Exact solutions to magnetized plasma flow

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Exact analytic solutions for steady-state magnetized plasma flow (MPF) using ideal magnetohydrodynamics formalism are presented. Several cases are considered. When plasma flow is included, a finite plasma pressure gradient ∇p can be maintained in a force-free state $\mathbf{J}\times\mathbf{B}=0$ by the velocity gradient. Both incompressible and compressible MPF examples are discussed for a Taylor-state spheromak $\mathbf{B}$ field. A new magnetized nozzle solution is given for compressible plasma when $\mathbf{U}\|\mathbf{B}$. Transition from a magnetized nozzle to a magnetic nozzle is possible when the $\mathbf{B}$ field is strong enough. No physical nozzle would be needed in the magnetic nozzle case. Diverging-, drum- and nozzle-shaped MPF solutions when $\mathbf{U}\perp\mathbf{B}$ are also given. The electric field is needed to balance the $\mathbf{U}\times\mathbf{B}$ term in Ohm's law. The electric field can be generated in the laboratory with the proposed conducting electrodes. If such electric fields also exist in stars and galaxies, such as through a dynamo process, then these solutions can be candidates to explain single and double jets.
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I. INTRODUCTION

High-speed plasma wind and cosmic jets are well-known phenomena in the universe.[1,2] In fusion experiments, when external energies and/or momentum are used to drive the plasma, plasma motion, such as rotation[3–5] and flow along the magnetic fields,[6] is observed routinely.[7–9] These diverse phenomena are examples of plasma flow within a magnetic field: the magnetized plasma flow (MPF). The MPF can be described by the magnetohydrodynamics (MHD) equations with the plasma momentum term included.

The MPF is also used to address technology concerns. Using a magnetic field instead of a physical boundary to guide the plasma fluid flow in a converging–diverging configuration leads to the concept of a "magnetic nozzle."[10,11] Magnetically nozzled plasma flow is more desirable over the materially nozzled flow because of the potentially longer lifetime and more controllable operation in the first case. Magnetic nozzles certainly can be used for propulsion and material processing.[12]

Theoretical studies of the MPF began in the mid 1950s.[13–15] Approximate axially symmetric steady-state solutions were obtained by Morozov and Solovev.[16] Exact incompressible solutions were given for a generalized symmetry with one ignorable spatial coordinate.[17] Special axisymmetric, nonsteady MPF was studied by Colwell.[18] In general, without a certain type of symmetry, the MPF problem is too complicated. Computational methods have to be used.[19,20]

We have obtained several exact solutions to axisymmetric MPF under various assumptions. Section II briefly presents the formulation of axisymmetric MPF. The formalism introduced will be used in Sec. III B, where purely rotating MPF will be discussed, and in Sec. V, where a class of MPF solutions with purely poloidal flow and toroidal magnetic field will be derived. In Sec. III, based on known solutions, force-free MPF solutions with finite pressure are given. Application of the solutions to a special force-free magnetic state—a Taylor-state spheromak[21–24] is discussed for both incompressible and compressible flows.

The general compressible MPF formalism was discussed in detail by Morozov and Solovev.[25] Transonic MPF with translational symmetry along the z axis were studied by Lifshitz and Goedbloed,[26] and works cited therein. We demonstrate the existence of a new axisymmetric magnetized nozzle solution in Sec. IV, and discuss the transition of a magnetized nozzle to a magnetic nozzle. The distinction between a magnetized nozzle and a magnetic nozzle is that the former relies on a material boundary—the physical nozzle in the conventional sense—to accelerate plasma, while the latter solely relies on the converging–diverging magnetic field to confine the plasma flow. One distinction between a magnetized nozzle and a conventional nozzle is that the former has a magnetic field within the flow. Another distinction between a magnetized nozzle and a conventional nozzle is that the conventional nozzle usually operates with neutral gas, and a magnetized nozzle operates most effectively using plasmas, or ionized gases.

In Sec. V, using the mathematical formalism introduced in Sec. II, new MPF solutions with purely poloidal flow (the MPF does not cross the r–z plane) and purely toroidal magnetic field are obtained. Realization of the flow in laboratory settings using conducting boundaries are emphasized. Three specific examples are given. The analytic solutions derived here may also be used to bench mark new computational codes.

II. PROBLEM FORMULATION

Steady-state MHD equations with flow have been studied in both fusion and astrophysics contexts. Ideal incompressible plasma flow was studied by many authors.[27–32]

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Ideal MHD flow equations were also derived by several authors independently.[25,33–36] The steady-state ideal MPF can be described by the ideal MHD equations with the plasma fluid momentum term included. These equations are Faraday's law in steady state

$$\nabla\times\mathbf{E}=0, \tag{1}$$

Ampere's law

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}, \tag{2}$$

divergence-free law for magnetic field

$$\nabla\cdot\mathbf{B}=0, \tag{3}$$

ideal Ohm's law

$$\mathbf{E}+\mathbf{U}\times\mathbf{B}=0, \tag{4}$$

steady-state single-fluid momentum equation

$$\rho\mathbf{U}\cdot\nabla\mathbf{U}=-\nabla p+\mathbf{J}\times\mathbf{B}, \tag{5}$$

and the steady-state continuity equation

$$\nabla\cdot(\rho\mathbf{U})=0, \tag{6}$$

where ρ is the mass density, $\mathbf{U}$ represents the flow velocity, and the other symbols have their usual meanings. By assuming axisymmetry in cylindrical coordinates $(r,\ \theta,\ z)$, θ can be ignored. (The symmetry need not be chosen to be cylindrical.[25]) From Faraday's law, Eq. (1), the electric field can be expressed as the gradient of a potential $\mathbf{E}=-\nabla\Phi$. The azimuthal electric field E_θ is zero from axisymmetry.

