NASA/TM—2004-212944

# The Extended Parabolic Equation Method and Implication of Results for Atmospheric Millimeter-Wave and Optical Propagation

Robert M. Manning
Glenn Research Center, Cleveland, Ohio

March 2004

The NASA STI Program Office . . . in Profile

Since its founding, NASA has been dedicated to the advancement of aeronautics and space science. The NASA Scientific and Technical Information (STI) Program Office plays a key part in helping NASA maintain this important role.

The NASA STI Program Office is operated by Langley Research Center, the Lead Center for NASA's scientific and technical information. The NASA STI Program Office provides access to the NASA STI Database, the largest collection of aeronautical and space science STI in the world. The Program Office is also NASA's institutional mechanism for disseminating the results of its research and development activities. These results are published by NASA in the NASA STI Report Series, which includes the following report types:

- TECHNICAL PUBLICATION. Reports of completed research or a major significant phase of research that present the results of NASA programs and include extensive data or theoretical analysis. Includes compilations of significant scientific and technical data and information deemed to be of continuing reference value. NASA's counterpart of peer-reviewed formal professional papers but has less stringent limitations on manuscript length and extent of graphic presentations.

- TECHNICAL MEMORANDUM. Scientific and technical findings that are preliminary or of specialized interest, e.g., quick release reports, working papers, and bibliographies that contain minimal annotation. Does not contain extensive analysis.

- CONTRACTOR REPORT. Scientific and technical findings by NASA-sponsored contractors and grantees.

- CONFERENCE PUBLICATION. Collected papers from scientific and technical conferences, symposia, seminars, or other meetings sponsored or cosponsored by NASA.

- SPECIAL PUBLICATION. Scientific, technical, or historical information from NASA programs, projects, and missions, often concerned with subjects having substantial public interest.

- TECHNICAL TRANSLATION. English-language translations of foreign scientific and technical material pertinent to NASA's mission.

Specialized services that complement the STI Program Office's diverse offerings include creating custom thesauri, building customized databases, organizing and publishing research results . . . even providing videos.

For more information about the NASA STI Program Office, see the following:

- Access the NASA STI Program Home Page at ***http://www.sti.nasa.gov***

- E-mail your question via the Internet to help@sti.nasa.gov

- Fax your question to the NASA Access Help Desk at 301–621–0134

- Telephone the NASA Access Help Desk at 301–621–0390

- Write to:
  NASA Access Help Desk
  NASA Center for AeroSpace Information
  7121 Standard Drive
  Hanover, MD 21076

NASA/TM—2004-212944

# The Extended Parabolic Equation Method and Implication of Results for Atmospheric Millimeter-Wave and Optical Propagation

Robert M. Manning
Glenn Research Center, Cleveland, Ohio

National Aeronautics and
Space Administration

Glenn Research Center

March 2004

Available from

NASA Center for Aerospace Information
7121 Standard Drive
Hanover, MD 21076

National Technical Information Service
5285 Port Royal Road
Springfield, VA 22100

Available electronically at http://gltrs.grc.nasa.gov

# The Extended Parabolic Equation Method and Implication of Results for Atmospheric Millimeter-Wave and Optical Propagation

## 1. Introduction

During the three decades since the introduction of the stochastic parabolic wave equation by Tatarskii and Klyatskin [*Tatarskii*, 1969], [*Klyatskin*, 1970], this theory of wave propagation through random media has found much success in the applications, from laser beam propagation through the atmosphere to image propagation. The applicability of this theory, however, is limited by the fact that it only describes situations in which the paraxial approximation prevails. Thus, the parabolic wave equation can be applied only when the smallest size of the permittivity fluctuations $l_0$ and the wavelength $\lambda$ satisfy the condition $l_0 >> \lambda$, i.e., the wavelength must be the smallest spatial scale in the problem. Although extensions of this theory have been advanced (see, for example, [*Saichev*, 1980a], [*Ostashev and Tatarskii*, 1995]), none have been analytically treated at the level where corrections to the paraxial approximation have been identified and compared to paraxial results.

The purpose of this memorandum is to provide a theoretical foundation for the extension of the theory which results in an 'extended' parabolic equation capable of treating wide-angle propagation in situations where the condition $l_0 >> \lambda$ is violated, thus potentially making the extended theory formally applicable to millimeter wavelengths. Such a formulation is potentially required in propagation situations in which the operating wavelength is on the order of the spatial size of the inhomogeneities which make up the random medium. Such a theory will find application in the study of, e.g., millimeter wave propagation through atmospheric turbulence, in which the size spectrum of the turbulent eddys subtends a few millimeters, or in optical propagation through dust and sand storms where the wavelength can be on the order of the size of the inclusion. Most importantly, however, is the fact that solutions to the extended theory will provide a measure of correctness of the paraxial theory.

In section 2, a first-principles derivation of the complete equations for forward and back scatter propagation are obtained from the stochastic Helmholtz equation, the various forms of which yield known results used in other such studies. Section 3 presents the derivation of an 'extended' stochastic parabolic equation from the general equations for forward scatter presented in Section 2. Here, the extended parabolic equation obtained is an operator equation which transcends the restrictions of the paraxial approximation but reduces to the latter in the approximation $l_0 >> \lambda$.

Since the electric field as described by the extended parabolic equation is a random quantity, one can only deal with the related statistical quantities such as the first and second order moments. This is done in Section 4 where a general operator method is developed to treat generalized statistical moments within the extended parabolic equation. Solutions of the specific equations for the first order moment and second order moment (i.e., the mutual coherence function) is dealt with is Section 5 for the case of the random atmospheric permittivity field as described by the Kolmogorov spectrum of fluctuations. An appendix is provided which makes contact with the developments of Section 2 and other operator representations employed in the treatment of wave propagation problems.

## 2. Derivation of the Equations for Forward and Back Scatter Propagation form the Stochastic Scalar Helmholtz Equation

Consider the propagation of a scalar electric field component of an electromagnetic wave in a random medium characterized by a stochastic permittivity $\varepsilon(\vec{r}) = 1 + \tilde{\varepsilon}(\vec{r})$ where $\tilde{\varepsilon}(\vec{r})$ is the random part. The Helmholtz equation governing the resulting stochastic electric field in the scalar approximation (i.e., neglecting wave depolarization) is then given by

$$\frac{\partial^2 E(x,\vec{\rho})}{\partial x^2} + \nabla_{\vec{\rho}}^2 E(x,\vec{\rho}) + k^2 E(x,\vec{\rho}) = k^2 \tilde{\varepsilon}(x,\vec{\rho}) E(x,\vec{\rho}) \tag{1}$$

where the direction of wave propagation along the otherwise arbitrary $x-$axis is separated out from the three dimensional coordinate $\vec{r} = (x,\vec{\rho})$. The total stochastic field $E(x,\vec{\rho})$ can be decomposed into a forward propagating wave field $E^+(x,\vec{\rho})$ and a backward propagating wave field $E^-(x,\vec{\rho})$, i.e.,

$$E(x,\vec{\rho}) = E^+(x,\vec{\rho}) + E^-(x,\vec{\rho}) \tag{2}$$

Similarly, one has for the derivative of the fields

$$\frac{\partial E(x,\vec{\rho})}{\partial x} = \frac{\partial E^+(x,\vec{\rho})}{\partial x} + \frac{\partial E^-(x,\vec{\rho})}{\partial x} \tag{3}$$

As shown in Appendix 1, one can write these wave field components as an expansion into inhomogeneous plane waves,

$$E^{\pm}(x,\vec{\rho}) = \int\int e^{\pm}(\vec{q}) \exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2 q \tag{4}$$

Thus,

$$\frac{\partial E^{\pm}(x,\vec{\rho})}{\partial x} = \int\int e^{\pm}(\vec{q}) \left[\pm i\left(k^2 - q^2\right)^{1/2}\right] \exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2 q \tag{5}$$

This expression can be rewritten by expanding the factor $\left(k^2 - q^2\right)^{1/2}$ in the integrand into a series,

$$\frac{\partial E^{\pm}(x,\vec{\rho})}{\partial x} = \int\int e^{\pm}(\vec{q})\left[\pm i\left(k^2 - \frac{1}{2}q^2 + \cdots\right)\right]\exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2q$$

$$= \int\int e^{\pm}(\vec{q})\left[\pm i\left(k^2 - \frac{1}{2}\nabla_\rho^2 + \cdots\right)\right]\exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2q$$

$$= \int\int e^{\pm}(\vec{q})\left[\pm i\left(k^2 - \nabla_\rho^2\right)^{1/2}\right]\exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2q$$

$$= \pm i\left(k^2 - \nabla_\rho^2\right)^{1/2} \int\int e^{\pm}(\vec{q})\exp\left[i\vec{q}\cdot\vec{\rho} \pm i\left(k^2 - q^2\right)^{1/2} x\right] d^2q$$

$$= \pm i\left(k^2 - \nabla_\rho^2\right)^{1/2} E^{\pm}(x,\vec{\rho}) \tag{6}$$

Therefore, applying this differential operator expression to eq. (3) gives

$$\frac{\partial E(x,\vec{\rho})}{\partial x} = i\left(k^2 + \nabla_\rho^2\right)^{1/2}\left[E^{+}(x,\vec{\rho}) - E^{-}(x,\vec{\rho})\right] \tag{7}$$

Differentiating this expression one more time,

$$\frac{\partial^2 E(x,\vec{\rho})}{\partial x^2} = i\left(k^2 + \nabla_\rho^2\right)^{1/2}\left[\frac{\partial E^{+}(x,\vec{\rho})}{\partial x} - \frac{\partial E^{-}(x,\vec{\rho})}{\partial x}\right] \tag{8}$$

Making the identification $\beta \equiv \left(k^2 + \nabla_\rho^2\right)^{1/2}$ and equating eqs. (3) and (7) yields

$$\frac{\partial E^{+}(x,\vec{\rho})}{\partial x} + \frac{\partial E^{-}(x,\vec{\rho})}{\partial x} = i\beta\left[E^{+}(x,\vec{\rho}) - E^{-}(x,\vec{\rho})\right] \tag{9}$$

Solving eq. (9) for $\partial E^{+}(x,\vec{\rho})/\partial x$ and substituting the result into eq. (8) gives

$$\frac{\partial^2 E(x,\vec{\rho})}{\partial x^2} = -\beta^2\left[E^{+}(x,\vec{\rho}) - E^{-}(x,\vec{\rho})\right] - 2i\beta\frac{\partial E^{-}(x,\vec{\rho})}{\partial x} \tag{10}$$

Similarly, solving eq. (9) for $\partial E^{-}(x,\vec{\rho})/\partial x$ and substituting the result into eq. (8),

$$\frac{\partial^2 E(x,\vec{\rho})}{\partial x^2} = \beta^2\left[E^{+}(x,\vec{\rho}) - E^{-}(x,\vec{\rho})\right] + 2i\beta\frac{\partial E^{+}(x,\vec{\rho})}{\partial x} \tag{11}$$

Finally, substituting eq. (10) into eq. (2) and using eq. (1) gives

$$2i\frac{\partial E^{-}(x,\vec{\rho})}{\partial x} - 2\beta E^{-}(x,\vec{\rho}) = -k^2\beta^{-1}\varepsilon(x,\vec{\rho})\left[E^{+}(x,\vec{\rho}) + E^{-}(x,\vec{\rho})\right] \tag{12}$$

Similarly, substituting eq. (11) into eq. (2) and using eq. (1),

$$2i\frac{\partial E^{+}(x,\vec{\rho})}{\partial x}+2\beta E^{+}(x,\vec{\rho})=k^{2}\beta^{-1}\varepsilon(x,\vec{\rho})\left[E^{+}(x,\vec{\rho})+E^{-}(x,\vec{\rho})\right] \tag{13}$$

