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by

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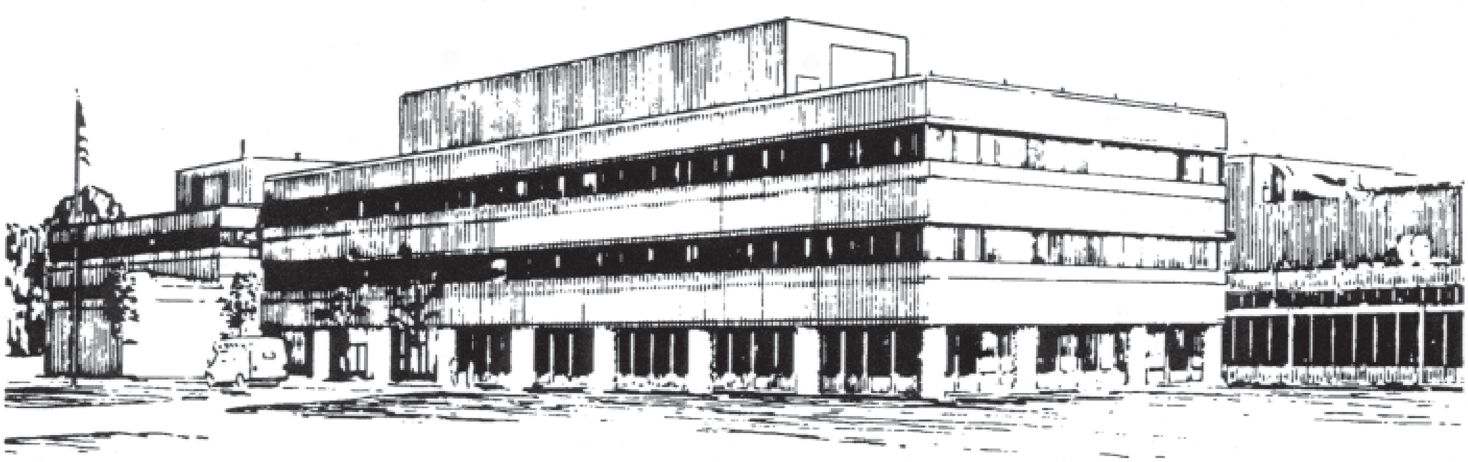

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Magnetic moment conservation in toroidal systems

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Abstract

Nonsecular variation of the magnetic moment in a large aspect ratio toroidal system is calculated. The cyclotron motion is shown to modify the bounce frequency of trapped particles by a factor of $\sqrt{2}$ over the value obtained using the guiding center approximation in the large aspect ratio limit. The increased bounce frequency causes a decrease of the radial diffusion by the same factor, and thus the standard expression for neoclassical diffusion at large aspect ratio is also incorrect by a factor of $\sqrt{2}$.

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Adiabatic conservation of the magnetic moment is held to be one of the fundamental principles of the motion of charged particles in a magnetic field, provided only that the variation of the field be small on the scale of the cyclotron radius and the temporal variation slow on the scale of the cyclotron frequency[1, 2, 3]. In this letter we show that for trapped particles in a toroidal configuration, the changes in the magnetic moment, although small, add in a coherent manner to produce a significant nonsecular variation of the magnetic moment during the bounce time, and hence a modification of the bounce frequency.

The energy in guiding center approximation for charged particle motion in an axisymmetric system is

$$E = \frac{mv_\parallel^2}{2} + \mu B \tag{1}$$

with μ the magnetic moment, $v_\parallel$ the velocity component parallel to the magnetic field $\vec{B}$, and m the particle mass. The field magnitude B is a function of r, a flux label, and θ, the poloidal angle. The guiding center motion is best studied using a Hamiltonian formulation[4, 5, 6, 7].

We directly calculate the nonsecular modification of the magnetic moment due to the cyclotron motion, ignored in the guiding center approximation. For simplicity consider a large aspect ratio circular equilibrium, with field given by

$$\frac{\vec{B}}{B_0} = \frac{\hat{\phi}R}{X} - \frac{z\hat{x}}{Rq} + \frac{x\hat{z}}{Rq} \tag{2}$$

with X, z, ϕ right handed cylindrical coordinates, $X = R + x$ the distance from the symmetry axis, R the major radius of the torus, ϕ the toroidal angle, and the safety factor q is a function of $r = \sqrt{x^2 + z^2}$, the minor radius variable, which also labels the flux surfaces.