For axisymmetric configurations, the magnetic field can be generally expressed in terms of two scalar functions Ψ and I, where[37] $\mathbf{B}=\nabla\Psi\times\nabla\theta+I\nabla\theta$. Since $\mathbf{B}\cdot\nabla\Psi=0$, $\Psi=$ constant defines a magnetic flux surface. Similarly, from the continuity equation, Eq. (6), the velocity $\mathbf{U}$ can be expressed in terms of scalar functions ξ and Γ as $\mathbf{U}=(1/\rho)\nabla\xi\times\nabla\theta+\Gamma\nabla\theta$, where $\xi=$ constant defines a plasma-flow surface, or so-called streamline.

Introduction of the functions Φ, Ψ, I, ξ, and Γ modify the ideal MHD equation set in the following ways: Equations (1), (2), (3), and (6) are satisfied automatically. Equations (1)–(6) reduce to Eq. (4), the Ohm's law, and Eq. (5), the momentum equation. This set of equations is not complete without inclusion of an equation of state relating pressure p and mass density ρ.

We can write a general form of the remaining equations by defining the Poisson's bracket for any two quantities Ψ and I as $[\Psi,I]=(\partial\Psi/\partial r)(\partial I/\partial z)-(\partial I/\partial r)(\partial\Psi/\partial z)$. Then, the radial and axial Ohm's laws can be expressed as

$$-\nabla\Phi+\frac{\Gamma}{r^2}\nabla\Psi-\frac{I}{\rho r^2}\nabla\xi=0, \tag{7}$$

$$\frac{1}{\rho}[\Psi,\xi]=0. \tag{8}$$

The momentum balance, Eq. (5), becomes two equations, one for axial momentum and the second a description of conservation of angular momentum

$$\nabla\left(\frac{\mathbf{U}^2}{2}+w\right)-\frac{1}{\rho r^2}(\boldsymbol{\Delta}^*_{[\rho]}\xi\nabla\xi+\rho\Gamma\nabla\Gamma)$$
$$=-\frac{1}{\mu_0\rho r^2}(\boldsymbol{\Delta}^*\Psi\nabla\Psi+I\nabla I), \tag{9}$$

$$[\xi,\Gamma]=\frac{1}{\mu_0}[\Psi,I]. \tag{10}$$

Here, the equation of state is assumed to be of the form $\nabla w=\nabla p/\rho$ with w usually known as the enthalpy. The generalized operator $\Delta^*_{[\rho]}$ with kernel ρ is defined as

$$\boldsymbol{\Delta}^*_{[\rho]}\xi\equiv\nabla\cdot\left(\frac{\nabla\xi}{\rho}\right)-\frac{2}{\rho r}\frac{\partial\xi}{\partial r}. \tag{11}$$

III. FORCE-FREE MPF

Force-free states are defined as plasma states within which the electromagnetic force $\mathbf{J}\times\mathbf{B}$ vanishes. Force-free conditions are believed widely applicable in astrophysical environments because forces other than electromagnetic are comparatively much smaller. Force-free states can also appear within a conducting boundary, a so-called flux conserver, in a laboratory environment. A typical example is a relaxed spheromak state, also known as a Taylor state.[38] A force-free equilibrium with mass flow and finite pressure exists for a constant density ρ.[39] Here, another type of force-free MPF with a finite-pressure profile is given. Assume $\mathbf{E}=0$ within a plasma. From the ideal Ohm's law (4), one obtains $\mathbf{U}\times\mathbf{B}=0$. That is, in a ideal MPF within which the electric field vanishes, the flow has to align with the magnetic field. This is also the known as the ''frozen-in law.'' A general incompressible solution $\nabla\cdot\mathbf{U}=0$ was worked out by Tataronis and Mond,[40] where

$$\mathbf{U}=\frac{\mathbf{B}}{\sqrt{\mu_0\rho}}, \tag{12}$$

for $\mathbf{B}$-aligned plasma flow and a finite pressure sustained by the flow, and

$$p+\frac{\rho\mathbf{U}^2}{2}=\text{constant}. \tag{13}$$

From the incompressible and $\mathbf{B}$-aligned $\mathbf{U}$ conditions, it can be shown that ρ is a function of flux surfaces only, $\rho\equiv\rho(\Psi)$. This solution was first derived for the $\mathbf{J}\times\mathbf{B}\neq 0$ case.[40] The well-known solution for a constant-density plasma with flow along a magnetic field due to Chandrasekhar[27] is a special case. We now apply the solution of Eqs. (12) and (13) to the force-free case, and point out the solution implies a finite plasma pressure with flow. In addition, we will find out that the flow-supported pressure gradient is usually different from the magnetic flux gradient. Therefore, equal-pressure surfaces do not coincide with the magnetic flux surfaces.