Thus, eqs. (12) and (13) are, upon reinstating the expression for $\beta$,

$$2i\frac{\partial E^{+}(x,\vec{\rho})}{\partial x}+2\left(k^{2}+\nabla_{\rho}^{2}\right)^{1/2}E^{+}(x,\vec{\rho})=k^{2}\left(k^{2}+\nabla_{\rho}^{2}\right)^{-1/2}\varepsilon(x,\vec{\rho})\left[E^{+}(x,\vec{\rho})+E^{-}(x,\vec{\rho})\right] \tag{14}$$

and

$$2i\frac{\partial E^{-}(x,\vec{\rho})}{\partial x}-2\left(k^{2}+\nabla_{\rho}^{2}\right)^{1/2}E^{-}(x,\vec{\rho})=-k^{2}\left(k^{2}+\nabla_{\rho}^{2}\right)^{-1/2}\varepsilon(x,\vec{\rho})\left[E^{+}(x,\vec{\rho})+E^{-}(x,\vec{\rho})\right] \tag{15}$$

in which $\left(k^{2}+\nabla_{\rho}^{2}\right)^{1/2}$ is a differential operator and $\left(k^{2}+\nabla_{\rho}^{2}\right)^{-1/2}$ is an integral operator. (See Appendix 1 for these identifications.) If eqs. (14) and (15) are added together, one obtains the additional relation

$$2i\frac{\partial E(x,\vec{\rho})}{\partial x}+2\left(k^{2}+\nabla_{\rho}^{2}\right)^{1/2}\left[E^{+}(x,\vec{\rho})-E^{-}(x,\vec{\rho})\right]=0 \tag{16}$$

Equations (14) and (15) can also be written in terms of the Fourier transform of the field. Defining

$$E(x,\vec{\kappa})\equiv\frac{1}{(2\pi)^{2}}\int_{-\infty}^{\infty}E(x,\vec{\rho})\exp(-i\vec{\kappa}\cdot\vec{\rho})d^{2}\rho \tag{17}$$

Eqs.(14) and (15) become

$$2i\frac{\partial E^{+}(x,\vec{\kappa})}{\partial x}+2\left(k^{2}-\kappa^{2}\right)^{1/2}E^{+}(x,\vec{\kappa})=k^{2}\left(k^{2}-\kappa^{2}\right)^{-1/2}\tilde{\varepsilon}(x,\vec{\kappa})*\left[E^{+}(x,\vec{\kappa})+E^{-}(x,\vec{\kappa})\right] \tag{18}$$

and

$$2i\frac{\partial E^{-}(x,\vec{\kappa})}{\partial x}-2\left(k^{2}-\kappa^{2}\right)^{1/2}E^{-}(x,\vec{\kappa})=-k^{2}\left(k^{2}-\kappa^{2}\right)^{-1/2}\tilde{\varepsilon}(x,\vec{\kappa})*\left[E^{+}(x,\vec{\kappa})+E^{-}(x,\vec{\kappa})\right] \tag{19}$$

where

$$\tilde{\varepsilon}(x,\vec{\kappa})*E\;\;(x,\vec{\kappa})\equiv\int\int\tilde{\varepsilon}(x,\vec{\kappa}')E\;\;(x,\vec{\kappa}-\vec{\kappa}')d^{2}\kappa' \tag{20}$$

denotes the convolution. Equations (14)-(16), or (18) and (19) provide the foundations of many wave propagation studies [*Frankenthal and Beran*, 1997], [*Saichev*, 1980a,b], [*Malakhov and Saichev*, 1980] and [*Ostashev and Tatarskii*, 1978,1979].

## 3. An Extended Stochastic Parabolic Equation for Wide-Angle Forward Propagation

Forward scatter propagation in which the backscattered component is negligible, i.e., $E^-(x,\vec{\rho}) << E^+(x,\vec{\rho})$ is described by eq.(14) in which $E^-(x,\vec{\rho}) = 0$, viz.,

$$2i\frac{\partial E^+(x,\vec{\rho})}{\partial x} + 2\left(k^2 + \nabla_\rho^2\right)^{1/2} E^+(x,\vec{\rho}) = k^2\left(k^2 + \nabla_\rho^2\right)^{-1/2} \varepsilon(x,\vec{\rho}) E^+(x,\vec{\rho}) \tag{21}$$

This relationship, although capable of describing propagation situations in which the wave is scattered at angles up to $\pi/2$ with respect to the propagation direction, is still, strictly speaking, limited in its application to cases in which $\lambda < l_0$ as noted in an analysis by [*Ostashev and Tatarskii*, 1995]. Thus, in this extension of the theory, the restriction $\lambda << l_0$ is lifted only to be replaced by the less stringent condition $\lambda < l_0$. However, in what is to follow, situation in which $\lambda > l_0$will also be considered as approximations. Equation (21) can be put into the form isomorphic to the standard paraxial parabolic equation by multiplying by $2ik$, rearranging some factors, and identifying a new set of differential and integral operators given by

$$\hat{U}_\rho \equiv 2k^2\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{1/2}, \qquad \hat{V}_\rho \equiv \frac{1}{2k^2}\left(1 + \frac{\nabla_\rho^2}{k^2}\right)^{-1/2} = \hat{U}_\rho^{\;-1} \tag{22}$$

which allow eq. (21) to be written as

$$2ik\frac{\partial E^+(x,\vec{\rho})}{\partial x} + \hat{U}_\rho E^+(x,\vec{\rho}) - 2k^4\hat{V}_\rho\left\{\tilde{\varepsilon}(x,\vec{\rho})E^+(x,\vec{\rho})\right\} = 0 \tag{23}$$

Equation (23) can be called a generalized or extended stochastic parabolic equation in the differential operator $\hat{U}$ and the integral operator $\hat{V}$. Reduction of this general relation to the paraxial parabolic equation is accomplished by expanding the operators $\hat{U}_\rho \approx 2k^2 + \nabla_\rho^2$ and $\hat{V}_\rho \approx 1/2k^2$. Equation (23) then becomes

$$2ik\frac{\partial E^+(x,\vec{\rho})}{\partial x} + 2k^2E^+(x,\vec{\rho}) + \nabla_\rho^2E^+(x,\vec{\rho}) - k^2\tilde{\varepsilon}(x,\vec{\rho})E^+(x,\vec{\rho}) = 0 \tag{24}$$

Letting $E^+(x,\vec{\rho}) = W(x,\vec{\rho})\exp(ikx)$ and substituting into eq. (24) gives

$$2ik\frac{\partial W(x,\vec{\rho})}{\partial x}+\nabla_{\rho}^{2}W(x,\vec{\rho})-k^{2}\tilde{\varepsilon}(x,\vec{\rho})W(x,\vec{\rho})=0 \tag{25}$$

which is the classical stochastic parabolic equation in the paraxial approximation.

The extended stochastic parabolic equation eq. (23), just as its specialized counterpart, eq. (25), can be written as an equation governing the statistical moments of the electric field; this will form the subject of the next section.

# 4. The Generalized Statistical Moments of the Electric Field Within the Extended Stochastic Parabolic Equation

## A. Derivation of the Fundamental Equations

Consider the $n+m^{\text{th}}$ statistical moment of the electric field defined by

$$\Gamma_{nm}(x;\vec{\rho}_1,\vec{\rho}_2,\vec{\rho}_n;\vec{\rho}_{n+1}\cdots,\vec{\rho}_{n+m})\equiv\left\langle\prod_{i=1}^{n}E(x,\vec{\rho}_i)\prod_{j=n+1}^{n+m}E^{*}\left(x,\vec{\rho}_j\right)\right\rangle \tag{26}$$

in which each of the transverse coordinates $\vec{\rho}_i$ exists in a plane located at a distance $x$ from the origin, at which a source is placed, in an otherwise arbitrary coordinate system. Defining the random operator

$$D_{x_i,\vec{\rho}_i}\equiv 2ik\frac{\partial}{\partial x}+\hat{U}_{\rho_i}+2k^{4}\hat{V}_{\rho_i}\tilde{\varepsilon}(x_i,\vec{\rho}_i) \tag{27}$$

Eq.(23) applied to $E(x,\vec{\rho}_i)$ becomes

$$D_{x,\vec{\rho}_i}E(x,\vec{\rho}_i)=0 \tag{28}$$

Similarly, the complex conjugate of this relationship taken at $\left(x,\vec{\rho}_j\right)$ gives

$$D_{x,\vec{\rho}_i}^{*}E^{*}\left(x,\vec{\rho}_j\right)=0 \tag{29}$$

Multiplying each of these equations by the remaining $n+m-1^{\text{th}}$ $E$ fields gives

$$\prod_{\substack{i=1\\ i\neq k}}^{n}E(x,\vec{\rho}_i)\left\{D_{x,\vec{\rho}_k}E(x,\vec{\rho}_k)\right\}\prod_{j=n+1}^{n+m}E^{*}\left(x,\vec{\rho}_j\right)=0 \tag{30}$$

and

$$\prod_{i=1}^{n} E(x,\vec{\rho}_i)\prod_{\substack{j=n+1\\ j\neq l}}^{n+m} E^*\left(x,\vec{\rho}_j\right)\left\{D^*_{x,\vec{\rho}_l} E^*(x,\vec{\rho}_l)\right\}=0 \tag{31}$$

Summing eq. (30) over the $n$ possible values of $k$, and summing eq. (31) over the $m$ possible values of $l$ and subtracting the second result from the first gives

$$\sum_{k=1}^{n}\left[\prod_{\substack{i=1\\ i\neq k}}^{n} E(x,\vec{\rho}_i)\left\{D_{x,\vec{\rho}_k} E(x,\vec{\rho}_k)\right\}\prod_{j=n+1}^{n+m} E^*\left(x,\vec{\rho}_j\right)\right]-$$

$$-\sum_{l=1+n}^{n+m}\left[\prod_{i=1}^{n} E(x,\vec{\rho}_i)\prod_{\substack{j=n+1\\ j\neq l}}^{n+m} E^*\left(x,\vec{\rho}_j\right)\left\{D^*_{x,\vec{\rho}_l} E^*(x,\vec{\rho}_l)\right\}\right]=0 \tag{32}$$

Using eq. (27) and rearranging operators as they operate only on $E$-fields with specific $(x,\vec{\rho})$ coordinates gives, after some algebraic manipulations,

$$2ik\frac{\partial}{\partial x}\left[\prod_{i=1}^{n} E(x,\vec{\rho}_i)\prod_{j=n+1}^{n+m} E^*\left(x,\vec{\rho}_j\right)\right]+$$

$$+\left[\sum_{j=1}^{n}\left(\hat{U}_{\rho_j}+2k^4\hat{V}_{\rho_j}\tilde{\varepsilon}\left(x,\vec{\rho}_j\right)\right)-\sum_{l=1+n}^{n+m}\left(\hat{U}^*_{\rho_l}+2k^4\hat{V}^*_{\rho_l}\tilde{\varepsilon}^*(x,\vec{\rho}_l)\right)\right]\cdot$$

$$\cdot\prod_{i=1}^{n} E(x,\vec{\rho}_i)\prod_{j=n+1}^{n+m} E^*\left(x,\vec{\rho}_j\right)=0 \tag{33}$$

Finally, defining the operator

$$\hat{L}_{nm}\equiv 2ik\frac{\partial}{\partial x}+\sum_{j=1}^{n}\left(\hat{U}_{\rho_j}+2k^4\hat{V}_{\rho_j}\tilde{\varepsilon}\left(x,\vec{\rho}_j\right)\right)-\sum_{l=1+n}^{n+m}\left(\hat{U}^*_{\rho_l}+2k^4\hat{V}^*_{\rho_l}\tilde{\varepsilon}^*(x,\vec{\rho}_l)\right) \tag{34}$$

Eq.(33) becomes

$$\hat{L}_{nm}g_{nm}(x;\vec{\rho}_1,\vec{\rho}_2;\cdots,\vec{\rho}_n;\vec{\rho}_{n+1};\cdots,\vec{\rho}_{n+m})=0 \tag{35}$$

where

$$g_{nm}(x;\vec{\rho}_1,\vec{\rho}_2;\cdots,\vec{\rho}_n;\vec{\rho}_{n+1},\cdots,\vec{\rho}_{n+m})\equiv\prod_{i=1}^{n} E(x,\vec{\rho}_i)\prod_{j=n+1}^{n+m} E^*\left(x,\vec{\rho}_j\right) \tag{36}$$

Equation (35) is an operator equation governing the product $g_{nm}$ of the $n+m$ field values within the random medium. It is desired to obtain the ensemble average of the product $\langle g_{nm}(\cdots)\rangle = \Gamma_{nm}(\cdots)$ given by eq. (26). Equation (35) cannot simply be ensemble averaged since, as the operator $\hat{L}_{nm}$ is itself a random function of $\tilde{\varepsilon}(x,\vec{\rho})$ of which the values of the $E$−field are coupled, one has that $\langle \hat{L}_{nm} g_{nm}\rangle \neq \langle \hat{L}_{nm}\rangle \Gamma_{nm}$. In order to obtain a closed equation for the field moment $\Gamma_{nm}$, a stochastic operator method can be applied to eq. (35) which will yield an equation for $\Gamma_{nm}$.

