Asymptotically Efficient Spherical Codes—Part II: Laminated Spherical Codes*

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Abstract

New spherical codes called *laminated spherical codes* are constructed in dimensions 2-49 using a technique similar to the construction of laminated lattices. Each spherical code is recursively constructed from existing spherical codes in one lower dimension. Laminated spherical codes outperform the best known spherical codes in the minimum distance sense for many code sizes. The density of a laminated spherical code approaches the density of the laminated lattice in one lower dimension, as the minimum distance approaches zero. In particular, the three-dimensional laminated spherical code is asymptotically optimal, in the sense that its density approaches the Fejes Tóth upper bound as the minimum distance approaches zero. Laminated spherical codes perform asymptotically as well as wrapped spherical codes in those dimensions where laminated lattices are optimal sphere packings. The laminated spherical codes are also structured, which leads to an efficient decoding algorithm in low dimensions.

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1 Introduction

Part I of this two paper series [1] described a technique to map any packing Λ onto the unit k-dimensional sphere S_k. In the present paper (Part II), a new technique is introduced to construct spherical codes called laminated spherical codes. Whereas wrapped spherical codes were described by an explicit function that maps $\mathbb{R}^{k-1}$ onto the unit sphere S_k for any $k \in \mathbb{Z}$, laminated spherical codes will be defined in a recursive manner using terminology and techniques from laminated lattice constructions. The laminated spherical codes improve upon previously known codes, and for low dimensions and many code sizes outperform the wrapped spherical codes described in Part I [1]. Most of the known best performing spherical codes in three dimensions with less than about 30,000 codepoints and in four and five dimensions with less than about 150 codepoints are due to Hardin and Sloane and are publically available over the Internet (see [2]). For codes larger than the Hardin-Sloane codes, the laminated spherical codes introduced in this paper often give the best known performance.

See [1] for applications, definitions, and background of spherical codes. The basic notation and definitions in [1] will not be repeated here. The remainder of Section 1 outlines some results and terminology regarding laminated lattices from [3, chapter 6] that will be useful in the remainder of the paper. Section 2 gives the formal construction of laminated spherical codes, derives their asymptotic density, and gives numerical comparisons. Section 3 gives efficient decoding algorithms.

A *lattice* is the closure under integer vector addition of a finite set of basis vectors. The 1-dimensional laminated lattice with minimum distance 1 is defined by

$$\Lambda_1 \equiv \mathbb{Z}. \tag{1}$$

For $k \geq 2$, a *k-dimensional laminated lattice* Λ_k is a lattice whose minimum distance is 1, which contains a $(k-1)$-dimensional laminated lattice as a sublattice, and whose density is the highest possible under these conditions. Λ_k is not unique, in general, although its density is unique.

Given a set of points $S \subset \mathbb{R}^k$, a *hole* of S is a point $X \in \mathbb{R}^k$ for which there exists an $\epsilon > 0$ such that for all $Y \in \mathbb{R}^k$, if $\|X - Y\| < \epsilon$, then $\inf_{Z \in S} \|X - Z\| \geq \inf_{Z \in S} \|Y - Z\|$. X is a *deep* *hole* if $\inf_{Z \in S} \|X - Z\| \geq \inf_{Z \in S} \|Y - Z\|$ for all $Y \in \mathbb{R}^k$. The *covering radius* of S is defined as $\sup_{X \in \mathbb{R}^k} \inf_{Z \in S} \|X - Z\|$.

Λ_k is constructed by stacking up layers of Λ_{k-1} as closely as possible, and can be decomposed as

$$\Lambda_k = \bigcup_{l=-\infty}^{\infty} \Lambda_{k-1}^{(l)}, \tag{2}$$

where $\Lambda_{k-1}^{(l)}$ is the lth layer of Λ_k (i.e., a translation of Λ_{k-1}). The *subcovering radius* c_{k-1} of Λ_{k-1} is defined such that $\sqrt{1 - c_{k-1}^2}$ is the distance between layers of Λ_k. Note that c_k is a lower bound on the covering radius of Λ_k. The values of c_k are known for $k \leq 47$, and are tabulated in [3, p. 158] for laminated lattices that are scaled by a factor of 2.

Let

$$l_k \equiv \min \left\{ i > 0 \colon (0, \ldots, 0, i\sqrt{1 - c_{k-1}^2}) \in \Lambda_k \right\} \tag{3}$$

denote the *layer number* of Λ_k. The layer number is the smallest number of consecutive layers of Λ_{k-1} stacked within Λ_k such that the top layer is "directly over" the bottom layer, i.e., differing in the last coordinate only. For example, $l_2 = 2$, $l_3 = 3$, and $l_4 = 2$. Since adjacent layers of Λ_k are separated by $\sqrt{1 - c_{k-1}^2}$ and the distance between distinct points of Λ_k is at least 1, we have $l_k \sqrt{1 - c_{k-1}^2} \geq 1$.

Corresponding to each point $X = (x_1, \ldots, x_{k-1}) \in \Lambda_{k-1}^{(l)}$ is an associated unique hole $h(X)$ whose distance from X is c_{k-1} and which is opposite a lattice point in $\Lambda_{k-1}^{(l+1)}$. Let n_{k-1} denote the number of points of $\Lambda_{k-1}^{(l)}$ which are nearest lattice points of $h(X)$, and let $n(X)$ denote the lattice point of Λ_{k-1}^{l+1} which is opposite $h(X)$. That is, $h(X)$ has n_{k-1} nearest neighbors in $\Lambda_{k-1}^{(l)}$, regardless of the choice of X (one of these nearest neighbors is X). These nearest neighbor lattice points are denoted by $D(X)_1, \ldots, D(X)_{n_{k-1}}$, and the convex hull of these points forms a *Delaunay cell* (e.g., [3, p. 35]). For any finite set of points $\mathcal{P} \subset \mathbb{R}^k$, define

$$H(\mathcal{P}) \equiv \arg \max_{Y \in CHULL(\mathcal{P})} \min_{X \in \mathcal{P}} \|X - Y\|. \tag{4}$$

$H(\mathcal{P})$ is a hole of $\mathcal{P}$ that lies within the convex hull of $\mathcal{P}$, and for each $X \in \mathcal{P}$,

$$h(X) = H(\{D(X)_1, \ldots, D(X)_{n_{k-1}}\}). \tag{5}$$

For $k \leq 8$, if $X \in \Lambda_{k-1}$, then $h(X)$ is the centroid of the set $\{D(X)_1, \ldots, D(X)_{n_{k-1}}\}$.

2 Laminated spherical codes

Any spherical code can be described by the projection of its codepoints to the interior of a sphere of one less dimension, via the mapping $(x_1, \ldots, x_{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} x_i^2}) \to (x_1, \ldots, x_{k-1})$. If the projected codepoints are constrained to lie on a small number of radii $\left(\sum_{i=1}^{k-1} x_i^2\right)^{1/2}$, then a structured packing can be obtained. Conversely, a k-dimensional spherical code may be obtained by placing codepoints on concentric $(k-1)$-dimensional spheres and projecting each codepoint onto S_k. We exploit this idea and pack the concentric spheres closely. We place codepoints of one sphere at the radial extension of the holes of codepoints of the next smaller radius sphere, and use a method similar to constructing laminated lattices (e.g., [3]) to construct new spherical codes, which we denote by $\mathcal{C}_L^\Lambda$. This is illustrated for $k = 2$ and $k = 3$ in Figure 1. Our method is similar to those of [4] and [5] in that a projection from $k-1$ dimensions to k dimensions is used; the difference lies in the placement of points prior to the projection. Our technique creates codes of any size and thus provides a lower bound on achievable minimum distance as a function of code size. We shall show that the spherical code density approaches the density of the laminated lattice Λ_{k-1}, as $d \to 0$.

2.1 Construction of laminated spherical codes

Let $k \leq 49$ and $d \in (0, 1]$. For $k = 2$, a largest spherical code with minimum distance d is obvious (although not unique); we denote it by $\mathcal{C}_L^\Lambda(2, d)$. For $k \geq 3$, we recursively define the k-dimensional *laminated spherical code* $\mathcal{C}_L^\Lambda(k, d)$ with minimum distance d as follows.

$$\mathcal{C}_L^\Lambda(k, d) \equiv \left\{ \left(x_1, \ldots, x_{k-1}, \pm\sqrt{1 - \sum_{i=0}^{k-1} x_i^2} \right) : (x_1, \ldots, x_{k-1}) \in \bigcup_{i=0}^{N} r_i \mathcal{C}_i(k-1, d/r_{s(i)}) \right\}, \tag{6}$$

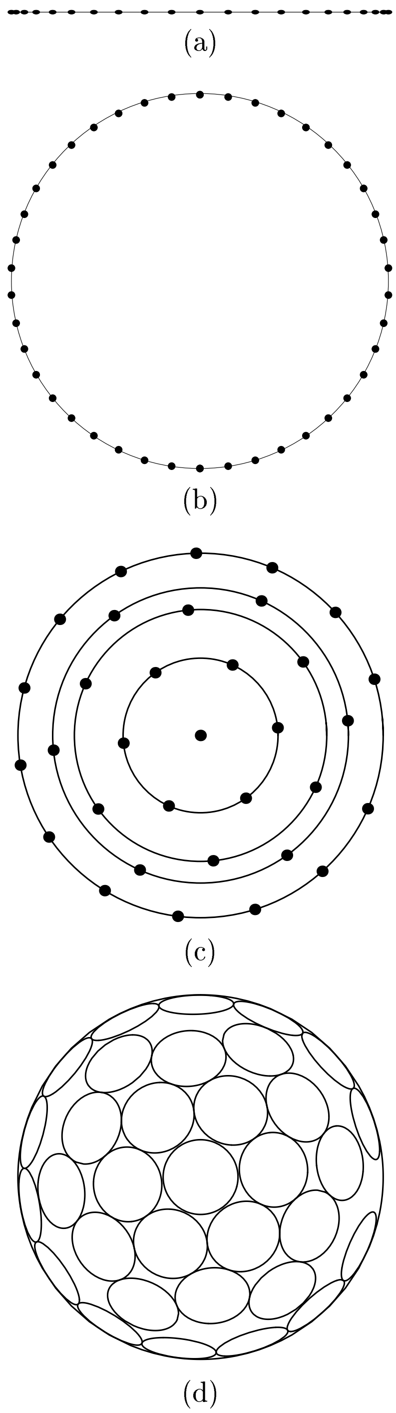


Figure 1: (a) 11 scaled 1-dimensional spherical codes. (b) 2-dimensional code derived from the scaled 1-dimensional codes by projecting codepoints up and down. (c) 5 scaled 2-dimensional codes. (d) 3-dimensional code derived from the 2-dimensional codes by projecting codepoints out of the page, where codepoints are the centers of the caps.