A. Incompressible MPF with finite pressure

Equations (12) and (13) give a finite-pressure profile for any force-free state,

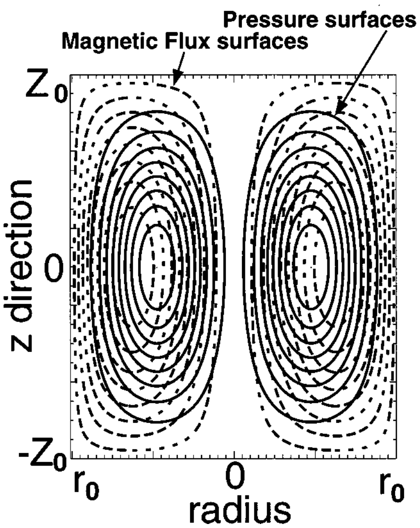

FIG. 1. Cross section of the magnetic flux surfaces and equal pressure surfaces for a Taylor-state spheromak with incompressible flow parallel to the magnetic field $\mathbf{U}\|\mathbf{B}$.

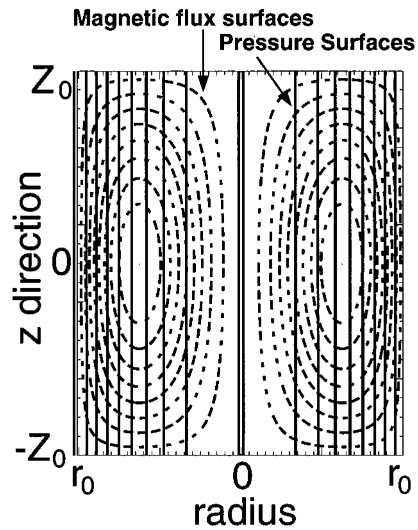

FIG. 2. Cross section of the magnetic flux surfaces and equal-pressure surfaces for a Taylor-state spheromak with purely toroidal rotation of compressible flow.

$$\frac{\mathbf{B}^2}{2\mu_0}+p=\frac{\mathbf{B}_0^2}{2\mu_0}+p_0, \tag{14}$$

where $\mathbf{B}_0$ and p_0 are integration constants that have the magnetic-field unit and pressure unit, respectively. This solution implies that the pressure distribution is independent of the density distribution. The shapes of equal-pressure surfaces are shown in Fig. 1 using a spheromak equilibrium magnetic field satisfying $\nabla\times\mathbf{B}=\lambda\mathbf{B}$ and constant λ. The boundary condition was chosen to be a perfectly conducting cylinder with radius r_0 and heights extending from $-Z_0$ to Z_0. The pressure and magnetic surfaces are no longer coincident with each other. There is a significant displacement between the axis of these surfaces. The displacement between the magnetic and pressure surfaces also exists for a different kind of plasma flow.[36] Assume that the ideal gas law $p=\rho k_B T/M$ is valid for the present case, where k_B is the Boltzmann constant, T is the plasma temperature, and M is the ion mass. Since the plasma density is a function of magnetic flux surfaces, and the plasma pressure is not a function of the flux surface, the plasma temperature T is generally not a function of the flux surface.

B. Compressible MPF with finite pressure

A rotation-only force-free MPF is defined by a flow with vanishing poloidal flow component $\xi=0$ and solely with finite toroidal rotational component $\Gamma\neq 0$. Ohm's law, Eq. (8), is satisfied identically. Ohm's law, Eq. (7), implies that $\Phi=\Phi(\Psi)$, and

$$\Phi'_\Psi=\frac{\Gamma}{r^2}, \tag{15}$$

where Φ'_Ψ stands for the first-order differentiation of Φ with respect to Ψ. Using the force-free condition, the only nontrivial equation left is the momentum Eq. (9)

$$\nabla\left(\frac{\mathbf{U}^2}{2}+w\right)-\frac{\Gamma}{r^2}\nabla\Gamma=0. \tag{16}$$