### B. Development of an Operator Method for the Equation Governing the Field Moments

It is expedient to adopt the methods of [*Tatarskii and Gertsenshtein*, 1963], and [*Manning*, 1989] and decompose the operator $\hat{L}_{nm}$ into its average and random parts, i.e.,

$$\hat{L}_{nm} = \langle \hat{L}_{nm}\rangle + \tilde{L}_{nm} \tag{37}$$

where

$$\langle \hat{L}_{nm}\rangle \equiv 2ik\frac{\partial}{\partial x} + \sum_{i=1}^{n}\hat{U}_{\rho_i} - \sum_{j=n+1}^{n+m}\hat{U}^{*}_{\rho_i} \tag{38}$$

and

$$\tilde{L}_{nm} \equiv 2k^4\left[\sum_{j=1}^{n}\hat{V}_{\rho_j}\tilde{\varepsilon}(x,\vec{\rho}_j) - \sum_{l=1+n}^{n+m}\hat{V}^{*}_{\rho_l}\tilde{\varepsilon}^{*}(x,\vec{\rho}_l)\right], \qquad \langle \tilde{L}_{nm}\rangle = 0. \tag{39}$$

Hence, eq. (35) becomes

$$\left(\langle \hat{L}_{nm}\rangle + \tilde{L}_{nm}\right)g_{nm} = 0. \tag{40}$$

Ensemble averaging this relation yields

$$\langle \hat{L}_{nm}\rangle \Gamma_{nm} + \langle \tilde{L}_{nm} g_{nm}\rangle = 0 \tag{41}$$

Similarly writing

$$g_{nm} = \Gamma_{nm} + \tilde{g}_{nm}, \qquad \langle \tilde{g}_{nm}\rangle = 0 \tag{42}$$

and substituting into eq. (41) gives

$$\langle \hat{L}_{nm}\rangle \Gamma_{nm} + \langle \tilde{L}_{nm}\tilde{g}_{nm}\rangle = 0. \tag{43}$$

Remembering that it is the goal of this development to obtain an expression for the general field moment $\Gamma_{nm}$, one follows the development given in [*Manning*, 1989] and subtracts eq. (43) from eq. (35) and using eq. (42) once again obtains

$$\hat{L}_{nm}\Gamma_{nm} + \hat{L}_{nm}\tilde{g}_{nm} - \langle \hat{L}_{nm} \rangle \Gamma_{nm} - \langle \tilde{L}_{nm}\tilde{g}_{nm} \rangle = 0 . \tag{44}$$

Combining the first and third members of this equation using the fact that $\left[\hat{L}_{nm} - \langle \hat{L}_{nm} \rangle\right]\Gamma_{nm} = \tilde{L}_{nm}\Gamma_{nm}$ gives

$$\hat{L}_{nm}\tilde{g}_{nm} + \tilde{L}_{nm}\Gamma_{nm} - \langle \tilde{L}_{nm}\tilde{g}_{nm} \rangle = 0. \tag{45}$$

One must now isolate the random quantity $\tilde{g}_{nm}$ by defining an operator $\hat{L}_{nm}^{-1}$ inverse to $\hat{L}_{nm}$, i.e., $\hat{L}_{nm}^{-1}\hat{L}_{nm} = 1$. Thus, operating on eq. (45) with $\hat{L}_{nm}^{-1}$ yields

$$\tilde{g}_{nm} + \hat{L}_{nm}^{-1}\tilde{L}_{nm}\Gamma_{nm} - \hat{L}_{nm}^{-1}\langle \tilde{L}_{nm}\tilde{g}_{nm} \rangle = 0 . \tag{46}$$

Finally, operating on this relation with $\tilde{L}_{nm}$, ensemble averaging, and solving the resulting expression for $\langle \tilde{L}_{nm}\tilde{g}_{nm} \rangle$ gives

$$\langle \tilde{L}_{nm}\tilde{g}_{nm} \rangle = -\left[1 - \langle \tilde{L}_{nm}\hat{L}_{nm}^{-1} \rangle\right]^{-1} \langle \tilde{L}_{nm}\hat{L}_{nm}^{-1}\tilde{L}_{nm} \rangle \Gamma_{nm} . \tag{47}$$

Substituting this result back into eq. (43), one obtains for the equation governing $\Gamma_{nm}$

$$\left\{ \langle \hat{L}_{nm} \rangle - \left[1 - \langle \tilde{L}_{nm}\hat{L}_{nm}^{-1} \rangle\right]^{-1} \langle \tilde{L}_{nm}\hat{L}_{nm}^{-1}\tilde{L}_{nm} \rangle \right\} \Gamma_{nm} = 0 . \tag{48}$$

The solution of this operator equation, using eqs. (22), (38), and (39), gives an exact solution for the arbitrary field moments for wide-angle propagation through a random medium characterized by the stochastic permittivity $\tilde{\varepsilon}(x,\vec{\rho})$.

The general relation given by eq. (48) can be reduced to the parabolic equation for the field moments in the paraxial approximation in the case where $\lambda << l$. In this case, one has the approximations for the operators $\hat{U}_\rho \approx 2k^2 + \nabla_\rho^2$ and $\hat{V}_\rho \approx 1/2k^2$ used earlier. Equations (38) and (39) then become

$$\langle \hat{L}_{nm} \rangle \approx 2ik\frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla_{\rho_j}^2 - \sum_{l=1+n}^{n+m} \nabla_{\rho_l}^2 \tag{49}$$

and

$$\tilde{L}_{nm} \equiv k^2 \left[ +\sum_{j=1}^{n} \tilde{\varepsilon}\left(x,\vec{\rho}_j\right) - \sum_{l=1+n}^{n+m} \tilde{\varepsilon}^*\left(x,\vec{\rho}_l\right) \right]. \tag{50}$$

One then employs two approximations; the first is given by

$$\hat{L}_{nm} = \left[ \left\langle \hat{L}_{nm} \right\rangle + \tilde{L}_{nm} \right]^{-1} \approx \left\langle \hat{L}_{nm} \right\rangle^{-1} \tag{51}$$

where

$$\begin{aligned} \left\langle \hat{L}_{nm} \right\rangle^{-1} &= \left[ 2ik\frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla^2_{\rho_j} - \sum_{l=1+n}^{n+m} \nabla^2_{\rho_l} \right]^{-1} \\ &\approx \left[ 2ik\frac{\partial}{\partial x} \right]^{-1} \\ &= \frac{1}{2ik} \int_0^x dx' \end{aligned} \tag{52}$$

and the second is given by

$$\left[ 1 - \left\langle \tilde{L}_{nm} \hat{L}^{-1}_{nm} \right\rangle \right]^{-1} \approx \left[ 1 - \left\langle \tilde{L}_{nm} \right\rangle \left\langle \hat{L}^{-1}_{nm} \right\rangle \right]^{-1} \approx 1. \tag{53}$$

under the assumption that the permittivity fluctuations which enter eq. (50) are characterized by a zero mean value, i.e., $\left\langle \tilde{\varepsilon}\left(x,\vec{\rho}_j\right) \right\rangle = 0$. Equation (48) then becomes

$$\left[ 2ik\frac{\partial}{\partial x} + \sum_{j=1}^{n} \nabla^2_{\rho_j} - \sum_{l=1+n}^{n+m} \nabla^2_{\rho_l} - \left\langle \tilde{L}_{nm} \hat{L}^{-1}_{nm} \tilde{L}_{nm} \right\rangle \right] \Gamma_{nm} = 0, \tag{54}$$

where

$$\begin{aligned} \left\langle \tilde{L}_{nm} \hat{L}^{-1}_{nm} \tilde{L}_{nm} \right\rangle \approx \frac{k^3}{2i} \int_0^x \Bigg[ &\sum_{i=1}^{n}\sum_{k=1}^{n} \left\langle \tilde{\varepsilon}(x,\vec{\rho}_i)\tilde{\varepsilon}(x',\vec{\rho}'_k) \right\rangle - \sum_{i=1}^{n}\sum_{l=1+n}^{n+m} \left\langle \tilde{\varepsilon}(x,\vec{\rho}_i)\tilde{\varepsilon}^*(x',\vec{\rho}'_l) \right\rangle - \\ &- \sum_{j=1+n}^{n+m}\sum_{k=1}^{n} \left\langle \tilde{\varepsilon}^*(x,\vec{\rho}_j)\tilde{\varepsilon}(x',\vec{\rho}'_k) \right\rangle + \sum_{j=1+n}^{n+m}\sum_{l=1+n}^{n+m} \left\langle \tilde{\varepsilon}^*(x,\vec{\rho}_j)\tilde{\varepsilon}^*(x',\vec{\rho}'_l) \right\rangle \Bigg] dx', \end{aligned} \tag{55}$$

which is the well known classical paraxial form for the problem [*Tatarskii*, 1971], [*Manning*, 1993]. It is interesting to note that the 'geometrical optics' approximation made in eq. (52) leads to the classical parabolic equation. Thus, one can envision a substantial extension of this development beyond that of the classical treatment if one is to use the entire form of the operator $\left\langle \hat{L}_{nm} \right\rangle^{-1}$, i.e., use the inverse of the operator $\left\langle \hat{L}_{nm} \right\rangle$ as solved in the paraxial approximation rather than the geometrical optics approximation which was used above. This will form the subject of the next section.

