase in technology of 3% only means that output will increase if it is accompanied by investment in the capital stock. Also, if the new investment represents only a small fraction of the existing stock, then the incremental production due to the new technology is far less than 3%. It can also be seen from the tables that in the vintage case, a higher $\bar{g}$ is required to induce any labor to be allocated to the research sector than in the non-vintage case.

Because the level of *m* determines the growth rate of the economy, economies with different labor allocations will not converge in the long run either in the σ-convergence or β-convergence sense. Since the allocation of labor between the sectors is affected by preferences, economies with different preferences will not converge. This model provides one plausible explanation for the low benchmark measurements of convergence of approximately 2% in the empirical literature.[2] It is possible that the cross-country regressions include economies with different preferences, thereby mixing economies that do converge with others that do not, resulting in a convergence rate which is below that shown in the theoretical literature.

For $\gamma \geq 1$, the savings rate of the vintage case is greater than the non-vintage case, as in the neoclassical model. The savings rate is higher in the vintage model because the marginal return to capital is higher in the vintage model. Investment provides not just additional units of capital but entrée to new technology.

In the vintage model, the quantity of labor applied to a vintage goes to zero as the age of the capital goes to infinity. Given an amount of labor, a newer vintage has a greater marginal product of labor than an older vintage. Hence, equalizing the marginal products to maximize output leads to less labor applied to older vintages, and given a reasonable labor growth rate, each period the capital of any existing vintage contributes less output than it did the previous period. Depreciation causes a similar effect but for a different reason. As capital ages, a particular vintage will contribute less output each year because there is less of that particular capital. The similarity in outcomes can be seen in the smaller difference between the savings rates of the non-vintage/depreciation case and the vintage/no-depreciation case versus comparisons of the other cases.

4. Model #2: Cobb-Douglas Research Sector

The non-vintage model of this section is the model examined in Eicher and Turnovsky

[2] A rate of convergence of 2% was reported in Barro and Sala-I-Martin (1992).

(1999, 2001). The model uses a Cobb-Douglas production function in the research sector. The two factors of technology production are labor and technology. While the qualitative results for m and s, relating the vintage case to the non-vintage case, are the same as in the previous section, the growth rates only depend on exogenous parameters and therefore do not differ between the two cases. Numerical results are given for the benchmark parameter values used in Eicher and Turnovksy, as well as a no depreciation case. The benchmark parameters are listed in Table 14. The benchmark parameters feature constant returns to scale in labor and capital for the final goods sector $(\sigma_N + \sigma_K = 1)$, and increasing returns to labor and technology in the research sector $(\eta_N + \eta_A > 1)$. The results, however, are robust for both constant and diminishing returns in the technology sector.

Non-Vintage Case

The central planner's problem can be summarized by the following equations:

$$\frac{1}{1-\gamma}\int_0^\infty C^{1-\gamma} e^{-\rho t} dt \qquad \rho > 0; \qquad \gamma > 0 \tag{57}$$

$$Y = d_Y A^{\sigma_A} (mN)^{\sigma_N} K^{\sigma_K} \qquad d_Y > 0, \quad 0 < \sigma_i < 1 \tag{58}$$

$$\dot{A} = d_A \left((1-m)N\right)^{\eta_N} A^{\eta_A} - \delta_A A \qquad d_A, \delta_A > 0 \qquad 0 < \eta_i < 1 \tag{59}$$

$$\dot{K} = Y - C - \delta_K K = d_Y A^{\sigma_A} (mN)^{\sigma_N} K^{\sigma_K} - C - \delta_K K \qquad \delta_K > 0 \tag{60}$$

The variables are the same as in the previous model except that η_N and η_A are the elasticities of production in the research sector, and d_Y and d_A are exogenous positive constants.

The optimality and transversality conditions resulting from this model are summarized by the following equations and (58)-(60):

$$C^{-\gamma} = \Lambda \tag{61}$$

$$\frac{\Lambda}{\Phi} = \frac{\eta_N d_A (1-m)^{\eta_N - 1} N^{\eta_N} A^{\eta_A}}{\sigma_N d_Y A^{\sigma_A} m^{\sigma_N - 1} N^{\sigma_N} K^{\sigma_K}} \tag{62}$$

$$\eta_A d_A (1-m)^{\eta_N} N^{\eta_N} A^{\eta_A - 1} - \delta_A + \sigma_A d_Y A^{\sigma_A - 1} (mN)^{\sigma_N} K^{\sigma_K} \frac{\Lambda}{\Phi} = \rho - \frac{\dot{\Phi}}{\Phi} \tag{63}$$

$$\sigma_K d_Y A^{\sigma_A} (mN)^{\sigma_N} K^{\sigma_K - 1} - \delta_K = \rho - \frac{\dot{\Lambda}}{\Lambda} \tag{64}$$

$$\lim_{t\to\infty} e^{-\rho t} \Phi A = \lim_{t\to\infty} e^{-\rho t} \Lambda K = 0 \tag{65}$$

The balanced growth path is defined as in the previous section, with labor allocation, the savings rate, and the growth rates constant. Under these assumptions, it is possible to solve for the equilibrium growth rates. Dividing (59) by A and taking the time derivative:

$$g_A = \frac{\dot{A}}{A} = \frac{\eta_N}{1-\eta_A} n \tag{66}$$

As shown in the previous section in the non-vintage case, the assumption of a constant savings rate implies that output, the capital stock, and consumption grow at the same rate. Therefore, dividing (60) by *K* and taking the derivative with respect to time:

$$g_K = g_Y = g_C = \frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\sigma_N n + \sigma_A g_A}{1-\sigma_K} = \frac{\sigma_A \eta_N + \sigma_N (1-\eta_A)}{(1-\sigma_K)(1-\eta_A)} n \tag{67}$$

Taking the time derivative of (61):

$$g_\Lambda = \frac{\dot{\Lambda}}{\Lambda} = -\gamma g_Y = -\gamma \frac{\sigma_A \eta_N + \sigma_N (1-\eta_A)}{(1-\sigma_K)(1-\eta_A)} n \tag{68}$$

Taking the time derivative of (62):

$$g_{\Phi} = \frac{\dot{\Phi}}{\Phi} = (\sigma_N - \eta_N)n + \sigma_K g_K + (\sigma_A - \eta_A)g_A + g_{\Lambda}$$
$$= \frac{(1-\eta_A)\sigma_N - (1-\sigma_A-\sigma_K)\eta_N - (1-\eta_A)\gamma\sigma_N - \gamma\eta_N\sigma_A}{(1-\eta_A)(1-\sigma_K)} n \tag{69}$$

It can be shown that:

$$g_{\Lambda} - g_{\Phi} = g_A - g_K \tag{70}$$

All of the growth rates depend only upon exogenous parameters. Furthermore, they are all proportional to the labor growth rate. This is a standard result in two sector Cobb-Douglas models. While this result may contradict empirical evidence for single countries, a defense of this result is if the economy being modeled is a "worldwide" economy. In that case, higher labor growth is "beneficial to the growth of worldwide knowledge: the larger the population is, the more people there are to make new discoveries."[3]