It can be proven for nontrival solutions, that is, $\Psi\neq\Psi(r)$, which requires

$$\frac{\Gamma}{r^2}=\omega_0, \tag{17}$$

where ω_0 is a constant angular velocity. This is the law of isorotation first discovered by Ferraro,[41] and discussed by many authors later on.[28] In general, ω_0 may be a function of the magnetic flux surface. However, in the force-free case discussed here, only constant ω_0 throughout the plasma is allowed for $\Psi\neq\Psi(r)$. Since we assumed $p=p(\rho)$ here, we can use the usual adiabatic or isothermal equation of state of the form

$$\frac{p}{p_0}=\frac{\rho^\gamma}{\rho_0^\gamma}, \tag{18}$$

with $\gamma=5/3$ for the adiabatic case and $\gamma=1$ for the isothermal case. The pressure profile is given by

$$\frac{\gamma}{\gamma-1}\frac{p}{\rho}-\frac{\omega_0^2 r^2}{2}=\frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0}-\frac{\omega_0^2 r_0^2}{2}, \tag{19}$$

for $\gamma\neq 1$, and

$$p=p_0\exp\left(\frac{\omega_0^2 r^2-\omega_0^2 r_0^2}{2}\right), \tag{20}$$

for $\gamma=1$, the isothermal case. In either case, both the pressure and the density are functions of radius only. An example of the equal-pressure surfaces is shown together with a Taylor-state magnetic equilibrium in Fig. 2. Again, it is noticeable that the pressure and magnetic surfaces are no longer coincident with each other. The difference from that of the incompressible situation is that the equal-pressure surfaces are open ended.

IV. NOZZLE-TYPE MPF

According to the well-known gas-dynamics theory, the continuity equation, Eq. (6), Bernoulli's equation $\mathbf{U}^2/2 + \int (dp/\rho) =$ constant and the equation of state, Eq. (18), together, when integrated over streamlines in a flow region of area A, give rise to the Hugonoit equation $[(U^2/C_s^2) - 1]\, dU/U = dA/A$, where $C_s \equiv \sqrt{\gamma p/\rho}$ is the sound speed. The Hugonoit equation leads to a nozzle type of solution for the flow only when the gas is compressible. Below, we demonstrate one kind of magnetized nozzle solution for the MPF. We will consider systems of ionized gas–plasma flow, not neutral gas flow, so that the magnetic field can be effectively confining to ions and electrons. Substantial external energy, either in the form of dc electric energy, rf wave energy, or any other form, is needed to maintain a gas in a plasma state. In another scenario, once a plasma is created upstream of the nozzle, if the plasma transit time through the nozzle system is much less than the electron–ion recombination time, then no extra energy is needed along the flow to maintain a plasma state.[42]

A. Magnetized nozzle

In this subsection, we identify the magnetized compressible steady flow conditions so that the equation of state, Eq. (18), and Bernoulli's equation $\mathbf{U}^2/2 + \int (dp/\rho) =$ constant along the streamlines are still valid. Assume the flow is magnetic-field aligned, $\mathbf{U} \parallel \mathbf{B}$. $\mathbf{U}$ dotted into the momentum equation, Eq. (5), gives

$$\mathbf{U}\cdot\nabla\left(\frac{\mathbf{U}^2}{2}+\int\frac{dp}{\rho}\right)=0, \tag{21}$$

which means Bernoulli's equation is still valid along the streamlines. To satisfy the continuity equation and the divergence-free condition for the magnetic field, one solution is that

$$\mathbf{B}=\lambda_0\rho\mathbf{U}, \tag{22}$$

with λ_0 a constant. Therefore, Eq. (22), together with the continuity equation, Bernoulli's equation, and the adiabatic equation of state, form a complete set of solutions for magnetized nozzles.

In the special case of vanishing internal plasma current, $\mathbf{J}=0$, one further has that the mass density gradient is along the flow $\nabla\rho\times\mathbf{B}=0$. The continuity Eq. (6); the equation of state, Eq. (18); and Bernoulli's equation govern the physical boundary that forms a nozzle-shaped object. Due to the absence of the electric current within the plasma, the magnetic field described is entirely produced by external current sources, such as the electric current flowing in the conducting coils. Also, due to the absence of the electric current within the plasma, there is no acceleration effect from the electromagnetic force in this type of magnetized nozzle. The inclusion of a background magnetic field, however, may be beneficial to lower the heat load on the nozzle walls due to the magnetic confinement of charged particles.

B. Magnetic nozzle

A magnetic nozzle is defined as a ''nozzle'' that uses the magnetic field instead of a physical boundary (a mechanical nozzle) to confine the fluid flow.[12] A nozzle type of flow solution is possible if the magnetic field is shaped in a conventional converging–diverging nozzle configuration, with plasma flow along the magnetic field. When the magnetic field is strong enough [i.e., $\lambda_0\to\infty$ in Eq. (22)], then the conventional physical boundary is no longer needed in a magnetized nozzle and we, therefore, achieve the magnetic nozzle operation. One requirement on the strength of the magnetic field is that the ion gyroradius be much less than the smallest dimension of the magnetic nozzle system. However, due to the fact that an electron gyroradius is much less than an ion gyroradius, the charge separation between ions and electrons could induce large electric fields that would eventually prevent the charge separation. In other words, ambipolar diffusion will be set up in the steady state. Therefore, to expect that the ion gyroradius is much less than the smallest dimension of the magnetic nozzle system is too strong a statement.[42] For the magnetic field to be effectively confining, the particle diffusion time across the magnetic field must be much greater than the transit time along the magnetic field, that is,

