final draft:

CLASSIFICATION OF NONEXPANSIVE SYMMETRIC EXTENSION TRANSFORMS FOR MULTIRATE FILTER BANKS*

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Abstract.

This paper describes and classifies a family of invertible discrete-time signal transforms, referred to as *symmetric extension transforms* (SET's), for finite-length signals. SET's are algorithms for applying perfect reconstruction multirate filter banks to symmetric extensions of finite-length signals, thereby avoiding the boundary artifacts introduced by simple periodic extension. A key point is when such symmetric decompositions can be formed with no increase in data storage requirements ("nonexpansive decompositions"). Transforms based on three types of symmetric extension and four classes of linear phase filters are analyzed in terms of their memory requirements for general M-channel perfect reconstruction filter banks. The classification is shown to be complete in the sense that it contains all possible nonexpansive SET's. Completeness is then used to deduce design constraints on the construction of nonexpansive M-channel SET's, including new obstructions to the existence of certain classes of filter banks. This paper also forms the principal technical reference on the SET algorithms incorporated in the Federal Bureau of Investigation's digital fingerprint image coding standard.

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1. Introduction.

A critically decimated M-channel multirate filter bank is shown in Figure 1 [18, 52, 64, 65, 67, 61]. It is called a *perfect reconstruction multirate filter bank* (PR MFB) if it has a linear translation-invariant system function satisfying

$$T(z) \equiv \frac{\hat{X}(z)}{X(z)} = Az^{-D} \quad . \tag{1.1}$$

The first construction of nontrivial finite impulse response (FIR) filter banks satisfying (1.1) was given by Smith and Barnwell [54] for power-complementary (or *paraunitary*) two-channel systems. The construction of compactly supported regular orthogonal wavelets [20, 15] corresponding to paraunitary FIR PR MFB's was demonstrated shortly thereafter by Daubechies [19]. Both of these constructions resulted in filters with asymmetric impulse responses since, as was already known [63], the only paraunitary two-channel FIR filter banks with linear phase are trivial. Subsequent constructions of two-channel FIR PR MFB's [45, 9] and wavelets [16] with linear phase were given by dropping orthogonality. Paraunitary linear phase two-channel PR MFB's are possible using infinite impulse response (IIR) filters, and symmetric extension methods have been studied using IIR filters [57, 12, 3, 4], but the present paper will only consider FIR filters. M-channel FIR PR MFB's have been designed in, e.g., [66, 43]; of particular interest for this paper, M-channel linear phase FIR PR MFB's with $M > 2$ have been constructed in [68, 46, 58, 59, 44, 37].

1.1. Symmetric Signal Extensions and Expansiveness.

A problem arises when we apply a PR MFB to finite-length signals, namely, handling the boundary conditions at the ends of the signal. Let $x(n)$ be a discrete-time signal of length N_0. One popular method is to apply the filters in Figure 1 to the infinite, periodic extension of period N_0. If the filters are FIR and have lengths less than or equal to N_0 then this is equivalent to circular convolution of period N_0. Even if the filters are longer than the input vector, however, when we convolve them with an infinite, N_0-periodic extension of the input, we obtain filter outputs, u_i, with period N_0. Thus, if M divides N_0 (indicated "$M|N_0$"), then this periodized input defines an invertible transform for finite-length vectors, the *periodic extension transform*, given by saving exactly one period of each of the downsampled subbands.

Note that we are viewing a periodic extension transform as a linear transformation of $\mathbf{R}^{N_0}$ and not as some sort of new, time-varying filter bank, as is done in other treatments of this subject. We regard the filter bank as fixed and the extension as a (linear) *preprocessing step* applied to the input vector. This seems much simpler conceptually and much closer to the perspective taken when developing implementations. Moreover, this paper will show how a single PR MFB can be used to define a number of different finite-dimensional linear transformations, which is a good reason for distinguishing between a filter bank (which is formally defined in terms of convolution on the integers) and the various transformations it can induce on $\mathbf{R}^{N_0}$.

Since the output channels have period N_0/M, a periodic extension transform is *nonexpansive*; i.e., it conserves storage costs by transforming an N_0-point sequence into $M \times (N_0/M)$ transformed data points. This is desirable for applications in which memory requirements are an issue, such as data storage or transmission. The periodic extension transform suffers, however, from the defect of introducing a jump discontinuity at the point where the ends of the signal are matched up. This artifact usually results in added variance in the high-frequency subband(s), a phenomenon that can degrade the performance of systems acting on the output, such as data quantizers.