Defining stationary variables using the same methodology as in section 3, one can then solve for those variables. The solution for the fraction of labor allocated to the final goods sector, *m* is:

$$m = \frac{x}{1+x} \tag{71}$$

where:

$$x = \frac{\sigma_N}{\eta_N \sigma_A}\left(\frac{(\rho + \delta_A - g_{\Phi})}{g_A + \delta_A} - \eta_A\right) \tag{72}$$

Using *m*, it is possible to solve for the savings rate:

[3] David Romer, (1990), *Advanced Macroeconomics*, McGraw Hill, p. 100.

$$s = \frac{\sigma_K\left(g_K + \delta_K\right)}{\rho + \delta_K - g_\Lambda} \tag{73}$$

Table 14 shows the benchmark parameter values used in this section. These values are the same values used in Eicher and Turnovsky (1999, 2001). Tables 15 and 16 give the values of selected stationary variables for the benchmark parameters case and the no depreciation case respectively.[4] The labor allocation to the final goods sector is increasing in γ, and the savings rate is decreasing in γ.

$$\frac{\partial x}{\partial \gamma} = \left(-\frac{\sigma_N}{\eta_N \sigma_A\left(g_A + \delta_A\right)}\right)\frac{\partial g_\Phi}{\partial \gamma} = \frac{\sigma_N^2\left(1-\eta_A\right) + \eta_N \sigma_A \sigma_N}{\eta_N \sigma_A\left(g_A + \delta_A\right)\left(1-\eta_A\right)\left(1-\sigma_K\right)} n > 0 \tag{74}$$

$$\frac{\partial m}{\partial \gamma} = \frac{\frac{\partial x}{\partial \gamma}}{\left(1+x\right)^2} > 0 \tag{75}$$

$$\frac{\partial s}{\partial \gamma} = \frac{\sigma_K\left(g_K + \delta_K\right)}{\left(\rho + \delta_K - g_\Lambda\right)^2}\frac{\partial g_\Lambda}{\partial \gamma} = \frac{\sigma_K\left(g_K + \delta_K\right)}{\left(\rho + \delta_K - g_\Lambda\right)^2}\left(-g_Y\right) < 0 \tag{76}$$

Vintage Case

The central planner's problem can be summarized by the following equations, where *Q* is the aggregate capital stock. As in the AK model, the differences between the vintage and non-vintage cases appear in the production and capital accumulation equations, (78) and (80).

$$\frac{1}{1-\gamma}\int_0^\infty C^{1-\gamma} e^{-\rho t} dt \qquad \rho > 0; \qquad \gamma > 0 \tag{77}$$

$$Y = d_Y (mN)^{\sigma_N} Q^{1-\sigma_N} \qquad 0<m<1, \quad 0<\sigma_i < 1 \tag{78}$$

$$\dot{A} = d_A\left((1-m)N\right)^{\eta_N} A^{\eta_A} - \delta_A A \qquad 0<\eta_i<1 \tag{79}$$

$$\dot{Q} = A^{\frac{\sigma_A}{1-\sigma_N}}(d_Y(mN)^{\sigma_N}Q^{1-\sigma_N} - C)^{\frac{\sigma_K}{1-\sigma_N}} - \left(\frac{\sigma_K}{1-\sigma_N}\right)\delta_K Q \tag{80}$$

The optimality and transversality conditions resulting from this model can be summarized by the following equations plus (78)-(80). Φ and Λ are the shadow prices of technology and capital respectively.

$$C^{-\gamma} = \frac{\sigma_K}{1-\sigma_N} A^{\frac{\sigma_A}{1-\sigma_N}} \left(Y-C\right)^{\frac{\sigma_K}{1-\sigma_N}-1} \Lambda \tag{81}$$

$$\frac{\Lambda}{\Phi} = \frac{\eta_N d_A (1-m)^{\eta_N - 1} N^{\eta_N} A^{\eta_A}}{\sigma_N A^{\frac{\sigma_A}{1-\sigma_N}} \left(Y-C\right)^{\frac{\sigma_K}{1-\sigma_N}-1} d_Y m^{\sigma_N - 1} N^{\sigma_N} Q^{1-\sigma_N}} \tag{82}$$

$$\eta_A d_A (1-m)^{\eta_N} N^{\eta_N} A^{\eta_A - 1} - \delta_A + \frac{\sigma_A}{1-\sigma_N} A^{\frac{\sigma_A}{1-\sigma_N}-1} \left(Y-C\right)^{\frac{\sigma_K}{1-\sigma_N}} \frac{\Lambda}{\Phi} = \rho - \frac{\dot{\Phi}}{\Phi} \tag{83}$$

$$\sigma_K A^{\frac{\sigma_A}{1-\sigma_N}} \left(d_Y (mN)^{\sigma_N} Q^{1-\sigma_N} - C\right)^{\frac{\sigma_K}{1-\sigma_N}-1} d_Y (mN)^{\sigma_N} Q^{-\sigma_N} - \left(\frac{\sigma_K}{1-\sigma_N}\right)\delta_K = \rho - \frac{\dot{\Lambda}}{\Lambda} \tag{84}$$

$$\lim_{t\to\infty} e^{-\rho t}\Phi A = \lim_{t\to\infty} e^{-\rho t}\Lambda Q = 0 \tag{85}$$

The growth rates of output, consumption, and the shadow value of the technology stock in the vintage case have the same functional forms as in the non-vintage case. The growth rate of the aggregate capital stock is:

$$g_Q = \frac{\dot{Q}}{Q} = \frac{\sigma_A \eta_N + \sigma_N \sigma_K \left(1-\eta_A\right)}{\left(1-\sigma_N\right)\left(1-\sigma_K\right)\left(1-\eta_A\right)} n \tag{86}$$

---

[4] Eicher and Turnovsky (1999,2001) have a computational error which causes their labor allocation variable to differ from mine.

The growth rate of the shadow value of the aggregate capital stock is:

$$g_\Lambda = \frac{\dot{\Lambda}}{\Lambda} = \left(1 - \gamma - \frac{\sigma_K}{1-\sigma_N}\right) g_Y - \frac{\sigma_A}{1-\sigma_N} g_A \qquad (87)$$

Unlike in the AK model, the growth rates are only functions of exogenous parameters. Hence the growth rates that share functional forms between the vintage and non-vintage cases will be the same. The levels of the stationary choice variables, however, will be different. Solving for the savings rate:

$$s = \frac{\sigma_K \left(g_Q + \left(\frac{\sigma_K}{1-\sigma_N} \right) \delta_K \right)}{\rho + \left(\frac{\sigma_K}{1-\sigma_N} \right) \delta_K - g_\Lambda} \qquad (88)$$

Solving for the labor allocation variable, *m*:

$$m = \frac{x}{1+x} \qquad (89)$$

where:

$$x = \frac{\left(\rho + \delta_A - g_\Phi - \eta_A (g_A + \delta_A)\right)(1-\sigma_N)\sigma_N \left(\rho + \left(\frac{\sigma_K}{1-\sigma_N} \right) \delta_K - g_\Lambda \right)}{\sigma_A \eta_N \sigma_K \left(g_Q + \left(\frac{\sigma_K}{1-\sigma_N} \right) \delta_K \right) (g_A + \delta_A)} \qquad (90)$$

Tables 17 and 18 give the values of selected stationary variables for the benchmark parameter case and no depreciation case respectively.

