Multipole-Expanded Soft-Collinear Effective Theory II

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Effective Lagrangian

- QCD are related to effective fields exactly by [hep-ph/0211358]:

$$A = A_c + A_{\rm us}\,,$$
$$\psi = \xi + WZ^\dagger q - \frac{1}{in_+D}\,\frac{\not{n}_+}{2}\left(i\not{D}_\perp\xi + \left[\left[i\not{D}_\perp WZ^\dagger\right]\right] q\right),$$

with

$$WZ^\dagger(x) \equiv P\exp\left(ig\int_{-\infty}^0 ds\, n_+A(x+sn_+)\right)\times$$
$$\bar{P}\exp\left(-ig\int_{-\infty}^0 ds\, n_+A_{\rm us}(x+sn_+)\right),$$

and

$$[[f(D)A]] \equiv f(D)A - Af(D_{\rm us}), \qquad [[Af(D)]] \equiv Af(D) - f(D_{\rm us})A.$$

- Can be shown by integrating out η and using EOM of ξ [hep-ph/0206152]

Effective Lagrangian cont'd

- Inserting A and ψ into the QCD Lagrangian gives [hep-ph/0211358]:

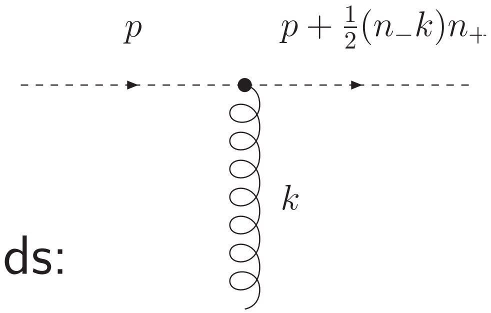
$$
\begin{aligned}
\mathcal{L} = {} & \bar{\xi}\left(in_- D + i\not{D}_\perp \frac{1}{in_+ D}\, i\not{D}_\perp\right)\frac{\not{n}_+}{2}\xi + \bar{q}\, i\not{D}_{\rm us}\, q + \bar{\xi}\left[\left[i\not{D}_\perp W Z^\dagger\right]\right] q + \bar{q}\left[\left[Z W^\dagger i\not{D}_\perp\right]\right]\xi \\
& + \bar{\xi}\,\frac{\not{n}_+}{2}\left[\left[in_- D\, W Z^\dagger\right]\right] q + \bar{\xi}\,\frac{\not{n}_+}{2}\, i\not{D}_\perp\, \frac{1}{in_+ D}\left[\left[i\not{D}_\perp W Z^\dagger\right]\right] q \\
& + \bar{q}\left[\left[Z W^\dagger\, in_- D\right]\right]\frac{\not{n}_+}{2}\xi + \bar{q}\left[\left[Z W^\dagger i\not{D}_\perp\right]\right]\frac{1}{in_+ D}\, i\not{D}_\perp\, \frac{\not{n}_+}{2}\xi \\
& - \bar{q}\left[\left[Z W^\dagger i\not{D}_\perp\right]\right]\frac{\not{n}_+}{2}\,\frac{1}{in_+ D}\left[\left[i\not{D}_\perp W Z^\dagger\right]\right] q \\
= {} & \bar{\xi}\left(in_- D + i\not{D}_\perp \frac{1}{in_+ D}\, i\not{D}_\perp\right)\frac{\not{n}_+}{2}\xi + \bar{q}\, i\not{D}_{\rm us}\, q + \bar{\xi} i\not{D}_\perp W Z^\dagger q + \bar{q} Z W^\dagger i\not{D}_\perp \xi \\
& + \bar{\xi}\,\frac{\not{n}_+}{2}\, in_- D\, W Z^\dagger\, q + \bar{\xi}\,\frac{\not{n}_+}{2}\, i\not{D}_\perp\, \frac{1}{in_+ D}\left(i\not{D}_\perp W Z^\dagger - W Z^\dagger i\not{D}_{\perp\rm us}\right) q \\
& + \bar{q}\, Z W^\dagger\, in_- D\, \frac{\not{n}_+}{2}\xi + \bar{q}\left(Z W^\dagger i\not{D}_\perp - i\not{D}_{\perp\rm us} Z W^\dagger\right)\frac{1}{in_+ D}\, i\not{D}_\perp\, \frac{\not{n}_+}{2}\xi \\
& + \mathcal{O}(\lambda^3 \mathcal{L}_c).
\end{aligned}
$$