where

$$r(i,b) \equiv \frac{r_{i-1}\left(1-\frac{d^2}{2}\right)\sqrt{1-b(\frac{c_{k-2}d}{r_{s(i-1)}})^2}+d\sqrt{(1-r_{i-1}^2)(1-\frac{d^2}{4}-b(\frac{r_{i-1}c_{k-2}}{r_{s(i-1)}})^2)}}{1-b\left(\frac{r_{i-1}c_{k-2}d}{r_{s(i-1)}}\right)^2} \tag{7}$$

$$r_i \equiv \begin{cases} id & \text{if } i=0 \text{ or } i=1 \\ r(i,0) & \text{if } i>1 \text{ and } \lfloor r_{i-1}d^{-2/k}\rfloor \neq \\ & \lfloor \max(r(i,1),r(i-l_{k-1},0))d^{-2/k}\rfloor \\ \max(r(i,1),r(i-l_{k-1},0)) & \text{otherwise} \end{cases} \tag{8}$$

$$g(i) \equiv \min\{j : r_j > id^{2/k}\} \tag{9}$$

$$s(i) \equiv g(\max\{j : r_{g(j)} \leq r_i\}) \tag{10}$$

$$N \equiv \max\left\{i : r_i \leq \sqrt{1-d^2/4}\right\} \tag{11}$$

$$w_l(X) \equiv n(w_{l-1}(f^{-1}(X))) \tag{12}$$

$$L(X) \equiv \{w_{l-1}^{-1}(D(w_{l-1}(X)_i) : 1 \leq i \leq n_{k-2}\} \tag{13}$$

$$f(X) \equiv \frac{H(L(X))}{\|H(L(X))\|} \tag{14}$$

$$\mathcal{C}_{g(j)+l}(k-1,d) \equiv \begin{cases} \mathcal{C}_L^{\Lambda}(k-1,d) & \text{if } l=0 \\ f(\mathcal{C}_{g(j)+l-1}(k-1,d)) & \text{if } 1 \leq l \leq g(j+1)-g(j)-1 \end{cases}. \tag{15}$$

Remarks:

- The sequence $\{r_i\}$ satisfies $0 = r_0 < \cdots < r_N \leq \sqrt{1-d^2/4}$.
- $\{r_i\}$ and $s(i)$ are defined in terms of each other, but each is well defined. In particular, $s(i-1)$ and r_i each depend only on $r_0, \ldots r_{i-1}$.
- $w_l(X)\colon \mathcal{C}_{g(j)+l}(k-1,d) \to \Lambda_{k-1}^l$ associates each point from a spherical code with a point on a laminated lattice
- $L(X)$ associates $X \in \mathcal{C}_{g(j)+l}(k-1,d)$ with a subset of $\mathcal{C}_{g(j)+l}(k-1,d)$ that is used to determine a point on the shell with radius $r_{g(j)+l+1}$.
- l_k is defined in (3) and c_k, $H(\cdot)$, and $n(\cdot)$ are defined in Section 1.

The following definitions are also used in the code construction:

$$
\begin{array}{lrcl}
i\text{th shell:} & & & r_i\mathcal{C}_i(k-1, d/r_{s(i)}) \\
i\text{th gap:} & T_i & \equiv & \left\{(x_1,\dots,x_k)\in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2}\in (r_{i-1}, r_i]\right\} \\
i\text{th annulus:} & A_i & \equiv & \bigcup_{j=g(i)+1}^{g(i+1)-1} T_j \\
i\text{th buffer zone:} & B_i & \equiv & \left\{(x_1,\dots,x_k)\in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2}\in (r_{g(i)-1}, r_{g(i)}]\right\} \\
 & W_1 & \equiv & \left\{(x_1,\dots,x_k)\in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} < d^{1/k}\right\} \\
 & W_2 & \equiv & \left\{(x_1,\dots,x_k)\in S_k\colon \sqrt{\sum_{i=1}^{k-1} x_i^2} > 1-d^{1/k}\right\} \\
\text{wasted region:} & W & \equiv & W_1\cup W_2 \\
 & T & \equiv & \bigcup_{i=0}^{\lceil d^{-2/k}\rceil -1} A_i \\
 & B & \equiv & \bigcup_{i=1}^{\lceil d^{-2/k}\rceil -1} B_i
\end{array}
\tag{16}
$$

The ith shell *shell* is a $(k-1)$-dimensional spherical code scaled to a sphere of radius r_i. The points between the spheres of the $(i-1)$th and ith shells constitute the ith *gap*. We refer to the *ith annulus* as the set of points in $\mathbb{R}^{k-1}$ whose distance to the origin is in the interval $(r_{g(i)}, r_{g(i+1)-1}]$. That is, an annulus is a collection of consecutive gaps. Note that $s(i)$ is the smallest integer for which the set of all points whose magnitude is in the interval $(r_{s(i)}, r_i]$ lies in a single annulus. The *ith buffer zone* is the set of points lying between the $(i-1)$th and ith annuli. We call shells, gaps, annuli, and buffer zones which are projected onto the unit k-dimensional sphere S_k again shells, gaps, annuli, and buffer zones, respectively. The sets W_1 and W_2 are referred to as "wasted" regions, in which codepoints are not necessarily as tightly packed as the rest of the sphere. The radii $\{r_i\}$ and the sets W, T, and B are determined by the parameters k and d. For any k and d, we have $S_k = W\cup T\cup B$, as seen in Figure 2.

Each point in $\mathcal{C}_L^{\Lambda}(k,d)$ corresponds to a unique point on the lattice Λ_{k-1}, since each point in the shell $\mathcal{C}_{g(j)+l-1}(k-1,d)$ corresponds to a unique point on the lattice Λ_{k-2}^{l-1}, via the function $w_{l-1}(\cdot)$. Recall that a point $X\in\Lambda_{k-2}^{l-1}$ gives rise to the point $n(X)\in\Lambda_{k-2}^{l}$ via the hole $H(D(X)_1,\dots,D(X)_{n_{k-1}})$. Likewise, a point $X\in\mathcal{C}_{g(j)+l-1}(k-1,d)$ gives rise to a point in $\mathcal{C}_{g(j)+l}(k-1,d)$ via the hole of the codepoints of

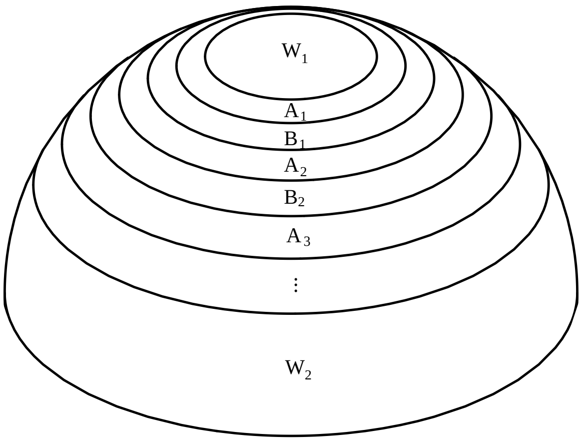


Figure 2: The sphere (here $k = 3$) is partitioned into annuli, buffer zones, and wasted regions. In general, W_1 and W_2 may contain one or more annuli.

$\mathcal{C}_{g(j)+l-1}(k-1,d)$ which *correspond* to $D(X)_1, \ldots, D(X)_{n_{k-1}}$, namely, $L(X)$. Thus, $\mathcal{C}_{g(j)+l}(k-1,d)$ is equal to a set of the (properly normalized) holes arising from $\mathcal{C}_{g(j)+l-1}(k-1,d)$, and $w_l(\cdot)$ is now also defined. For $k = 3$, the 2-dimensional code layers can explicitly be written as

$$\mathcal{C}_{g(j)+i}(2,d) \equiv \begin{cases} \left\{\left(\cos\left(i\theta_{g(j)+i}\right), \sin\left(i\theta_{g(j)+i}\right)\right)\right\} & \text{if } i \text{ even} \\ \left\{\left(\cos\left((i+\frac{1}{2})\theta_{g(j)+i}\right), \sin\left((i+\frac{1}{2})\theta_{g(j)+i}\right)\right)\right\} & \text{if } i \text{ odd} \end{cases} \tag{17}$$

where $\theta_m \equiv 2\sin^{-1}\left(\frac{d}{2r_{s(m)}}\right)$, $j \in \{0, \ldots, \lfloor 2\pi/\theta_i \rfloor - 1\}$ and $i \in \{0, \ldots, N\}$.

The laminated spherical code construction ensures that each $(k-1)$-dimensional code has minimum distance d. The sequence $\{r_i\}$ must be defined such that codepoints from different shells are at least distance d from each other. This constraint is analogous to the separation of layers of Λ_{k-2} in Λ_{k-1}, except that here we have the added complication that each layer (i.e., $(k-1)$-dimensional spherical code) is projected onto S_k.