The Cobb-Douglas research sector model produces similar qualitative results of the equilibrium labor allocation and savings rates as the AK research sector model. The non-vintage case has uniformly higher allocations to the research sector, reflecting the greater marginal returns to technology in that case. The allocation of labor for both the vintage and non-vintage cases can be written in the form $m = \frac{x}{1+x}$. It can be shown using (72), (90), and (70) that:

$$x_V = x_{NV} \frac{(1-\sigma_N)\left[\rho + \frac{\sigma_K}{1-\sigma_N}\delta_K + g_Q - g_\Phi - g_A\right]}{\sigma_K\left(g_Q + \frac{\sigma_K}{1-\sigma_N}\delta_K\right)} \tag{91}$$

where the subscripts refer to the vintage and non-vintage cases (x_V and x_{NV} respectively). For $\gamma \geq 1$, $(-g_\Phi - g_A) \geq 0$. Therefore, the vintage case will have a higher fraction of labor allocated to the final goods sector if the final goods production function has constant returns to scale in labor and capital. The result is in fact more robust and will hold for most reasonable parameter values found in the literature, but it does depend on the elasticities of labor and capital, the discount rate, and gamma.

On the other hand, the savings rate is higher in the vintage case, reflecting the greater marginal returns to investment in that case. If the production function has constant returns in labor and capital, it can be shown using (73) and (88), that $s_V \geq s_{NV}$, because $g_Q \geq g_K$ (these terms are in the numerator of the savings rate equations) and $-g_{\Lambda_V} \leq -g_{\Lambda_{NV}}$ (these terms are in the denominator of the savings rate equations).

Higher levels of γ encourage earlier consumption; and therefore, as is seen in the neoclassical model, the savings rate declines in the vintage model as γ increases. As in the non-vintage case, the sign of $\frac{\partial s}{\partial \gamma}$ is the sign of $\frac{\partial g_\Lambda}{\partial \gamma}$. Since $\frac{\partial g_\Lambda}{\partial \gamma}$ is the same for both cases:

$$\frac{\partial s}{\partial \gamma} < 0 \tag{92}$$

For the same reason, increasing γ also leads to less labor allocated to the research sector. In order for *m* to be positive and less than one, *x* must be greater than zero. The denominator of *x* is positive and not a function of γ. The numerator can be written as:

$$h(\gamma) = af_1(g_\Phi)\, f_2(g_\Lambda) \tag{93}$$

where *a* is a positive scalar, f_1 and f_2 are positive functions of the above growth rates, and $\frac{df_1}{dg_\Phi} = \frac{df_2}{dg_\Lambda} = -1$. g_Φ and g_Λ are functions of γ. So $\frac{\partial x}{\partial \gamma}$ has the same sign as $\frac{\partial h}{\partial \gamma}$. Also $\frac{\partial g_\Phi}{\partial \gamma} < 0$ and $\frac{\partial g_\Lambda}{\partial \gamma} < 0$. Hence:

$$\frac{\partial h}{\partial \gamma} = a\left(-\frac{\partial g_\Phi}{\partial \gamma} f_2 - \frac{\partial g_\Lambda}{\partial \gamma} f_1 \right) > 0 \tag{94}$$

and therefore $\frac{\partial x}{\partial \gamma} > 0$ and $\frac{\partial m}{\partial \gamma} > 0$.

Eicher and Turnovsky (2001) show for the non-vintage case that the convergence rates of the two sectors depend on the allocation of labor and savings rates. Therefore it is likely that the convergence rates will differ between the vintage and non-vintage cases.

5. Conclusion

This paper demonstrates the significant impact on growth theory results of simplifying assumptions of disembodied and exogenous technological. Growth theory is primarily concerned with long-run growth rates, and the AK research sector model shows that equilibrium growth rates can differ between vintage and non-vintage models. Given the same parameter values, the equilibrium growth rate of the vintage two-sector AK model is lower than in the non-vintage model. Mine is the first demonstration that the equilibrium growth rate can be affected by whether technology is modeled as embodied or disembodied.

Furthermore, vintage and non-vintage cases have differing returns to technology production and investment. Therefore the choice variables related to these decisions have different equilibrium values. The central planners of the vintage cases allocate less labor to research and save a greater fraction of their output. It is the smaller fraction of labor allocated to research in the vintage case that results in the lower steady state growth rate in the AK model. Also, because the preference parameter affects the allocation of labor, preferences affect the growth rate of output in the AK model. The implication is that economies with different preferences will not converge in the long run. Eicher and Turnovsky (2001) show that stationary choice variables affect the transitional dynamics in the two-sector, Cobb-Douglas, non-vintage case. Having different optimal stationary values, it is likely that future research will demonstrate that the vintage and non-vintage cases have different convergence rates, as Berger (2001) finds in the neoclassical model.

I demonstrate that the manner of technological growth, embodied or disembodied, affects the optimal choices of actors in an economy. Since their choices affect the equilibrium or the transition to equilibrium, the form of technological growth can have important implications for the central questions in growth economics.

Appendix 1. Vintage Capital Cobb-Douglas Production Maximization

At each instant, a static optimization takes place in which given the stocks of technology, labor, and capital, labor must be allocated to the spectrum of vintages of capital in order to maximize production. The formal production maximization problem is

$$\begin{aligned}&\text{Maximize} \quad Y = \int_{v=0}^{V} A_v^{\sigma_A} N_v^{\sigma_N} K_v^{\sigma_K} dv \qquad 0 < \sigma_A, \sigma_N, \sigma_K < 1 \\ &\text{over} \quad N_v \\ &\text{subject to:} \\ &\qquad N_v \geq 0 \\ &\qquad \int_{v=0}^{V} N_v dv \leq N\end{aligned} \tag{95}$$

V is the newest vintage available. A_v is the level of technology associated with vintage *v*. N_v, the sole choice variable, is the quantity of labor assigned to capital of vintage *v*. *N* is the total amount of labor available. K_v is the amount of capital of vintage *v* available. Assume that the function $g(v) = A_v^{\sigma_A} K_v^{\sigma_K}$ is a positive, bounded, measurable function.

At a solution, the total labor constraint will hold with equality since the objective function rewards the use of labor with each vintage and there is no cost of using labor. It is also true that at a solution $N_v > 0$ almost everywhere, because the marginal product of labor of each vintage goes to infinity as N_v goes to zero.

