Statistical Hamiltonian Theory of Convective Cell Generation[a]

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— Outline —

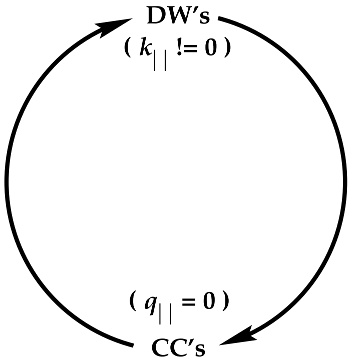
The goal of this work
is to obtain a general formula
for the convective-cell growth rate $\gamma_{\vec{q}}$
for systems of nonlinearly coupled fields.

- Introductory remarks on previous work for scalar fields.
- Hamiltonian formulation for coupled fields:
  - Role of scalar "plasmon density" for multiple coupled fields?
  - Hamiltonian bracket formulation of Eulerian field equations
  - Tensor wave kinetic equation
  - General formula for $\gamma_{\vec{q}}$
  - Remarks

We will begin with some introductory remarks on previous calculations for scalar fields...

Drift waves and convective cells form a coupled turbulent system

- **Drift wave:** $k_{||} \neq 0$.
- **Convective cell:** $q_{||} = 0$
  - **Zonal flow:** $q_y = 0$, $q_{||} = 0$
  - **Streamer:** $q_x = 0$, $q_{||} = 0$



In general, the DW–CC interaction is self-consistent. We focus here on the CC generation question (the right-hand arrow in the figure).

Example: Generalized Hasegawa–Mima equation (accounts for nonadiabatic electron response for CC's).

$$(\alpha - \nabla^2)\partial_t \varphi + \vec{V}_E \cdot \vec{\nabla}(\alpha - \nabla^2)\varphi = 0. \tag{1}$$

$$\alpha = \begin{cases} 1 & \text{(DW's)} \\ 0 & \text{(CC's)} \end{cases} \tag{2}$$

Statistical formulation

Correlation matrix: $C_{\vec{k}}(t) \doteq \langle \delta\varphi_{\vec{k}}(t)\delta\varphi^*_{\vec{k}}(t)\rangle$. Spectral balance eqn:

$$\partial_t C_{\vec{k}} - \underbrace{2\gamma^{\text{lin}}_{\vec{k}} C_{\vec{k}}}_{\text{linear growth}} + \underbrace{2\eta_{\vec{k}} C_{\vec{k}}}_{\text{nonlinear damping}} = \underbrace{2F_{\vec{k}}}_{\text{nonlinear noise}}, \tag{3}$$

or

$$\partial_t C_{\vec{k}} = (2\gamma^{\text{lin}}_{\vec{k}} + 2\underbrace{\gamma^{\text{nlin}}_{\vec{k}}}_{-\eta_k})C_{\vec{k}} + 2F_{\vec{k}}. \tag{4}$$

When applied to convective cell's,

$$\partial_t C_{\vec{q}} = \cdots + 2\gamma^{\text{nlin}}_{\vec{q}} C_{\vec{q}} + \cdots \tag{5}$$

Frequently, it is assumed that $q \ll k$. Analogy:

$$\begin{aligned} q &\text{ — resolved modes} \\ k &\text{ — unresolved modes} \end{aligned} \tag{6}$$

If, for example,

$$\gamma_{\vec{q}} \sim +q^2\mu_{\vec{q}} \Rightarrow \text{negative eddy viscosity!} \tag{7}$$

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Previous work

Extensive work on the statistics of the Hasegawa–Mima equation was performed by Krommes and Kim (2000) (motivated by original work on zonal-flow generation by Diamond et al. (1998).

- $q \ll k$ assumed.
- Systematic derivation of $\gamma_{\vec{q}}$ from Markovian closure theory.
- Physical mechanisms elucidated and paradoxes resolved.
- Modulation algorithm justified (with proper triad interaction time $\theta_{\vec{k},-\vec{k},\vec{q}}$).

But an important issue remained:

What does one do for systems with coupled fields (common in practice)?

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Multiple-field descriptions

Examples of applications with $n > 1$ fields:

- curvature-driven ITG ($n = 2$)
- electrostatic collisional drift waves (Hasegawa–Wakatani) ($n = 2$)
- finite-β collisional drift waves ($n = 3$)

Apparently, one requires a tensor wave kinetic equation.

Is there a scalar "plasmon density"?

We will aim for a compact general formula for $\gamma_{\vec{q}}$.