The magnetic moment is given by

$$\mu = \frac{E}{B} - \frac{mv_\parallel^2}{2B} \tag{3}$$

where for this large aspect ratio field

$$v_\parallel = \frac{\vec{v} \cdot \vec{B}}{B} = -\frac{z\dot{x}}{qRB} + \frac{x\dot{z}}{qRB} + \frac{\dot{\phi}R}{B}, \tag{4}$$

and E is the conserved particle energy.

Separate the guiding center motion from the cyclotron motion, writing

$$
\begin{aligned}
x &= x_{gc} + \rho cos(\xi) \\
z &= z_{gc} + \rho sin(\xi) \\
\phi &= \phi_{gc}
\end{aligned}
\tag{5}
$$

where the subscript refers to guiding center, ξ is the gyro phase, and ρ the gyro radius. There is also a term proportional to ρ in ϕ which is higher order in the inverse aspect ratio $\epsilon = r/R$, which we neglect. Now calculate the time derivative of the magnetic moment

$$\dot{\mu} = -\frac{E\dot{B}}{B^2} + \frac{\dot{B}mv_\parallel^2}{2B^2} - \frac{mv_\parallel \dot{v}_\parallel}{B}, \tag{6}$$

and substitute the expressions from Eqs. 4, 5, neglecting the higher order time derivatives of ρ and keeping only the lowest order term in the derivative of ξ, $\dot{\xi} = \omega_0$, the on-axis cyclotron frequency. Then average Eq. 6 over the cyclotron period time scale, and note that terms involving the guiding center variables alone vanish, since μ is conserved by the guiding center equations, and terms linear in ρ average to zero. Thus there remain only terms in ρ^2, *ie* the terms which describe the effect of the finite extent of the gyro radius on the particle motion. We refer to transport theory which retains both neoclassical (banana) effects and cyclotron radius effects as omniclassical[9], since the result is not simply an addition of neoclassical and classical efffects. We keep only leading order contributions in the quantities ϵ and ρ/R. Regarding the ordering note that we are interested in trapped particles, and thus the parallel velocity at the midplane ($\theta = 0$) is $v_\parallel(0) \sim v\sqrt{\epsilon}$. Other orderings are $x_{gc} \sim \epsilon R$, $\dot{x}_{gc} \sim \epsilon^{3/2} v$, and similarly for z_{gc}, and $R\dot{\phi} \sim v\sqrt{\epsilon}$, with $v \sim \rho\omega_0$. We also neglect terms arising from the shear, dq/dr.

Both the second and third terms of Eq. 6 contribute to leading order, and the sum of these contributions gives, (using $B_b \mu_b = E$ where the subscript b refers to the bounce point where $v_\parallel = 0$),

$$\dot{\mu} = \frac{z_{gc}\dot{\phi}_{gc}}{qR}\mu_b. \tag{7}$$

The order of this term is $\epsilon^{3/2}\mu\omega_0\rho/R$. The scale length of the field variation is $L = R$, and thus this term is of order $(\rho/L)^{5/2}$. The proof of the invariance of the magnetic moment[1], shows that the time derivative of μ is zero to within terms of order $(\rho/L)^2$ and thus Eq. 7 does not contradict the classical proof of the conservation of the magnetic moment. However this expression changes sign only on the time scale of the bounce motion, and thus during a bounce it adds a coherent correction to μ. Furthermore the bounce time $T_b \sim 1/(v\sqrt{\epsilon})$ and thus during a bounce time the order of the change in μ is $\Delta\mu \sim \epsilon\mu$, the scaling of the bounce time reducing the ordering of the final result by one factor of ρ. It is this coherent modification of μ over many cyclotron periods (infinitely many in the limit of zero energy) which produces a significant modification of μ.