First I will show that $N_v > 0$ a.e. Let *D* and *E* be sets of equal positive measure, where $D, E \in [1, V]$, with $N_v > 0$ and finite for $v \in D$; and $N_v = 0$ for $v \in E$. It will be shown there exists an $\varepsilon > 0$ such that shifting ε from each $N_{v \in D}$ to $N_{v \in E}$, which is feasible, results in an increase in production.

The addition and reduction to production from the shift of ε are $\int_{v\in E} g(v)\varepsilon^{\sigma_N} dv$ and $\int_{v\in D} g(v)\left(N_v^{\sigma_N} - (N_v - \varepsilon)^{\sigma_N}\right) dv$ respectively. Let $\varepsilon = a \sup_{v\in D} N_v = aN$, where a is a scalar and $N = \sup_{v\in D} N_v$. The supremum exists since *D* is a compact set. Let $G = \sup_{v\in D} g(v)$, $g = \inf_{v\in E} g(v)$, and $m(D)$ equal the measure of set *D*.

$$\begin{aligned}
\int_{v\in D} g(v)\left(N_v^{\sigma_N} - (N_v - \varepsilon)^{\sigma_N}\right) dv &= \int_{v\in D} g(v)\left(N_v^{\sigma_N} - (N_v - aN)^{\sigma_N}\right) dv \\
&\leq \int_{v\in D} G\left(N^{\sigma_N} - (N - aN)^{\sigma_N}\right) dv \\
&= GN^{\sigma_N}\left(1 - (1-a)^{\sigma_N}\right) \cdot m(D)
\end{aligned}$$

$$\begin{aligned}
\int_{v\in E} g(v)\varepsilon^{\sigma_N} dv &= \int_{v\in E} g(v)(aN)^{\sigma_N} dv \\
&\geq \int_{v\in E} g a^{\sigma_N} N^{\sigma_N} dv \\
&= g a^{\sigma_N} N^{\sigma_N} \cdot m(D)
\end{aligned}$$

Thus, $\int_{v\in E} g(v)\varepsilon^{\sigma_N} dv > \int_{v\in D} g(v)\left(N_v^{\sigma_N} - (N_v - \varepsilon)^{\sigma_N}\right) dv$ if $g a^{\sigma_N} > G\left(1 - (1-a)^{\sigma_N}\right)$. This statement is true if $\frac{a^{\sigma_N}}{1-(1-a)^{\sigma_N}} > \frac{G}{g}$. This inequality is true for small enough *a*, since $\frac{a^{\sigma_N}}{1-(1-a)^{\sigma_N}} \to \infty$ as $a \to 0$. Therefore, since *E* was an arbitrary set, there can be no set of positive measure such that the labor applied to the vintages of that set equals zero.

The lagrangian for (95) is:

$$L(N_v;\lambda) = \int_{v=0}^{V} A_v^{\sigma_A} N_v^{\sigma_N} K_v^{\sigma_K} dv + \lambda\left(N - \int_{v=0}^{V} N_v dv\right) \tag{96}$$

A representative first order condition is:

$$\frac{\partial L}{\partial N_v} = \sigma_N A_v^{\sigma_A} N_v^{\sigma_N - 1} K_v^{\sigma_K} - \lambda = 0 \tag{97}$$

(97) must hold almost everywhere, and an equality can be used instead of an inequality because $N_v > 0$ a.e. at a solution. (97) states that the marginal product of labor of each vintage must be equal to the shadow value of labor. Labor, being homogeneous, has a unique shadow value regardless of the vintage to which it is applied. Setting two first order conditions for different vintages equal to each other we obtain:

$$A_i^{\sigma_A} N_i^{\sigma_N - 1} K_i^{\sigma_K} = A_j^{\sigma_A} N_j^{\sigma_N - 1} K_j^{\sigma_K} \tag{98}$$

Solving for N_i

$$N_i = \left(\frac{A_i}{A_j}\right)^{\frac{\sigma_A}{1-\sigma_N}} \left(\frac{K_i}{K_j}\right)^{\frac{\sigma_K}{1-\sigma_N}} N_j \tag{99}$$

This equation must hold true for every *i* given fixed A_j, N_j, and K_j. Therefore, integrating over the N_i:

$$N = \int_{i=0}^{V} N_i di = \frac{N_j}{A_j^{\frac{\sigma_A}{1-\sigma_N}} K_j^{\frac{\sigma_K}{1-\sigma_N}}} \int_{i=0}^{V} A_i^{\frac{\sigma_A}{1-\sigma_N}} K_i^{\frac{\sigma_K}{1-\sigma_N}} di \tag{100}$$

Solving for N_j in terms of the given parameters we obtain a closed form solution for each choice variable. This solution must hold almost everywhere.

$$N_j^* = \frac{N A_j^{\frac{\sigma_A}{1-\sigma_N}} K_j^{\frac{\sigma_K}{1-\sigma_N}}}{\int_{i=0}^{V} A_i^{\frac{\sigma_A}{1-\sigma_N}} K_i^{\frac{\sigma_K}{1-\sigma_N}} di} \tag{101}$$

Let

$$Q = \int_{i=0}^{V} A_i^{\frac{\sigma_A}{1-\sigma_N}} K_i^{\frac{\sigma_K}{1-\sigma_N}} di \tag{102}$$

Q can be viewed as an aggregated capital stock with the weights of each vintage a function of the technology inherent in each vintage and the production elasticities.

Substituting N_j^* into the direct production function, we obtain the indirect production function, which is a function only of given parameters.

$$
\begin{aligned}
Y^*(A, N, K, \sigma_A, \sigma_N, \sigma_K) &= \int_{v=0}^{V} A_v^{\sigma_A} K_v^{\sigma_K} \left(\frac{N A_v^{\frac{\sigma_A}{1-\sigma_N}} K_v^{\frac{\sigma_K}{1-\sigma_N}}}{Q} \right)^{\sigma_N} dv \\
&= N^{\sigma_N} Q^{-\sigma_N} \int_{v=0}^{V} A_v^{\frac{\sigma_A}{1-\sigma_N}} K_v^{\frac{\sigma_K}{1-\sigma_N}} dv \\
&= N^{\sigma_N} Q^{1-\sigma_N}
\end{aligned}
\tag{103}
$$