The radius r_i is recursively chosen as small as possible and yet large enough so that the points at radius r_i are at least distance d from the points at radius r_{i-1}, after the projection. Suppose $r_0, \ldots, r_{i-1}$ have been determined. Then r_i is defined as the smallest positive number such that for each $X \equiv (x_1, \ldots, x_{k-1}) \in r_i\mathcal{C}_i(k-1, d/r_{s(i)})$ and $Y \equiv (y_1, \ldots, y_{k-1}) \in r_{i-1}\mathcal{C}_{i-1}(k-1, d/r_{s(i)})$, the distance between the corresponding codepoints $X' = \left(x_1, \ldots x_{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} x_i^2}\right)$ and $Y' =$

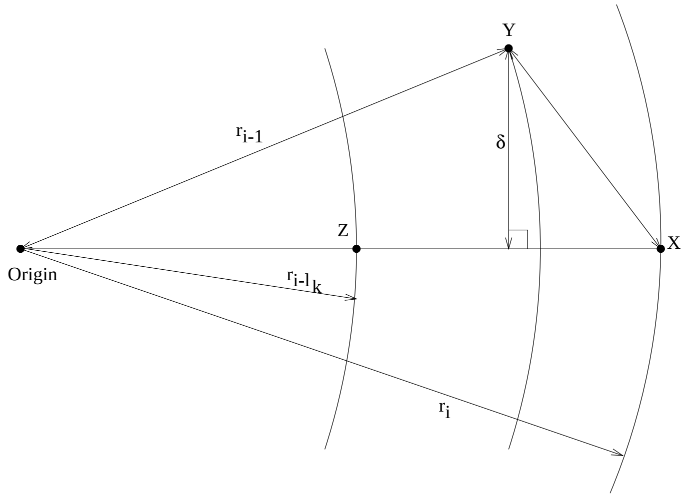


Figure 3: Relation between r_i and r_{i-1}.

$\left(y_1, \dots y_{k-1}, \sqrt{1 - \sum_{i=1}^{k-1} y_i^2}\right)$ in $\mathcal{C}_L^\Lambda(k,d)$ is at least d (see Figure 3). That is, r_i is chosen such that $\|X' - Y'\| \geq d$.

Let δ be the distance from Y to the hole associated with Y. Note that $\mathcal{C}_{i-1}(k-1, d/r_{s(i-1)})$ and $\mathcal{C}_{s(i)}(k-1, d/r_{s(i)})$ each have codepoints with angular separation θ_i. Thus, the distance from Y to its associated hole is $\frac{r_{i-1}}{r_{s(i-1)}}$ times greater than the distance from a point in $\mathcal{C}_{s(i)}$ to its associated hole. Hence,

$$\delta \geq \frac{r_{i-1} c_{k-2} d}{r_{s(i-1)}}, \tag{18}$$

where the inequality comes from the fact that $c_{k-2}d$ is less than or equal to the distance from a point of $\mathcal{C}_{s(i)}(k-1, d)$ to its associated hole. Therefore, r_i is chosen such that

$$\begin{aligned} d^2 &\leq \|X' - Y'\|^2 & (19) \\ &= \left(r_i - \sqrt{r_{i-1}^2 - \delta^2}\right)^2 + \delta^2 + \left(\sqrt{1 - r_{i-1}^2} - \sqrt{1 - r_i^2}\right)^2. & (20) \end{aligned}$$

Rearranging terms gives

$$\sqrt{(1 - r_{i-1}^2)(1 - r_i^2)} \leq 1 - d^2/2 - r_i \sqrt{r_{i-1}^2 - \delta^2} \tag{21}$$

which upon squaring both sides and solving the quadratic for r_i gives

$$r_i \geq \frac{\left(1-\frac{d^2}{2}\right)\sqrt{r_{i-1}^2-\delta^2}+\sqrt{(1-r_{i-1}^2)(d^2-\frac{d^4}{4}-\delta^2)}}{1-\delta^2} \tag{22}$$

The negative solution for the quadratic equation is smaller than r_{i-1} and is omitted. Taking the derivative of the right-hand side of (22) with respect to δ reveals that it is a decreasing function of δ when δ is the range for which (22) produces a real value. Thus, using (18) and setting

$$r_i = \frac{r_{i-1}\left(1-\frac{d^2}{2}\right)\sqrt{1-(\frac{c_{k-2}d}{r_{s(i-1)}})^2}+d\sqrt{(1-r_{i-1}^2)(1-\frac{d^2}{4}-(\frac{r_{i-1}c_{k-2}}{r_{s(i-1)}})^2)}}{1-\left(\frac{r_{i-1}c_{k-2}d}{r_{s(i-1)}}\right)^2} \tag{23}$$

ensures that $\|X'-Y'\| \geq d$.

The recursion for r_i in (23) is used only when Y belongs to the same annulus as X. If r_{i-1} is the radius of the outermost shell in an annulus, then r_i is defined such that every point on $r_i S_{k-1}$ is at least a distance d from every point on the circle of radius r_{i-1}. This is equivalent to taking $\delta = 0$ in the solution above (i.e., when $\overrightarrow{YX}$ is directed radially outward), and gives

$$r_i = r_{i-1}\left(1-\frac{d^2}{2}\right)+d\sqrt{(1-r_{i-1}^2)\left(1-\frac{d^2}{4}\right)}. \tag{24}$$

While (23) ensures that the minimum distance between codepoints in a pair of adjacent shells is no more than d, it does not insure this condition for codepoints in nonadjacent shells. It is possible that (23) would result in codepoints from shell i and shell $i+l_{k-1}$ which are closer than d. Let X', Y', and Z' respectively be the projections of X, Y, and Z shown in Figure 3, onto S_k. Then $Y', X' \in \mathcal{C}_L^{\Lambda}(k,d)$. If $\|Z'-X'\| < d$, then (23) is not used, in which case r_i is set to the value which produces $\|Z'-X'\| = d$ and from (24) gives

$$r_i = r_{i-l_{k-1}}\left(1-\frac{d^2}{2}\right)+d\sqrt{(1-r_{i-l_{k-1}}^2)\left(1-\frac{d^2}{4}\right)}. \tag{25}$$

In summary, the r_i's may be determined by the following algorithm:

$r_0 := 0;$

$i := 1;$

$r_s := r_1 := d;$

while $r_i \leq \sqrt{1 - d^2/4}$ {

  $i := i + 1;$

  if $\lfloor r_{i-1} d^{-2/k} \rfloor \neq \lfloor \max(r(i,1), r(i - l_{k-1}, 0)) d^{-2/k} \rfloor$

    then $r_s := r_i := r(i, 0);$ /* begin new annulus */

  else $r_i := \max(r(i,1), r(i - l_{k-1}, 0));$ /* regular solution */

}

$N := i - 1;$

2.2 Example of a laminated spherical code

We now illustrate the construction of a laminated spherical code for $k = 3$ and $d = 0.3$. First, the radii $\{r_i\}$ are determined. Since in Λ_2 lattice points in one layer differ in the second coordinate from lattice points two layers away, we have $l_2 = 2$. In Λ_1, holes are a distance $1/2$ from two lattice points, and hence $c_1 = 1/2$, and $n_1 = 2$. An iteration of the algorithm above gives $(r_0, \ldots r_N) = (0, .3, .569, .752, .872, .978)$, $(s(0), \ldots s(5)) = (0, 1, 2, 2, 2, 5)$, and $(g(0), \ldots, g(3)) = (0, 1, 2, 5)$. Next, we determine the 2-dimensional spherical codes to be projected onto S_3.

$i = 0$: $\mathcal{C}_i(k-1, d/r_{s(i)}) = \mathcal{C}_0(2, \infty) = \mathcal{C}_{g(0)+0}(2, \infty) = \mathcal{C}_L^{\Lambda}(2, \infty) = \{(1, 0)\}$.

$i = 1$: $\mathcal{C}_i(k-1, d/r_{s(i)}) = \mathcal{C}_{g(1)+0}(2, 1) = \mathcal{C}_L^{\Lambda}(2, 1) = \{(\cos(i\pi/3), \sin(i\pi/3)) \colon 0 \leq i \leq 5\}$.

$i = 2$: $\mathcal{C}_i(k-1, d/r_{s(i)}) = \mathcal{C}_{g(2)+0}(2, .527)$. Thus, $\theta_i = 2\sin^{-1}(.527/2) = .533$, and
$\mathcal{C}_i(k-1, d/r_{s(i)}) = \{(\cos(.533i), \sin(.533i)) \colon 0 \leq i \leq 10\}$.

$i = 3$: $\mathcal{C}_i(k-1, d/r_{s(i)}) = \mathcal{C}_{g(2)+1}(2, .527) = f(\mathcal{C}_i(2, .527)) = \{\frac{H(L(X))}{\|H(L(X))\|} : X \in f(\mathcal{C}_i(2, .527))\}$. If $Y \in \Lambda_1^0$, then $D(Y)_1 = Y$, $D(Y)_2 = Y + (1, 0)$, and so $n(Y) = H(D(Y)_1, D(Y)_2) + (0, \sqrt{1 - c_{k-2}^2}) =$

$Y + (1/2, \sqrt{3}/2)$. Let $X = (\cos(.533i), \sin(.533i)) \in \mathcal{C}_2(2, .527)$. Then $w_0(X) \equiv (i, 0)$, and hence

$$\begin{aligned} L(X) &= \{w_0^{-1}(i,0), w_0^{-1}(i+1,0)\} \\ &= \{(\cos(.533i), \sin(.533i)), (\cos(.533(i+1)), \sin(.533(i+1)))\}. \end{aligned}$$

From this, we obtain, $H(L(X))/\|H(L(X)\| = (\cos(.533(i+1/2)), \sin(.533(i+1/2)))$. In the next shell, we use the fact that $w_1(X) = (i + 1/2, \sqrt{3}/2)$. Letting X range over $\mathcal{C}_2(2, .527)$ gives $\mathcal{C}_i(k-1, d/r_{s(i)}) = \{(\cos(.533(i+1/2)), \sin(.533(i+1/2))) \colon 0 \le i \le 10\}$.