Substituting the lowest order expressions $z_{gc} = rsin(\theta)$, $\dot{\phi}_{gc} = q\dot{\theta}$ we find

$$\mu = \mu_b \left[1 - \frac{r}{R}(cos(\theta) - cos(\theta_b))\right]. \tag{8}$$

Thus we find the maximum modification of μ from the value at the bounce point to occur at $\theta = 0$ with $\delta\mu/\mu = -\epsilon(1 - cos(\theta_b))$. Energy is of course conserved, so there is a resulting increase in the parallel velocity, and for a particle with bounce angle $\theta_b = \pi/2$, to leading order $v_\parallel = v\sqrt{2\epsilon cos(\theta)}$, a factor of $\sqrt{2}$ increase over the guiding center result. Since the functional form is unchanged, this also leads to a similar increase in the bounce frequency.

We find it remarkable that an effect which depends on the ratio of the gyro radius to the scale length of the field variation persists in changing the bounce time by an energy independent factor with a limit of $\sqrt{2}$ at large aspect ratio. Furthermore this effect results from a term held to be negligible in the standard proofs of the invariance of the magnetic moment.

We have verified these results using two numerical codes; ORBIT[6] for guiding center motion and GYROXY[8] for full cyclotron motion, using large aspect ratio circular cross section toroidal geometry. The $q(r)$ profile was taken to vary from $q = 0.8$ at the magnetic axis to $q = 4$ at the plasma edge. In Fig. 1 is shown the variation of μ in time over one bounce period for a trajectory on a flux surface with $r/R = 1/8$. The ratio of μ to μ_b is plotted, with μ_b the value at the bounce point. The solid line is the code result, and the dashed line the first order prediction, Eq. 8. The orbit is initiated at $\theta = \pi/2$ with zero parallel velocity and low energy, with $\rho/R \sim 10^{-3}$. In Fig. 2 is shown the magnitude of the maximum variation of μ at $\theta = 0$ as a function of the inverse aspect ratio, ϵ, for a series of orbits initiated at different radii. Excellent agreement with Eq. 8 is obtained in the limit of large aspect ratio.

In Fig. 3 is shown the ratio of the actual bounce frequency resulting from following the complete cyclotron orbits to the value obtained using the guiding center code, seen to approximate $\sqrt{2}$ for large aspect ratio. The same series of orbits were launched at $\theta = \pi/2$ with zero parallel velocity in both codes.

Radial diffusion is obtained by depositing particles uniformly on a flux surface with a uniform pitch distribution, and observing the spread of the resulting Gaussian under the action of energy conserving pitch angle scattering. The results are shown in Fig. 4. In the case of full cyclotron motion, the initial conditions also included random gyro phase. These results were obtained with $r/R = 1/8$, again using a circular cross section equilibrium. Plotted is the square of the deviation from the initial flux surface vs time. The initial jump, occuring on the order of a bounce time, reflects the banana width in the case of guiding center motion and the cyclotron orbit modified banana width in the case of full cyclotron orbits. After this, the slope of the resulting straight line gives the diffusion in the flux coordinate. The upper curve (G) is the result using the guiding center code, and the lower curve (C) is that resulting from the complete cyclotron motion. The relevant number determining the diffusion is ν^*, the ratio of the collision frequency to the bounce frequency[10], and the two diffusion rates are observed to differ by the factor $\sqrt{2}$.

Care was taken to use exactly the same collision frequency, since the collision operators in the two representations are substantially different. Energy conserving pitch angle scattering was used in both codes. In the guiding center code we used the Lorentz collision operator[11]

$$\lambda' = \lambda(1 - \nu\tau) \pm [(1 - \lambda^2)\nu\tau]^{1/2} \tag{9}$$

with $\lambda = v_\parallel/v$ the pitch and τ the magnitude of the time step. In the cyclotron orbit code the velocity vector was moved in a random direction by a small angle on the surface $v = constant$. The equivalance of these operators was checked by observing the spread in velocity space of a distribution initially with $v_\parallel = 0$ up to the point of obtaining a distribution with uniform pitch, the two curves of mean square pitch, $< \lambda^2 >$, matching within statistical accuracy of a few percent over the entire history. The pitch angle scattering frequency used was $\nu = 10^{-6}\omega_0$ giving $\nu^* = 0.05$. Also shown is the neoclassical analytic banana regime result (A) for large aspect ratio,

$d\Psi^2 = 2D_\Psi t$, with Ψ the normalized poloidal flux, and

$$D_\Psi = \nu \left(\frac{\rho}{R}\right)^2 \sqrt{r/R}. \tag{10}$$

The intercept at $t = 0$ is simply to avoid confusion with the other curves, only the slope of this plot is relevant.