TABLES

Table 1. Benchmark parameter values: AK research sector

ρ	n	σ_K	δ_A	δ_K
.03	.015	.35	.01	.05

Table 2. Labor Allocation to Final Goods (m), benchmark parameters, AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**	**4.5**	**5**
0.01	2.925	1.95	1.625	1.4625	1.365	1.3	1.2536	1.2188	1.1917	1.17
0.03	0.3083	0.65	0.7639	0.8208	0.855	0.8778	0.894	0.9063	0.9157	0.9233
0.05	-0.215	0.39	0.5917	0.6925	0.753	0.7933	0.8221	0.8438	0.8606	0.874
0.07	-0.4393	0.2786	0.5179	0.6375	0.7093	0.7571	0.7913	0.817	0.8369	0.8529
0.09	-0.5639	0.2167	0.4769	0.6069	0.685	0.737	0.7742	0.8021	0.8238	0.8411
0.11	-0.6432	0.1773	0.4508	0.5875	0.6695	0.7242	0.7633	0.7926	0.8154	0.8336
0.13	-0.6981	0.15	0.4327	0.574	0.6588	0.7154	0.7558	0.7861	0.8096	0.8285
0.15	-0.7383	0.13	0.4194	0.5642	0.651	0.7089	0.7502	0.7812	0.8054	0.8247
0.17	-0.7691	0.1147	0.4093	0.5566	0.645	0.7039	0.746	0.7776	0.8021	0.8218
0.19	-0.7934	0.1026	0.4013	0.5507	0.6403	0.7	0.7427	0.7747	0.7996	0.8195
0.21	-0.8131	0.0929	0.3948	0.5458	0.6364	0.6968	0.74	0.7723	0.7975	0.8176
0.23	-0.8293	0.0848	0.3895	0.5418	0.6333	0.6942	0.7377	0.7704	0.7958	0.8161
0.25	-0.843	0.078	0.385	0.5385	0.6306	0.692	0.7359	0.7688	0.7943	0.8148
0.27	-0.8546	0.0722	0.3812	0.5356	0.6283	0.6901	0.7343	0.7674	0.7931	0.8137
0.29	-0.8647	0.0672	0.3779	0.5332	0.6264	0.6885	0.7329	0.7662	0.792	0.8128
0.31	-0.8734	0.0629	0.375	0.531	0.6247	0.6871	0.7317	0.7651	0.7911	0.8119
0.33	-0.8811	0.0591	0.3725	0.5292	0.6232	0.6859	0.7306	0.7642	0.7903	0.8112
0.35	-0.8879	0.0557	0.3702	0.5275	0.6219	0.6848	0.7297	0.7634	0.7896	0.8106

Table 3. Labor Allocation to Final Goods (m), no depreciation,
AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	1.925	1.95	1.9583	1.9625	1.965	1.9667	1.9679	1.9688	1.9694	1.97
0.03	-0.025	0.65	0.875	0.9875	1.055	1.1	1.1321	1.1563	1.175	1.19
0.05	-0.415	0.39	0.6583	0.7925	0.873	0.9267	0.965	0.9938	1.0161	1.034
0.07	-0.5821	0.2786	0.5655	0.7089	0.795	0.8524	0.8934	0.9241	0.948	0.9671
0.09	-0.675	0.2167	0.5139	0.6625	0.7517	0.8111	0.8536	0.8854	0.9102	0.93
0.11	-0.7341	0.1773	0.4811	0.633	0.7241	0.7848	0.8282	0.8608	0.8861	0.9064
0.13	-0.775	0.15	0.4583	0.6125	0.705	0.7667	0.8107	0.8437	0.8694	0.89
0.15	-0.805	0.13	0.4417	0.5975	0.691	0.7533	0.7979	0.8312	0.8572	0.878
0.17	-0.8279	0.1147	0.4289	0.586	0.6803	0.7431	0.788	0.8217	0.8479	0.8688
0.19	-0.8461	0.1026	0.4189	0.577	0.6718	0.7351	0.7803	0.8141	0.8405	0.8616
0.21	-0.8607	0.0929	0.4107	0.5696	0.665	0.7286	0.774	0.808	0.8345	0.8557
0.23	-0.8728	0.0848	0.404	0.5636	0.6593	0.7232	0.7688	0.803	0.8296	0.8509
0.25	-0.883	0.078	0.3983	0.5585	0.6546	0.7187	0.7644	0.7988	0.8254	0.8468
0.27	-0.8917	0.0722	0.3935	0.5542	0.6506	0.7148	0.7607	0.7951	0.8219	0.8433
0.29	-0.8991	0.0672	0.3894	0.5504	0.6471	0.7115	0.7575	0.792	0.8189	0.8403
0.31	-0.9056	0.0629	0.3858	0.5472	0.644	0.7086	0.7547	0.7893	0.8162	0.8377
0.33	-0.9114	0.0591	0.3826	0.5443	0.6414	0.7061	0.7523	0.7869	0.8139	0.8355
0.35	-0.9164	0.0557	0.3798	0.5418	0.639	0.7038	0.7501	0.7848	0.8118	0.8334

Table 4. Savings Rate (s), benchmark parameters,
AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0.298	0.2404	0.2212	0.2115	0.2058	0.2019	0.1992	0.1971	0.1955	0.1943
0.05	0	0.267	0.2241	0.2027	0.1898	0.1812	0.1751	0.1705	0.1669	0.164
0.07	0	0.2833	0.2259	0.1972	0.18	0.1686	0.1604	0.1542	0.1495	0.1456
0.09	0	0.2942	0.2271	0.1936	0.1735	0.1601	0.1505	0.1433	0.1377	0.1333
0.11	0	0.302	0.228	0.191	0.1688	0.154	0.1434	0.1355	0.1293	0.1244
0.13	0	0.3079	0.2287	0.189	0.1652	0.1494	0.1381	0.1296	0.123	0.1177
0.15	0	0.3126	0.2292	0.1875	0.1625	0.1458	0.1339	0.1249	0.118	0.1124
0.17	0	0.3163	0.2296	0.1862	0.1602	0.1429	0.1305	0.1212	0.114	0.1082
0.19	0	0.3193	0.2299	0.1852	0.1584	0.1406	0.1278	0.1182	0.1108	0.1048
0.21	0	0.3218	0.2302	0.1844	0.1569	0.1386	0.1255	0.1157	0.108	0.1019
0.23	0	0.324	0.2304	0.1837	0.1556	0.1369	0.1235	0.1135	0.1057	0.0995
0.25	0	0.3258	0.2306	0.1831	0.1545	0.1355	0.1219	0.1117	0.1037	0.0974
0.27	0	0.3274	0.2308	0.1825	0.1535	0.1342	0.1204	0.1101	0.102	0.0956
0.29	0	0.3288	0.231	0.1821	0.1527	0.1331	0.1192	0.1087	0.1005	0.094
0.31	0	0.3301	0.2311	0.1816	0.152	0.1322	0.118	0.1074	0.0992	0.0926
0.33	0	0.3312	0.2312	0.1813	0.1513	0.1313	0.117	0.1063	0.098	0.0914
0.35	0	0.3321	0.2313	0.181	0.1507	0.1306	0.1162	0.1054	0.097	0.0902