$$\frac{R^2}{D}\gg\frac{L_z}{C_s}, \tag{23}$$

in which R and L_z are characteristic dimensions in the radial and axial directions, respectively. D is the averaged cross-field diffusion coefficient, and C_s is the sound speed at the nozzle throat. In an ideal case, one can use the classical diffusion coefficient $D=r_{ic}^2\nu_{ie}$, where $r_{ic}=\sqrt{2m_iT_i}/eB_0$ is the singly charged ion gyroradius and $\nu_{ie} = n_e e^4 \ln\Lambda/3\varepsilon_0^2\sqrt{(2\pi)^3 m_i T_e^3}$, while all the symbols have their usual meanings in plasma physics, e is the electron charge, n_e is the plasma density, etc. Equation (23) implies $(R^2/L_z)\,C_s \gg 2e^2 \ln\Lambda/3\varepsilon_0^2\sqrt{[1/(2\pi)^3 m_i T_i]}\,(\rho/B_0^2)$, $T_i=T_e$ is a characteristic temperature, B_0 is an average magnetic field, and $\rho=n_e m_i$ is the mass density. Since the diffusion mechanism depends on many aspects of the problem, such as the initial spatial distribution of the injection plasma in the upstream region of the nozzle and the boundary conditions, using the classical diffusion coefficient here only serves to demonstrate the concepts.[42] In real experiments, it should not be surprising when other diffusion forms of the coefficient work better. An additional constraint is that the plasma must be collisional enough (the density high enough) for the fluid approximation to be valid. Otherwise, if the plasma is collisionless both along and across the magnetic field, a magnetic nozzle turns into a mirror-type magnetic-confinement device, where the total particle energy and the magnetic momentum are conserved for each ion. Since the cross-field diffusion time is much greater than the transit time along the magnetic field, the least requirement on the plasma collisionality is that the plasma must be collisional across the magnetic field (radial diffusion time is much greater than the collision time) while it may be collisionless along the magnetic field.[42]

V. MPF WITH POLOIDAL FLOW

Now we consider cases of MPF with poloidal flow only, $\xi \neq 0$, and $\Gamma = 0$. Some emphasis is put on how to realize these flows in a laboratory environment using conducting electrodes. Assume that the plasma flow is perpendicular to the magnetic field, $\mathbf{U}\cdot\mathbf{B}=0$. Using the axisymmetric formulation with stream function ξ, magnetic flux Ψ, poloidal electric current I, and plasma rotation Γ, then the $\mathbf{U}\cdot\mathbf{B}=0$ condition reduces to $\nabla\Psi\cdot\nabla\xi=0$. Since ξ is nonzero, therefore $\Psi=0$ (or a constant) is a simple solution, which will be studied below. Other solutions with I nonzero but constant for this type of MPF have been obtained previously.[36] In the present case, our solutions are more general.

A. Equation reduction

Ohm's law (7) under the above assumptions gives

$$\nabla\Phi + \frac{I}{\rho r^2}\nabla\xi = 0, \tag{24}$$

which means $\Phi=\Phi(\xi)$, and $I/\rho r^2 = -\Phi'_\xi$. Combining this new form of Ohm's law and the equation of motion (9), one obtains

$$\nabla\left(\frac{\mathbf{U}^2}{2} + w + \frac{I^2}{\mu_0 \rho r^2}\right) - \frac{\mathbf{\Delta}^*_{[\rho]}\xi\nabla\xi}{\rho r^2} = -\frac{I\nabla\Phi'_\xi}{\mu_0}. \tag{25}$$

Therefore,

$$\frac{\mathbf{U}^2}{2} + w + \frac{I^2}{\mu_0\rho r^2} = G(\xi), \tag{26}$$

is a function of ξ only. Therefore, Eq. (25) can be written in a scalar form:

$$\frac{\mathbf{\Delta}^*_{[\rho]}\xi}{\rho r^2} = \frac{dG(\xi)}{d\xi} - \frac{\rho r^2\Phi'_\xi}{\mu_0}\frac{d\Phi'_\xi}{d\xi}. \tag{27}$$

B. Solutions

Assume the plasma fluid is incompressible with constant density ρ. Equation (27) reduces to

$$\frac{\mathbf{\Delta}^*\xi}{\rho^2 r^2} = \frac{dG(\xi)}{d\xi} - \frac{\rho r^2\Phi'_\xi}{\mu_0}\frac{d\Phi'_\xi}{d\xi}. \tag{28}$$

Equation (28) can have the so-called self-similar solutions[17,43] by introducing a new variable

$$t \equiv r^2\omega, \tag{29}$$

where $\omega\equiv\omega(z)$ is a function of z only, and the streamline function is a function of t only, $\xi=\xi(t)\equiv\xi(r^2\omega)$. Substituting this expression into Eq. (28), one has

$$\frac{\omega}{\rho^2 t}\left[4\omega t\xi_{tt} + \xi_{tt}t^2\left(\frac{\omega_z}{\omega}\right)^2 + \xi_t t\frac{\omega_{zz}}{\omega}\right] = \frac{dG(\xi)}{d\xi} - \frac{\rho t\Phi'_\xi}{\mu_0\omega}\frac{d\Phi'_\xi}{d\xi}, \tag{30}$$

in which, ξ_t stands for differentiation of ξ with respect to t, and so forth. Now, seek the solution of the following form

$$\left(\frac{\omega_z}{\omega}\right)^2 = a\omega + b + \frac{c}{\omega} + \frac{d}{\omega^2}. \tag{31}$$