We have also verified numerically that the guiding center trajectory of the orbit, including cyclotron motion, is not changed, as is expected. The result described in this paper has no effect on the banana width nor on the toroidal precession, these both being due to drift motion, which can be calculated directly and is correctly given by the guiding center equations. All that is changed by the order ρ^2 cyclotron orbit corrections is the parallel velocity along the orbit, and hence the bounce frequency. However, this modification applies also to passing particles near the trapped-passing boundary, and thus we expect also an increase in the bootstrap current by the same factor of $\sqrt{2}$, although we have not calculated this. Other effects include of course a modification of any resonance phenomena which involve the bounce frequency or the trapped particle velocity.

In conclusion, we find the assumption of μ conservation to be incorrect, in that terms thought previously to be inconsequential in fact modify the trapped particle bounce time by a factor of $\sqrt{2}$ in the large aspect ratio limit. This modification also has a direct effect on radial diffusion, the standard neoclassical expression being incorrect by the same factor.

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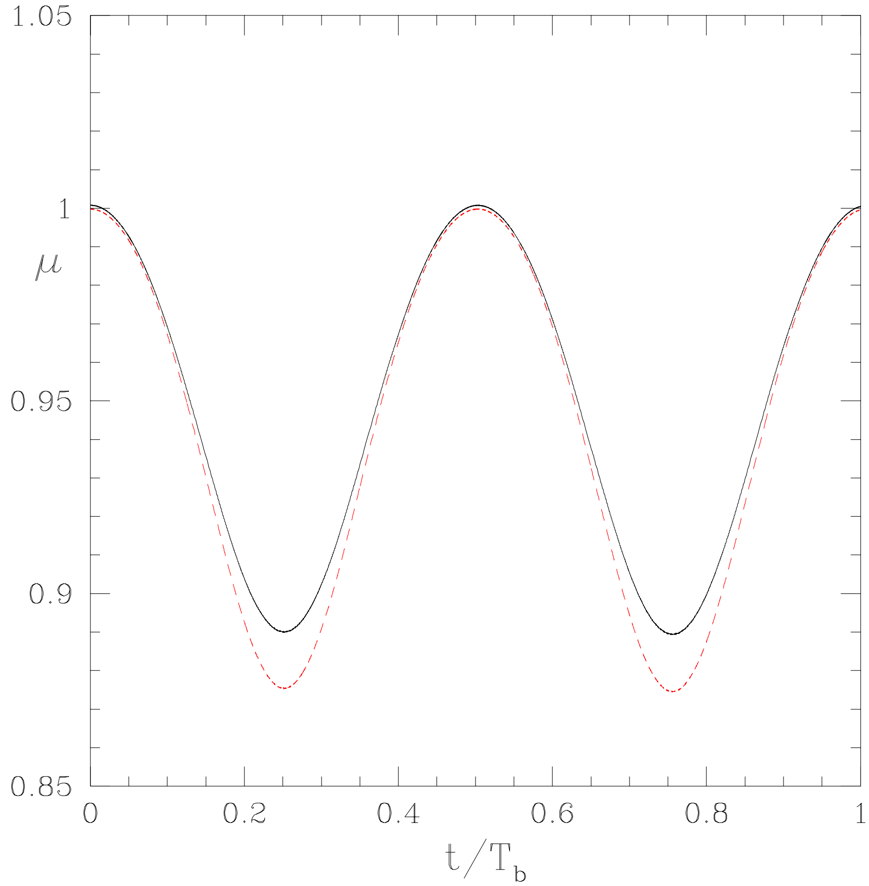

Figure 1: Magnetic moment vs time over one bounce period T_b, for an orbit initiated at $\theta = \pi/2$ and $r/R = 1/8$ with zero parallel velocity. the dashed curve is the lowest order analytical result.

