GPPII Homework Problem Set 5

Due March 16, 2009

1. Maximum growth rate of magnetorotational instability (MRI).
   Suppose that the base state is given by $\mathbf{B} = (0, 0, B)$ and $\mathbf{V} = (0, r\Omega(r), 0)$. Perturb only B_r, B_θ, V_r, and V_θ in the form proportional to $\exp(\gamma t - ikz)$.

   (a) Invoke the thin disk approximation ($k \gg 1/r$) to show that the perturbation is incompressible. Also verify that the perturbed field satisfies $\nabla \cdot \mathbf{B} = 0$.

   (b) Derive the linearized equations from the induction equation and equation of motion in the thin disk limit:

$$\gamma B_r = -ikV_r B \tag{1}$$

$$\gamma B_\theta = -ikV_\theta B + \frac{d\Omega}{d\ln r} B_r \tag{2}$$

$$\gamma V_r - 2\Omega V_\theta = -i\frac{kB}{\mu_0 \rho} B_r \tag{3}$$

$$\gamma V_\theta + \frac{\kappa^2}{2\Omega} V_r = -i\frac{kB}{\mu_0 \rho} B_\theta \tag{4}$$

   where $\kappa^2 = (1/r^3)d(r^4\Omega^2)/dr$ and ρ is mass density.

   (c) Prove that the dispersion relation is given by

$$\gamma^4 + \left(\kappa^2 + 2(kV_A)^2\right)\gamma^2 + (kV_A)^2\left((kV_A)^2 + \frac{d\Omega^2}{d\ln r}\right) = 0 \tag{5}$$

   where $V_A = B/\sqrt{\mu_0 \rho}$.

   (d) MRI exists only when $d\Omega^2/d\ln r < 0$, but it can be stabilized by a strong B or V_A. What is the minimum V_A to stabilize MRI?

(e) Prove that the maximum growth rate is given by

$$\gamma_{max} = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right| \tag{6}$$

when

$$(kV_A)^2 = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln r}. \tag{7}$$

What is the maximum growth rate for Keplerian flows?