Table 5. Savings Rate (s), no depreciation,
AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**	**4.5**	**5**
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0	0.1783	0.1189	0.0892	0	0	0	0	0	0
0.05	0	0.2358	0.1572	0.1179	0.0943	0.0786	0.0674	0.0589	0	0
0.07	0	0.2644	0.1763	0.1322	0.1058	0.0881	0.0755	0.0661	0.0588	0.0529
0.09	0	0.2816	0.1877	0.1408	0.1126	0.0939	0.0805	0.0704	0.0626	0.0563
0.11	0	0.293	0.1953	0.1465	0.1172	0.0977	0.0837	0.0733	0.0651	0.0586
0.13	0	0.3012	0.2008	0.1506	0.1205	0.1004	0.086	0.0753	0.0669	0.0602
0.15	0	0.3073	0.2049	0.1536	0.1229	0.1024	0.0878	0.0768	0.0683	0.0615
0.17	0	0.312	0.208	0.156	0.1248	0.104	0.0892	0.078	0.0693	0.0624
0.19	0	0.3158	0.2106	0.1579	0.1263	0.1053	0.0902	0.079	0.0702	0.0632
0.21	0	0.3189	0.2126	0.1595	0.1276	0.1063	0.0911	0.0797	0.0709	0.0638
0.23	0	0.3215	0.2144	0.1608	0.1286	0.1072	0.0919	0.0804	0.0715	0.0643
0.25	0	0.3237	0.2158	0.1619	0.1295	0.1079	0.0925	0.0809	0.0719	0.0647
0.27	0	0.3256	0.2171	0.1628	0.1302	0.1085	0.093	0.0814	0.0724	0.0651
0.29	0	0.3272	0.2182	0.1636	0.1309	0.1091	0.0935	0.0818	0.0727	0.0654
0.31	0	0.3287	0.2191	0.1643	0.1315	0.1096	0.0939	0.0822	0.073	0.0657
0.33	0	0.3299	0.2199	0.165	0.132	0.11	0.0943	0.0825	0.0733	0.066
0.35	0	0.331	0.2207	0.1655	0.1324	0.1103	0.0946	0.0828	0.0736	0.0662

Table 6. Growth Rate of Output (g_y), benchmark parameters,
AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**	**4.5**	**5**
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0.0315	0.0158	0.0105	0.0079	0.0063	0.0053	0.0045	0.0039	0.0035	0.0032
0.05	0	0.0465	0.031	0.0233	0.0186	0.0155	0.0133	0.0116	0.0103	0.0093
0.07	0	0.0773	0.0515	0.0387	0.0309	0.0258	0.0221	0.0193	0.0172	0.0155
0.09	0	0.1081	0.0721	0.054	0.0432	0.036	0.0309	0.027	0.024	0.0216
0.11	0	0.1388	0.0926	0.0694	0.0555	0.0463	0.0397	0.0347	0.0309	0.0278
0.13	0	0.1696	0.1131	0.0848	0.0678	0.0565	0.0485	0.0424	0.0377	0.0339
0.15	0	0.2004	0.1336	0.1002	0.0802	0.0668	0.0573	0.0501	0.0445	0.0401
0.17	0	0.2312	0.1541	0.1156	0.0925	0.0771	0.066	0.0578	0.0514	0.0462
0.19	0	0.2619	0.1746	0.131	0.1048	0.0873	0.0748	0.0655	0.0582	0.0524
0.21	0	0.2927	0.1951	0.1463	0.1171	0.0976	0.0836	0.0732	0.065	0.0585
0.23	0	0.3235	0.2156	0.1617	0.1294	0.1078	0.0924	0.0809	0.0719	0.0647
0.25	0	0.3542	0.2362	0.1771	0.1417	0.1181	0.1012	0.0886	0.0787	0.0708
0.27	0	0.385	0.2567	0.1925	0.154	0.1283	0.11	0.0963	0.0856	0.077
0.29	0	0.4158	0.2772	0.2079	0.1663	0.1386	0.1188	0.1039	0.0924	0.0832
0.31	0	0.4465	0.2977	0.2233	0.1786	0.1488	0.1276	0.1116	0.0992	0.0893
0.33	0	0.4773	0.3182	0.2387	0.1909	0.1591	0.1364	0.1193	0.1061	0.0955
0.35	0	0.5081	0.3387	0.254	0.2032	0.1694	0.1452	0.127	0.1129	0.1016

Table 7. Growth Rate of Output (g_y), no depreciation,
AK research sector, non-vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0	0.0312	0.0208	0.0156	0	0	0	0	0	0
0.05	0	0.0619	0.0413	0.031	0.0248	0.0206	0.0177	0.0155	0	0
0.07	0	0.0927	0.0618	0.0463	0.0371	0.0309	0.0265	0.0232	0.0206	0.0185
0.09	0	0.1235	0.0823	0.0617	0.0494	0.0412	0.0353	0.0309	0.0274	0.0247
0.11	0	0.1542	0.1028	0.0771	0.0617	0.0514	0.0441	0.0386	0.0343	0.0308
0.13	0	0.185	0.1233	0.0925	0.074	0.0617	0.0529	0.0463	0.0411	0.037
0.15	0	0.2158	0.1438	0.1079	0.0863	0.0719	0.0616	0.0539	0.0479	0.0432
0.17	0	0.2465	0.1644	0.1233	0.0986	0.0822	0.0704	0.0616	0.0548	0.0493
0.19	0	0.2773	0.1849	0.1387	0.1109	0.0924	0.0792	0.0693	0.0616	0.0555
0.21	0	0.3081	0.2054	0.154	0.1232	0.1027	0.088	0.077	0.0685	0.0616
0.23	0	0.3388	0.2259	0.1694	0.1355	0.1129	0.0968	0.0847	0.0753	0.0678
0.25	0	0.3696	0.2464	0.1848	0.1478	0.1232	0.1056	0.0924	0.0821	0.0739
0.27	0	0.4004	0.2669	0.2002	0.1602	0.1335	0.1144	0.1001	0.089	0.0801
0.29	0	0.4312	0.2874	0.2156	0.1725	0.1437	0.1232	0.1078	0.0958	0.0862
0.31	0	0.4619	0.3079	0.231	0.1848	0.154	0.132	0.1155	0.1026	0.0924
0.33	0	0.4927	0.3285	0.2463	0.1971	0.1642	0.1408	0.1232	0.1095	0.0985
0.35	0	0.5235	0.349	0.2617	0.2094	0.1745	0.1496	0.1309	0.1163	0.1047

