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On The Causality of Plasma Permittivity

C. R. Cockrell

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National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665-5225

ON THE CAUSALITY OF PLASMA PERMITTIVITY

ABSTRACT

Causal solutions for effective relative complex permittivities of plasmas are derived. Both lossless and lossy plasmas are considered.

INTRODUCTION

The analysis of aperture antennas in the presence of plasma media is usually approached by assuming steady-state plane wave incidence on a plane plasma layer. For inhomogeneous plasma, the differential equations describing the fields within the plasma must be solved numerically in most instances. The inhomogeneity is expressed through the effective complex permittivity of the plasma media [1-3]. The permittivity is derived from the equation of motion of the electrons within the media for an applied electric field [1,3]. The permittivity depends on the operating, collision, and plasma frequencies.

Whenever the initial time of the input is important or a spectrum of frequencies is considered, one must be concerned with causality; i.e., the response (electric flux) of a system (permittivity) cannot precede the input (electric field) [4]. The effective complex permittivity for an inhomogeneous plasma as derived in the literature, which is based on steady-state plane wave analysis, can be easily shown to be *noncausal.*

In this paper a causal solution for plasma permittivity is derived. The solutions are determined by using familiar mathematical transforms. Causality solutions for both lossless and lossy plasmas are considered.

NONCAUSAL AND CAUSAL PERMITTIVITIES

The effective relative complex permittivity for a plasma media is given as [1,2]

$$\varepsilon(\omega) = \left(1 - \frac{\omega_p^2}{\omega^2 + \nu^2} - j \frac{\omega_p^2 \; (\nu/\omega)}{\omega^2 + \nu^2} \right) \tag{1}$$

where ω_p is the plasma frequency, ν is the collision frequency (corresponds to losses), and ω is the operating frequency. For the permittivity given by equation (1) to be causal, its inverse Fourier transform must be zero for times less than zero. The inverse Fourier transform is given as

$$\bar{\varepsilon}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varepsilon(\omega) \; e^{j\omega t} d\omega \tag{2}$$

With the expression for $\varepsilon(\omega)$ substituted, equation (2) becomes

$$\bar{\varepsilon}(t) = \delta(t) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega_p^2}{\omega^2+\nu^2} \; e^{j\omega t} \; d\omega - j \; \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega_p^2 \left(\frac{\nu}{\omega}\right)}{\omega^2+\nu^2} \; e^{j\omega t} \; d\omega \tag{3}$$

where $\delta(t)$ is the impulse function. For times greater than zero, equation (3) is evaluated by contour integration and becomes

$$\bar{\varepsilon}_{>}(t) = \frac{\omega_p^2}{2\nu} \left(1 - 2e^{-\nu t} \right) , \; t > 0 \tag{4}$$

where $\bar{\varepsilon}_{>}(t)$ represents $\bar{\varepsilon}(t)$ for times greater than zero. Evaluating equation (3) for times less than zero yields

$$\bar{\varepsilon}_{<}(t) = - \frac{1}{2} \frac{\omega_p^2}{\nu} , \quad t < 0 \tag{5}$$

where $\bar{\varepsilon}_{<}(t)$ represents $\bar{\varepsilon}(t)$ for times less than zero. The total function $\bar{\varepsilon}(t)$ is sketched in figure 1. Since $\bar{\varepsilon}(t)$ exists for times less than zero, the function $\varepsilon(\omega)$ is *not* causal. However, by adding the quantity $\pi \frac{\omega_p^2}{\nu} \delta(\omega)$ to the right-hand side of equation (1), the system function can be made causal; i.e., the equation now representing $\varepsilon(\omega)$ becomes

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2+\nu^2} + \pi \frac{\omega_p^2}{\nu} \delta(\omega) - j \frac{\omega_p^2 \left(\frac{\nu}{\omega}\right)}{\omega^2+\nu^2} \tag{6}$$

The inverse Fourier transform of equation (6) readily verifies that the function $\varepsilon(\omega)$ is indeed causal; i.e.,

$$\bar{\varepsilon}(t) = \delta(t) + \begin{cases} \frac{\omega_p^2}{\nu} \left(1 - e^{-\nu t} \right) & t > 0 \\ 0 & t < 0 \end{cases} \tag{7}$$

which is sketched in figure 2. The effective relative complex permittivity for a plasma with losses of a causal system is, therefore, given by equation (6).