Hamiltonian description of Eulerian PDE's

Key reference:

P. J. Morrison, Hamiltonian description of the ideal fluid, Rev. Mod. Phys. 70, 467 (1998).

Very briefly: A nonlinear system of PDE's is Hamiltonian if

$$\partial_t \vec{\psi} = \{\vec{\psi}, \mathcal{H}\}. \tag{8}$$

That is, we need

- a **Hamiltonian functional** $\mathcal{H}[\vec{\psi}]$,
- a **Poisson bracket** $\{\cdot\,,\,\cdot\}$.

The bracket must satisfy

- **antisymmetry:** $\{A, B\} = -\{B, A\}$
- **Jacobi identity:**
  $\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0.$

Let overline denote spatial integral. We take

$$\textbf{Hamiltonian:} \quad \mathcal{H}[\vec{\psi}] = \frac{1}{2}\overline{\psi^i \widehat{g}_{ij} \psi^j},$$

$$\textbf{Bracket:} \quad \{A, B\} = \overline{S^{rs}\left[\frac{\delta A}{\delta \psi^r}, \frac{\delta B}{\delta \psi^s}\right]}, \tag{9}$$

where S is symmetric, $S^{ij}[\vec{\psi}] = S^{ij}{}_k \psi^k$, and

$$[A, B] \doteq \partial_x A\, \partial_y B - \partial_x B\, \partial_y A = \widehat{\mathrm{z}} \cdot \vec{\nabla} A \times \vec{\nabla} B. \tag{10}$$

Thus

$$\dot{\vec{\psi}} = \{\psi^i, \mathcal{H}\} = -[S^{ij}, \delta\mathcal{H}/\delta\psi^j]. \tag{11}$$

Example:

$$\mathcal{H}[\omega] = \frac{1}{2}\overline{\omega(-\nabla^{-2})\omega}, \quad \{A, B\} = \overline{\omega\left[\frac{\delta A}{\delta \omega}, \frac{\delta B}{\delta \omega}\right]} \tag{12a,b}$$

lead to the 2D Euler equation for vorticity ω,

$$\partial_t \omega = \{\omega, \mathcal{H}\} = -\vec{V} \cdot \vec{\nabla} \omega, \tag{13}$$

where $\omega \doteq \nabla^2 \varphi$ and $\vec{V} \doteq \widehat{\mathrm{z}} \times \vec{\nabla} \varphi$. Here there is a single structure constant: $S^{\omega\omega}{}_{\omega} = 1$.

Covariant formulation

$\widehat{g}$ [a linear integro-differential operator $\widehat{g} = \widehat{g}(\nabla)$] serves as a metric tensor to lower indices:

$$\psi_i = \widehat{g}_{ij}\psi^j. \tag{14}$$

Thus

$$\mathcal{H} = \frac{1}{2}\overline{\psi^i\psi_i}. \tag{15}$$

Note

$$\frac{\delta\mathcal{H}}{\delta\psi^k} \equiv \widehat{\psi}_k = \widehat{g}_{kj}\psi^j = \psi_k. \tag{16}$$

Thus the dynamics are naturally (nonrelativistically) covariant:

$$\dot{\vec{\psi}}^i = -S^{ij}{}_k[\psi^k, \psi_j]. \tag{17}$$

$\mathcal{H}$ is clearly conserved:

$$\dot{\mathcal{H}} = \overline{\psi_i\dot{\psi}^i} = -S^{ij}{}_k\overline{[\psi^k, \psi_j]\psi_i} = -S^{ij}{}_k\overline{\psi^k[\psi_j, \psi_i]} = 0 \tag{18}$$

(sym. $\times$ antisym.). When $S^{ij}{}_k = S^{ik}{}_j$, $Z \doteq \frac{1}{2}\overline{\psi^i\psi^i}$ is also conserved. Z is a Casimir invariant (conserved independently of the form of $\mathcal{H}$). There may be other Casimirs as well.