Table 8. Labor Allocation to Final Goods (m), benchmark parameters,
AK research sector, vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	1.4019	1.3168
0.03	0.4504	0	0	0	0	0	0	1.0729	1.0535	1.0426
0.05	-0.1924	0.4759	0.7732	0.9036	0.9544	0.9725	0.98	0.9839	0.9862	0.9878
0.07	-0.3609	0.3146	0.6314	0.7857	0.865	0.908	0.9327	0.9477	0.9575	0.9643
0.09	-0.445	0.2368	0.5663	0.7333	0.8238	0.8759	0.9076	0.928	0.9416	0.9513
0.11	-0.4957	0.1902	0.5277	0.7021	0.7989	0.8562	0.892	0.9155	0.9315	0.943
0.13	-0.5297	0.159	0.5018	0.6813	0.7823	0.8429	0.8813	0.9069	0.9246	0.9372
0.15	-0.554	0.1366	0.4833	0.6663	0.7703	0.8332	0.8736	0.9006	0.9194	0.933
0.17	-0.5724	0.1198	0.4694	0.6551	0.7612	0.8259	0.8677	0.8958	0.9155	0.9298
0.19	-0.5866	0.1066	0.4585	0.6463	0.7541	0.8202	0.863	0.892	0.9124	0.9273
0.21	-0.5981	0.0961	0.4498	0.6392	0.7484	0.8156	0.8593	0.889	0.9099	0.9252
0.23	-0.6075	0.0875	0.4426	0.6334	0.7437	0.8118	0.8562	0.8864	0.9079	0.9235
0.25	-0.6153	0.0803	0.4367	0.6286	0.7398	0.8086	0.8536	0.8843	0.9061	0.9221
0.27	-0.6219	0.0742	0.4316	0.6245	0.7365	0.8059	0.8514	0.8825	0.9046	0.9209
0.29	-0.6276	0.0689	0.4273	0.621	0.7336	0.8036	0.8495	0.881	0.9034	0.9198
0.31	-0.6326	0.0644	0.4235	0.6179	0.7311	0.8016	0.8479	0.8796	0.9023	0.9189
0.33	-0.6369	0.0604	0.4202	0.6152	0.729	0.7998	0.8464	0.8784	0.9013	0.9181
0.35	-0.6407	0.0568	0.4173	0.6129	0.727	0.7983	0.8451	0.8774	0.9004	0.9174

Table 9. Labor Allocation to Final Goods (m), no depreciation,
AK research sector, vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	0	0
0.03	-0.0246	0	0	0	0	0	0	0	0	0
0.05	-0.3437	0.4806	0	0	0	0	0	0	0	0
0.07	-0.4567	0.3156	0.7379	0	0	0	0	0	0	0
0.09	-0.5154	0.2372	0.6328	0.8833	0	0	0	0	0	0
0.11	-0.5514	0.1904	0.5775	0.7974	0	0	0	0	0	0
0.13	-0.5757	0.1591	0.5419	0.755	0.893	0	0	0	0	0
0.15	-0.5933	0.1367	0.517	0.7271	0.8561	0	0	0	0	0
0.17	-0.6066	0.1198	0.4984	0.7069	0.8327	0.9199	0	0	0	0
0.19	-0.6169	0.1067	0.4841	0.6916	0.8158	0.898	0	0	0	0
0.21	-0.6253	0.0961	0.4726	0.6795	0.8028	0.8829	0.9426	0	0	0
0.23	-0.6322	0.0875	0.4633	0.6697	0.7924	0.8713	0.9273	0	0	0
0.25	-0.6379	0.0803	0.4555	0.6616	0.7839	0.8622	0.9164	0.9605	0	0
0.27	-0.6428	0.0742	0.4489	0.6548	0.7768	0.8546	0.908	0.9483	0	0
0.29	-0.6469	0.0689	0.4433	0.6489	0.7708	0.8483	0.9011	0.9398	0.9762	0
0.31	-0.6506	0.0644	0.4384	0.6439	0.7656	0.8429	0.8953	0.9332	0.9641	0
0.33	-0.6538	0.0604	0.4341	0.6395	0.7611	0.8383	0.8904	0.9277	0.9568	0
0.35	-0.6566	0.0569	0.4304	0.6356	0.7572	0.8342	0.8861	0.9231	0.9513	0.977

Table 10. Savings Rate (s), benchmark parameters,
AK research sector, vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0.2948	0	0	0	0	0	0	0	0	0
0.05	0	0.2868	0.2268	0.1865	0.1654	0.1566	0.1527	0.1506	0.1493	0.1484
0.07	0	0.3099	0.2556	0.218	0.1924	0.1755	0.1644	0.1568	0.1516	0.1478
0.09	0	0.3203	0.267	0.2294	0.2026	0.1836	0.1699	0.1601	0.1528	0.1474
0.11	0	0.3263	0.2735	0.2357	0.2083	0.1882	0.1733	0.1621	0.1536	0.1471
0.13	0	0.3302	0.2777	0.2398	0.212	0.1912	0.1755	0.1635	0.1542	0.1469
0.15	0	0.3331	0.2806	0.2427	0.2145	0.1933	0.1771	0.1645	0.1546	0.1468
0.17	0	0.3352	0.2828	0.2448	0.2164	0.1949	0.1782	0.1652	0.1549	0.1467
0.19	0	0.3368	0.2845	0.2464	0.2179	0.1961	0.1792	0.1658	0.1552	0.1466
0.21	0	0.3381	0.2859	0.2477	0.2191	0.1971	0.1799	0.1663	0.1554	0.1465
0.23	0	0.3392	0.287	0.2488	0.22	0.1979	0.1805	0.1667	0.1555	0.1465
0.25	0	0.3401	0.2879	0.2497	0.2208	0.1986	0.181	0.167	0.1557	0.1464
0.27	0	0.3409	0.2886	0.2504	0.2215	0.1991	0.1815	0.1673	0.1558	0.1464
0.29	0	0.3415	0.2893	0.2511	0.2221	0.1996	0.1818	0.1675	0.1559	0.1463
0.31	0	0.3421	0.2899	0.2516	0.2226	0.2	0.1821	0.1677	0.156	0.1463
0.33	0	0.3426	0.2904	0.2521	0.223	0.2004	0.1824	0.1679	0.1561	0.1463
0.35	0	0.343	0.2908	0.2525	0.2234	0.2007	0.1827	0.1681	0.1561	0.1462

Table 11. Savings Rate (s), no depreciation,
AK research sector, vintage capital

$\bar{g} \backslash \gamma$	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**	**4.5**	**5**
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0	0	0	0	0	0	0	0	0	0
0.05	0	0.284	0	0	0	0	0	0	0	0
0.07	0	0.3089	0.2273	0	0	0	0	0	0	0
0.09	0	0.3197	0.2513	0.175	0	0	0	0	0	0
0.11	0	0.3259	0.2624	0.2057	0	0	0	0	0	0
0.13	0	0.33	0.269	0.2179	0.1658	0	0	0	0	0
0.15	0	0.3329	0.2735	0.2252	0.1812	0	0	0	0	0
0.17	0	0.3351	0.2768	0.2303	0.1898	0.1482	0	0	0	0
0.19	0	0.3367	0.2793	0.234	0.1956	0.1595	0	0	0	0
0.21	0	0.3381	0.2812	0.2368	0.1998	0.1665	0.1308	0	0	0
0.23	0	0.3391	0.2828	0.2391	0.2031	0.1715	0.1406	0	0	0
0.25	0	0.3401	0.2841	0.2409	0.2057	0.1752	0.1468	0.1142	0	0
0.27	0	0.3408	0.2852	0.2425	0.2078	0.1782	0.1513	0.1238	0	0
0.29	0	0.3415	0.2862	0.2437	0.2095	0.1806	0.1548	0.1298	0.0962	0
0.31	0	0.342	0.287	0.2448	0.211	0.1827	0.1577	0.1342	0.1084	0
0.33	0	0.3425	0.2877	0.2458	0.2123	0.1844	0.16	0.1376	0.1147	0
0.35	0	0.343	0.2883	0.2466	0.2134	0.1858	0.162	0.1403	0.1191	0.0929