# 5. Solutions for the First- and Second-Order Statistical Moments: The Average Field and the Mutual Coherence Function for Wide-Angle Scattering

## A. First-Order Moment: The Average Field

The first order moment of the random electric field in a plane transverse to the direction of propagation is defined through eq. (26) to be given by

$$\Gamma_{10}(x;\vec{\rho}_1) = \left\langle E(x;\vec{\rho}_1)\right\rangle \tag{56}$$

which is a solution of the operator relation of eq. (48), viz.,

$$\left\{\left\langle \hat{L}_{10}\right\rangle - \left[1-\left\langle \tilde{L}_{10}\hat{L}_{10}^{-1}\right\rangle\right]^{-1}\left\langle \tilde{L}_{10}\hat{L}_{10}^{-1}\tilde{L}_{10}\right\rangle\right\}\Gamma_{10} = 0 \tag{57}$$

where, from eqs. (38) and (39),

$$\left\langle \hat{L}_{10}\right\rangle \equiv 2ik\frac{\partial}{\partial x} + \hat{U}_{\rho}\,, \qquad \tilde{L}_{10} \equiv 2k^4\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}), \quad \vec{\rho}\equiv\vec{\rho}_1\,. \tag{58}$$

The ability to proceed in an analytical fashion is dependent upon some simplifying approximations. Since $\hat{L}_{10} \equiv \left\langle \hat{L}_{10}\right\rangle + \tilde{L}_{10}$, one has that

$$\hat{L}_{10}^{-1} = \left[\left\langle \hat{L}_{10}\right\rangle + \tilde{L}_{10}\right]^{-1} \approx \left\langle \hat{L}_{10}\right\rangle^{-1} \tag{59}$$

which allows one to approximately write

$$\left[1-\left\langle \tilde{L}_{10}\hat{L}_{10}^{-1}\right\rangle\right]^{-1} \approx \left[1-\left\langle \tilde{L}_{10}\right\rangle\left\langle \hat{L}_{10}\right\rangle^{-1}\right]^{-1} \approx 1 \tag{60}$$

where, analogous to the case met with in eq. (53), one has $\left\langle \hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho})\right\rangle \approx 0$. Using the results of eqs. ( 59) and (60), eq. (57) becomes

$$\left\{\left\langle \hat{L}_{10}\right\rangle - \left\langle \tilde{L}_{10}\left\langle \hat{L}_{10}\right\rangle^{-1}\tilde{L}_{10}\right\rangle\right\}\Gamma_{10} = 0 \tag{61}$$

Employing the appropriate definitions of the operators, eq. (61) gives

$$\left[2ik\frac{\partial}{\partial x}+\hat{U}_\rho-\left\langle\left(2k^4\hat{V}_\rho\,\tilde{\varepsilon}(x,\vec{\rho})\right)\left\{2ik\frac{\partial}{\partial x}+\hat{U}_\rho\right\}^{-1}\left(2k^4\hat{V}_\rho\tilde{\varepsilon}(x,\vec{\rho})\right)\right\rangle\right]\Gamma_{10}(x,\vec{\rho})=0 \qquad (62)$$

This differential equation in the operators $\hat{U}_\rho$ and $\hat{V}_\rho$ must now be solved for the first-order moment $\Gamma_{10}(x,\vec{\rho})$.

To this end, one must first deal with the factor

$$\left\{2ik\frac{\partial}{\partial x}+\hat{U}_\rho\right\}^{-1}\equiv G(x,\vec{\rho}) \qquad (63)$$

which is the Green function of the operators $2ik\,\partial/\partial x+\hat{U}_\rho$, operationally defined by

$$\left\{2ik\frac{\partial}{\partial x}+2k^2\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{1/2}\right\}G(x,\vec{\rho})=\delta(x-x')\delta(\vec{\rho}-\vec{\rho}\,') \qquad (64)$$

where the definition of $\hat{U}_\rho$ is used. Applying the approximation $\hat{U}_\rho\approx 2k^2+\nabla_\rho^2$ as well as the Fourier Transform relationship

$$g(x,\vec{\kappa})=\left(\frac{1}{2\pi}\right)^2\int_{-\infty}^{\infty}G(x,\vec{\rho})\exp(-i\vec{\kappa}\cdot\vec{\rho})d^2\rho, \qquad (65)$$

eq. (64) can be written as

$$\frac{\partial g}{\partial x}+\frac{1}{2ik}\left(2k^2-\kappa^2\right)g=\left(\frac{1}{2\pi}\right)^2\frac{\delta(x-x')\exp(-i\vec{\kappa}\cdot\vec{\rho}\,')}{2ik} \qquad (66)$$

The solution to this equation is

$$g(x,\vec{\kappa})=\frac{\exp\left[-i\left(2k^2-\kappa^2\right)(x-x')/2k\right]\exp(-ik\rho')}{(2\pi)^2(2ik)} \qquad (67)$$

Finally, applying the inverse transform

$$G(x,\vec{\rho})=\int_{-\infty}^{\infty}g(x,\vec{\kappa})\exp(i\vec{\kappa}\cdot\vec{\rho})d^2\kappa$$

yields, upon evaluating the resulting integral in plane polar coordinates,

$$
\begin{aligned}
G(x,\vec{\rho}) &= \left(\frac{1}{2\pi}\right)^2\left(\frac{1}{2ik}\right)\int_0^{2\pi}\int_0^{\infty}\exp\left[-i\left(2k^2-\kappa^2\right)(x-x')/2k\right]\exp(i\kappa\rho\cos\theta)\kappa\,d\kappa\,d\theta \\
&= \left(\frac{1}{2\pi}\right)\left(\frac{1}{2ik}\right)\int_0^{\infty}\exp\left[-i\left(2k^2-\kappa^2\right)(x-x')/2k\right]\mathrm{J}_0(\kappa\rho)\kappa d\kappa \\
&= \left(\frac{1}{4\pi}\right)\exp\left[-ik(x-x')\right]\frac{\exp\left[-ik(\vec{\rho}-\vec{\rho}')^2/2(x-x')\right]}{x-x'} \\
&= G(x,\vec{\rho};x',\vec{\rho}')
\end{aligned}
\tag{68}
$$

Thus, the third term within the brackets of eq. (62) can be written

$$
\begin{aligned}
\langle\cdots\rangle &\equiv \left\langle\left(2k^4\hat{V}_\rho\,\tilde{\varepsilon}(x,\vec{\rho})\right)\left\{2ik\frac{\partial}{\partial x}+\hat{U}_\rho\right\}^{-1}\left(2k^4\hat{V}_\rho\tilde{\varepsilon}(x,\vec{\rho})\right)\right\rangle = \\
&= \left(2k^4\right)^2\int_0^x\int_{-\infty}^{\infty}G(x,\vec{\rho};x',\vec{\rho}')\left\langle V_\rho\tilde{\varepsilon}(x,\vec{\rho})V_{\rho'}\tilde{\varepsilon}(x,\vec{\rho}')\right\rangle d^2\rho' dx'
\end{aligned}
\tag{69}
$$

Proceeding further, one now must deal with the operator products

$$
V_\rho\tilde{\varepsilon}(x,\vec{\rho}) = \left(\frac{1}{2k^2}\right)\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{-1/2}\tilde{\varepsilon}(x,\vec{\rho}) \tag{70}
$$

Since $\tilde{\varepsilon}(x,\vec{\rho})$ is a random function, it can be represented in the form of a Fourier-Stieltjes integral [*Manning*, 1993], i.e.,

$$
\tilde{\varepsilon}(x,\vec{\rho}) = \int\exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}) \tag{71}
$$

in which the spectral amplitude $dZ(x,\vec{\rho})$ is endowed with the same statistical properties as is the random function $\tilde{\varepsilon}(x,\vec{\rho})$ as will be shown in what is to follow. Applying eq. (71) to eq. (70) results in the following development:

$$
\begin{aligned}
V_\rho\tilde{\varepsilon}(x,\vec{\rho}) &= \left(\frac{1}{2k^2}\right)\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{-1/2}\int\exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}) \\
&= \left(\frac{1}{2k^2}\right)\left(1-\frac{1}{2}\frac{\nabla_\rho^2}{k^2}+\cdots\right)\int\exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}) \\
&= \left(\frac{1}{2k^2}\right)\int\left(1-\frac{1}{2}\frac{(i\kappa)^2}{k^2}+\cdots\right)\exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa}) \\
&= \left(\frac{1}{2k^2}\right)\int\left(1-\frac{\kappa^2}{k^2}\right)^{-1/2}\exp(i\vec{\kappa}\cdot\vec{\rho})dZ(x,\vec{\kappa})
\end{aligned}
\tag{72}
$$

Thus, the ensemble averaged product appearing in right side of eq. (69) becomes,

$$\left\langle V_\rho \tilde{\varepsilon}(x,\vec{\rho}) V_{\rho'} \tilde{\varepsilon}(x',\vec{\rho}')\right\rangle = \left(\frac{1}{2k^2}\right)^2 \int \int \left(1-\frac{\kappa^2}{k^2}\right)^{-1/2}\left(1-\frac{\kappa'^2}{k^2}\right)^{-1/2} \cdot$$

$$\cdot \exp(i\vec{\kappa}\cdot\vec{\rho}+i\vec{\kappa}'\cdot\vec{\rho}')\left\langle dZ(x,\vec{\kappa})dZ(x',\vec{\kappa}')\right\rangle \tag{73}$$

One now makes use of the fact that the atmospheric permittivity fluctuation field $\tilde{\varepsilon}(x,\vec{\rho})$ is taken to be statistically homogeneous, characterized by a power spectral density $\Phi_\varepsilon(x,\vec{\kappa})$ in the transverse plane, and $\delta$-correlated in the longitudinal direction; these circumstances allow one to write [*Manning*, 1993]

$$\left\langle dZ(x,\vec{\kappa})dZ(x',\vec{\kappa}')\right\rangle = 2\pi\delta(x-x')\delta(\vec{\kappa}+\vec{\kappa}')\Phi_\varepsilon(x,\vec{\kappa})d^2\kappa d^2\kappa' \tag{74}$$

Using this in eq. (73) and performing the integrations where possible yields

$$\left\langle V_\rho \tilde{\varepsilon}(x,\vec{\rho}) V_{\rho'} \tilde{\varepsilon}(x',\vec{\rho}')\right\rangle = 2\pi\left(\frac{1}{2k^2}\right)^2 \delta(x-x')\int_{-\infty}^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\exp[i\vec{\kappa}\cdot\vec{\rho}_d]\Phi_\varepsilon(x,\vec{\kappa})d^2\kappa \tag{75}$$

where $\vec{\rho}_d \equiv \vec{\rho}-\vec{\rho}'$ is the difference coordinate.

Equation (69) can now finally be evaluated by substituting into it eqs. (68) and (75); converting the integration in the $\vec{\rho}_d$-plane into one in plane polar coordinates and performing the associated angular integration gives

$$\langle\cdots\rangle = \pi k^4 \int_{-\infty}^{\infty}\int_0^x \delta(x-x')\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\exp[-ik(x-x')]\cdot$$

$$\cdot\int_0^{\infty} \mathrm{J}_0(\kappa\rho_d)(x-x')^{-1}\exp\left[-ik\rho_d^2/2(x-x')\right]\rho_d d\rho_d \Phi_\varepsilon(x,\vec{\kappa})\,dx' d^2\kappa \tag{76}$$

Continuing on and performing the $\rho_d$ and $x'$ integrations gives

$$\langle\cdots\rangle = -\left(\frac{i}{k}\right)\left(\frac{\pi}{2}\right)k^4\int_{-\infty}^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(x,\vec{\kappa})d^2\kappa \tag{77}$$

where the $\delta$-function relation

$$\int_0^x \delta(x-x')dx' = \frac{1}{2} \tag{78}$$

is employed. Finally, noting that the statistics governing the random field $\tilde{\varepsilon}(x,\vec{\rho})$ are not only homogeneous but also isotropic [*Manning*, 1993], one has that $\Phi_\varepsilon(x,\vec{\kappa}) = \Phi_\varepsilon(x,\kappa)$ thus allowing the integral above to also be evaluated in plane polar coordinates and, performing the angular integration, yields

$$\langle\cdots\rangle = -i\pi^2 k^3 \int_0^\infty \left(1-\frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(x,\kappa)\kappa d\kappa \tag{79}$$

Returning to eq. (62) and, substituting eq. (79) into eq. (62) gives

$$\left[2ik\frac{\partial}{\partial x} + 2k^2\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{1/2} + i\pi^2 k^3 \int_0^\infty \left(1-\frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(x,\kappa)\kappa d\kappa\right]\Gamma_{10}(x,\vec{\rho}) = 0 \tag{80}$$

The general solution to this equation is unknown. However, for the plane-wave case, one has that

$$\left(1+\frac{\nabla_\rho^2}{k^2}\right)^{1/2}\Gamma_{10}(x,\vec{\rho}) = \left(1+\frac{\nabla_\rho^2}{k^2}\right)^{1/2}\Gamma_{10}(x) = \Gamma_{10}(x) \tag{81}$$

since the plane wave will not possess any transverse variations. In this special case, eq. (80) becomes

$$\left[2ik\frac{\partial}{\partial x} + 2k^2 + i\pi^2 k^3 \int_0^\infty \left(1-\frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(x,\kappa)\kappa d\kappa\right]\Gamma_{10}(x) = 0 \tag{82}$$

the solution of which is

$$\Gamma_{10}(x) = \Gamma_{10}(0)\exp\left[ikx - \frac{\pi^2}{2}k^2 x \int_0^\infty \left(1-\frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(x,\kappa)\kappa d\kappa\right] \tag{83}$$

In the limit where $\left(1-\kappa^2/k^2\right)^{-1} \approx 1$, eq. (83) becomes the well-known small-angle scattering result from the parabolic equation method [*Tatarskii*, 1971], [*Manning*, 1993].