Effective Lagrangian cont'd

- Terms containing A_c, $A_{\rm us}$ and W do not have a simple scaling in λ
- Gauge transformations mix terms of different order in the λ-expansion
- To derive effective Lagrangian "momentum conservation" has been used
- Procedure can be rendered gauge invariant by choosing a reference gauge in which momentum conservation is applied
- Then, in any gauge, one first uses U_c to go to the reference gauge, applies momentum conservation, and then goes back to the original gauge with $U_c^\dagger$
- For example, choosing collinear light-cone gauge, $n_+ A_c = 0$, in which case $U_c = Z W^\dagger$ and $W Z^\dagger = 1$, allows one to add or drop terms such as

$$\bar{\xi} W Z^\dagger i \not{D}_{\perp \rm us} q, \qquad \bar{q} Z W^\dagger \left[\left[i \not{D}_\perp W Z^\dagger \right] \right] q.$$

Homogeneous Gauge Transformations



- One must account for the fact that momentum is not conserved at collinear-usoft vertices

- Hence one has to perform a Taylor-expansion of usoft fields:

$$\phi_{\rm us}(x) = \phi_{\rm us}(x_-) + \Big[x_\perp \partial \phi_{\rm us}\Big](x_-) + \frac{1}{2}\, n_- x \Big[n_+ \partial\, \phi_{\rm us}\Big](x_-) + \frac{1}{2}\Big[x_{\mu\perp} x_{\nu\perp} \partial^\mu \partial^\nu \phi_{\rm us}\Big](x_-) + \mathcal{O}(\lambda^3 \phi_{\rm us}),$$

with $x_- = \frac{1}{2}(n_+ x)\, n_-$

- However, under multipole-expanded usoft gauge transformations all $\phi_{\rm c}$ transform inhomogeneous:

$$\xi(x) \to U_{\rm us}(x)\,\xi(x) = U_{\rm us}(x_-)\,\xi(x) + [(x_\perp \partial)\, U_{\rm us}](x_-)\,\xi(x) + \ldots,$$

$$A_c(x) \to U_{\rm us}(x_-)\, A_c(x)\, U^\dagger_{\rm us}(x_-) + [(x_\perp \partial)\, U_{\rm us}](x_-)\, A_c(x)\, U^\dagger_{\rm us}(x_-) + U_{\rm us}(x_-)\, A_c(x)\, [(x_\perp \partial)\, U^\dagger_{\rm us}](x_-) + \ldots.$$

Homogeneous Gauge Transformations cont'd

- Homogenized version of the gauge symmetry [hep-ph/0211358]:

collinear:

$$n_+\hat{A}_c \to U_c\, n_+\hat{A}_c\, U_c^\dagger + \frac{i}{g} U_c \left[n_+\partial, U_c^\dagger\right], \qquad \hat{\xi} \to U_c\, \hat{\xi},$$

$$\hat{A}_{\perp c} \to U_c\, \hat{A}_{\perp c}\, U_c^\dagger + \frac{i}{g} U_c \left[\partial_\perp, U_c^\dagger\right],$$

$$n_-\hat{A}_c \to U_c\, n_-\hat{A}_c\, U_c^\dagger + \frac{i}{g} U_c \left[n_- D_{\rm us}(x_-), U_c^\dagger\right],$$

$$A_{\rm us} \to A_{\rm us}, \qquad q \to q,$$

usoft:

$$\hat{A}_c \to U_{\rm us}(x_-)\, \hat{A}_c\, U_{\rm us}^\dagger(x_-), \qquad \hat{\xi} \to U_{\rm us}(x_-)\, \hat{\xi},$$

$$A_{\rm us} \to U_{\rm us}\, A_{\rm us}\, U_{\rm us}^\dagger + \frac{i}{g} U_{\rm us} \left[\partial, U_{\rm us}^\dagger\right], \qquad q \to U_{\rm us}\, q.$$

Homogeneous Gauge Transformations cont'd

- The collinear Wilson line

$$W_c(x) \equiv P \exp\left(ig \int_{-\infty}^{0} ds\, n_+ \hat{A}_c(x + sn_+)\right),$$

transforms as

$$W_c \to U_c W_c, \qquad W_c \to U_{\rm us}(x_-) W_c U_{\rm us}^\dagger(x_-),$$

because the arguments of collinear fields in the path-ordered product correspond to the same $(x + sn_+)_- = x_-$

- The other objects with simple transformation properties are:

$$\hat{\xi}, \quad q, \quad F_{\rm us}^{\mu\nu}, \quad in_+\hat{D}_c, \quad i\hat{D}_{\perp c}, \quad in_-\hat{D}, \quad iD_{\rm us}^{\mu}.$$