The additional term needed for causality can be determined in a more rigorous way. If equation (1) is derived first in terms of the Laplace transform variable s, the causality solution given in equation (6) follows directly upon applications of the inverse Laplace transform and the Fourier transform. The equivalent expression for equation (1) in terms of the Laplace transform variable s is

$$\varepsilon(s) = 1 + \frac{\omega_p^2}{s(s+\nu)} \tag{8}$$

The inverse Laplace transform is written as

$$\bar{\varepsilon}(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \varepsilon(s)\, e^{st}\, ds \quad , \qquad t \geq 0 \tag{9}$$

which when evaluated becomes

$$\bar{\varepsilon}(t) = \delta(t) + \begin{cases} \frac{\omega_p^2}{\nu} \left[1 - e^{-\nu t} \right] & t > 0 \\ 0 & t < 0 \end{cases} \tag{10}$$

Equation (10) is identical to equation (7) which has been shown to produce a causal result. The frequency representation given by equation (6) is found by taking the Fourier transform of equation (10); i.e.,

$$\varepsilon(\omega) = \int_{-\infty}^{\infty} \bar{\varepsilon}(t)\ e^{-j\omega t} dt \tag{11}$$

With equation (10) substituted,

$$\varepsilon(\omega) = \int_{0}^{\infty} \delta(t)\ e^{-j\omega t} dt + \frac{\omega_p^2}{\nu} \int_{0}^{\infty} e^{-j\omega t} dt - \frac{\omega_p^2}{\nu} \int_{0}^{\infty} e^{-(\nu+j\omega)t} dt \tag{12}$$

The first and third integrals are readily evaluated so that equation (12) is written as

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\nu} \int_{0}^{\infty} e^{-j\omega t} dt - \frac{\omega_p^2}{\nu} \frac{1}{\nu+j\omega} \tag{13}$$

The remaining integral cannot be evaluated as it stands. However, rewriting equation (13) as

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\nu} \frac{1}{2} \int_{-\infty}^{\infty} \left[1 + \mathrm{sgn}(t)\right] e^{-j\omega t} dt - \frac{\omega_p^2}{\nu} \frac{1}{\nu+j\omega} \tag{14}$$

where

$$\mathrm{sgn}(t) = \begin{cases} +1 & t > 0 \\ -1 & t < 0 \end{cases} \tag{15}$$

equation (13) can be evaluated as

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{\nu} \pi\ \delta(\omega) - j\ \frac{\omega_p^2}{\nu\omega} - \frac{\omega_p^2}{\nu} \frac{1}{\nu+j\omega} \tag{16}$$

or equivalently,

$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2+\nu^2} + \frac{\omega_p^2}{\nu} \pi\ \delta(\omega) - j\ \frac{\omega_p^2\left[\frac{\nu}{\omega}\right]}{\omega^2+\nu^2} \tag{17}$$

which is identical to equation (6).

Thus far only permittivity with losses has been considered. To examine the lossless case ($\nu = 0$), one cannot simply evaluate equation (17) with $\nu = 0$. However, one can substitute $\nu = 0$ in equation (8) yielding

$$\varepsilon(s) = 1 + \frac{\omega_p^2}{s^2} \tag{18}$$

The inverse Laplace transform of equation (18) becomes

$$\bar{\varepsilon}(t) = \delta(t) + \omega_p{}^2 t \qquad , \qquad t \geq 0 \tag{19}$$

which is sketched in figure 3. The Fourier transform of equation (19) is

$$\varepsilon(\omega) = 1 + \omega_p^2 \int_0^\infty t\, e^{-j\omega t}\, dt \tag{20}$$

which cannot be evaluated in its present form. Rewriting the t part of the integrand as

$$t = \frac{1}{2}\, t \left[1 + \text{sgn}(t) \right] \tag{21}$$

where sgn(t) has been defined earlier in equation (15), the expression for $\varepsilon(\omega)$ becomes

$$\varepsilon(\omega) = 1 + \frac{\omega_p^2}{2} \int_{-\infty}^\infty t\, e^{-j\omega t} dt + \frac{\omega_p^2}{2} \int_{-\infty}^\infty t\, \text{sgn}(t)\, e^{-j\omega t} dt \tag{22}$$