The spatial spectrum of the atmospheric permittivity fluctuations is given by the Kolmogorov spectrum

$$\Phi_\varepsilon(\kappa) = 0.033 C_\varepsilon^2 \kappa^{-11/3}, \quad \frac{2\pi}{L_0} < \kappa < \frac{2\pi}{l_0} \tag{84}$$

where $C_\varepsilon^2$ is the structure constant governing the strength or level of the turbulent fluctuations of the permittivity $\varepsilon$, $L_0$ is the outer (i.e., largest) spatial scale of fluctuations and $l_0$ is the inner (i.e., smallest) spatial scale. [*Tatarskii*, 1971], [*Manning*, 1993] Typically in the open atmosphere, $L_0$~ 100 meters and $l_0$~1 millimeter. In order to avoid convergence problems with integrals such as that of eq. (83), and at the same time, incorporate the effects of the limits to the spectral interval, the Kolmogorov spectrum is augmented with a lower and upper 'cutoff' reflecting the finite outer and inner scales of turbulence. In this instance, one uses the associated modified von Karman spectrum

$$\Phi_\varepsilon(\kappa) = 0.033 C_\varepsilon^2 \left(K_0^2 + \kappa^2\right)^{-11/6} \exp\left(-\kappa^2/\kappa_m^2\right) \tag{85}$$

where the spatial frequency corresponding to the outer scale of turbulence is $K_0 \equiv 2\pi/L_0$ and that corresponding to the inner scale of turbulence is given by $\kappa_m = 5.92/l_0$ [*Manning*, 1993]. The use of this spectrum in eq. (83) results in an integration that is not analytically amenable. To this end, the following approximation procedure can be used [*Ishimaru*, 1977]

$$\begin{aligned}
\Phi_\varepsilon(\kappa) &= 0.033 C_\varepsilon^2 \left(K_0^2 + \kappa^2\right)^{-11/6} \exp\left(-\kappa^2/\kappa_m^2\right) \\
&= 0.033 C_\varepsilon^2 \left[\left(K_0^2 + \kappa^2\right)^{-11/6} - \left(K_0^2 + \kappa^2\right)^{-11/6} + \left(K_0^2 + \kappa^2\right)^{-11/6} \exp\left(-\kappa^2/\kappa_m^2\right)\right] \\
&= 0.033 C_\varepsilon^2 \left[\left(K_0^2 + \kappa^2\right)^{-11/6} - \left(K_0^2 + \kappa^2\right)^{-11/6} \left\{1 - \exp\left(-\kappa^2/\kappa_m^2\right)\right\}\right] \\
&\approx 0.033 C_\varepsilon^2 \left[\left(K_0^2 + \kappa^2\right)^{-11/6} - \kappa^{-11/3} \left\{1 - \exp\left(-\kappa^2/\kappa_m^2\right)\right\}\right]
\end{aligned} \tag{86}$$

where the last line results from the fact that $\left\{1 - \exp\left(-\kappa^2/\kappa_m^2\right)\right\}$ has a non zero contribution only when $\kappa >> K_0$. Using this representation in the integral indicated in eq. (83) and evaluating, using Mathematica [*Wolfram*, 1999], the resulting expression about the singularity at $\kappa = k$ and retaining the Cauchy principal value yields

$$\begin{aligned}
\Gamma_{10}(x) = \Gamma_{10}(0) \exp\Bigg[ ikx - C_\varepsilon^2 x \Bigg\{ & 0.5439 k^{1/3} (-1)^{1/6} \exp\left(-\frac{k^2}{\kappa_m^2}\right) \Gamma\left(\frac{11}{6}, -\frac{k^2}{\kappa_m^2}\right) - \\
& -0.0814 k^{1/3} i\pi \exp\left(-\frac{k^2}{\kappa_m^2}\right) + 0.0977 k^2 K_0^{-5/3} \; {}_2F_1\left(1,1;\frac{1}{6};-\frac{K_0^2}{k^2}\right) \Bigg\} \Bigg]
\end{aligned} \tag{87}$$

where $\Gamma(\cdots,\cdots)$ is the incomplete gamma function. At the outset, this expression can be simplified by noting that one always has that $k >> K_0$. The hypergeometric function ${}_2F_1(\cdots)$ thus can be approximated by unity, the first term in its series expansion. Also, for the same reason, the second term within the braces is negligible with respect to the third term. One then has

$$\begin{aligned}
\Gamma_{10}(x) = \Gamma_{10}(0) \exp\Bigg[ ikx - C_\varepsilon^2 x \Bigg\{ & 0.5439 k^{1/3} (-1)^{1/6} \exp\left(-\frac{k^2}{\kappa_m^2}\right) \Gamma\left(\frac{11}{6}, -\frac{k^2}{\kappa_m^2}\right) + \\
& +0.0977 k^2 K_0^{-5/3} \Bigg\} \Bigg]
\end{aligned} \tag{88}$$

Finally, replacing the incomplete gamma function in terms of the slightly more manageable Kummer confluent hypergeometric function [*Gradshteyn and Ryzhik*, 1980, eq. (8.351.4)],

$$\Gamma\left(\frac{11}{6},-\frac{k^2}{\kappa_m^2}\right)=\left(-\frac{k^2}{\kappa_m^2}\right)^{11/6}\exp\left(\frac{k^2}{\kappa_m^2}\right)\mathrm{U}\left(1,\frac{17}{6},-\frac{k^2}{\kappa_m^2}\right) \tag{89}$$

yields

$$\Gamma_{10}(x)=\Gamma_{10}(0)\exp\left[ikx-C_\varepsilon^2 x\left\{0.5439k^4\kappa_m^{-11/3}\mathrm{U}\left(1,\frac{17}{6},-\frac{k^2}{\kappa_m^2}\right)+ +0.0977k^2K_0^{-5/3}\right\}\right] \tag{90}$$

This expression will now be examined in the limits $k>\kappa_m$ and $k<\kappa_m$. In the first case, the confluent hypergeometric function becomes [*Abramowitz and Stegun*, 1965, eq. (13.5.2)]

$$\mathrm{U}\left(1,\frac{17}{6},-\frac{k^2}{\kappa_m^2}\right)\approx\left(-\frac{k^2}{\kappa_m^2}\right)^{-1},\ k>\kappa_m$$

In this case, eq. (90) becomes

$$\Gamma_{10}(x)=\Gamma_{10}(0)\exp\left[ikx-0.0977k^2K_0^{-5/3}C_\varepsilon^2x+0.5439k^2\kappa_m^{-5/3}C_\varepsilon^2x\right],\quad k>\kappa_m \tag{91}$$

This result agrees with that obtained from the small-angle scattering form of the theory; see, e.g., [*Ishimaru*, 1977] in which the second term of eq. (91) is neglected since $L_0>>l_0$ for atmospheric turbulence.

In the other limit in which $k<\kappa_m$, one now employs the approximation [*Abramowitz & Stegun*,1965, eq. (13.5.6)]

$$\mathrm{U}\left(1,\frac{17}{6},-\frac{k^2}{\kappa_m^2}\right)\approx\Gamma\left(\frac{11}{6}\right)\left(-\frac{k^2}{\kappa_m^2}\right)^{-11/6},\quad k<\kappa_m$$

Substituting this result into eq. (90) yields, after simplification,

$$\Gamma_{10}(x)=\Gamma_{10}(0)\exp\left[ikx-0.0977k^2C_\varepsilon^2K_0^{-5/3}x-0.4431k^{1/3}C_\varepsilon^2x- -0.2558\,i\,k^{1/3}C_\varepsilon^2x\right],\ k<\kappa_m \tag{92}$$

The third and fourth terms within the exponential represent the wide-angle scattering corrections to the mean field. The imaginary fourth term describes the diffraction phenomena inherent in the scattering mechanism. These correction terms are only appreciable in the event that the wavelength $\lambda\approx L_0$. In the case of atmospheric turbulence, in which $\lambda<<L_0$, these correction

terms become negligible. This result is, unfortunately, not so much a testament to the extended theory as presented here as it is for the applicability of the small-angle scattering approach within the context of Kolmogorov turbulence spectra. In situations in which the outer scale of turbulence is on the order of the wavelength, these additional perturbations could easily be identified.

In many instances within the literature, one finds use of the refractive index structure constant $C_n^2$ which is related to that of the permittivity $C_\varepsilon^2$ by the relation $C_\varepsilon^2 = 4C_n^2$. In addition to this, these results also appear in the form of the coherent intensity $I_C(x)$ defined by

$$I_C(x) \equiv \left|\langle E(x,0)\rangle\right|^2 = \left|\Gamma_{10}(x)\right|^2 \tag{93}$$

Thus, in terms of $C_n^2$, the coherent intensity associated with eqs. (91) and (92) is given by

$$I_C(x) = \left|\Gamma_{10}(0)\right|^2 \exp\left[-0.7816k^2 C_\varepsilon^2 K_0^{-5/3} x + 4.3512k^2 C_\varepsilon^2 \kappa_m^{-5/3} x\right], \quad k > \kappa_m \tag{94}$$

and

$$I_C(x) = \left|\Gamma_{10}(0)\right|^2 \exp\left[-0.7816k^2 C_\varepsilon^2 K_0^{-5/3} x - 3.5448k^{1/3} C_\varepsilon^2 x - \right.$$
$$\left. -2.0464\, i\, k^{1/3} C_\varepsilon^2 x\right], \; k < \kappa_m \tag{95}$$

### B. Second-Order Moment: The Mutual Coherence Function

The second order moment of the random electric field in a plane transverse to the direction of propagation is, from eq. (26), given by

$$\Gamma_{11}(x;\vec{\rho}_1,\vec{\rho}_2) \equiv \left\langle E(x;\vec{\rho}_1) E^*(x;\vec{\rho}_2)\right\rangle \tag{96}$$

which is a solution eq. (48), in this case given by

$$\left\{\left\langle \hat{L}_{11}\right\rangle - \left[1 - \left\langle \tilde{L}_{11}\hat{L}_{11}^{-1}\right\rangle\right]^{-1} \left\langle \tilde{L}_{11}\hat{L}_{11}^{-1}\tilde{L}_{11}\right\rangle\right\}\Gamma_{11} = 0 \tag{97}$$

where from the definitions of eqs. (38) and (39),

$$\left\langle \hat{L}_{11}\right\rangle \equiv 2ik\frac{\partial}{\partial x} + \hat{U}_{\rho_1} - \hat{U}_{\rho_2}^* \tag{98}$$

and

$$\tilde{L}_{11} \equiv 2k^4\left[\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}_1) - \hat{V}_{\rho_2}^*\tilde{\varepsilon}^*(x,\vec{\rho}_2)\right], \qquad \left\langle \tilde{L}_{11}\right\rangle = 0 \tag{99}$$