- The multipole-expanded Lagrangian will be composed of these objects

Field Redefinitions

- First choose light-cone gauge, $n_+ A_c = n_+ \hat{A}_c = 0$
- Then define [hep-ph/0211358]:

$$\xi \equiv R\,\hat{\xi}, \qquad A_c \equiv R\hat{A}_c R^\dagger,$$

with

$$R(x) \equiv P\exp\left(ig\int_C dy_\mu A^\mu_{\rm us}(y)\right),$$

where C is a straight path from x_- to x

- If $\hat{A}_c$ is not in the light-cone gauge, $U_c = W_c^\dagger$ will transform $\hat{A}_c$ to it
- Hence one should replace R by $RW_c^\dagger$ in the collinear field redefinitions:

$$\begin{aligned} \xi &= RW_c^\dagger\hat{\xi}, \\ gA_{\perp c} &= R\left(W_c^\dagger i\hat{D}_{\perp c}W_c - i\partial_\perp\right)R^\dagger, \\ gn_-A_c &= R\left(W_c^\dagger in_-\hat{D}W_c - in_-D_{\rm us}(x_-)\right)R^\dagger. \end{aligned}$$

Multipole-Expanded Quark Lagrangian

- To order λ^2 the result takes the form

$$\mathcal{L} = \bar{\xi}\left(in_- D + i\not{D}_{\perp c}\frac{1}{in_+ D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \bar{q}(x)\, i\not{D}_{\rm us}(x) q(x) + \mathcal{L}_\xi^{(1)} + \mathcal{L}_\xi^{(2)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\xi q}^{(2)}\,,$$

where the power-suppressed terms are given by [hep-ph/0211358]:

$$\begin{aligned}
\mathcal{L}_\xi^{(1)} &= \bar{\xi}\left(x_\perp^\mu n_-^\nu\, W_c\, gF_{\mu\nu}^{\rm us} W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi, \\
\mathcal{L}_\xi^{(2)} &= \frac{1}{2}\,\bar{\xi}\left((n_- x)\, n_+^\mu n_-^\nu\, W_c\, gF_{\mu\nu}^{\rm us} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu W_c \left[D_{\rm us}^\rho, gF_{\mu\nu}^{\rm us}\right] W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi \\
&\quad + \frac{1}{2}\,\bar{\xi}\left(i\not{D}_{\perp c}\,\frac{1}{in_+ D_c}\, x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF_{\mu\nu}^{\rm us} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF_{\mu\nu}^{\rm us} W_c^\dagger\,\frac{1}{in_+ D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi, \\
\mathcal{L}_{\xi q}^{(1)} &= \bar{q}\, W_c^\dagger i\not{D}_{\perp c}\,\xi - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c} W_c\, q, \\
\mathcal{L}_{\xi q}^{(2)} &= \bar{q}\, W_c^\dagger\left(in_- D + i\not{D}_{\perp c}\,(in_+ D_c)^{-1}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \bar{q}\, x_{\perp\mu}\overleftarrow{D}_{\rm us}^\mu W_c^\dagger\, i\not{D}_{\perp c}\xi \\
&\quad - \bar{\xi}\,\frac{\not{n}_+}{2}\left(in_-\overleftarrow{D} + i\overleftarrow{\not{D}}_{\perp c}\,\left(in_+\overleftarrow{D}_c\right)^{-1} i\overleftarrow{\not{D}}_{\perp c}\right) W_c\, q - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c} W_c\, x_{\perp\mu} D_{\rm us}^\mu q.
\end{aligned}$$

Multipole-Expanded Yang-Mills Lagrangian

- Including terms of order λ^2 the result is

$$\mathcal{L}_{\rm YM} = -\frac{1}{2}\,\mathsf{tr}\left(F_c^{\mu\nu}F^c_{\mu\nu}\right) - \frac{1}{2}\,\mathsf{tr}\left(F_{\rm us}^{\mu\nu}(x)F^{\rm us}_{\mu\nu}(x)\right) + \mathcal{L}^{(1)}_{\rm YM} + \mathcal{L}^{(2)}_{\rm YM},$$

where the power-suppressed terms are given by [hep-ph/0211358]:

$$\mathcal{L}^{(1)}_{\rm YM} = \mathsf{tr}\left(n_+^\mu F^c_{\mu\nu_\perp} W_c\, i\Big[x_\perp^\rho n_-^\sigma F^{\rm us}_{\rho\sigma}, W_c^\dagger\,[iD_c^{\nu_\perp}W_c]\Big]W_c^\dagger\right) - \mathsf{tr}\left(n_{+\mu}F_c^{\mu\nu_\perp}W_c n_-^\rho F^{\rm us}_{\rho\nu_\perp}W_c^\dagger\right),$$