Statistical determination of $\gamma_{\vec{q}}$

$\gamma_{\vec{q}}$ describes the mean growth rate of the CC energy spectrum after averaging over an ensemble of random DW's and CC's

Define the correlation matrix as

$$C^{ij}_{\vec{k}} \doteq \langle \psi^i_{\vec{k}} \psi^j_{\vec{k}} \rangle. \tag{19}$$

This evolves according to

$$\partial_t \mathsf{C}_{\vec{k}} = [(\underbrace{\mathsf{L}_{\vec{k}}}_{\text{lin.}} - \underbrace{\Sigma_{\vec{k}}}_{\text{nlin.}}) \cdot \mathsf{C}_{\vec{k}}]^H + \cdots\,, \tag{20}$$

The nonlinear effects (responsible for $\gamma_{\vec{q}}$) are represented by the mass operator Σ. It can be shown (from field theory) that

$$\boxed{\Sigma(t;t') = \delta\vec{G}(t)/\delta\langle\vec{\psi}\rangle(t')\Big|_{\vec{\eta}=\vec{0}}}\,, \tag{21}$$

where

- $\vec{G} \doteq \langle \widetilde{\vec{G}} \rangle$,
- $\widetilde{\vec{G}} \doteq [\mathsf{S}; \hat{\vec{\psi}}]$ (the divergence of a generalized flux),
- and $\hat{\vec{\eta}}$ is a source added to the right-hand side of the dynamical equation (17).

The role of $\hat{\vec{\eta}}$ is to break the symmetry of homogeneous statistics, since all mean fields vanish for $\hat{\vec{\eta}} = 0$.

Now $\gamma_{\vec{q}}$ is determined by $\Sigma_{\vec{q}}$:

$$\gamma_{\vec{q}} \doteq \dot{\underline{\mathcal{E}}}_{\vec{q}}/2\underline{\mathcal{E}}_{\vec{q}} = -[(\underline{\Sigma}^{H}_{\vec{q}})^{ij}\underline{C}_{ji}(\vec{q})]/\underline{C}^{k}_{k}(\vec{q}). \tag{22}$$

To examine the energy evolution, we project the fluctuations into

DW subspace — $\vec{\psi}'$,

CC subspace — $\underline{\vec{\psi}}$.

We obtain

$$\dot{\vec{\psi}}' = -[\mathsf{S}'; \underline{\hat{\vec{\psi}}}] - [\underline{\mathsf{S}}; \hat{\vec{\psi}}'] - ([\mathsf{S}'; \hat{\vec{\psi}}'] - \underline{\widetilde{\vec{G}}}) + \hat{\vec{\eta}}', \tag{24a}$$

$$\dot{\underline{\vec{\psi}}} = -\underline{\widetilde{\vec{G}}} + \underline{\hat{\vec{\eta}}}, \tag{24b}$$

Generalized Poynting theorem

The short-wavelength energy $\widetilde{\mathcal{E}} \doteq \mathcal{H}[\vec{\psi}']$ obeys $\dot{\widetilde{\mathcal{E}}} = \overline{\vec{\hat{\psi}}' \cdot \dot{\vec{\psi}}'}$, or

$$\dot{\widetilde{\mathcal{E}}} = -\overline{\vec{\hat{\psi}}' \cdot [\mathsf{S}'; \underline{\vec{\hat{\psi}}}']} - \overline{\vec{\hat{\psi}}' \cdot [\underline{\mathsf{S}}; \vec{\hat{\psi}}']} + \overline{\vec{\hat{\psi}}' \cdot \vec{\hat{\eta}}'} \tag{25a}$$

$$= \underbrace{\overline{[\mathsf{S}'; \vec{\hat{\psi}}'] \cdot \underline{\vec{\hat{\psi}}}}}_{\widetilde{G}} - \operatorname{Tr}\left(\underline{\mathsf{S}} \cdot \overline{\underbrace{[\vec{\hat{\psi}}', \vec{\hat{\psi}}']}_{0\ \text{(antisym.)}}}\right) + \overline{\vec{\hat{\psi}}' \cdot \vec{\hat{\eta}}'}. \tag{25b}$$

Now average Eq. (25b). Define

$$\underline{\vec{Q}} \doteq \langle \underline{\vec{\psi}} \rangle, \quad \underline{\vec{P}} \doteq \langle \underline{\vec{\hat{\psi}}} \rangle, \quad \underline{\vec{G}} \doteq \langle \underline{\widetilde{\vec{G}}} \rangle, \quad \overline{\mathcal{E}} \doteq \langle \widetilde{\mathcal{E}} \rangle. \tag{26a,b,c,d}$$

Then we are led to a multiple-field generalization of the Poynting theorem (Krommes and Kim, 2000):

$$\boxed{\dot{\overline{\mathcal{E}}} \approx \overline{\underline{\vec{G}} \cdot \underline{\vec{P}}}.} \tag{27}$$

Use $\vec{X}$ to denote the space dependence of slowly varying CC quantities and recall Eq. (21). Then

$$\frac{\delta\dot{\overline{\mathcal{E}}}}{\delta\delta\underline{P}_i(\vec{X})} = \underline{G}^i(\vec{X}) + \overline{\underline{\Sigma}^{ki}(\overline{\vec{X}};\vec{X})\cdot\underline{P}_k(\overline{\vec{X}})}. \tag{28}$$