Since

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-j\omega t} dt \tag{23}$$

and

$$\delta'(\omega) = \frac{1}{2\pi} \int_{-\infty}^\infty (-jt)\, e^{-j\omega t} dt \tag{24}$$

where the prime denotes the derivative with respect to ω, equation (22) becomes

$$\varepsilon(\omega) = 1 + j\, \omega_p^2\, \pi\, \delta'(\omega) + \frac{\omega_p^2}{2} \int_{-\infty}^\infty t\, \text{sgn}(t)\, e^{-j\omega t} dt \tag{25}$$

Since the Fourier transform of the product of two functions is proportional to the convolution of their transforms, equation (25) can be written as

$$\varepsilon(\omega) = 1 + j\,\omega_p^2\,\pi\,\delta'(\omega) + \frac{1}{2\pi}\,\mathcal{F}\left\{\frac{t}{2}\right\} * \mathcal{F}\left\{\mathrm{sgn}(t)\right\} \tag{26}$$

where the symbol * denotes the convolution operation and $\mathcal{F}$ denotes the Fourier transform. With the transforms substituted, equation (26) can be written explicitly as

$$\varepsilon(\omega) = 1 + j\,\omega_p^2\,\pi\,\delta'(\omega) + \omega_p^2 \int_{-\infty}^{\infty} \delta'(\beta)\,\frac{1}{\omega - \beta}\,d\beta \tag{27}$$

which upon evaluation becomes

$$\boxed{\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega^2} + j\,\pi\,\omega_p^2\,\delta'(\omega)} \tag{28}$$

Equation (28), therefore, represents the effective relative complex permittivity for plasmas without losses for a causal system.

CONCLUDING REMARKS

By using familiar mathematical transforms, causal representations for the effective relative complex permittivities of plasmas have been derived. The causal permittivities for lossy and lossless plasmas were found to be highly singular at zero frequency. Permittivities which obey the principle of causality could be an important consideration in time-domain numerical solutions of Maxwell's equations.