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm YM} = {} & \frac{1}{2}\,\mathsf{tr}\left(n_+^\mu F^c_{\mu\nu_\perp} W_c\, i\Big[n_- x\, n_+^\rho n_-^\sigma F^{\rm us}_{\rho\sigma} + x_\perp^\rho x_{\perp\omega} n_-^\sigma\left[D^\omega_{\rm us}, F^{\rm us}_{\rho\sigma}\right], W_c^\dagger\,[iD_c^{\nu_\perp}W_c]\Big]W_c^\dagger\right) \\
& - \frac{1}{2}\,\mathsf{tr}\left(n_{+\mu}F_c^{\mu\nu_\perp}W_c\, i\Big[x_\perp^\rho n_-^\sigma F^{\rm us}_{\rho\nu_\perp}, W_c^\dagger\, in_- D W_c - in_- D_{\rm us}\Big]W_c^\dagger\right) \\
& + \mathsf{tr}\left(F_c^{\mu_\perp\nu_\perp}W_c\, i\Big[x_\perp^\rho F^{\rm us}_{\rho\mu_\perp}, W_c^\dagger\,[iD_{c\,\nu_\perp}W_c]\Big]W_c^\dagger\right) \\
& + \frac{1}{2}\,\mathsf{tr}\left(n_+^\mu n_-^\nu F^c_{\mu\nu}W_c n_+^\rho n_-^\sigma F^{\rm us}_{\rho\sigma}W_c^\dagger\right) - \mathsf{tr}\left(F_c^{\mu_\perp\nu_\perp}W_c F^{\rm us}_{\mu_\perp\nu_\perp}W_c^\dagger\right) \\
& - \mathsf{tr}\left(n_{+\mu}F_c^{\mu\nu_\perp}W_c n_-^\rho x_{\perp\sigma}\left[D^\sigma_{\rm us}, F^{\rm us}_{\rho\nu_\perp}\right]W_c^\dagger\right).
\end{aligned}$$

No Renormalization

- Lagrangian does not yet include the effect of hard and soft fluctuations
- Effects are included by matching the QCD to the effective Lagrangian:

$$\mathcal{L}_{\rm tree} = \sum_i \mathcal{O}_i \quad \longrightarrow \quad \mathcal{L} = \sum_{i'} C_{i'} \mathcal{O}_{i'} \, .$$

- In general, this modifies the coefficients of the effective operators, and may induce new operators
- One can show that the tree-level SCET Lagrangian is not renormalized at any order in α_s, and at any order in the λ-expansion [hep-ph/0206152]

No Renormalization cont'd

- Initial N-loop diagram takes the form

$$\int \prod_{i=1}^{N} d^d l_i \prod_k \frac{1}{(\sum l_i + \sum p_j + \sum k_{m,\mathrm{s}} + \sum k_{n,\mathrm{us}})^2} \times \text{polynomial},$$

- For the line-specific linear combinations of momenta one finds, assuming that all l_i are hard:

$$(L_{k,\mathrm{h}} + P_{k,c} + K_{k,\mathrm{s}} + K_{k,\mathrm{us}})^2 = L_{k,\mathrm{h}}^2 + (n_+ P_{k,c})(n_- L_{k,\mathrm{h}}) + \mathcal{O}(\lambda).$$

- After Taylor-expansion in quantities that are subleading in λ one has

$$\int \prod_{i=1}^{N} d^d l_i \prod_k \frac{1}{(L_{k,\mathrm{h}}^2 + (n_+ P_{k,c})(n_- L_{k,\mathrm{h}}))^{a_k}} \times \text{polynomial},$$

which vanishes because $n_-^2 = 0$

- It follows that the SCET Lagrangian is not corrected by hard loops

No Renormalization cont'd

- In the soft case one has instead

$$(L_{k,\mathrm{s}} + P_{k,c} + K_{k,\mathrm{us}})^2 = (n_+ P_{k,c})(n_- L_{k,\mathrm{s}}) + \left[L_{k,\mathrm{s}}^2 + L_{\perp k,\mathrm{s}} P_{\perp k,c} + P_{k,c}^2 + (n_+ P_{k,c})(n_- K_{k,\mathrm{us}})\right] + \mathcal{O}(\lambda^3).$$

- For generic collinear momenta, $n_+ P_{k,c} \sim 1$, or if no collinear momentum flows through the line m, $n_+ P_{m,c} = 0$, one finds

$$\int \prod_{i=1}^{N} d^d l_i \prod_k \frac{1}{((n_+ P_{k,c})(n_- L_{k,\mathrm{s}}) + i\epsilon)^{a_k}} \prod_m \frac{1}{(L_{m,\mathrm{s}}^2 + i\epsilon)^{a_m}} \times \text{polynomial} = 0.$$