Upon Fourier transforming ($\vec{X}-\vec{X}'\to\vec{q}$), we finally obtain for $\vec{\overline{\eta}}=\vec{0}$

$$\frac{\delta^2\dot{\overline{\mathcal{E}}}}{\delta\underline{P}^*_{\vec{q},i}\delta\underline{P}_{\vec{q},j}} = 2(\underline{\Sigma}^H_{\vec{q}})^{ij}, \qquad \frac{\delta^2\dot{\overline{\mathcal{E}}}}{\delta\underline{P}^*_{\vec{q},i}\delta\underline{Q}^j_{\vec{q}}} = 2(\underline{\Sigma}^H_{\vec{q}})^i{}_j, \tag{29}$$

Thus we have, for example,

$$\boxed{\gamma_{\vec{q}} = -\frac{1}{2}\left(\frac{\delta^2\dot{\overline{\mathcal{E}}}}{\delta\underline{P}^*_{\vec{q},i}\delta\underline{P}_{\vec{q},j}}\underline{C}_{ji}(\vec{q})\right)\Bigg/\ \underline{C}^k_k(\vec{q}).} \tag{30}$$

(Note that in this form it is the fully covariant C that enters.)

The next step is to determine $\dot{\overline{\mathcal{E}}}$ due to modulation of the drift waves by the convective cells. . .

Determining $\dot{\mathcal{E}}$ for the convective cells

Convective cells weakly modulate the drift waves (Diamond et al., 1998). Thus we must generalize the spectral evolution equation (20) to weakly inhomogeneous statistics.

Write $A(\vec{x}, t, \vec{x}', t') = A(\vec{\rho}, \tau \mid \vec{X}, T)$, where $\vec{\rho} \doteq \vec{x} - \vec{x}'$, $\vec{X} \doteq \frac{1}{2}(\vec{x} + \vec{x}')$, $\tau \doteq t - t'$, and $T \doteq \frac{1}{2}(t + t')$. Note that

$$\overline{A} \doteq \int d\vec{x}\, A(\vec{x}, \vec{x}) = \sum_{\vec{k}} \int d\vec{X}\, A_{\vec{k}}(\vec{X}), \tag{31}$$

where the Fourier transform is with respect to $\vec{\rho}$.

If under CC modulation the DW's evolve according to

$$\partial_t \vec{\psi}' + i\widehat{\Omega} \cdot \vec{\psi}' = \vec{0}, \tag{32}$$

where $\widehat{\Omega}$ is a linear operator (possibly a product of two noncommuting operators $\widehat{\mathrm{A}}$ and $\widehat{\mathrm{B}}$) whose Fourier transform is $\Omega_{\vec{k}}(\vec{X})$, then the tensor WKE for the DW spectrum is

$$\partial_T \mathsf{C}_{\vec{k}}(\vec{X},T) = 2(\Omega_{\vec{k}} \cdot \mathsf{C}_{\vec{k}})^A - (\underbrace{\{\Omega_{\vec{k}}; \mathsf{C}\} + \{\mathsf{A}_{\vec{k}}; \mathsf{B}_{\vec{k}}\} \cdot \mathsf{C}_{\vec{k}}}_{\text{effects of inhomogeneities}})^H, \quad (33)$$

where $\{A_{\vec{k}}, B_{\vec{k}}\} \doteq \vec{\nabla} A_{\vec{k}} \cdot \partial_{\vec{k}} B_{\vec{k}} - \partial_{\vec{k}} A_{\vec{k}} \cdot \vec{\nabla} B_{\vec{k}}$ ($\vec{\nabla} \equiv \partial_{\vec{X}}$). Note that $\overline{\{A_{\vec{k}}, B_{\vec{k}}\}} = 0$. The first two terms of Eq. (24a) define $\Omega_{\vec{k}} = \underline{\Omega}_{\vec{k}} + \underline{\Omega}'_{\vec{k}}$, where

$$\underline{\Omega}^i{}_k[\underline{\vec{P}}] \doteq -S^{ij}{}_k \widehat{D} \underline{P}_j, \quad \underline{\Omega}'^i{}_k[\underline{\vec{Q}}] \doteq (\underbrace{S^{i\bar{k}}{}_j \widehat{D} \underline{Q}^j}_{\mathsf{A}})(\underbrace{g_{\bar{k}k}}_{\mathsf{B}}), \quad (34a,b)$$