Differentiate Eq. (31) with respect to z to obtain

$$\frac{\omega_{zz}}{\omega} = \frac{3}{2}a\omega + b + \frac{c}{2\omega}. \tag{32}$$

Using expressions (31) and (32) for $(\omega_z/\omega)^2$ and ω_{zz}/ω, and collecting terms with equal power of ω, we find that Eq. (30) corresponds to four ordinary differential equations:

$$(4+ta)\xi_{tt} + \tfrac{3}{2}a\xi_t = 0, \tag{33}$$

$$b(\xi_t + t\xi_{tt}) = 0, \tag{34}$$

$$\frac{c}{\rho^2}\left(\xi_{tt}t + \frac{1}{2}\xi_t\right) = \frac{dG(\xi)}{d\xi}, \tag{35}$$

and

$$\frac{d\xi_{tt}}{\rho^2} = -\frac{\rho\Phi'_\xi}{\mu_0}\frac{d\Phi'_\xi}{d\xi}. \tag{36}$$

From Eqs. (33) and (34), it can be proven that if $b\neq 0$, then we must have $\xi_t=0$, which is a trivial solution with zero-flow velocity throughout. For nontrivial solutions we conclude that $b=0$. From Eq. (33), the solution for ξ is obtained as

$$\xi = \frac{\xi_0}{\sqrt{4+ta}}. \tag{37}$$

There is an additive integration constant labeling streamlines that can be set to zero. It is also understood from the self-similar solution assumption that $t=r^2\omega(z)$ with an arbitrary dependence on z. The velocity field is described by

$$\mathbf{U} = \frac{-a}{2\xi_0^2}\xi^3\nabla(r^2\omega)\times\nabla\theta. \tag{38}$$

The solution to the total energy $G(\xi)$ is[44]

$$G(\xi) = \frac{c}{\rho^2}\frac{a}{8\xi_0^2}\xi^4 - \frac{c}{\rho^2}\frac{a}{2\xi_0^4}\xi^6 + G_0, \tag{39}$$

where G_0 is a integration constant. The electric potential is described by[45]

$$\Phi = \Phi_0 \pm \int d\xi\sqrt{F_0 - d\frac{\mu_0}{\rho^3}\frac{a^2}{4\xi_0^4}\xi^6}. \tag{40}$$

Both Φ_0 and F_0 are integration constants.

Three examples of this type of MPF are shown: in Fig. 3, for diverging $t=(r/r_0)^2\exp(-z/z_0)$, in Fig. 4, for drum-shaped $t=(r/r_0)^2\cosh(-z/z_0)$, and in Fig. 5, for nozzle shaped $t=(r/r_0)^2\exp[-(z/z_0)^2]$. These flow configurations could be realized in a laboratory environment by setting up conducting boundaries, which are both streamlines and equal potential surfaces at the same time. The conducting electrode

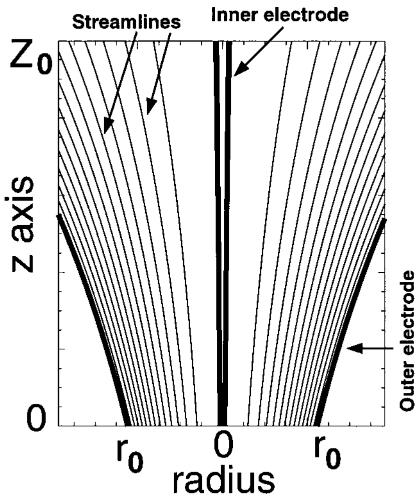

FIG. 3. Cross section of an ideal diverging axisymmetric MPF. Streamlines are shown with diverging–conducting-electrode boundaries (marked inner and outer electrode). $t=(r/r_0)^2 \exp(-z/Z_0)$. r_0, Z_0 are characteristic dimensions.

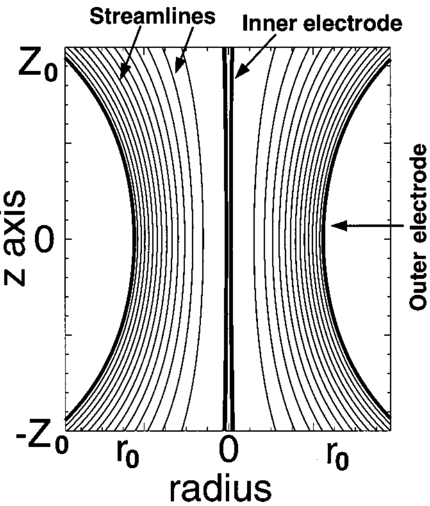

FIG. 5. Cross section of an ideal nozzle-shaped axisymmetric MPF. Conducting boundaries and streamlines are marked. $t=(r/r_0)^2 \exp[-(z/Z_0)^2]$. r_0, Z_0 are characteristic dimensions.

boundaries are marked in Figs. 3, 4, and 5. Solutions may also be used to explain astrophysics flow phenomena. The diverging configuration corresponds to the single-jet case, the nozzle configuration corresponds to the double-jet case. Since electric-field generation is equivalent to the driving electric current within a plasma, which can be realized through a dynamo process, the laboratory electrode boundary conditions could be replaced by internal dynamo processes within stars or galaxies.