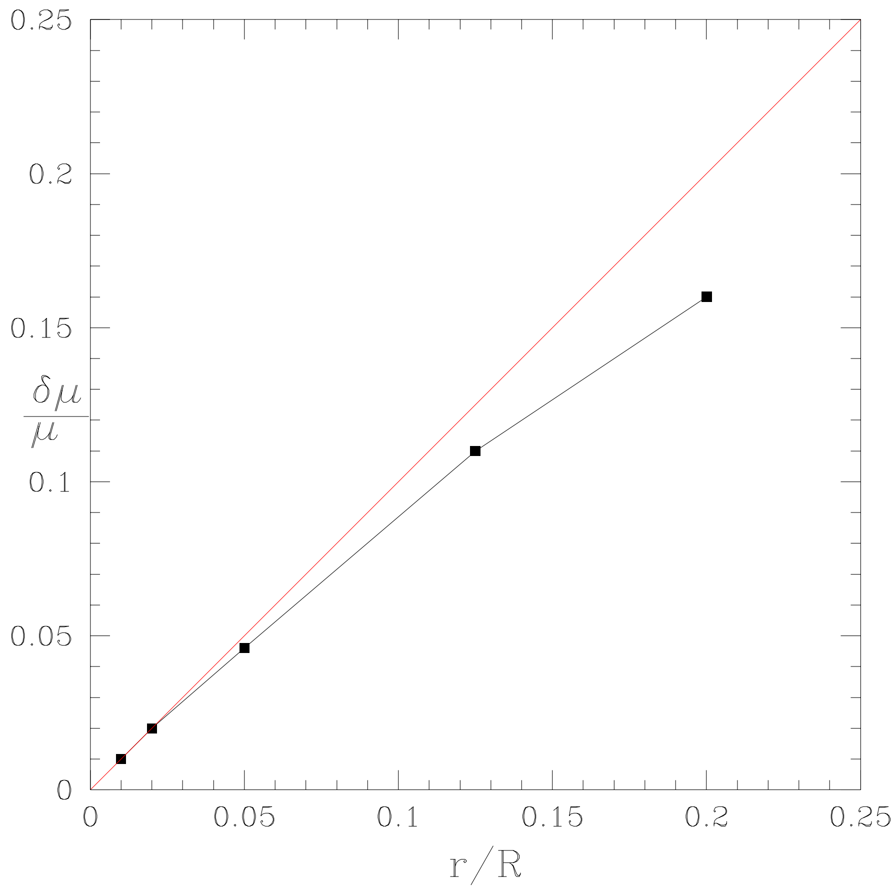

Figure 2: Maximum change in the magnitude of the magnetic moment vs r/R, with orbits initiated at $\theta = \pi/2$ with zero parallel velocity.