Table 12. Growth Rate of Output (g_y), benchmark parameters,
AK research sector, vintage capital

$\bar{g} \backslash \gamma$	**0.5**	**1**	**1.5**	**2**	**2.5**	**3**	**3.5**	**4**	**4.5**	**5**
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0.025	0	0	0	0	0	0	0	0	0
0.05	0	0.0399	0.0171	0.007	0.0031	0.0017	0.0012	0.0009	0.0007	0.0006
0.07	0	0.0734	0.0393	0.0227	0.0141	0.0095	0.0069	0.0052	0.0042	0.0035
0.09	0	0.1053	0.0597	0.0365	0.024	0.0168	0.0124	0.0096	0.0077	0.0064
0.11	0	0.1367	0.0796	0.05	0.0336	0.0239	0.0179	0.0139	0.0112	0.0093
0.13	0	0.1678	0.0993	0.0634	0.0432	0.031	0.0233	0.0182	0.0147	0.0122
0.15	0	0.1989	0.1189	0.0766	0.0526	0.0381	0.0288	0.0226	0.0182	0.0151
0.17	0	0.2298	0.1384	0.0898	0.0621	0.0451	0.0342	0.0269	0.0217	0.018
0.19	0	0.2607	0.1579	0.103	0.0715	0.0522	0.0397	0.0312	0.0252	0.0209
0.21	0	0.2916	0.1774	0.1162	0.0809	0.0592	0.0451	0.0355	0.0287	0.0238
0.23	0	0.3225	0.1968	0.1293	0.0903	0.0662	0.0505	0.0398	0.0322	0.0267
0.25	0	0.3534	0.2163	0.1425	0.0997	0.0732	0.0559	0.0441	0.0357	0.0296
0.27	0	0.3842	0.2357	0.1556	0.1091	0.0802	0.0613	0.0484	0.0392	0.0325
0.29	0	0.415	0.2551	0.1687	0.1185	0.0872	0.0668	0.0527	0.0427	0.0354
0.31	0	0.4458	0.2746	0.1818	0.1278	0.0942	0.0722	0.057	0.0462	0.0383
0.33	0	0.4767	0.294	0.195	0.1372	0.1012	0.0776	0.0613	0.0497	0.0412
0.35	0	0.5075	0.3134	0.2081	0.1466	0.1082	0.083	0.0656	0.0532	0.0441

Table 13. Growth Rate of Output (g_y), no depreciation,
AK research sector, vintage capital

$\bar{g} \setminus \gamma$	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
0.01	0	0	0	0	0	0	0	0	0	0
0.03	0	0	0	0	0	0	0	0	0	0
0.05	0	0.055	0	0	0	0	0	0	0	0
0.07	0	0.0887	0.0432	0	0	0	0	0	0	0
0.09	0	0.1206	0.0658	0.0312	0	0	0	0	0	0
0.11	0	0.152	0.0865	0.0493	0	0	0	0	0	0
0.13	0	0.1832	0.1066	0.064	0.0364	0	0	0	0	0
0.15	0	0.2142	0.1265	0.078	0.0482	0	0	0	0	0
0.17	0	0.2452	0.1462	0.0916	0.0587	0.0359	0	0	0	0
0.19	0	0.2761	0.1658	0.1051	0.0688	0.0448	0	0	0	0
0.21	0	0.307	0.1854	0.1185	0.0787	0.0528	0.0335	0	0	0
0.23	0	0.3379	0.2049	0.1319	0.0885	0.0605	0.0407	0	0	0
0.25	0	0.3687	0.2244	0.1452	0.0981	0.068	0.0471	0.0302	0	0
0.27	0	0.3996	0.2439	0.1584	0.1077	0.0754	0.0532	0.0365	0	0
0.29	0	0.4304	0.2634	0.1716	0.1173	0.0827	0.0591	0.0418	0.0256	0
0.31	0	0.4612	0.2828	0.1848	0.1268	0.0899	0.0649	0.0469	0.0321	0
0.33	0	0.492	0.3023	0.198	0.1363	0.0971	0.0707	0.0517	0.0369	0
0.35	0	0.5228	0.3217	0.2112	0.1458	0.1043	0.0763	0.0564	0.0412	0.0274

Table 14. Benchmark parameter values, Cobb-Douglas research sector

d_Y	d_A	σ_A	σ_N	σ_K	η_A	η_N	ρ	n	δ_A	δ_K
1	1	.3	.6	.4	.6	.5	.04	.015	.01	.05

Table 15. Equilibrium Values, benchmark parameters,
Cobb-Douglas research sector, non-vintage capital

γ	m	s	g_C, g_Y, g_K	g_A	g_Φ	g_Λ
.5	.8454	.2911	.0244	.0188	-.0066	-.0122
1	.8775	.2601	.0244	.0188	-.0188	-.0244
2	.9135	.2144	.0244	.0188	-.0431	-.0488
3	.9331	.1824	.0244	.0188	-.0675	-.0731
5	.9540	.1404	.0244	.0188	-.1163	-.1219

Table 16. Equilibrium Values, no depreciation,
Cobb-Douglas research sector, non-vintage capital

γ	m	s	g_C, g_Y, g_K	g_A	g_Φ	g_Λ
.5	.8828	.1868	.0244	.0188	-.0066	-.0122
1	.9102	.1515	.0244	.0188	-.0188	-.0244
2	.9388	.1099	.0244	.0188	-.0431	-.0488
3	.9536	.0862	.0244	.0188	-.0675	-.0731
5	.9687	.0602	.0244	.0188	-.1163	-.1219

Table 17. Equilibrium Values, benchmark parameters,
Cobb-Douglas research sector, vintage capital

γ	m	s	g_C, g_Y	g_A	g_Q	g_Φ	g_Λ
.5	.8779	.3043	.0244	.0188	.0384	-.0066	-.0263
1	.9123	.2754	.0244	.0188	.0384	-.0188	-.0384
2	.9480	.2315	.0244	.0188	.0384	-.0431	-.0628
3	.9655	.1996	.0244	.0188	.0384	-.0675	-.0872
5	.9815	.1566	.0244	.0188	.0384	-.1163	-.1359

Table 18. Equilibrium Values, no depreciation,
Cobb-Douglas research sector, vintage capital

γ	m	s	g_C, g_Y	g_A	g_Q	g_Φ	g_Λ
.5	.9285	.2321	.0244	.0188	.0384	-.0066	-.0263
1	.9539	.1960	.0244	.0188	.0384	-.0188	-.0384
2	.9762	.1495	.0244	.0188	.0384	-.0431	-.0628
3	.9855	.1209	.0244	.0188	.0384	-.0675	-.0872
5	.9930	.0874	.0244	.0188	.0384	-.1163	-.1359

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