- But, if the collinear momentum of line n scales as $n_+ P_{n,c} \sim \lambda$, one gets

$$\int \prod_{i=1}^{N} d^d l_i \prod_n \frac{1}{((n_+ P_{n,c})(n_- L_{n,\mathrm{s}}) + L_{n,\mathrm{s}}^2 + L_{\perp n,\mathrm{s}} P_{\perp n,c} + P_{n,c}^2 + i\epsilon)^{a_n}} \times \prod_k \frac{1}{((n_+ P_{k,c})(n_- L_{k,\mathrm{s}}) + i\epsilon)^{a_k}} \prod_m \frac{1}{(L_{m,\mathrm{s}}^2 + i\epsilon)^{a_m}} \times \text{polynomial},$$

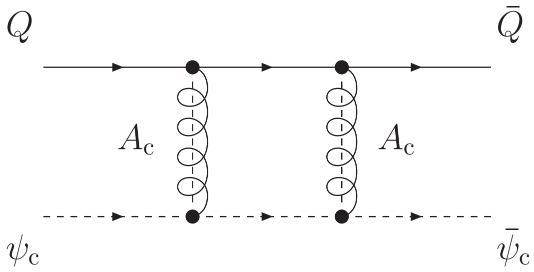
  which is not obviously zero

- No renormalization is proven only for "non-exceptional" soft configurations

Heavy Quark Lagrangian

- Do the newly present collinear modes require to add terms to $\mathcal{L}_{\text{HQET}}$ or is the combined system simply given by $\mathcal{L} = \mathcal{L}_{\text{HQET}} + \mathcal{L}_{\text{SCET}}$?

- At first sight diagrams with intermediate off-shell Q lines might result in new interactions of Q, A_c and ψ_c not present in $\mathcal{L}_{\text{HQET}}$



- However, the effective theory diagrams correspond exactly to those configurations where the QCD diagram develops on-shell singularities

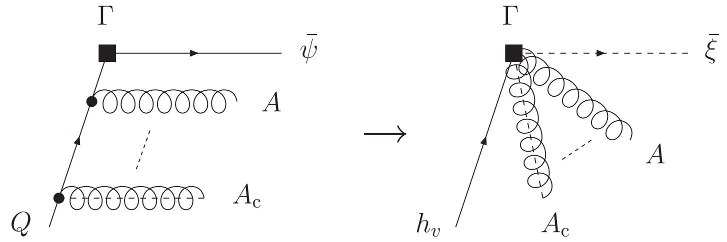
- According to a theorem by Coleman and Norton these singularities correspond to classically allowed scattering processes

- A heavy Q cannot emit or absorb two A_c without going off-shell, so that in a classical scattering process one A_c must be in $|i\rangle$ and the other one in $|f\rangle$

- But this is simply in contradiction to the kinematical assumption, which excludes collinear particles in $|i\rangle$

Effective Heavy-To-Light Current

- Once a A_c has been emitted from Q, it stays off-shell when further A are radiated until it decays into a light quark



- In position space one has

$$\begin{aligned}
J_{\rm QCD}(x) &= \bar{\psi}(x)\,\Gamma \sum_{n=0}^{\infty} \int dz_1 \cdots dz_n \, D_F(x-z_1)\, ig\slashed{A}(z_1) \cdots D_F(z_{n-1}-z_n)\, ig\slashed{A}_c(z_n)\, Q(z_n) \\
&= \bar{\psi}(x)\,\Gamma \sum_{n=0}^{\infty} \frac{1}{i\slashed{\partial}-m} \Big(-g\slashed{A}(x)\Big) \cdots \frac{1}{i\slashed{\partial}-m} \Big(-g\slashed{A}_c(x)\Big)\, Q(x) \\
&= e^{-im\,v\cdot x}\;\bar{\psi}\,\Gamma \left(1-\frac{1}{i\slashed{D}-m\,(1-\slashed{v})}\,g\slashed{A}_c\right) Q_v \equiv e^{-im\,v\cdot x}\;\bar{\psi}\,\Gamma\mathcal{Q},
\end{aligned}$$

which is obviously gauge invariant:

$$g\,\slashed{A}_c\,Q_v = [i\slashed{D}-m\,(1-\slashed{v})]\,Q_v - [i\slashed{D}_{\rm us}-m\,(1-\slashed{v})]\,Q_v = [i\slashed{D}-m\,(1-\slashed{v})]\,Q_v\,.$$