$i = 4$: $\mathcal{C}_i(k-1, d/r_{s(i)}) = \mathcal{C}_{g(2)+2}(2, .527) = \{(\cos(.533i), \sin(.533i)) \colon 0 \le i \le 10\}$.

$i = 5$: A new annulus begins with $\mathcal{C}_{g(3)+0}(2, .307)$. Thus, $\theta_i = 2\sin^{-1}(.307/2) = .308$, and $\mathcal{C}_i(k-1, d/r_{s(i)}) = \{(\cos(.308i), \sin(.308i)) \colon 0 \le i \le 19\}$.

The resulting code $\mathcal{C}_L^\Lambda(3, .3)$ is defined using (6) and has six shells. The shells contain $1, 6, 11, 11, 11$, and 20 points, respectively, and thus, the entire code has 120 points. The unprojected 2-dimensional codes are shown in Figure 4(a). The spherical caps of the final code $\mathcal{C}_L^\Lambda(3, .3)$ are shown in Figure 4(b). As d approaches 0, the advantage of the laminating technique becomes more apparent. We compare the apple-peeling spherical code $\mathcal{C}^A(3, 0.05)$ to the laminated spherical code $\mathcal{C}_L^\Lambda$(3,0.05) in Figure 5.

There are a number of improvements that may be made to the general construction. First, there may be points on the sphere that are not within d of any codepoint– and thus these points may be added to the codebook. For example, on shells 3 and 4 of $\mathcal{C}_L^\Lambda(3, .3)$, it appears that an extra codepoint may be added without reducing the minimum distance. Also, we may modify the width of the annuli, which would alter the number of shells that may be fit on the sphere, as well as their placement. The annulus width used above was $d^{2/k} \approx .448$. If the annulus width is set to zero instead, the apple-peeling code results. When the annulus width is .5, a code of size 128 may be obtained. In fact, each annulus width can be optimized separately.

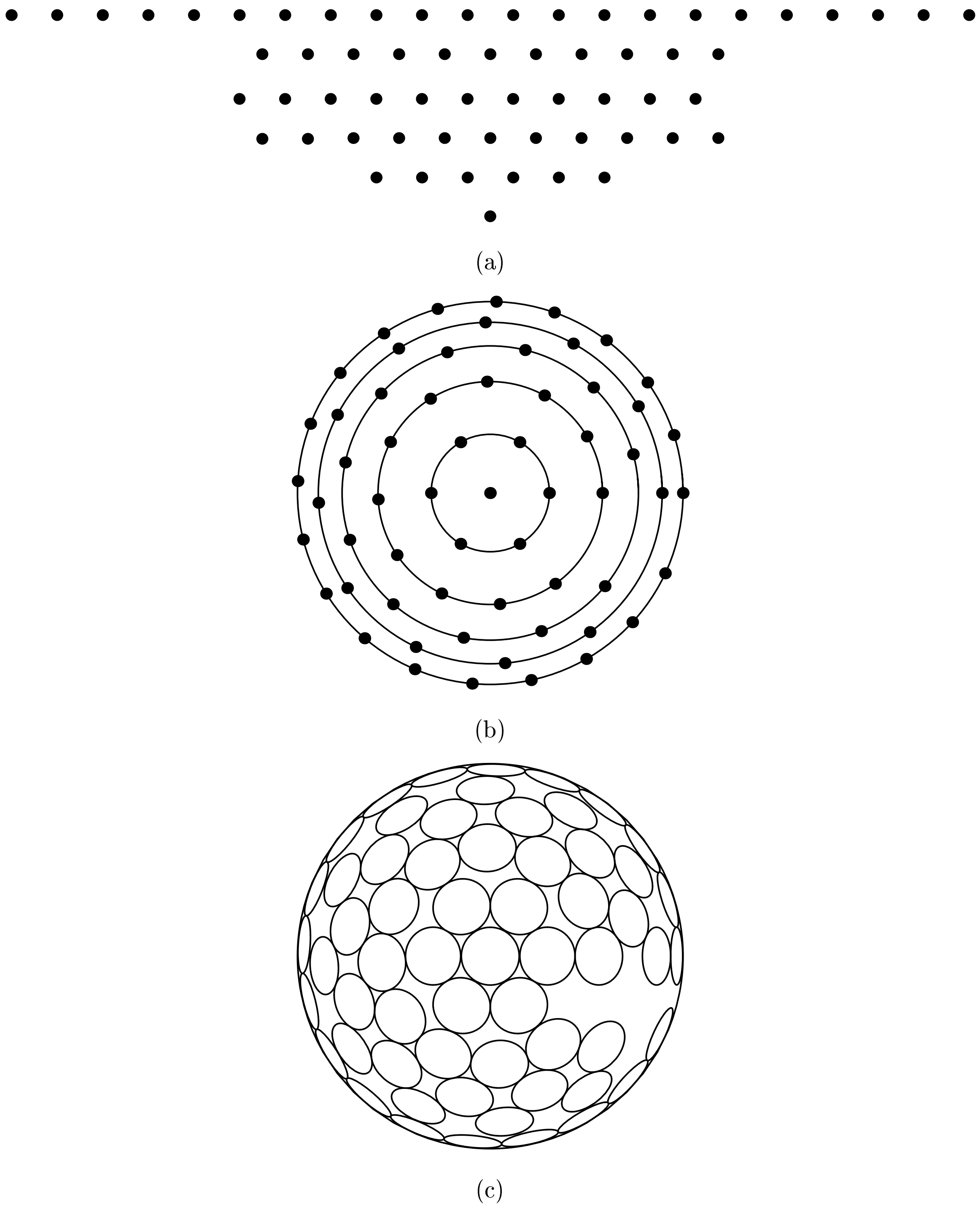
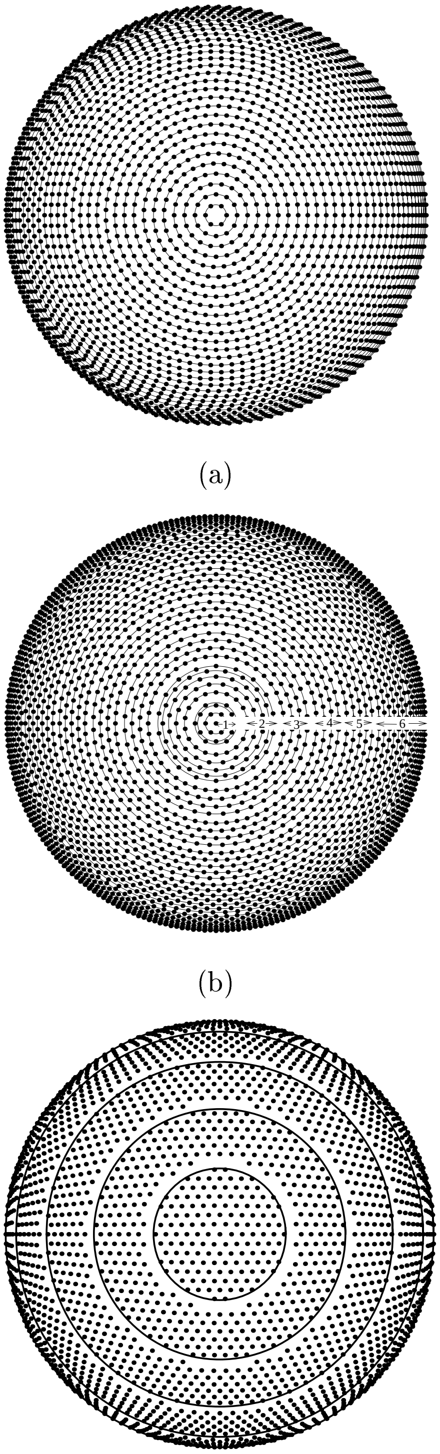


Figure 4: (a) A finite subset of Λ_2. (b) $\mathcal{C}_L^{\Lambda}(3,.3)$, before projection. (c) $\mathcal{C}_L^{\Lambda}(3,.3)$ after projection.



(a)

(b)

(c)

Figure 5: Comparison of apple-peeling, laminated, and wrapped spherical codes. To obtain the spherical codes, the points shown in the circles are projected straight out of the page onto the surface of a sphere. (a) Apple-peeling code $\mathcal{C}^A(3, 0.05)$ has 4764 codepoints. (b) Laminated code $\mathcal{C}_L^\Lambda(3, 0.05)$ has 5244 codepoints. (c) Wrapped code $\mathcal{C}^\Lambda$ constructed from the hexagonal lattice with minimum distance 0.05 has 4802 codepoints.

2.3 Asymptotic density of the laminated spherical code

Let $\Delta_{\mathcal{C}_L^\Lambda}(k,d)$ be the density of $\mathcal{C}_L^\Lambda(k,d)$, let $\Delta_{\mathcal{C}_L^\Lambda}(k) = \limsup_{d\to 0} \Delta_{\mathcal{C}_L^\Lambda}(k,d)$, and let Δ_{Λ_k} be the density of the sphere packing with spheres of radius $1/2$ and centers in Λ_k. Within an annulus, layers of shells are stacked similarly to layers of lattices in a laminated lattice. Theorem 1 establishes that $\Delta_{\mathcal{C}_L^\Lambda}(k)$ is asymptotically equal to the density of the sphere packing generated by Λ_{k-1}.

Theorem 1 *The density of a k-dimensional laminated spherical code $\mathcal{C}_L^\Lambda(k,d)$ with minimum distance d is no more than $O(d^{1/k})$ less than the density of the $(k-1)$-dimensional laminated lattice Λ_{k-1}, for all $k \le 49$. That is, $\Delta_{\mathcal{C}_L^\Lambda}(k,d) = \Delta_{\Lambda_{k-1}} - O(d^{1/k})$.*

Proof: See Appendix.