As with the case for the first order moment, an assumption is made at the outset to render the problem analytically tractable. In particular, one has that

$$\left[1-\left\langle \tilde{L}_{11}\hat{L}_{11}^{-1}\right\rangle\right]^{-1} \approx \left[1-\left\langle \tilde{L}_{11}\right\rangle\left\langle \hat{L}_{11}\right\rangle^{-1}\right]^{-1} \approx 1 \tag{100}$$

which allows eq. (97) to become

$$\left\{\left\langle \hat{L}_{11}\right\rangle - \left\langle \tilde{L}_{11}\hat{L}_{11}^{-1}\tilde{L}_{11}\right\rangle\right\}\Gamma_{11} = 0 \tag{101}$$

which, upon using the operator definitions, is

$$\begin{aligned}\Bigg[2ik\frac{\partial}{\partial x} + \hat{U}_{\rho_1} - \hat{U}^*_{\rho_2} - \Bigg\langle\Big(2k^4\left\{\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}_1) - \hat{V}^*_{\rho_2}\tilde{\varepsilon}^*(x,\vec{\rho}_2)\right\}\Big) \cdot \\ \cdot\left\{2ik\frac{\partial}{\partial x} + \hat{U}_{\rho_1} - \hat{U}^*_{\rho_2}\right\}^{-1}\Big(2k^4\left\{\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}_1) - \hat{V}^*_{\rho_2}\tilde{\varepsilon}^*(x,\vec{\rho}_2)\right\}\Big)\Bigg\rangle\Bigg]\Gamma_{11} = 0\end{aligned} \tag{102}$$

Solving this equation commences with obtaining an expression for the Green function

$$\left\{2ik\frac{\partial}{\partial x} + \hat{U}_{\rho_1} - \hat{U}^*_{\rho_2}\right\}^{-1} \equiv G(x,\vec{\rho}_1,\vec{\rho}_2) \tag{103}$$

given by the solution of

$$\left\{2ik\frac{\partial}{\partial x} + \hat{U}_{\rho_1} - \hat{U}^*_{\rho_2}\right\}G(x,\vec{\rho}_1,\vec{\rho}_2) = \delta(x-x')\delta(\vec{\rho}_1-\vec{\rho}_1')\delta(\vec{\rho}_2-\vec{\rho}_2') \tag{104}$$

where

$$\hat{U}_{\rho_1} \equiv 2k^2\left(1+\frac{\nabla^2_{\rho_1}}{k^2}\right)^{1/2}, \qquad \hat{U}_{\rho_2} \equiv 2k^2\left(1+\frac{\nabla^2_{\rho_2}}{k^2}\right)^{1/2} \tag{105}$$

Proceeding as in Section C.1 and using the approximations $\hat{U}_\rho \approx 2k^2 + \nabla^2_\rho$ for these operators, eq. (104) becomes

$$\left\{2ik\frac{\partial}{\partial x} + \nabla^2_{\rho_1} - \nabla^2_{\rho_2}\right\}G(x,\vec{\rho}_1,\vec{\rho}_2) = \delta(x-x')\delta(\vec{\rho}_1-\vec{\rho}_1')\delta(\vec{\rho}_2-\vec{\rho}_2') \tag{106}$$

Defining the two-dimensional Fourier Transform $g(x,\vec{\kappa}_1,\vec{\kappa}_2)$ of $G(x,\vec{\rho}_1,\vec{\rho}_2)$ by

$$g(x,\vec{\kappa}_1,\vec{\kappa}_2) = \left(\frac{1}{2\pi}\right)^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,\vec{\rho}_1,\vec{\rho}_2)\exp(-i\vec{\kappa}_1\cdot\vec{\rho}_1 - i\vec{\kappa}_2\cdot\vec{\rho}_2)d^2\rho_1\, d^2\rho_2 \tag{107}$$

the transform of eq. (106) is

$$\frac{\partial g}{\partial x} - \left(\frac{1}{2ik}\right)\left(\kappa_1^2 - \kappa_2^2\right)g = \left(\frac{1}{2\pi}\right)^4 \frac{\delta(x-x')}{2ik}\exp(i\vec{\kappa}_1\cdot\vec{\rho}_1' + i\vec{\kappa}_2\cdot\vec{\rho}_2') \tag{108}$$

and the solution of which is

$$g(x,\vec{\kappa}_1,\vec{\kappa}_2) = \frac{\exp\left[-i\left(\kappa_1^2-\kappa_2^2\right)(x-x')/2k\right]}{(2\pi)^4(2ik)}\exp(i\vec{\kappa}_1\cdot\vec{\rho}_1' + i\vec{\kappa}_2\cdot\vec{\rho}_2') \tag{109}$$

Applying the transform inverse to that of eq. (107) to eq. (109), converting to plane polar coordinates, and evaluating the associated angular integrals gives

$$G(x,\vec{\rho}_1,\vec{\rho}_2) = \left(\frac{1}{2\pi}\right)^2\left(\frac{1}{2ik}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-i(x-x')\kappa_1^2/2k\right]\mathrm{J}_0(\kappa_1\rho_{d1})\cdot$$
$$\cdot\exp\left[-i(x-x')\kappa_2^2/2k\right]\mathrm{J}_0(\kappa_2\rho_{d2})\kappa_1\,\kappa_2\, d\kappa_1\, d\kappa_2 \tag{110}$$

where $\rho_{di} \equiv \left|\vec{\rho}_i - \rho_i'\right|$. Finally, performing the remaining integrals over the spatial frequencies yields for the Green function

$$G(x,\vec{\rho}_1,\vec{\rho}_2) = -\left(\frac{ik}{2}\right)\left(\frac{1}{2\pi}\right)^2\left(\frac{1}{x-x'}\right)^2 \exp\left[ik\left((\vec{\rho}_1-\vec{\rho}_1') - (\vec{\rho}_2-\vec{\rho}_2')\right)/2(x-x')\right]$$
$$= G(x,\vec{\rho}_1,\vec{\rho}_2;x',\vec{\rho}_1',\vec{\rho}_2') \tag{111}$$

Therefore, the fourth term in eq. (102) is given by

$$\langle\cdots\rangle \equiv \left\langle\left(2k^4\left\{\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}_1) - \hat{V}_{\rho_2}^*\tilde{\varepsilon}^*(x,\vec{\rho}_2)\right\}\right)\{\cdots\}^{-1}\left(2k^4\left\{\hat{V}_{\rho_1}\tilde{\varepsilon}(x,\vec{\rho}_1) - \hat{V}_{\rho_2}^*\tilde{\varepsilon}^*(x,\vec{\rho}_2)\right\}\right)\right\rangle$$
$$= \left(2k^4\right)^2 \int_0^x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,\vec{\rho}_1,\vec{\rho}_2;x',\vec{\rho}_1',\vec{\rho}_2')\left\langle \hat{V}_{\rho_1}\hat{V}_{\rho_1'}\tilde{\varepsilon}(x,\vec{\rho}_1)\tilde{\varepsilon}(x,\vec{\rho}_1') - \right.$$
$$-\hat{V}_{\rho_1}\hat{V}_{\rho_2'}^*\tilde{\varepsilon}(x,\vec{\rho}_1)\tilde{\varepsilon}^*(x',\vec{\rho}_2') - \hat{V}_{\rho_2}^*\hat{V}_{\rho_1'}\tilde{\varepsilon}^*(x,\vec{\rho}_2)\tilde{\varepsilon}(x',\vec{\rho}_1') +$$
$$\left. +\hat{V}_{\rho_2}^*\hat{V}_{\rho_2'}^*\tilde{\varepsilon}^*(x,\vec{\rho}_2)\tilde{\varepsilon}^*(x',\vec{\rho}_2')\right\rangle d^2\rho_1'\, d^2\rho_2'\, dx' \tag{112}$$

As was done in Section (5A) above, one now uses the Fourier-Stieltjes representation for the random functions $\varepsilon(x,\vec{\rho})$; thus, the result of eq. (72) is now employed in eq. (112). Additionally, in order to carry out the evaluation of the ensemble averages over the spectral amplitudes, one assumes statistical homogeneity in the transverse plane and $\delta$-correlatedness in the longitudinal direction. The result of implementing this procedure on eq. (112) gives

$$\langle\cdots\rangle = 2\pi k^4 \int_0^x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,\vec{\rho}_1,\vec{\rho}_2;x',\vec{\rho}_1',\vec{\rho}_2')\delta(x-x')\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(x,\vec{\kappa})\cdot$$
$$\cdot\left[\exp\left(i\vec{\kappa}\cdot(\vec{\rho}_1-\vec{\rho}_1')\right)-\exp\left(i\vec{\kappa}\cdot(\vec{\rho}_1-\vec{\rho}_2')\right)-\right.$$
$$\left.-\exp\left(i\vec{\kappa}\cdot(\vec{\rho}_2-\vec{\rho}_1')\right)+\exp\left(i\vec{\kappa}\cdot(\vec{\rho}_2-\vec{\rho}_2')\right)\right]d^2\kappa\, d^2\rho_1' d^2\rho_2' dx' \qquad (113)$$

Substitution of eq. (111) into eq. (113) results in an integral which is unwieldy but nonetheless straightforward to evaluate and follows along the same lines as the treatment of eq. (76). Thus, integrating over the spatial variables and using eq. (78) gives

$$\langle\cdots\rangle = -i\pi k^3 \int_{-\infty}^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(\vec{\kappa})\left[1-\exp\left(-i\vec{\kappa}\cdot(\vec{\rho}_2-\vec{\rho}_1)\right)\right]d^2\kappa$$
$$= -i2\pi^2 k^3 \int_0^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(\kappa)\left[1-J_0\left(\kappa|\vec{\rho}_2-\vec{\rho}_1|\right)\right]\kappa\, d\kappa \qquad (114)$$

where statistical isotropy is assumed in arriving at the last result.

Finally, substituting eq. (114) into eq. (102) gives for the differential equation governing the wide-angle mutual coherence function

$$\left[2ik\frac{\partial}{\partial x}+2k^2\left(1+\frac{\nabla^2_{\rho_1}}{k^2}\right)^{1/2}-2k^2\left(1+\frac{\nabla^2_{\rho_2}}{k^2}\right)^{1/2}+\right.$$
$$\left.+i2\pi^2k^3\int_0^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(\kappa)\left[1-J_0\left(\kappa|\vec{\rho}_2-\vec{\rho}_1|\right)\right]\kappa\, d\kappa\right]\Gamma(x;\vec{\rho}_1,\vec{\rho}_2)=0 \qquad (115)$$

The solution to this operator equation is not known. However, in the plane wave case in which there are no transverse variations, expressions analogous to eq. (81) prevail, i.e.,

$$\left(1+\frac{\nabla^2_{\rho_1}}{k^2}\right)^{1/2}=\left(1+\frac{\nabla^2_{\rho_2}}{k^2}\right)^{1/2}\to 1 \qquad (116)$$

thus allowing eq. (115) to become

$$\left[2ik\frac{\partial}{\partial x}+i2\pi^2k^3\int_0^{\infty}\left(1-\frac{\kappa^2}{k^2}\right)^{-1}\Phi_\varepsilon(\kappa)\left[1-J_0\left(\kappa|\vec{\rho}_2-\vec{\rho}_1|\right)\right]\kappa\, d\kappa\right]\Gamma(x;\vec{\rho}_1,\vec{\rho}_2)=0 \qquad (117)$$

the solution of which is

$$\Gamma(x;\vec{\rho}_d) = \Gamma(0;\vec{\rho}_d)\exp\left[-\pi^2 k^3 x \int_0^\infty \left(1-\frac{\kappa^2}{k^2}\right)^{-1} \Phi_\varepsilon(\kappa)[1-\mathrm{J}_0(\kappa\rho_d)]\,\kappa d\kappa\right] \qquad (118)$$