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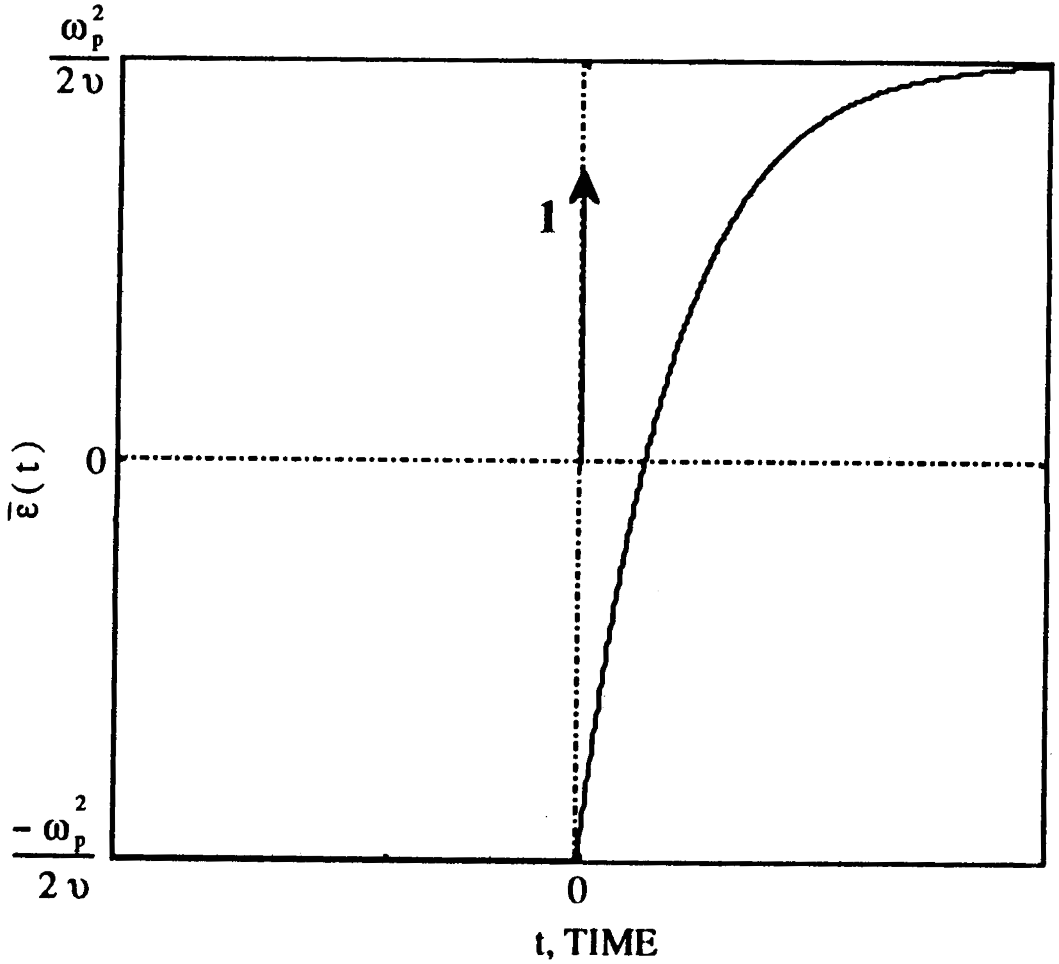

Fig. 1 Noncausal impulse response for plasma with losses.