Corollary 1 follows from Lemma 1 of [1], Theorem 1, and the fact that Λ_2 is the densest possible packing in two dimensions. It also establishes the fact that the Fejes Tóth upper bound in Lemma 1 of [1] is asymptotically tight.

Corollary 1 *The 3-dimensional laminated spherical codes $\mathcal{C}_L^\Lambda(3,d)$ are asymptotically optimal in the minimum distance sense as the minimum distance decreases.*

Figure 6 shows the laminated spherical code density $\Delta_{\mathcal{C}_L^\Lambda}(3,d)$ versus d. All code densities are normalized by the Fejes Tóth upper bound, i.e., $\Delta_{\mathcal{C}_L^\Lambda}(3,d)/\Delta_F(3,d)$ is plotted versus d, where $\Delta_F(3,d)$ is the upper bound on density for a 3-dimensional spherical code with minimum distance d. Figure 6 includes unstructured codes found by a computer program of Hardin and Sloane [2]. This program has produced many of the best known codes. For some larger code sizes, we obtained codes using a simulating annealing approach which slightly improves upon [5]. This method produces good codes, but its computational complexity limits the code size that can be constructed. Spherical codes can also be generated from shells of lattices (e.g., [3, 6]). Figure 6 shows the best codes generated among the first 1000 point-centered shells of the face-centered cubic and Z_3 lattices, whose minimum distances were obtained exactly. Figure 6 also shows spherical codes formed from concatenations of MPSK and BPSK codes. For $d < 0.7$, the laminated spherical code $\mathcal{C}_L^\Lambda(3,d)$ outperforms known codes derived from shells

d	Coxeter bound	$\mathcal{C}_L^\Lambda$	$\mathcal{C}^\Lambda$	$\mathcal{C}^A$
10^{-1}	1450	1294	1070	1236
10^{-2}	145,103	134,422	130,682	125,504
10^{-3}	1.45×10^7	1.43×10^7	1.40×10^7	1.26×10^7
10^{-4}	1.45×10^9	1.45×10^9	$1.44 \times 10^9*$	1.26×10^9
10^{-5}	1.45×10^{11}	1.45×10^{11}	$1.45 \times 10^{11}*$	1.26×10^{11}

* estimated

Table 1: Table of 3-dimensional code sizes at various minimum distances.

d	Coxeter bound	$\mathcal{C}_L^\Lambda$	$\mathcal{C}^\Lambda$	$\mathcal{C}^A$
10^{-1}	29,364	16,976	17,198	22,740
10^{-2}	2.94×10^7	2.31×10^7	$2.31 \times 10^7*$	2.28×10^7
10^{-3}	2.94×10^{10}	2.59×10^{10}	$2.59 \times 10^{10}*$	2.28×10^{10}
10^{-4}	2.94×10^{13}	2.72×10^{13}	$2.72 \times 10^{13}*$	2.28×10^{13}
10^{-5}	2.94×10^{16}	2.77×10^{16}	$2.77 \times 10^{16}*$	2.28×10^{16}

* estimated

Table 2: Table of 4-dimensional code sizes at various minimum distances. The Coxeter bound is not asymptotically tight: using the best known upper bound on packing density in three dimensions and Lemma 1 of [1], an asymptotic upper bound of $2.79 \times 10^{3n+1}$ is achieved, for $d = 10^{-n}$ and large n.

of lattices, and is comparable to the apple-peeling code. For $d < 0.1$, $\mathcal{C}_L^\Lambda(3, d)$ is the best code known, and convergence to the upper bound is apparent as $d \to 0$.

Figure 7 shows $\Delta_{\mathcal{C}_L^\Lambda}(4, d)$ versus d, compared to other codes. The code densities are normalized by the Coxeter upper bound, and the qualitative performance is the same as for three dimensions, although the convergence of the density to its asymptotic value is slower. Still, for $d \leq 0.5$, the laminated spherical code outperforms spherical codes derived from shells of lattices Z_4, A_4, and D_4, or concatenated MPSK. Note that the Coxeter upper bound is not asymptotically tight, since the bound $\Delta_3^{pack} < 0.7784$ [7] is tighter.

Figure 8 shows code size versus d, for various code constructions in eight dimensions. Here, the laminated spherical code $\mathcal{C}_L^\Lambda(8, d)$ is slower to converge towards the upper bound. The wrapped spherical code, presented in [1] outperforms most of the other codes for minimum distances less than about 0.4.

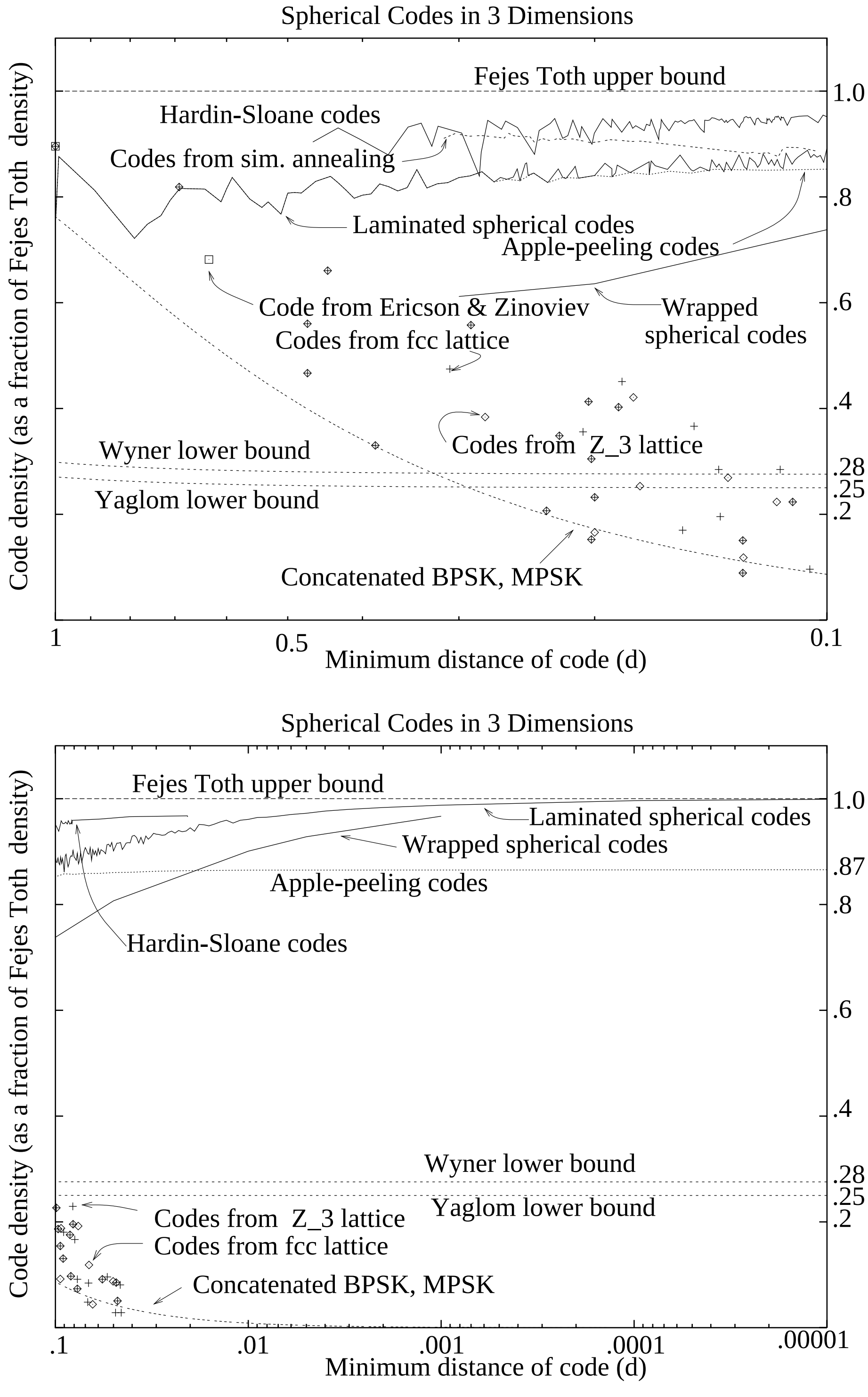


Figure 6: Comparison of 3-dimensional spherical codes.