The application of the Kolmogorov spectrum given by eq. (84) can be used in eq. (118) since the structure of the integrand is such that no convergence problems will arise due to the neglect of the spectral cutoffs inherent in the use of the modified von Karman spectrum, eq. (85). (Besides, the use of the complete modified von Karman spectrum in eq. (118) results in an integral that cannot be analytically evaluated.) Thus, substituting eq. (85) into eq. (118) and performing the required integration in Mathematica [*Wolfram*, 1999] gives

$$\begin{aligned}\Gamma_{11}(x,\rho_d) = \Gamma_{11}(0,\rho_d)\exp\Big[ & -(0.8861-0.5116i)k^{1/3}C_\varepsilon^2 x(1-\mathrm{J}_0(k\rho_d)) - \\ & -0.0271k^4 C_\varepsilon^2 \rho_d^{11/3} x\; {}_1F_2\left(1;\frac{17}{6},\frac{17}{6};-\frac{k^2\rho_d^2}{4}\right)\Big]\end{aligned} \qquad (119)$$

in which ${}_1F_2(\cdots)$ is a generalized hypergeometric function. Relating this generalized hypergeometric function to a more familiar function, one makes use of the relation [*Gradshteyn and Ryzhik* (1980), eq. (8.574.3)],

$${}_1F_2\left(1;\frac{17}{6},\frac{17}{6};-\frac{k^2\rho_d^2}{4}\right) = \left(\frac{11}{3}\right)^2 (k\rho_d)^{-11/3}\, \mathrm{s}_{8/3,0}(k\rho_d) \qquad (120)$$

where $\mathrm{s}_{8/3,0}(\cdots)$ is a Lommel function. Hence, eq. (119) becomes

$$\begin{aligned}\Gamma_{11}(x,\rho_d) = \Gamma_{11}(0,\rho_d)\exp\Big[ & -(0.8861-0.5116i)k^{1/3}C_\varepsilon^2 x(1-\mathrm{J}_0(k\rho_d)) - \\ & -0.3643k^{1/3}C_\varepsilon^2 x\,\mathrm{s}_{8/3,0}(k\rho_d)\Big]\end{aligned} \qquad (121)$$

This expression will now be examined in the limits $k\rho_d > 1$ and $k\rho_d < 1$. In the first limit, one employs an asymptotic result obtained from [*Gradshteyn and Ryzhik*, 1980, eq. (8.570.2)], viz.

$$\begin{aligned}\mathrm{s}_{8/3,0}(k\rho_d) &\approx (k\rho_d)^{5/3}\; {}_3F_0\left(1,-\frac{5}{6},-\frac{5}{6};;-\frac{4}{k^2\rho^2}\right) \quad k\rho_d > 1 \\ &\approx (k\rho_d)^{5/3}\end{aligned}$$

and, of course, the result $1-\mathrm{J}_0(k\rho_d) \approx 1$ to obtain (neglecting terms on the order of $k^{1/3}$ since $\lambda < \rho_d$)

$$\Gamma_{11}(x,\rho_d) = \Gamma_{11}(0,\rho_d)\exp\left[-0.3643k^2 C_\varepsilon^2 \rho_d^{5/3} x\right],\ k\rho_d > 1 \qquad (122)$$

In terms of the refractive index structure constant $C_n^2$, this yields

$$\Gamma_{11}(x,\rho_d)=\Gamma_{11}(0,\rho_d)\exp\left[-1.4572k^2C_n^2\rho_d^{5/3}x\right],\ k\rho_d>1 \qquad (123)$$

which is the familiar plane wave result from small-angle scattering theory [*Tatarskii*, 1971], [*Manning*, 1993].

In the other extreme where $k\rho_d<1$, one has

$$1-\mathrm{J}_0(k\rho_d)\approx\frac{k^2\rho_d^2}{4}$$

as well as [*Gradshteyn and Ryzhik*, 1980, eq. (8.570.1)]

$$\mathrm{s}_{8/3,0}(k\rho_d)\approx\left(\frac{3}{11}\right)^2(k\rho_d)^{11/3}$$

from which eq. (121) gives

$$\Gamma_{11}(x,\rho_d)=\Gamma_{11}(0,\rho_d)\exp\left[-0.2215k^{7/3}C_\varepsilon^2\rho_d^2x+0.1279\,i\,k^{7/3}C_\varepsilon^2\rho_d^2x-\right.$$
$$\left.-0.0271k^4C_\varepsilon^2\rho_d^{11/3}x\right] \qquad (124)$$

or, in terms of $C_n^2$,

$$\Gamma_{11}(x,\rho_d)=\Gamma_{11}(0,\rho_d)\exp\left[-0.886k^{7/3}C_n^2\rho_d^2x+0.5116\,i\,k^{7/3}C_n^2\rho_d^2x-\right.$$
$$\left.-0.1084k^4C_n^2\rho_d^{11/3}x\right] \qquad (125)$$

Hence, as with the case of the first order moment of the wave field, the corrections to the second order moment due to the use of this extended theory are negligible in the case of atmospheric turbulence at optical frequencies. Only in the case of millimeter waves could these small corrections be noted.

## 6. Summary and Conclusions

Beginning with the stochastic Helmholtz equation, relations are obtained which transcend the paraxial approximation as far as diffraction phenomena are concerned but are limited to situations prescribed by $l_0>\lambda$ as far as propagation phenomena is concerned. With this situation notwithstanding, a resulting extended stochastic parabolic equation for wide angle propagation has been derived which goes beyond the paraxial approximation of the classical small angle equation and reduces to it in the appropriate limits; this equation is an operator equation which is isomorphic to the classical small angle scattering equation. After the development of an operator method to treat such an equation for the statistical moments of the wave field, solutions of first and second moments were obtained in the case of atmospheric turbulence. It is found that the corrections made to the classical results are insignificant in the atmospheric case.

The Kolmogorov spectrum which is used to represent atmospheric turbulence is such that the contribution of the spectral frequencies which approximately correspond to the wavelength is relatively small as compared to those at the smaller spatial frequencies (which correspond the outer scale of turbulence). This gives rise to the insignificant levels of the correction terms at nominal operating wavelengths. These results support the use of the paraxial approximation of the stochastic parabolic wave equation at optical and millimeter wavelengths propagating through atmospheric turbulence. This may not be the case for propagation through atmospheric aerosols which possess an entirely different spectral behavior [*Manning*, 1993]. This case should be considered in further works on this subject.

# APPENDIX 1
## Plane Wave Expansions and Various Operator Representations

The expansion of an outgoing spherical wave

$$g_0(x,\vec{\rho}) = -\frac{\exp(ikr)}{4\pi r}, \quad \vec{r} = x\hat{x} + \vec{\rho} \tag{A1}$$

which is a solution of the Helmholtz equation

$$\left(\nabla^2 + k^2\right)g_0(x,\vec{\rho}) = 0 \tag{A2}$$

is given by

$$g_0(x,\vec{\rho}) = \int_{-\infty}^{\infty} A(\vec{q})\exp(i\vec{q}\cdot\vec{r})d^3q \tag{A3}$$

the spectrum $A(\vec{q})$ of which can be found by inverting eq. (A3), i.e.,

$$\begin{aligned}\int_{-\infty}^{\infty} g_0(x,\vec{\rho})\exp(-i\vec{q}\,'\cdot\vec{r})d^3r &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A(\vec{q})\exp(i\vec{q}\cdot\vec{r} - i\vec{q}\,'\cdot\vec{r})\,d^3q\,d^3r \\ &= (2\pi)^3\int_{-\infty}^{\infty} A(\vec{q})\delta(\vec{q}-\vec{q}\,')d^3q \\ &= (2\pi)^3 A(\vec{q}\,')\end{aligned} \tag{A4}$$

The left side of eq. (A4) is easily found by converting it to an integral in spherical coordinates, viz.,

$$\begin{aligned}\int_{-\infty}^{\infty} g_0(x,\vec{\rho})\exp(-i\vec{q}\,'\cdot\vec{r})d^3r &= \int_{-\infty}^{\infty}\left(-\frac{\exp(ikr)}{4\pi r}\right)\exp(-i\vec{q}\,'\cdot\vec{r})\,d^3r \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\frac{\exp(ikr)}{r}\exp(-iq'r\cos\theta)r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= \frac{1}{2}\int_{-1}^{1}\int_0^{\infty}\exp(ikr - iq'r\alpha)r\,dr\,d\alpha \\ &= -\frac{1}{2iq}\int_0^{\infty} 2i\exp(ikr)\sin(q'r)r\,dr \\ &= -\frac{1}{q'^2 - k^2}\end{aligned} \tag{A5}$$

Thus, from eq. (A4),

$$A(\vec{q}) = A(q) = -\left(\frac{1}{2\pi}\right)^3 \left(\frac{1}{q^2 - k^2}\right) \tag{A6}$$

Thus, eq. (A3) can be written

$$g_0(x,\vec{\rho}) = -\left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} \frac{\exp(i\vec{q}\cdot\vec{r})}{q^2 - k^2} d^3q \tag{A7}$$

From this relationship, many operator representations can be obtained which are used in this work as well in other works.

To this end, writing eq. (A7) in Cartesian coordinates gives

$$g_0(x,\vec{\rho}) = -\left(\frac{1}{2\pi}\right)^3 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\exp\left[i\left(q_x x + \vec{q}_\rho \cdot \vec{\rho}\right)\right]}{q_x^2 + q_\rho^2 - k^2} dq_x \, d^2q_\rho$$

performing the $q_x$-integration by closing the resulting contour integral in the complex plane in the $x > 0$ region yields

$$g_0(x,\vec{\rho}) = -\frac{i}{8\pi^2} \iint_{q_\rho^2 < k^2} \frac{\exp\left[i\vec{q}_\rho \cdot \vec{\rho} + ix\left(k^2 - q_\rho^2\right)^{1/2}\right]}{k^2 - q_\rho^2} d^2q_\rho \tag{A8}$$

Thus, one has in terms of the two-dimensional plane-wave spectrum

$$g_0(x,\vec{\rho}) = \iint \tilde{g}_0(\vec{q}_\rho) \exp\left[i\vec{q}_\rho \cdot \vec{\rho} + ix\left(k^2 - q_\rho^2\right)^{1/2}\right] d^2q_\rho \tag{A9}$$

where

$$\tilde{g}_0(\vec{q}_\rho) \equiv \left(\frac{1}{8\pi^2 i}\right) \frac{1}{\left(k^2 - q_\rho^2\right)^{1/2}} \tag{A10}$$

In the ‘transverse’ case where the spectrum is considered across the interval $\vec{\rho} - \vec{\rho}_0$ in the transverse plane at $x = 0$, eq. (A9) becomes

$$
\begin{aligned}
g_0(0,\vec{\rho}-\vec{\rho}_0) &= \iint \tilde{g}_0(\vec{q}_\rho)\exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{8\pi^2 i}\right)\iint \left(k^2-q_\rho^2\right)^{-1/2}\exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{8\pi^2 i}\right)\iint \left(k^2+\frac{1}{2}q_\rho^2-\cdots\right)\exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{8\pi^2 i}\right)\iint \left(k^2+\frac{1}{2}\left(\frac{1}{i}\right)^2\nabla_\rho^2+\cdots\right)\exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{8\pi^2 i}\right)\iint \left(k^2+\nabla_\rho^2\right)^{-1/2}\exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{8\pi^2 i}\right)\left(k^2+\nabla_\rho^2\right)^{-1/2}\iint \exp\left[i\vec{q}_\rho\cdot(\vec{\rho}-\vec{\rho}_0)\right]d^2q_\rho \\
&= \left(\frac{1}{2i}\right)\left(k^2+\nabla_\rho^2\right)^{-1/2}\delta(\vec{\rho}-\vec{\rho}_0)
\end{aligned}
$$