Effective Heavy-To-Light Current cont'd

- The λ-expansion of $\mathcal{Q}$ reads [hep-ph/0206152]:

$$\mathcal{Q} = WZ^\dagger \left(1 + \frac{i\not{D}_{\rm us}}{2m}\right) h_v - \frac{1}{n_- v}\,\frac{\not{n}_-}{2m}\left(i\not{D}WZ^\dagger - WZ^\dagger\, i\not{D}_{\rm us}\right) h_v$$

$$- \frac{1}{n_- v}\,\frac{1}{i n_+ D}\left(\left\{2ivD + \frac{i\not{D}_\perp i\not{D}_\perp}{m}\right\} WZ^\dagger - WZ^\dagger \left\{2ivD_{\rm us} + \frac{i\not{D}_{\perp\rm us} i\not{D}_{\perp\rm us}}{m}\right\}\right) h_v$$

$$+ \mathcal{O}(\lambda^3 h_v),$$

which is manifestly gauge-covariant and for $A_c \to 0$ reduces to Q_v

- $\mathcal{Q}$ can also be derived by solving $(i\not{D} - m)Q = 0$ in the presence of A_c together with the boundary condition $Q(x) \to e^{-im\, v\cdot x} Q_v(x)$ for $A_c \to 0$, which implies that $\mathcal{Q}$ is determined by:

$$\left[i\not{D} - m(1 - \not{v})\right]\mathcal{Q} = 0.$$

Effective Heavy-To-Light Current cont'd

- In terms of $\mathcal{Q}$ the effective current now reads:

$$e^{-im\,v\cdot x} J_{\text{QCD}} \equiv J = (\bar{\xi} + \bar{\eta} + \bar{q})\Gamma\mathcal{Q}.$$

- Different forms of the effective Lagrangian which are equivalent by the EOM give different results for J beyond leading order in λ
- In fact, J is constructed such that it reproduces the on-shell matrix elements with a current insertion in QCD
- Correct J is determined by matching the matrix elements in the full and the effective theory
- In the effective theory this amounts to compute insertions of J together with time-ordered products with terms of higher order in λ

Effective Heavy-To-Light Current cont'd

- Explicit calculation of time-ordered product terms can be avoided by coupling J to a weak external field B:

$$\mathcal{L}_j = e^{-im\, v\cdot x}\, \bar{\psi}\Gamma \mathcal{Q} B + \text{“h.c.”} \equiv (\bar{\xi} + \bar{\eta} + \bar{q}) j + \text{“h.c.”}\,.$$

- After integrating out η by use of the EOM one obtains [hep-ph/0206152]:

$$\mathcal{L}_j = \left(\bar{\xi} - \bar{\xi}\; i\overleftarrow{\not{D}}_\perp \left(i n_+ \overleftarrow{D} \right)^{-1} \frac{\not{n}_+}{2} + \bar{q} Z W^\dagger \right) j + \text{“h.c.”},$$

which implies:

$$J = \left(\bar{\xi} - \bar{\xi}\; i\overleftarrow{\not{D}}_\perp \left(i n_+ \overleftarrow{D} \right)^{-1} \frac{\not{n}_+}{2} + \bar{q} Z W^\dagger \right) \Gamma \mathcal{Q}.$$

- J is manifestly gauge invariant, and up to λ^2 not affected by usoft quarks:

$$\bar{q} Z W^\dagger \Gamma \mathcal{Q} \approx \bar{q} Z W^\dagger \Gamma\, W Z^\dagger h_v = \bar{q}\, \Gamma\, h_v \simeq 0.$$

Effective Heavy-To-Light Current cont'd

- Including terms up to λ^2 the effective current is given by [hep-ph/0206152]:

$$\left[\bar{\psi}(x)\,\Gamma\,Q(x)\right]_{\rm QCD} = e^{-imv\cdot x}\left[J^{(0)} + J^{(1)} + J^{(2)} + \ldots\right],$$

with

$$\begin{aligned}
J^{(0)} &= \bar{\xi}\,\Gamma\,WZ^\dagger h_v\,,\\
J^{(1)} &= -\bar{\xi}\,i\overleftarrow{\not{D}}_\perp\left(in_+\overleftarrow{D}\right)^{-1}\frac{\not{n}_+}{2}\,\Gamma\,WZ^\dagger h_v - \frac{1}{n_-v}\,\bar{\xi}\,\Gamma\,\frac{\not{n}_-}{2m}\left[\left[i\not{D}_\perp WZ^\dagger\right]\right]h_v\,,\\
J^{(2)} &= \bar{\xi}\,\Gamma\,WZ^\dagger\,\frac{i\not{D}_{\rm us}}{2m}\,h_v + \frac{1}{n_-v}\,\bar{\xi}\,i\overleftarrow{\not{D}}_\perp\left(in_+\overleftarrow{D}\right)^{-1}\frac{\not{n}_+}{2}\,\Gamma\,\frac{\not{n}_-}{2m}\left[\left[i\not{D}_\perp WZ^\dagger\right]\right]h_v\\
&\quad - \frac{1}{n_-v}\,\bar{\xi}\,\Gamma\,\frac{\not{n}_-\not{n}_+}{4m}\left[\left[in_-D\,WZ^\dagger\right]\right]h_v\\
&\quad - \frac{1}{n_-v}\,\bar{\xi}\,\Gamma\,\frac{1}{in_+D}\left[\left[\left\{2ivD + \frac{i\not{D}_\perp i\not{D}_\perp}{m}\right\}WZ^\dagger\right]\right]h_v\,.
\end{aligned}$$