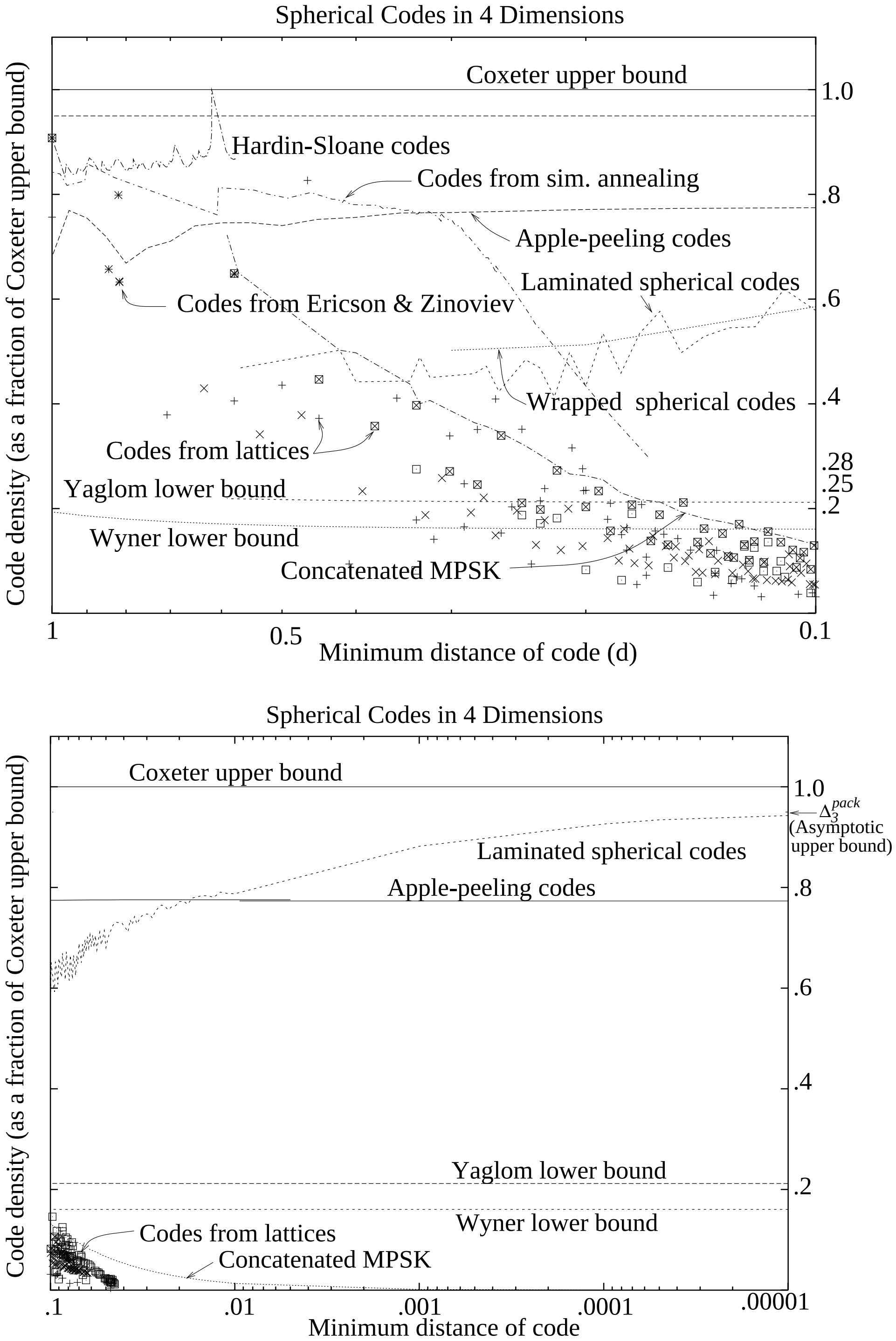


Figure 7: Comparison of 4-dimensional spherical codes.

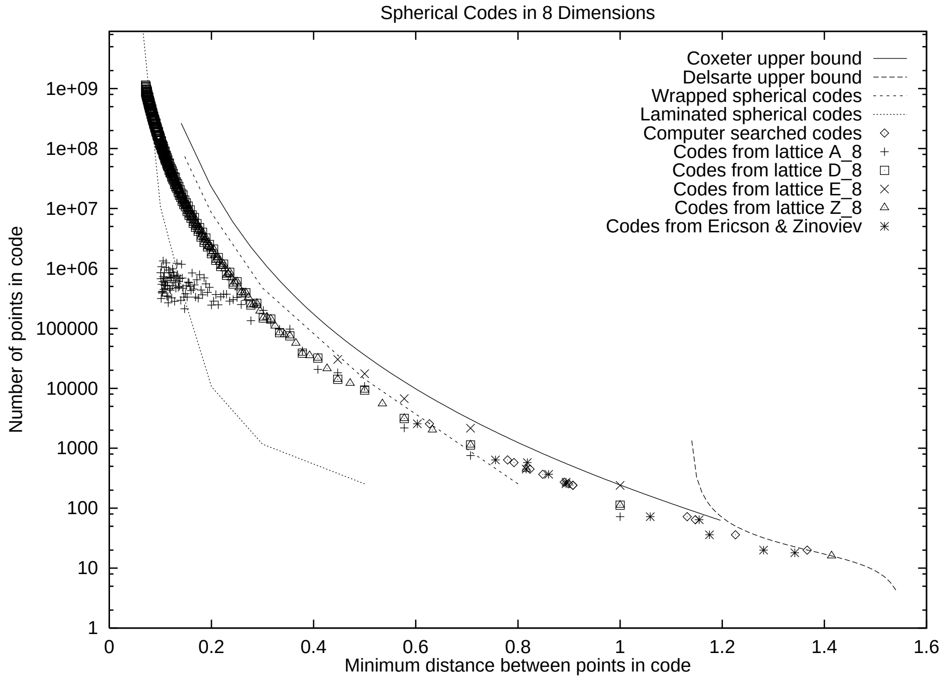


Figure 8: Comparison of 8-dimensional spherical codes.

3 Decoding laminated spherical codes

We consider two approaches to decoding, reflecting a tradeoff in the time and space complexity of the decoding: namely, (I) the decoder stores the radii used to construct the code, or (II) the decoder stores only the dimension k and minimum distance d. For any $R \in \mathbb{R}^k$ and any spherical code $\mathcal{C}(k,d)$, the nearest codepoint of $\mathcal{C}(k,d)$ to R is the same as the nearest codepoint of $\mathcal{C}(k,d)$ to $R/\|R\|$. Hence, in the following, we assume $R \in S_k$. Recall from (11) that $N = \max\left\{i\colon r_i \leq \sqrt{1-d^2/4}\right\}$, i.e., N is the number of radii needed to construct a laminated spherical code (not counting the radii used to construct lower dimensional laminated codes).

Lemma 1 $N = O(|\mathcal{C}_L^\Lambda(k,d)|^{1/(k-1)})$.

Proof: By construction of $\mathcal{C}_L^\Lambda(k,d)$, codepoints arising from $\mathcal{C}_i$ and $\mathcal{C}_{i-l_{k-1}}$ have angular separation at least θ, for all $i \geq l_{k-1}$. Thus, $(0,\dots,0,r_i,\sqrt{1-r_i^2})$ and $(0,\dots,0,r_{i-l_{k-1}},\sqrt{1-r_{i-l_{k-1}}^2})$ also have angular separation θ (even though these points might not be codepoints). The intersection of S_k and the plane in $\mathbb{R}^k$ spanned by $(0,\dots,0,1)$ and $(0,\dots,1,0)$ is a unit circle, and thus, $l_{k-1}N \leq \pi/(2\theta)$. From the definition of density, $|\mathcal{C}_L^{\Lambda}(k,d)| \cdot A(c(k,\theta/2)) = \Delta_{\mathcal{C}_L^{\Lambda}}(k,d) \cdot A_k$. Using equation (11) of [1] and the fact that $\Delta_{\mathcal{C}_L^{\Lambda}}(k,d) \cdot A_k$ is bounded away from zero (in both k and d), it follows that $|\mathcal{C}_L^{\Lambda}(k,d)| = O(1/d^{k-1})$, and hence $N = O(1/d) = O(|\mathcal{C}_L^{\Lambda}(k,d)|^{1/(k-1)})$. ■

Lemma 2 *The sequences of radii needed to construct $\mathcal{C}_L^{\Lambda}(k,d)$ can be written with storage complexity $O(\sqrt{|\mathcal{C}_L^{\Lambda}(k,d)|})$.*

Proof: Let $N(k,d)$ denote the *total* number of radii needed to construct $\mathcal{C}_L^{\Lambda}(k,d)$, i.e., N plus the total number of radii needed to construct all of the $(k-1)$-dimensional laminated spherical codes used in the recursive definition of $\mathcal{C}_L^{\Lambda}(k,d)$. There are $O(d^{-2/k})$ annuli in $\mathcal{C}_L^{\Lambda}(k,d)$, each of which has shells defined via a $(k-1)$-dimensional laminated spherical code with minimum distance at least d. Thus,

$$\begin{aligned}
N(k,d) &= O(d^{-1}) + O(d^{-2/k})N(k-1,d) && (26)\\
&= O(d^{-1}) + O(d^{-2/k})(O(d^{-1}) + O(d^{2/(k-1)}))N(k-2,d) && (27)\\
&= O(d^{-(1+(2/k))}) + O(d^{\frac{2}{k(k-1)}})N(k-2,d) && (28)\\
&= O(d^{-(1+(2/k)+2/(k-1)+\cdots+(2/4))}) && (29)\\
&= O(M^{\frac{1}{k-1}(1+2\sum_{i=4}^{k} 1/i)}) && (30)\\
&= O(\sqrt{M}). && (31)
\end{aligned}$$

■

Lemma 3 *For $k \leq 49$, the distance from any point on S_k to the nearest codepoint in $\mathcal{C}_L^{\Lambda}(k,d)$ is $O(d)$.*

Proof: The width of any shell is no more than d. Thus, the distance from an arbitrary point $R \in S_k$ to the nearest point, say R', on the boundary of the shell containing R is at most d. This shell boundary contains a scaled spherical code $r_i\mathcal{C}_i(k-1, d/r_{s(i)})$, for some i. Since $r_i d/r_{s(i)} \leq 2d$, R' is at most $2d$ from the nearest point on the boundary of some shell *used to construct* $\mathcal{C}_i(k-1, r_i d/r_{s(i)})$. By computing the distance to the boundary of a shell in successively lower dimensions, it follows that R is at most a distance $d + 2d + 4d + \cdots + 2^{k-2}d = O(d)$ from a codepoint of $\mathcal{C}_L^{\Lambda}(k,d)$. ■

Let $R \in S_k$. By construction of $\mathcal{C}_L^{\Lambda}(k,d)$, codepoints arising from $\mathcal{C}_i$ and $\mathcal{C}_{i-l_{k-1}}$ have angular separation at least θ, for all $i \geq l_{k-1}$. Hence, there are only a constant number of shells that contain points whose angular separation from R is $O(d)$. By Lemma 3, we may thus restrict the nearest neighbor search to a constant number of candidate shells, recursively find the nearest neighbor on each of those shells, and compare the distances to R. The algorithm may be written explicitly as follows:

$Nearest\_Neighbor(k, d, R, \{r_i\})$ {

  $(x_1, \ldots, x_k) := R$;

  $i := \min\left\{j : r_j > \sqrt{x_1^2 + \cdots + x_{k-1}^2}\right\}$;

  if $(k = 2)$ {

    $X := (r_{i-1}, \sqrt{1 - r_{i-1}})$;

    if $\left(\left\|\left(r_i, \sqrt{1-r_i^2}\right) - R\right\| < \|X - R\|\right)$ **then** $X := \left(r_i, \sqrt{1-r_i^2}\right)$;

  }

  if $(k > 2)$ {

    $X := (3, \ldots, 3)$;

    for $j := i - l_{k-1} \cdot 2^{k-1}$ **to** $i + l_{k-1} \cdot 2^{k-1}$ {

      $\{r'_n\}$:= sequence of radii used to construct $\mathcal{C}_{s(j)}(k-1, d/s(j))$;

      $X' = (x'_1, \ldots, x'_{k-1}) := Nearest\_Neighbor\left(k-1, d/r_{s(j)}, \frac{(x_1, \ldots, x_{k-1})}{\|(x_1, \ldots, x_{k-1})\|}, \{r'_n\}\right)$;

      if $\left(\| \left(x'_1, \ldots x'_{k-1}, \sqrt{1 - (x_1^2 + \cdots x_{k-1}^2)}\right) - R\| < \|X - R\|\right)$

        then $X := \left(x'_1, \ldots x'_{k-1}, \sqrt{1 - (x_1^2 + \cdots x_{k-1}^2)}\right)$;

    }

  }

  return (X);

}

Theorem 2 *There exists a nearest codepoint algorithm for $\mathcal{C}_L^{\Lambda}(k,d)$ using $O\left(\sqrt{|\mathcal{C}_L^{\Lambda}(k,d)|}\right)$ space and $O\left(\log |\mathcal{C}_L^{\Lambda}(k,d)|\right)$ time, and there exists a nearest codepoint algorithm using $O(1)$ space and $O\left(\sqrt{|\mathcal{C}_L^{\Lambda}(k,d)|}\right)$ time.*

Proof: In the algorithm above, storing the sequences of radii requires $O(\sqrt{|\mathcal{C}_L^{\Lambda}(k,d)|})$, by Lemma 2. To find the index i used in the algorithm requires $O(\log N) = O(\log |\mathcal{C}_L^{\Lambda}(k,d)|^{1/(k-1)}) = O(\log |\mathcal{C}_L^{\Lambda}(k,d)|)$

time, by Lemma 1. The other lines of the algorithm require constant time plus a constant number of recursive calls to the decoding algorithm. Since the dimension k in $\mathcal{C}_L^\Lambda(k,d)$ is a constant, the algorithm performs only a constant number of recursive calls. Thus, the algorithm uses $O(\sqrt{|\mathcal{C}_L^\Lambda(k,d)|})$ space and $O(\log|\mathcal{C}_L^\Lambda(k,d)|)$ time.

Alternatively, the sequences of radii need not be stored. If only k and d are stored, $O(1)$ storage space is required. The algorithm may then generate a sequence of radii when needed, using the technique outlined in the construction of $\mathcal{C}_L^\Lambda(k,d)$. This generation takes linear time in the size of the sequences, i.e., $O(\sqrt{|\mathcal{C}_L^\Lambda(k,d)|})$. ■

4 Conclusions

A new technique was presented that constructs laminated spherical codes in dimensions up to 49. The 3-dimensional laminated spherical codes are asymptotically optimal, in the sense that the ratio of the minimum distance of the constructed code to the upper bound given in [8] approaches one as the number of codepoints increases. This proves the upper bound is tight, asymptotically, and that previous lower bounds are not asymptotically optimal. The codes generated also compare favorably to other codes, for a wide variety of minimum distances. Good asymptotic performance is also achieved in higher dimensions, where the k-dimensional laminated spherical code density approaches the density of Λ_{k-1}. The question of whether the asymptotic density of the k-dimensional laminated spherical code is optimal is equivalent to the question of whether the Λ_{k-1} is the densest sphere packing.

Both wrapped spherical codes and laminated spherical codes presented in [1] improve upon the asymptotic performance of previous spherical codes. Similarly, the density of a wrapped spherical code with respect to a packing Λ approaches the density of Λ, and hence, any densest lattice Λ gives rise to an asymptotically optimal spherical code. The wrapped codes are not restricted to laminated lattices; any lattice or packing in $\mathbb{R}^{k-1}$ may be used to construct a k-dimensional wrapped code.

The comparison of the nonasymptotic performance reveals that both the wrapped and laminated spherical codes perform better than other constructions, i.e., have larger code sizes for a given minimum distance. In three and four dimensions, the laminated spherical codes perform better than the wrapped codes. In higher dimensions, the advantages of wrapped spherical become more apparent: the wrapped codes are easier to construct than laminated spherical codes because an explicit mapping is specified instead of a recursive one, and the decoding algorithm reduces to a decoding algorithm for the underlying lattice, a well-studied problem. The decoding algorithm for the laminated codes is recursive, and though efficient for low dimensions, is in fact exponential in the number of dimensions. Additionally, the asymptotic density of wrapped spherical codes is higher than the asymptotic density of laminated spherical codes in any dimension for which the laminated lattice in the previous dimension is not the best packing, e.g., dimensions 10-13.

Acknowledgements: The authors thank N. J. A. Sloane for pointing out some of the best spherical codes in three dimensions for codes up to 33,002 codepoints, and to A. Vardy for helpful discussions.

5 Appendix. Proof of Theorem 1

Proof of Theorem 1: Use induction on the dimension k. The claim holds for $k = 2$, since $\Delta_{\mathcal{C}_L^\Lambda}(2,d) > 1 - \frac{d}{2\pi}$ and $\Delta_{\Lambda_1} = 1$. Now suppose $k \geq 3$ and let

$$J \equiv \left\{ i \colon d^{1/k} < r_i < 1 - d^{1/k}, \text{ where } r_i \text{ is determined by (23)} \right\}. \tag{32}$$

For each $i \in J$, the density Δ_{T_i} of gap T_i of $\mathcal{C}_L^\Lambda$ shall be computed. The $(k-1)$-dimensional content (surface area) of T_i is

$$A(T_i) \quad = \quad \int_{\sqrt{1-r_i^2}}^{\sqrt{1-r_{i-1}^2}} A_{k-1}\left(1 - x^2\right)^{(k-3)/2} dx. \tag{33}$$

The integrand in (33) is monotonically nonincreasing in x, and hence

$$\left(\sqrt{1-r_{i-1}^2} - \sqrt{1-r_i^2}\right) r_{i-1}^{k-3} \leq \frac{A(T_i)}{A_{k-1}} \leq \left(\sqrt{1-r_{i-1}^2} - \sqrt{1-r_i^2}\right) r_i^{k-3}. \tag{34}$$

Using (23), we shall remove the occurrences of r_i in (34). First (23) will be put into an asymptotic form. In the following, constants encompassed by the O-notation do *not* depend on i. Since $r_i > d^{1/k}$ for all $i \in J$, $r_{s(i)} \geq d^{1/k} - d^{2/k} = \Omega(d^{1/k})$, and so

$$\left(1 - \frac{d^2}{2}\right)\sqrt{1 - \left(\frac{c_{k-2}}{r_{s(i)}}\right)^2} = \left(1 - \frac{d^2}{2}\right)\sqrt{1 - O(d^{2(1-(1/k))})} = 1 - O(d^{2(k-1)/k}). \tag{35}$$

Also,

$$\sqrt{1 - \frac{d^2}{4} - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}} \quad = \quad \sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}}\sqrt{1 - \frac{d^2}{4\left(1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}\right)}} \tag{36}$$

$$= \quad \sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}} - O(d^2), \tag{37}$$

where we have used the fact that

$$\frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2} \leq c_{k-2}^2 \left[\frac{r_{s(i)} + d^{2/k}}{r_{s(i)}}\right]^2 \tag{38}$$

$$= c_{k-2}^2 \left[1 + O(d^{1/k})\right]^2 \tag{39}$$

$$= c_{k-2}^2 + O(d^{1/k}) \tag{40}$$

$$< 0.99, \tag{41}$$

where (41) holds for sufficiently small d because $c_{k-2}^2 \leq 31/32$ for $k \leq 49$ [3]. Similarly,

$$\frac{r_{i-1}^2 d^2}{r_{s(i)}^2} = O(d^2(1 + d^{(2/k)-(1/k)})^2) = O(d^2), \tag{42}$$

and thus, (23) can be rewritten as

$$r_i = \frac{r_{i-1}(1 - O(d^{2(k-1)/k})) + d\sqrt{1 - r_{i-1}^2}\left(\sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}} - O(d^2)\right)}{1 - O(d^2)} \tag{43}$$

$$= r_{i-1} + d\sqrt{(1 - r_{i-1}^2)\left(1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}\right)} - O(d^{2(k-1)/k}), \tag{44}$$

and

$$\sqrt{1 - r_i^2} = \left(1 - \left(r_{i-1} + d\sqrt{(1 - r_{i-1}^2)\left(1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}\right)} - O(d^{2(k-1)/k})\right)^2\right)^{1/2} \tag{45}$$

$$= \sqrt{1 - r_{i-1}^2}\left(1 - \frac{2r_{i-1}d\sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}}}{\sqrt{1 - r_{i-1}^2}} + \frac{O(d^{2(k-1)/k})}{1 - r_{i-1}^2}\right)^{1/2} \tag{46}$$

$$= \sqrt{1 - r_{i-1}^2} - r_{i-1}d\sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}}} + O(d^{2(k-1)/k}) \tag{47}$$