Thus, one has the operator representation

$$
2ig_0(0,\vec{\rho}-\vec{\rho}_0) = \left(k^2+\nabla_\rho^2\right)^{-1/2}\delta(\vec{\rho}-\vec{\rho}_0) \tag{A11}
$$

Additionally, operating on eq. (A11) with the transverse Helmholtz operator $k^2+\nabla_\rho^2$ gives

$$
2i\left(k^2+\nabla_\rho^2\right)g_0(0,\vec{\rho}-\vec{\rho}_0) = \left(k^2+\nabla_\rho^2\right)^{1/2}\delta(\vec{\rho}-\vec{\rho}_0) \tag{A12}
$$

Finally, returning to eq. (A10), one can consider the derivative

$$
\begin{aligned}
\frac{\partial g_0(x,\vec{\rho})}{\partial x} &= \iint \tilde{g}_0(\vec{q}_\rho)\left\{i\left(k^2-q_\rho^2\right)^{1/2}\right\}\exp\left[i\vec{q}_\rho\cdot\vec{\rho}+ix\left(k^2-q_\rho^2\right)^{1/2}\right]d^2q_\rho \\
&= i\iint \tilde{g}_0(\vec{q}_\rho)\left(k^2-\nabla_\rho^2\right)^{1/2}\exp\left[i\vec{q}_\rho\cdot\vec{\rho}+ix\left(k^2-q_\rho^2\right)^{1/2}\right]d^2q_\rho \\
&= i\left(k^2-\nabla_\rho^2\right)^{1/2}g_0(x,\vec{\rho}-\vec{\rho}_0)
\end{aligned} \tag{A13}
$$

Equations (A11)-(A13) form the basis of the theory developed by [*Babkin and Klyatskin*, 1980] and [*Babkin, Klyatskin, and Lyubavin*, 1980] to which the methods of invariant imbedding were applied.

The operators $\left(k^2+\nabla_\rho^2\right)^{1/2}$ and $\left(k^2+\nabla_\rho^2\right)^{-1/2}$ can be given analytical representations using the foregoing. In particular, consider an arbitrary function $F(\vec{\rho})$ and the definition of the $\delta$-function to form

$$\left(k^2+\nabla_\rho^2\right)^{1/2}F(\vec{\rho})=\int_{-\infty}^{\infty}\left(k^2+\nabla_\rho^2\right)^{1/2}\delta(\vec{\rho}-\vec{\rho}')F(\vec{\rho}')\,d^2\rho'$$
$$=\int_{-\infty}^{\infty}K(\vec{\rho}-\vec{\rho}')F(\vec{\rho}')d^2\rho' \quad \text{(A14)}$$

where

$$K(\vec{\rho}-\vec{\rho}')\equiv\left(k^2+\nabla_\rho^2\right)^{1/2}\delta(\vec{\rho}-\vec{\rho}') \quad \text{(A15)}$$

But using the result of eq. (A12) in eq. (A15) yields

$$K(\vec{\rho}-\vec{\rho}')=2i\left(k^2+\nabla_\rho^2\right)g_0(0,\vec{\rho}-\vec{\rho}')$$

Thus, returning to eq. (A14) and using the definition of eq. (A1),

$$i\left(k^2+\nabla_\rho^2\right)^{1/2}F(\vec{\rho})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(k^2+\nabla_\rho^2\right)\frac{\exp\left(ik|\vec{\rho}-\vec{\rho}'|\right)}{|\vec{\rho}-\vec{\rho}'|}F(\vec{\rho}')\,d^2\rho' \quad \text{(A16)}$$

which is a differential operator. Similarly, for the operator $\left(k^2+\nabla_\rho^2\right)^{-1/2}$,

$$\left(k^2+\nabla_\rho^2\right)^{-1/2}F(\vec{\rho})=\int_{-\infty}^{\infty}\left(k^2+\nabla_\rho^2\right)^{-1/2}\delta(\vec{\rho}-\vec{\rho}')F(\vec{\rho}')\,d^2\rho'$$
$$=\int_{-\infty}^{\infty}K'(\vec{\rho}-\vec{\rho}')F(\vec{\rho}')d^2\rho' \quad \text{(A17)}$$

where

$$K'(\vec{\rho}-\vec{\rho}')\equiv\left(k^2+\nabla_\rho^2\right)^{1/2}\delta(\vec{\rho}-\vec{\rho}')=2ig_0(0,\vec{\rho}-\vec{\rho}') \quad \text{(A18)}$$

where the last result is by eq. (A11). Again, remembering the definition of eq. (A1), eq. (A17) becomes

$$i\left(k^2+\nabla_\rho^2\right)^{-1/2}F(\vec{\rho})=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\exp\left(ik|\vec{\rho}-\vec{\rho}'|\right)}{|\vec{\rho}-\vec{\rho}'|}F(\vec{\rho}')\,d^2\rho' \quad \text{(A19)}$$

which is an integral operator.

Both eqs. (A16) and (A19) are used extensively by [*Saichev*, 1980a,b] and [*Malakhov and Saichev*, 1980].

## References


Abramowitz, M., and I. A. Stegun (1965), *Handbook of Mathematical Functions*, Dover Publications, Inc., New York.

Babkin, G. I., V. I. Klyatskin (1980), "Theory of Wave Propagation in Nonlinear Inhomogeneous Media", **Sov. Phys. JETP 52** (3), pp. 416-420.

Babkin, G. I., V. I. Klyatskin, and L. Ya. Lyubavin (1980), "Theory of Invariant Imbedding and Waves in Statistically Nonuniform Media", **Sov. Phys. Dokl. 25** (2), pp. 98-100.

Frankenthal, S., and M. J. Beran (1997), "Transverse Spectra of Waves in Random Media", **Waves in Random Media 7** (2), pp. 257-282.

Gradshteyn, I. S., and I. M. Ryzhik (1980), *Table of Integrals, Series, and Products*, Academic Press, New York.

Ishimaru, A. (1977), "The Beam Wave Case and Remote Sensing", in *Laser Beam Propagation in the Atmosphere*, edited by J. W. Strohbehn, Springer-Verlag, Berlin,

Klyatskin, V. I. (1970), "Applicability of the Approximation of a Markov Random Process in Problems Relating to the Propagation of Light in a Medium with Random Inhomogeneities", **Sov. Phys. JETP 30** (3), pp. 520-523.

Malakhov, A. N. and A. I. Saichev (1980), "Representation of Wave Reflected from a Randomly Inhomogeneous Layer in the Form of a Series Satisfying the Causality Condition", **Radiophys. Quantum Electron. 22** (11) pp. 922-928.

Manning, R. M. (1989), "Radiative Transfer Theory for Inhomogeneous Media with Random Extinction and Scattering Coefficients", **J. Math. Phys. 30** (10), pp. 2432-2440.

Manning, R. M. (1993), *Stochastic Electromagnetic Image Propagation*, McGraw-Hill, New York.

Ostashev, V. E. and V. I. Tatarskii (1978), "A Series Based on Multiple Backscattering in Problems of Wave Propagation in Inhomogeneous Media", **Radiophys. Quantum Electron. 21** (4), pp. 504-513.

Ostashev, V. E. and V. I. Tatarskii (1979), "Multiple Forward Scattering Method in Media with Random Inhomogeneities", **Radiophys. Quantum Electron. 22** (2), pp. 116-122.

Ostashev, V. E. and V. I. Tatarskii (1995), "Representation of the Helmholtz equation solution in the form of a series based on backscattering multiplicity", **Waves in Random Media 5**, pp. 125-135.

Saichev, A. I. (1980a), “Allowance for Some Corrections to the Parabolic Quasi-optical Approximation in the Statistical Description of Waves Propagating in Randomly Inhomogeneous Media”, **Radiophys. Quantum Electron. 22** (11), pp. 980-982.

Saichev, A. I. (1980b), “Causal Formulation of the Problem of Backscattering from a layer of a Randomly Inhomogeneous Medium”, **Radioeng. Electron. Physics 22** (1), pp. 31-39.

Tatarskii, V. I. (1969),”Light Propagation in a Medium with Random Refractive Index Inhomogeneities in the Markov Random Process Approximation”, **Sov. Phys. JETP 29** (6), pp.1133-1138.

Tatarskii, V. I. (1971), *The Effect of the Turbulent Atmosphere on Wave Propagation*, U.S. Dept. of Commerce, National Technical Information Service, Springfield, VA.

Tatarskii, V. I. and M. E. Gertsenshtein (1963), Propagation of Waves in a Medium with Strong Fluctuations of the Refractive Index”, **Sov. Phys. JETP 17** (3), pp. 458-463.

Wolfram, S. (1999), *The Mathematica Book*, Fourth Edition, Mathematica Version 4, Cambridge University Press, Cambridge.

# REPORT DOCUMENTATION PAGE

*Form Approved*
*OMB No. 0704-0188*



| 1. AGENCY USE ONLY (*Leave blank*) | 2. REPORT DATE | 3. REPORT TYPE AND DATES COVERED |
|---|---|---|
| | March 2004 | Technical Memorandum |

**4. TITLE AND SUBTITLE**

The Extended Parabolic Equation Method and Implication of Results for Atmospheric Millimeter-Wave and Optical Propagation

**5. FUNDING NUMBERS**

WBS–22–302–20–2F

**6. AUTHOR(S)**

Robert M. Manning

**7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)**

National Aeronautics and Space Administration
John H. Glenn Research Center at Lewis Field
Cleveland, Ohio 44135–3191

**8. PERFORMING ORGANIZATION REPORT NUMBER**

E–14384

**9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)**

National Aeronautics and Space Administration
Washington, DC 20546–0001

**10. SPONSORING/MONITORING AGENCY REPORT NUMBER**

NASA TM—2004-212944

**11. SUPPLEMENTARY NOTES**

Responsible person, Robert M. Manning, organization code 5640, 216–433-6750.

**12a. DISTRIBUTION/AVAILABILITY STATEMENT**

Unclassified - Unlimited
Subject Categories: 32, 59, and 74 Distribution: Nonstandard

Available electronically at http://gltrs.grc.nasa.gov

This publication is available from the NASA Center for AeroSpace Information, 301–621–0390.

**12b. DISTRIBUTION CODE**

**13. ABSTRACT *(Maximum 200 words)***

The extended wide-angle parabolic wave equation applied to electromagnetic wave propagation in random media is considered. A general operator equation is derived which gives the statistical moments of an electric field of a propagating wave. This expression is used to obtain the first and second order moments of the wave field and solutions are found that transcend those which incorporate the full paraxial approximation at the outset. Although these equations can be applied to any propagation scenario that satisfies the conditions of application of the extended parabolic wave equation, the example of propagation through atmospheric turbulence is used. It is shown that in the case of atmospheric wave propagation and under the Markov approximation (i.e., the $\delta$-correlation of the fluctuations in the direction of propagation), the usual parabolic equation in the paraxial approximation is accurate even at millimeter wavelengths. The methodology developed here can be applied to any qualifying situation involving random propagation through turbid or plasma environments that can be represented by a spectral density of permittivity fluctuations.

**14. SUBJECT TERMS**

Diffraction propagation; Electromagnetic wave transmission; Optical communication

**15. NUMBER OF PAGES**

36

**16. PRICE CODE**

| 17. SECURITY CLASSIFICATION OF REPORT | 18. SECURITY CLASSIFICATION OF THIS PAGE | 19. SECURITY CLASSIFICATION OF ABSTRACT | 20. LIMITATION OF ABSTRACT |
|---|---|---|---|
| Unclassified | Unclassified | Unclassified | |