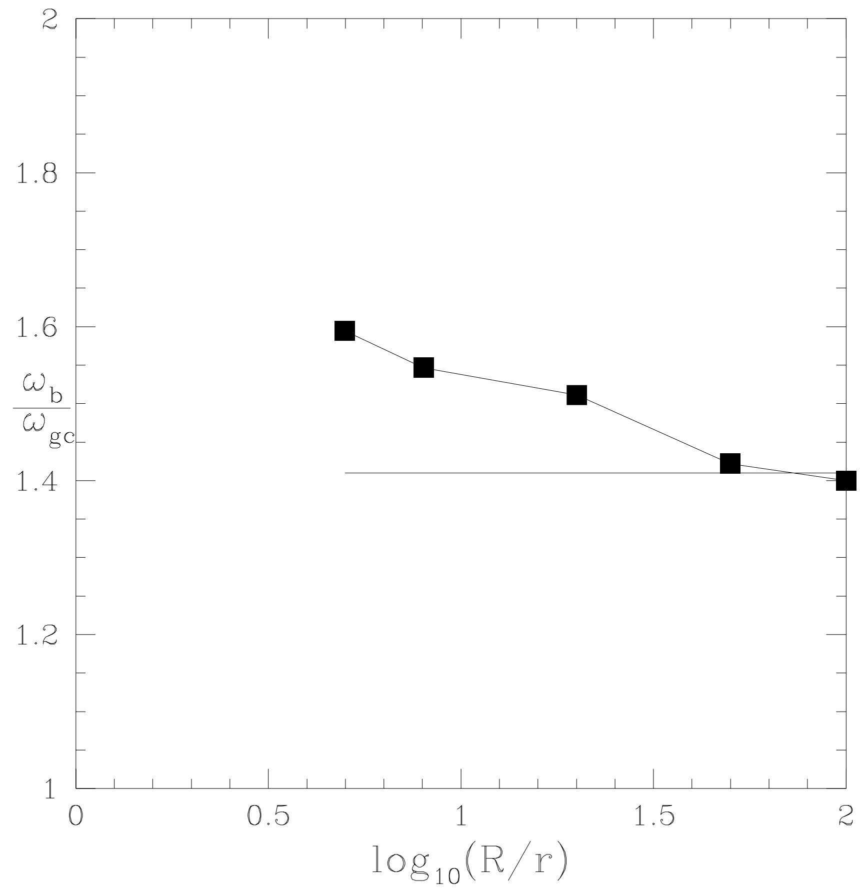

Figure 3: Ratio of the bounce frequency ω_b obtained using the full cyclotron equations, to ω_{gc}, resulting from the guiding center equations, vs the aspect ratio.

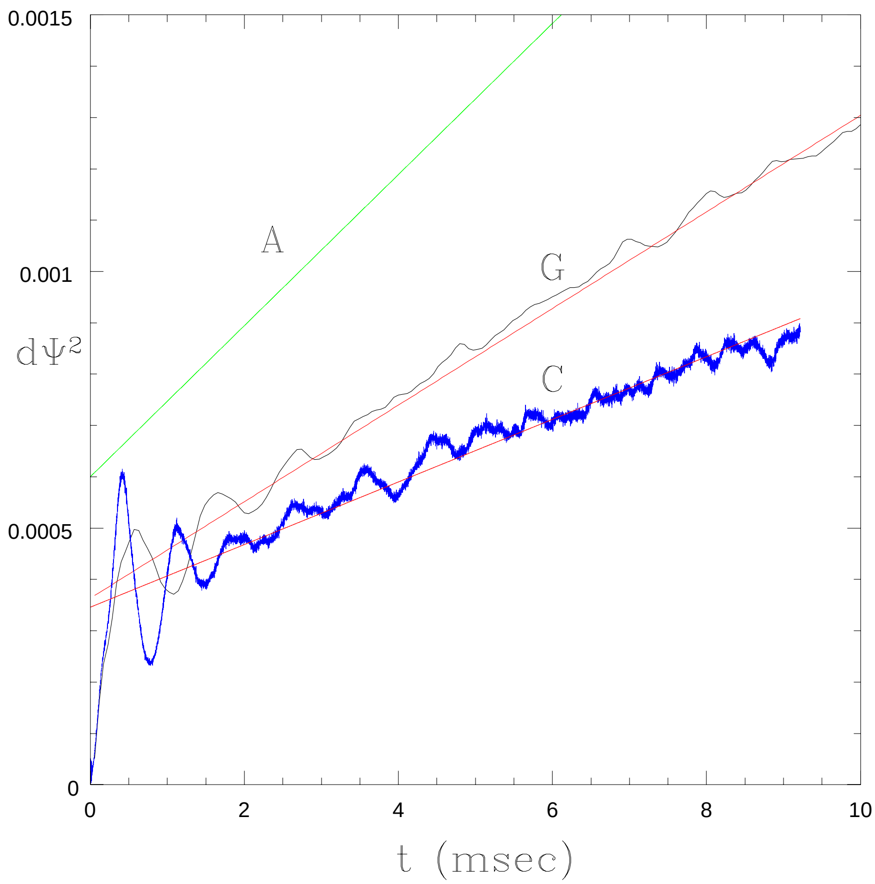

Figure 4: Diffusion from the flux surface $r/R = 1/8$ for full cyclotron motion (C), guiding center motion (G), and the large aspect ratio analytic estimate (A).

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