Analytic Evaluation of an Integral of Needed in GExp Model with non-zero $\Delta\Gamma$

R. Cahn

November 2001

Shahram Rahatlou has shown how the previous analytic calculation of a Gaussian, an exponential and an exponential of $|t|$ can be extended to do integrals like

$$F_n(t,\Gamma,\tau) = \int_{-\infty}^{\infty} dt' t'^n e^{-\Gamma|t'|} GExp(t'|\sigma,\tau) \tag{1}$$

where

$$GExp(t|\sigma,\tau) = \int_o^{\infty} \frac{dt''}{\tau} e^{-t''/\tau} \frac{e^{-\frac{(t-t'')^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \tag{2}$$

My earlier result was

$$\begin{aligned}
\int_{-\infty}^{\infty} dt' e^{-\Gamma|t'|} e^{i\Delta m t} \int_0^{\infty} & \frac{dt''}{\tau} e^{-t''/\tau} \frac{e^{-(t-t'-t'')^2/2\sigma^2}}{\sqrt{2\pi}\sigma} \\
&= \int_{-\infty}^{\infty} dt' e^{-\Gamma|t'|} e^{i\Delta m t'} GExp(t-t'|\sigma,\tau) \\
&= \frac{G(t,\Gamma-i\Delta m) - G(t,1/\tau)}{1-(\Gamma-i\Delta m)\tau} + \frac{G(t,1/\tau)}{1+(\Gamma+i\Delta m)\tau} + \frac{G(-t,\Gamma+i\Delta m)}{1+(\Gamma+i\Delta m)\tau}
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
G(t,\Gamma) &= \int_0^{\infty} \frac{dt'}{\sqrt{2\pi}\sigma} e^{-(t-t')^2/2\sigma^2} e^{-\Gamma t'} \\
&= \frac{1}{2} e^{(\Gamma\sigma^2-t)^2/2\sigma^2} e^{-t/2\sigma^2} e^{-\Gamma t} \mathrm{erfc}\left(\frac{-t+\Gamma\sigma^2}{\sqrt{2}\sigma}\right)
\end{aligned} \tag{4}$$

The integral of interest in studying $\Delta\Gamma$ is

$$\int_{-\infty}^{\infty} dt' e^{-\Gamma|t'|} \cosh\frac{\Delta\Gamma}{2} t\, GExp(t-t'|\sigma,\tau) \tag{5}$$

which we can obtain by taking half the sum of the results with Δm replaced by $i\Delta\Gamma/2$ and $-i\Delta\Gamma/2$. Therefore we have

$$\begin{aligned}
\int_{-\infty}^{\infty} dt' \qquad & e^{-\Gamma|t'|}\cosh\frac{\Delta\Gamma}{2}t\, GExp(t-t'|\sigma,\tau) \\
= \quad & \frac{1}{2}\left[\frac{G(t,\Gamma+\Delta\Gamma/2)-G(t,1/\tau)}{1-(\Gamma+\Delta\Gamma/2)\tau}+\frac{G(t,\Gamma-\Delta\Gamma/2)-G(t,1/\tau)}{1-(\Gamma-\Delta\Gamma/2)\tau}\right. \\
& +\frac{G(t,1/\tau)}{1+(\Gamma+\Delta\Gamma/2)\tau}+\frac{G(t,1/\tau)}{1+(\Gamma-\Delta\Gamma/2)\tau} \\
& \left.+\frac{G(-t,\Gamma+\Delta\Gamma/2)}{1+(\Gamma+\Delta\Gamma/2)\tau}+\frac{G(-t,\Gamma-\Delta\Gamma/2)}{1+(\Gamma-\Delta\Gamma/2)\tau}\right] \qquad (6)
\end{aligned}$$