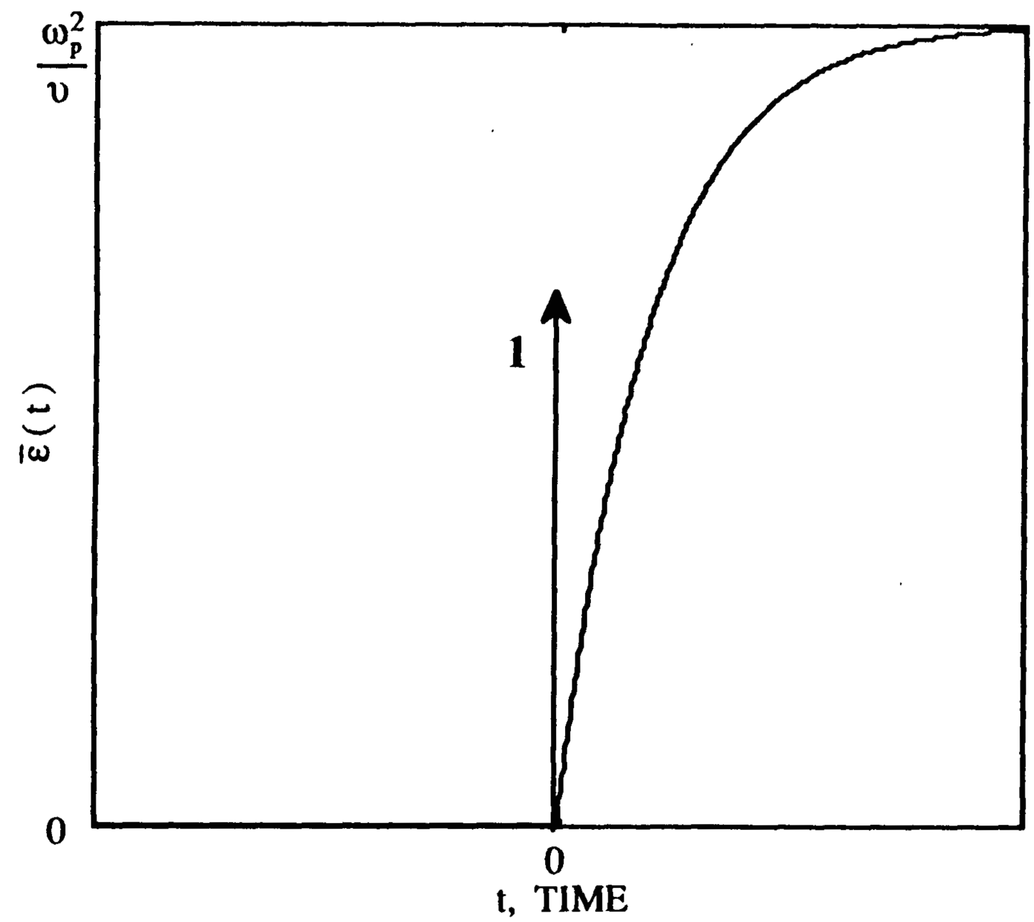

Fig. 2 Causal impulse response for plasma with losses.

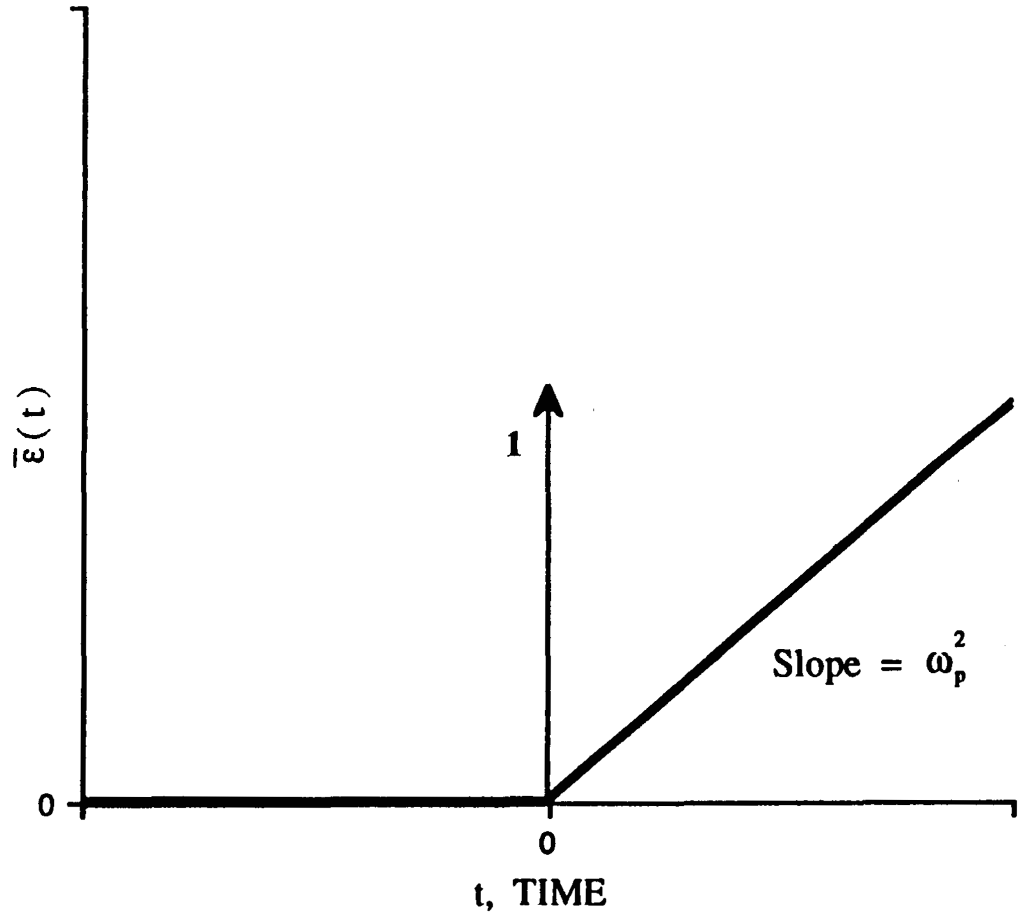

Fig. 3 Causal impulse response for plasma without losses.

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16. Abstract

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