Realization of these MPF solutions in the laboratory would be of great interest since it is now promising to have a detailed comparison of the exact theoretical results with experiments when the conditions prescribed here are met. The MPF has the potential to be used widely, such as in plasma-based electric propulsion and in alternative fusion concepts.[46] In addition, generation of the MPF in the laboratory is essential to studying fundamental MPF physics itself. In order to apply the results to astrophysical phenomena, an essential step is to understand how the dynamo process leads to the needed electric field and how a dynamo process would arise. Further efforts such as identifying proper astrophysical objects are also expected.

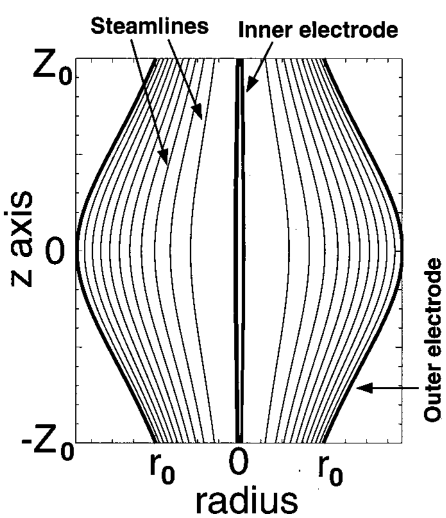

FIG. 4. Cross section of an ideal drum-shaped axisymmetric MPF. Conducting boundaries and streamlines are marked. $t=(r/r_0)^2 \cosh(-z/Z_0)$. r_0, Z_0 are characteristic dimensions.

VI. SUMMARY

The magnetized plasma flow is formulated using steady-state ideal MHD equations. Exact MPF solutions are obtained under various assumptions. When one assumes that the plasma fluid is in a equipotential state, the internal electric field $\mathbf{E}$ vanishes. Then the MPF is restricted along the magnetic field $\mathbf{U}\|\mathbf{B}$. When a finite electric field is produced by external electrodes at different electric potentials or by internal processes within a star or a galaxy, the flow velocity $\mathbf{U}$ does not need to align with the magnetic field.

Based on known solutions to incompressible steady-state MPF, we discussed the force-free magnetic field MPF with finite pressure gradients, which can be sustained by velocity gradients. Both incompressible MPF and compressible MPF examples are given for a Taylor-state spheromak magnetic structure. In the incompressible case, pressure surfaces are closed concentric axisymmetric toroids offset from the flux surface. In the compressible case, the pressure surfaces are open-ended concentric cylinders.

Magnetized nozzle solutions are obtained with the magnetic field relating to the mass density and flow velocity as $\mathbf{B}=\lambda_0\rho\mathbf{U}$. λ_0 is a constant proportionality parameter. A very special case in which the magnetic field is entirely generated by external currents outside the plasma, $\mathbf{J}=0$, and in which plasma flow is compressible, is discussed in detail. The transition from a magnetized nozzle to a magnetic nozzle, i.e., from one with a material confining boundary (mechanical nozzle) to one without it, is possible when the magnetic field is strong enough and shaped in a converging–diverging configuration. This type of magnetic nozzle relies on the internal energy to accelerate particles to supersonic speed with no electromagnetic energy consumption. The electromagnetic force effect in the derived general nozzle solution with nonvanishing plasma current **J** will be the topic of future work.

MPF solutions are also given when the magnetic field is purely toroidal, that is, only B_θ is nonvanishing in cylindrical symmetry, and the flow is purely poloidal, that is, only in the $r-z$ plane under cylindrical symmetry. Three representative cases, termed diverging-, drum-, and nozzle-shaped solutions, are given explicitly. The way to realize these flows in a laboratory environment is to shape the conducting electrodes at different electric potentials. We expect that when the electric field can be generated by the internal processes of a star or galaxy, these MPF may explain observed astrophysical flow phenomena.