Reparameterization Invariance

- Contrary to $\mathcal{L}_{\rm SCET}$, $\mathcal{L}_{\rm HQET}$ does receive radiative corrections, which may induce non-trivial Wilson coefficients as well as new operators at order α_s
- However, various operators are related by reparameterization of v and $n_\pm$, which can be used to reduce the number of independent Wilson coefficients
- J must be invariant under $v \to v' = v + \delta v$ with $(v')^2 = 1$ and $\delta v \sim \lambda^2$
- Manifest velocity reparameterization invariance is achieved by expressing Q in terms of Q_v and

$$\mathcal{V}^\mu \equiv v^\mu + \frac{iD^\mu}{m},$$

since

$$Q_v(x) \to e^{im\delta v\cdot x}\, Q_v(x), \qquad f(\mathcal{V}^\mu)\, Q_v(x) \to e^{im\delta v\cdot x}\, f(\mathcal{V}^\mu)\, Q_v(x).$$

Reparameterization Invariance cont'd

- Indeed $\mathcal{Q}$ can be cast into this form [hep-ph/0206152]:

$$\mathcal{Q} = WZ^\dagger Q_v - \frac{1}{\mathcal{V}^2 - 1} \left[\left[\not{\mathcal{V}} \not{\mathcal{V}} WZ^\dagger\right]\right] Q_v + \mathcal{O}(\lambda^3 Q_v).$$

- In consequence, velocity reparameterization invariance ties

$$WZ^\dagger \, \frac{i\not{D}_{\rm us}}{2m} \, h_v \sim \lambda^2 Q_v,$$

to $WZ^\dagger \, h_v \sim Q_v$, but no other relations are enforced at $\mathcal{O}(\lambda^2 Q_v)$

- Allowed reparameterizations of $n_\pm$ can be divided into three classes:

(a) longitudinal boosts: $n_+ \to \alpha \, n_+, \quad n_- \to \alpha^{-1} \, n_-,$

(b) "minus-transverse" boosts: $n_+ \to n_+, \quad n_- \to n_- + 2e_\perp - e_\perp^2 \, n_+,$

(c) "plus-transverse" boosts: $n_- \to n_-, \quad n_+ \to n_+ + 2f_\perp - f_\perp^2 \, n_-,$

with $\alpha \sim 1$, and $e_\perp$, $f_\perp \sim \lambda$ or smaller

Reparameterization Invariance cont'd

- Under the reparameterizations of $n_\pm$ the components of a vector a, ξ and $WZ^\dagger$ transform as [hep-ph/0206152]:

longitudinal boosts (a):

$$n_+ a \to \alpha\, n_+ a, \quad n_- a \to \alpha^{-1}\, n_- a, \quad a_\perp \to a_\perp, \quad \xi \to \xi, \quad WZ^\dagger \to WZ^\dagger,$$

"minus-transverse" boosts (b):

$$n_+ a \to n_+ a, \quad n_- a \to n_- a + 2 e_\perp a - e_\perp^2\, n_+ a,$$

$$a_\perp^\mu \to a_\perp^\mu - n_+ a\, e_\perp^\mu - e_\perp a\, n_+^\mu + e_\perp^2 n_+ a\, n_+^\mu, \quad \xi \to \xi + \frac{\not{e}_\perp \not{n}_+}{2}\, \xi, \quad WZ^\dagger \to WZ^\dagger,$$

"plus-transverse" boosts (c):

$$n_+ a \to n_+ a + 2 f_\perp a - f_\perp^2\, n_- a, \quad n_- a \to n_- a,$$

$$a_\perp^\mu \to a_\perp^\mu - n_- a\, f_\perp^\mu - f_\perp a\, n_-^\mu + f_\perp^2 n_+ a\, n_-^\mu, \quad \xi \to \xi + \not{f}_\perp \frac{1}{i n_+ D}\, i\not{D}_\perp \xi,$$

$$WZ^\dagger \to WZ^\dagger - \frac{2}{i n_+ D} \left[\left[i f_\perp D_\perp WZ^\dagger \right]\right] + \mathcal{O}(\lambda^3).$$