Hence, the left and right-hand sides of (34) differ by $O(d^2)$. This gives

$$\begin{aligned}
\frac{A(T_i)}{A_{k-1}} &= \left(\sqrt{1-r_{i-1}^2} - \sqrt{1-r_i^2}\right) r_{i-1}^{k-3} + O(d^2) && (48) \\
&= r_{i-1}^{k-2} d \sqrt{1 - \frac{c_{k-2}^2 r_{i-1}^2}{r_{s(i)}^2}} - O(d^{2(k-1)/k}) && (49) \\
&\leq r_{i-1}^{k-2} d \sqrt{1 - c_{k-2}^2}, && (50)
\end{aligned}$$

where (50) holds for sufficiently small d. From equation (11) of [1],

$$A(c(k, \theta/2)) = V_{k-1}(d/2)^{k-1}(1 - O(d^2)). \tag{51}$$

and thus the number of codepoints in each of shell $\mathcal{C}_{i-1}(k-1, d/r_{s(i)})$ and shell $\mathcal{C}_i(k-1, d/r_{s(i)})$ is

$$\begin{aligned}
N_{s(i)} &= \frac{\Delta_{\mathcal{C}_L^\Lambda}\left(k-1, \frac{d}{r_{s(i)}}\right) \cdot A_{k-1}}{A\left(c\left(k-1, \sin^{-1}\left(\frac{d}{2r_s(i)}\right)\right)\right)} && (52) \\
&\geq \frac{\Delta_{\mathcal{C}_L^\Lambda}\left(k-1, \frac{d}{r_{s(i)}}\right) \cdot A_{k-1}}{V_{k-2}\left(\frac{d}{2r_s(i)}\right)^{k-2}} && (53) \\
&= \frac{\Delta_{\mathcal{C}_L^\Lambda}(k-1, O(d^{(k-1)/k}))A_{k-1}}{V_{k-2}\left(\frac{d}{2r_s(i)}\right)^{k-2}}, && (54)
\end{aligned}$$

where (53) holds for sufficiently small d, by (51). The density of spherical caps in T_i is thus

$$\begin{aligned}
\Delta_{T_i} &= \frac{N_{s(i)}A(c(k,\theta/2))}{A(T_i)} && (55)\\
&\geq \frac{\left(\frac{\Delta_{\mathcal{C}_L^\Lambda}\left(k-1,O(d^{(k-1)/k})\right)\cdot A_{k-1}}{V_{k-2}\left(\frac{d}{2r_{s(i)}}\right)^{k-2}}\right)\cdot V_{k-1}(d/2)^{k-1}\cdot(1-O(d^2))}{A_{k-1}r_{i-1}^{k-2}d\sqrt{1-c_{k-2}^2}} && (56)\\
&= \frac{\Delta_{\mathcal{C}_L^\Lambda}(k-1,O(d^{(k-1)/k}))V_{k-1}}{2V_{k-2}\sqrt{1-c_{k-2}^2}} - O(d^{1/k}) && (57)\\
&= \frac{\Delta_{\Lambda_{k-2}}V_{k-1}}{2V_{k-2}\sqrt{1-c_{k-2}^2}}(1-O(d^{1/k})) - O(d^{1/k}) && (58)\\
&= \frac{\Delta_{\Lambda_{k-2}}V_{k-1}}{2V_{k-2}\sqrt{1-c_{k-2}^2}} - O(d^{1/k}), && (59)
\end{aligned}$$

where (58) follows by induction on k. The above density applies for all $i \in J$, i.e., all i determined by (23) that are not in a wasted region. Now we repeat the argument for r_i determined by (25). Let

$$J' \equiv \left\{i\colon d^{1/k} < r_i < 1 - d^{1/k}, \text{ where } r_i \text{ is determined by (25)}\right\}. \tag{60}$$

If $J' \neq \phi$, let $i \in J'$, and let

$$R_i \equiv \bigcup_{j=i-l_{k-1}+1}^{i} T_j. \tag{61}$$

Then from (25),

$$r_i = r_{i-l_{k-1}} + d\sqrt{1-r_{i-l_{k-1}}^2} - O(d^2), \tag{62}$$

and

$$\begin{aligned}
\sqrt{1-r_i^2} &= \sqrt{1-r_{i-l_{k-1}}^2 - 2r_{i-l_{k-1}} d\sqrt{1-r_{i-l_{k-1}}^2} \pm O(d^2)} && (63)\\
&= \sqrt{1-r_{i-l_{k-1}}^2}\left(1-\frac{r_{i-l_{k-1}}d}{\sqrt{1-r_{i-l_{k-1}}^2}} \pm \frac{O(d^2)}{1-r_{i-l_{k-1}}^2}\right) && (64)\\
&= \sqrt{1-r_{i-l_{k-1}}^2} - r_{i-l_{k-1}}d \pm O(d^{(2k-1)/k}). && (65)
\end{aligned}$$

From (34), with r_{i-1} replaced by $r_{i-l_{k-1}}$, we obtain

$$\left(\sqrt{1-r_{i-l_{k-1}}^2}-\sqrt{1-r_i^2}\right) r_{i-l_{k-1}}^{k-3} \leq \frac{A(R_i)}{A_{k-1}} \leq \left(\sqrt{1-r_{i-l_{k-1}}^2}-\sqrt{1-r_i^2}\right) r_i^{k-3}. \quad (66)$$

Again, the left and right-hand side differ by $O(d^2)$. Thus,

$$\begin{aligned}
\frac{A(R_i)}{A_{k-1}} &= \left(\sqrt{1-r_{i-l_{k-1}}^2}-\sqrt{1-r_i^2}\right) r_{i-l_{k-1}}^{k-3} + O(d^2) && (67)\\
&= r_{i-l_{k-1}}^{k-2} d \pm O(d^{(2k-1)/k}) && (68)\\
&\leq r_{i-l_{k-1}}^{k-2} d(1 \pm O(d^{1/k})). && (69)
\end{aligned}$$

The density of spherical caps in R_i is thus

$$\begin{aligned}
\Delta_{R_i} &= \frac{l_{k-1}N_{s(i)}A(c(k,\theta/2))}{A(R_i)} && (70)\\
&\geq \frac{l_{k-1}\left(\frac{\Delta_{\mathcal{C}_L^\Lambda}\left(k-1,O(d^{(k-1)/k})\right)\cdot A_{k-1}}{V_{k-2}\left(\frac{d}{2r_{s(i)}}\right)^{k-2}}\right)\cdot V_{k-1}(d/2)^{k-1}\cdot(1-O(d^2))}{A_{k-1}r_{i-l_{k-1}}^{k-2}d(1\pm O(d^{1/k}))} && (71)\\
&\geq \frac{\left(\frac{\Delta_{\mathcal{C}_L^\Lambda}\left(k-1,O(d^{(k-1)/k})\right)\cdot A_{k-1}}{V_{k-2}\left(\frac{d}{2r_{s(i)}}\right)^{k-2}}\right)\cdot V_{k-1}(d/2)^{k-1}\cdot(1\pm O(d^{1/k}))}{A_{k-1}r_{i-l_{k-1}}^{k-2}d\sqrt{1-c_{k-2}^2}} && (72)\\
&\geq \frac{\Delta_{\Lambda_{k-2}}V_{k-1}}{2\sqrt{1-c_{k-2}^2}V_{k-2}} - O(d^{1/k}), && (73)
\end{aligned}$$

where (72) follows from $l_{k-1} \geq 1/\sqrt{1-c_{k-2}^2}$ and (73) follows from (56)-(59).

Since (59) and (73) are independent of i, the density of T can be bounded as

$$\Delta_T \geq \frac{\Delta_{\Lambda_{k-2}} V_{k-1}}{2\sqrt{1-c_{k-2}^2}V_{k-2}} - O(d^{1/k}). \tag{74}$$

Since $r_j - r_{j-1} < d$, for all j, the $(k-1)$-dimensional content of any buffer zone is bounded above by $A_{k-1}d$. The number of buffer zones in this region is no more than $2\lceil d^{-2/k}\rceil$, where buffer zones with both positive and negative kth coordinates are included. Hence, the total $(k-1)$-dimensional content of $B = \cup_i B_i$ is bounded as

$$A(B) < 2A_{k-1}d\lceil d^{-2/k}\rceil = O(d^{(k-2)/k}), \tag{75}$$

and $A(W) = O(d^{1/k})$. Thus,

$$\begin{aligned}
\Delta_{\mathcal{C}_L^\Lambda}(k,d) &\geq \frac{\Delta_T A(T)}{A_k} && (76)\\
&= \Delta_T \frac{A_k - A(W) - A(B)}{4\pi} && (77)\\
&= \left(\frac{\Delta_{\Lambda_{k-2}} V_{k-1}}{2V_{k-2}\sqrt{1-c_{k-2}^2}} - O(d^{1/k})\right)\left(\frac{A_k - O(d^{1/k})}{A_k}\right) && (78)\\
&= \frac{\Delta_{\Lambda_{k-2}} V_{k-1}}{2V_{k-2}\sqrt{1-c_{k-2}^2}} - O(d^{1/k}), && (79)
\end{aligned}$$

where (78) follows from (74). Since layers of Λ_{k-1} within Λ_k are separated by a distance of $\sqrt{1-c_{k-1}^2}$ and each lattice point is distance 1 from an adjacent point, we have

$$\Delta_{\Lambda_k} = \frac{\Delta_{\Lambda_{k-1}} V_k (\frac{1}{2})^k}{V_{k-1}(\frac{1}{2})^{k-1}\sqrt{1-c_{k-1}^2}} = \frac{\Delta_{\Lambda_{k-1}} V_k}{2V_{k-1}\sqrt{1-c_{k-1}^2}}. \tag{80}$$

Thus, $\Delta_{\mathcal{C}_L^\Lambda}(k,d) \geq \Delta_{\Lambda_{k-1}} - O(d^{1/k})$. ■

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