and $\widehat{D} \doteq \vec{k} \cdot \widehat{\mathrm{z}} \times \vec{\nabla}$. We note the appearance of the operators A and B, which do not commute because $\widehat{D} = \widehat{D}[\vec{\nabla}]$, $\widehat{\mathrm{g}} = \widehat{\mathrm{g}}[\vec{\nabla}]$, and $\underline{\vec{Q}} = \underline{\vec{Q}}_{\vec{k}}(\vec{X})$. The contribution from $\{\mathsf{A}; \mathsf{B}\} = \vec{\nabla} \mathsf{A} \cdot \partial_{\vec{k}} \widehat{\mathrm{g}}$ adds to a part of $\{\underline{\Omega}'; \mathsf{C}\}$ to give a term proportional to $\partial_{\vec{k}}(g_{kl} C^{lj}) = \partial_{\vec{k}} C_k{}^j$. One obtains the tensor wave kinetic equation

$$\partial_T C^{ij} = 2(\Omega^i_k C^{kj})^A - (\{\underline{\Omega}^i{}_k[\underline{\vec{P}}], C^{kj}\}^H + \underbrace{\{\underline{\Omega}'^{ik}[\underline{\vec{Q}}], C_k{}^j\}^H}_{\text{small}}). \quad (35)$$

To determine $\dot{\overline{\mathcal{E}}}$, we perform the following steps (details not shown):

- lower the second index of Eq. (35) with $g_{\bar{\jmath}j}$ and take the trace;
- write the resulting equation in conservative form by passing $g_{\bar{\jmath}j}$ through the Poisson brackets, obtaining correction terms (Krommes and Kim, 2000) that combine to an antisymmetric form that vanishes under the trace;
- bar the equation and vary $\dot{\overline{\mathcal{E}}}$ according to Eq. (29);
- calculate $\delta \mathsf{C}/\delta \underline{\vec{P}}$ from the steady-state form of Eq. (35);
- note that the omitted linear and nonlinear DW dynamics contribute a term $\widehat{\theta}^{-1} \cdot \mathsf{C}$, where the fourth-rank tensor $\widehat{\theta}$ generalizes the triad interaction time of scalar Markovian theory (Krommes and Kim, 2000).

— Final result —

The final result is $(\Sigma^{ij})^H = H^{(ij)}(\Sigma^{ij})$, where $H^{(ij)}$ denotes the Hermitian part with respect to the specified indices and

$$\Sigma^{ij}_{\vec{q}} = -\sum_{\vec{k}} d^2 S^{ir}{}_k[(\partial + 2i)g_{rs}] \times \widehat{\theta}^{ks}{}_{\overline{k}\overline{s}} H^{(\overline{k}\overline{s})}[S^{j\overline{k}}{}_l(\partial + 2i)C^{l\overline{s}}_{\vec{k}}], \qquad (36)$$

$d \doteq \widehat{\mathrm{z}} \cdot \vec{q} \times \vec{k}$, and $\partial \doteq \vec{q} \cdot \partial_{\vec{k}}$.

- Properly reduces to scalar cases (e.g., tensor Hasegawa–Wakatani equations $\to$ electrostatic generalized Hasegawa–Mima equation). (This limit is subtle; see the next item.)
- The fourth-rank tensor $\widehat{\theta}$ is totally nontrivial. In general, it is not diagonal. For further discussion, see adjacent poster by Kolesnikov and Krommes.
- Representation is not unique. One C^{ij} could be eliminated for every Casimir invariant (plasmon density).

— Summary —

- We obtained a general formula for the convective-cell growth rate $\gamma_{\vec{q}}$ appropriate for systems of coupled PDE's.
  - Covariant Hamiltonian/Lie-Poisson bracket formulation of the nonlinearities.
  - Generalized Poynting theorem for multidimensional systems relates $\gamma_{\vec{q}}$ to second-order variations of energy functional.
  - Must be very careful about the proper formulation of wave kinetic equation for weak inhomogeneities.
  - $\gamma_{\vec{q}}$ expressed compactly in terms of bracket structure constants.
  - Formula incorporates mode–mode interactions through a fourth-rank triad-interaction tensor $\widehat{\theta}$. Use of this tensor is subtle.
  - Scalar plasmon density $\Rightarrow$ multiple Casimirs that constrain $\partial_t C^{ij}$.
- Possible applications include
  - curvature-driven ITG (electrostatic)
  - finite-β collisional drift waves (for current status, see adjacent poster by Kolesnikov and Krommes).

References

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