Reparameterization Invariance cont'd

- Longitudinal boosts constrain the allowed operators in $\mathcal{L}$, but do not relate different operators, since each term of J is separately invariant
- In contrast, transverse boosts lead to non-trivial relations between different operators in J
- In particular, the leading order current $\bar{\xi}\,\Gamma\,WZ^\dagger h_v$ is not invariant under transverse boosts
- It is easy to see that the transformation of $\bar{\xi}$ is compensated by the one of

$$-\bar{\xi}\,i\overleftarrow{\not{D}}_\perp \left(in_+\overleftarrow{D}\right)^{-1}\frac{\not{n}_+}{2},$$

which suggests that one should express J in terms of

$$\bar{\Xi} \equiv \bar{\xi} - \bar{\xi}\,i\overleftarrow{\not{D}}_\perp \left(in_+\overleftarrow{D}\right)^{-1}\frac{\not{n}_+}{2},$$

which is simply $\bar{\xi} + \bar{\eta}$ in the absence of usoft quark fields

Reparameterization Invariance cont'd

- To exploit transverse boost invariance further, one writes [hep-ph/0206152]:

$$J = \bar{\Xi}\Gamma W Z^\dagger Q_v - \frac{1}{n_- v}\bar{\Xi}\Gamma \frac{\not{n}_-}{2m}\left[\left[i\not{D}_\perp W Z^\dagger\right]\right] Q_v - \frac{1}{n_- v}\bar{\Xi}\Gamma \frac{\not{n}_-\not{n}_+}{4m}\left[\left[i n_- D\, W Z^\dagger\right]\right] Q_v$$

$$- \frac{1}{n_- v}\bar{\Xi}\Gamma \frac{1}{i n_+ D}\left[\left[2 i v D\, W Z^\dagger\right]\right] Q_v - \frac{1}{n_- v}\bar{\Xi}\Gamma \frac{1}{i n_+ D}\left[\left[\frac{i\not{D}_\perp i\not{D}_\perp}{m} W Z^\dagger\right]\right] Q_v + \ldots \equiv \sum_{i=0}^{4} J_i .$$

- J_0 is invariant under minus-transverse boosts (b), but J_1 is not:

$$\delta_{(b)} J_1 = -\frac{1}{n_- v}\bar{\Xi}\Gamma \frac{1}{2m}\left[\left[\{2\,\not{e}_\perp i\not{D}_\perp - \not{n}_-\not{n}_+ i e_\perp D\}\, W Z^\dagger\right]\right] Q_v + \ldots ,$$

  which is canceled by $\delta_{(b)} J_4$ and $\delta_{(b)} J_2$, respectively

- J_0 is not invariant under plus-transverse boosts (c), but $\delta_{(c)}(J_0 + J_3) = 0$, whereas $\delta_{(c)} J_{1,2,4} = \mathcal{O}(\lambda^3 J_0)$

- One thus concludes, that the Wilson coefficients of the operators in the sets $\{J_0, J_3\}$ and $\{J_1, J_2, J_4\}$ are related by reparameterization invariance of $n_\pm$

Multipole-Expanded Heavy-To-Light Current

- The multipole-expanded heavy-to-light current reads [hep-ph/0211358]:

$$\left[\bar{\psi}(x)\,\Gamma\,Q(x)\right]_{\rm QCD} = e^{-imv\cdot x}\left\{J^{(A0)} + J^{(A1)} + J^{(A2)} + J^{(B1)} + J^{(B2)}\right\},$$

with

$$\begin{aligned}
J^{(A0)} &= \bar{\xi}\,\Gamma W_c h_v\,,\\
J^{(A1)} &= \bar{\xi}\,\Gamma W_c\, x_{\perp\mu} D^\mu_{\rm us} h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c h_v\,,\\
J^{(A2)} &= \bar{\xi}\,\Gamma W_c\left(\frac{1}{2}\,n_- x\, n_+ D_{\rm us} h_v + \frac{1}{2}\,x_{\mu\perp} x_{\nu\perp} D^\mu_{\rm us} D^\nu_{\rm us} h_v + \frac{i\not{D}_{\rm us}}{2m}\,h_v\right)\\
&\quad - \bar{\xi}\,\Gamma\frac{1}{in_+ D_c}\left[in_- D\, W_c - W_c\, in_- D_{\rm us}\right] h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c\, x_{\perp\mu} D^\mu_{\rm us} h_v\,,\\
J^{(B1)} &= -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c} W_c\right] h_v\,,\\
J^{(B2)} &= -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c} W_c\right] x_{\mu\perp} D^\mu_{\rm us} h_v - \bar{\xi}\,\Gamma\,\frac{\not{n}_-}{2m}\left[in_- D\, W_c - W_c\, in_- D_{\rm us}\right] h_v\\
&\quad - \bar{\xi}\,\Gamma\frac{1}{in_+ D_c}\left[\frac{i\not{D}_{\perp c} i\not{D}_{\perp c}}{m} W_c\right] h_v + \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c} W_c\right] h_v\,.
\end{aligned}$$

- $J^{(A)}$ and $J^{(B)}$ are separately invariant under reparameterizations of $n_\pm$