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[1] R. D. Blandford, M. C. Begelman, and M. J. Rees, Sci. Am. **246**, 124 (1981).
[2] M. I. Pudovkin and V. S. Semenov, Space Sci. Rev. **41**, 1 (1986).
[3] S. Suckewer, H. P. Eubank, R. J. Goldston, E. Hinnov, and N. R. Sauthoff, Phys. Rev. Lett. **43**, 207 (1979).
[4] A. Bondeson and D. J. Ward, Phys. Rev. Lett. **72**, 2709 (1994).
[5] M. S. Chu, J. M. Greene, T. H. Jensen, R. L. Miller, A. Bondeson, R. W. Johnson, and M. E. Mauel, Phys. Plasmas **2**, 2236 (1995).
[6] N. Sato, Y. Watanabe, R. Hatakeyama, and T. Mieno, Phys. Rev. Lett. **61**, 1615 (1988).
[7] F. Zonca, S. A. Cohen, J. Cuthbertson, J. Timberlake, and D. Ruzic, J. Nucl. Mater. **176**, 746 (1990).
[8] A. F. Almagri, J. T. Chapman, C. S. Chiang, D. Craig, D. J. Hartog, C. C. Hegna, and S. C. Prager, Phys. Plasmas **5**, 3982 (1998).
[9] N. Asakura, S. Sakurai, M. Shimada, Y. Koide, N. Hosogane, and K. Itami, Phys. Rev. Lett. **84**, 3093 (2000).
[10] K. F. Schoenberg, R. A. Gerwin, I. Henins, R. A. Mayo, J. T. Scheuer, and G. A. Wurden, IEEE Trans. Plasma Sci. **21**, 625 (1993).
[11] J. T. Scheuer, K. F. Schoenberg, R. A. Gerwin, R. P. Hoyt, I. Henins, D. C. Black, R. M. Mayo, and R. W. Moses, IEEE Trans. Plasma Sci. **22**, 1015 (1993).
[12] K. F. Schoenberg, R. A. Gerwin, R. W. Moses, Jr., J. T. Scheuer, and H. P. Wagner, Phys. Plasmas **5**, 2090 (1998).
[13] L. Woltjer, Astrophys. **130**, 405 (1959).
[14] H. Grad, Rev. Mod. Phys. **32**, 830 (1960).
[15] E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
[16] A. I. Morozov and L. S. Solovev, Sov. Phys. Tech. Phys. **9**, 337 (1964).
[17] L. Del Zanna and C. Chiuderi, Astron. Astrophys. **310**, 341 (1996).
[18] D. J. Colwell, J. Appl. Math. Phys. **25**, 55 (1974).
[19] R. Zelazny, R. Stankiewicz, A. G. Ikowski, and S. Potempski, Plasma Phys. Controlled Fusion **35**, 1215 (1993).
[20] C. P. T. Groth, D. L. DeZeeuw, T. I. Gombosi, and K. G. Powell, Space Sci. Rev. **87**, 193 (1999).
[21] S. Chandrasekhar and P. C. Kendall, Astrophys. J. **126**, 457 (1957).
[22] J. B. Taylor, Phys. Rev. Lett. **33**, 139 (1974).
[23] M. N. Rosenbluth and M. N. Bussac, Nucl. Fusion **19**, 489 (1979).
[24] L. J. Porter, J. A. Klimchuk, and P. A. Sturrock, Astrophys. J. **385**, 738 (1992).
[25] A. I. Morozov and L. S. Solovev, *Review of Plasma Physics* (Consultants Bureau, New York, 1980), Vol. 8, p. 1.
[26] A. Lifschitz and J. P. Goedbloed, J. Plasma Phys. **58**, 61 (1997).
[27] S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **42**, 273 (1956).
[28] K. C. Tsinganos, Astrophys. J. **245**, 764 (1981).
[29] E. Hameiri, Phys. Fluids **26**, 230 (1983).
[30] R. V. E. Lovelace, C. Mehanian, C. M. Mobarry, and M. E. Sulkanen, Astrophys. J. **62**, 1 (1986).
[31] U. Gebhardt and M. Kiessling, Phys. Fluids B **4**, 1689 (1992).
[32] G. M. Webb, M. Brio, and J. P. Zank, J. Plasma Phys. **52**, 141 (1994).
[33] Y. A. Baransky, Ph.D. thesis, Columbia University, 1987.
[34] K. Brushlinskii and A. Morozov, *Review of Plasma Physics* (Consultants Bureau, New York, 1980), Vol. 8, p. 105.
[35] P. Rosenau, J. Tataronis, and G. Conn, J. Plasma Phys. **21**, 385 (1979).
[36] C. Copenhaver, Phys. Fluids **26**, 2635 (1983).
[37] S. Chandrasekhar, Proc. Natl. Acad. Sci. U.S.A. **42**, 1 (1956).
[38] T. R. Jarboe, Plasma Phys. Controlled Fusion **36**, 945 (1994).
[39] R. N. Sudan, Phys. Rev. Lett. **42**, 1277 (1979).
[40] J. A. Tataronis and M. Mond, Phys. Fluids **30**, 84 (1987).
[41] V. C. A. Ferraro, Mon. Not. R. Astron. Soc. **97**, 1458 (1937).
[42] R. A. Gerwin (private communication).
[43] A. A. Solovev and E. A. Soloveva, Astron. Lett. **23**, 316 (1997).
[44] $\xi_t = (-a/2\xi_0^2)\,\xi^3$, and $\xi_{tt} = (3a^2/4\xi_0^4)\,\xi^5$.
[45] $\Phi'_\xi = \pm\sqrt{F_0 - d\,(\mu_0/\rho^3)(a^2/4\xi_0^4)\,\xi^6}$.
[46] Z. Wang, Phys. Rev. Lett. (submitted).