In image coding, a popular technique is to quantize the output of the (two-dimensional) discrete cosine transform (DCT) [51]. The DCT-II can be identified with a half-sample phase shift of the first half of the discrete Fourier transform (DFT) of the signal, y, of length $2N_0$ defined by symmetrically extending (or reflecting) x:

$$y(n) = \begin{cases} x(n) \ ; & n = 0, \ldots, N_0 - 1 \ ; \\ x(2N_0 - 1 - n) \ ; & n = N_0, \ldots, 2N_0 - 1 \ . \end{cases} \tag{1.2}$$

Periodization artifacts are greatly reduced in the DCT by the fact that the extension, y, is an even signal. We can employ a similar strategy for transforming finite-length signals using a PR MFB. Given x of length N_0, form a symmetric extension, y, of length $N \approx 2N_0$ (there are other possibilities besides the extension (1.2) of length $N = 2N_0$). The extension, y, can then be transformed by N-periodic convolution, eliminating the jump discontinuity that would have resulted from simple periodic extension of x. This approach does for discrete wavelet transforms what the cosine transform does for the discrete Fourier transform. If the system transfer function is given by (1.1), we can recover x by projecting off N_0 samples from the output of the synthesis bank (see Figure 2):

$$\hat{x}(n) \ = \ \hat{y}(n + D) \ = \ Ax(n) \ ; \ \ n = 0, \ldots, N_0 - 1 \ .$$

Beneath each signal component in Figure 2 we have indicated the length (or period) of that component; note that the subbands, $a_0, \ldots, a_{M-1}$, have been given (possibly) different lengths, $\rho_0, \ldots, \rho_{M-1}$. We call the mapping $x \mapsto \{a_0, \ldots, a_{M-1}\}$ a *symmetric extension transform* (SET) with *range-space dimension* $N_1 = \rho_0 + \cdots + \rho_{M-1}$, and insist that this mapping preserve the perfect reconstruction property of the filter bank. Clearly, we do not want $\rho_i = N/M$ since the SET would then have range-space dimension $N_1 = N \approx 2N_0$ and the price we would pay for the symmetric extension would be a doubling of storage costs. For real signals, the DCT avoids this by using an even extension, so that only half of the DFT output (i.e., only the cosine modes) needs to be saved. In a SET, the output, a_i, will consist of one-half of a symmetric or antisymmetric subband, b_i. Following Martucci [38], a SET will be called *nonexpansive* if $N_1 = N_0$. Deriving the details of such procedures, which for clarity at this stage have been omitted from Figure 2, is the objective of this paper.

Image coding is one source of SET applications, since coding images (or higher-dimensional data) involves transforming many relatively short row or column vectors. If the filter bank is implemented with several levels of cascade, boundary effects can propagate across a significant fraction of the signal, so it is imperative to handle boundary conditions in a manner that preserves perfect reconstructability without increasing the variance of highpass-filtered subbands. Symmetric extrapolation meets these needs and furnishes an additional important benefit. Because the period of y is always even, a two-channel SET forms nonexpansive decompositions for both even- and odd-length inputs; periodic extension transforms are applicable only to even-length inputs. Similar advantages apply to M-channel SET's.

1.2. Historical Development.

The problem of handling boundary conditions is apparent to anyone using LTI filter banks for coding finite-duration data, and it appears that the symmetric extension technique has been discovered independently by a number of different researchers, particularly in image coding. The following survey of the subject's historical development benefitted from the assistance of an anonymous referee, who brought several additional references to the author's attention.

The first published description of a nonexpansive two-channel SET was due to Smith and Eddins [56], who observed that if a $2N_0$-periodic symmetric extension is filtered with even-length linear phase filters, the resulting subbands are symmetric and can be reconstructed losslessly by saving only $N_0/2$ samples of each. (In the present notation, $\rho_0 = \rho_1 = N_0/2$.) The thesis of Eddins [26] contains PSNR tabulations for SET's that demonstrate improvements (over circular convolution) in image coding rate-distortion performance on the order of 0.1–1.0 dB, typically around 0.5 dB. Eddins also observed that both the circular and symmetric extension methods work with IIR filters (linear phase IIR filters in the case of SET's) provided initial conditions are used that are consistent with the chosen extension; this technique was introduced in [26, 57]. A nice example of a simple IIR approach for image coding applications using non-symmetric extension was presented in [12]. Karlsson and Vetterli [35] proposed several extension methods as alternatives to circular convolution, but in spite of having access to [56], they were unable to obtain perfect reconstruction using symmetric extensions, and their method introduced boundary distortions in the signals being coded.

Around the same time, Simoncelli [53] described the use of symmetric extension with both even- and odd-length filters and analyzed the problem of preserving orthogonality using symmetric extension techniques. For second-order Markov data, Simoncelli reported improvements on the order of 0.1–0.2 dB in theoretical coding gain over PCM for symmetric versus circular extension. Bamberger [2] also treated SET's for odd-length one-dimensional filters and derived the first SET's for nonseparable two-dimensional filter banks by using separable symmetric extensions. Implementation of SET's using FIR filter banks was studied by Martucci in the context of symmetric convolution and generalized discrete trigonometric transforms [38, 41, 40, 39].

SET's for two-channel filter banks were developed independently by the author [5, 6] as part of his work on the FBI's image coding standard for digitized fingerprint records [27]. The FBI standard appears to be the first major application to have exploited an important feature of two-channel SET's, namely, that they allow for nonexpansive, error-free transformation of signals of arbitrary (not just even) lengths. Because many different image scanners are used within the justice community to digitize fingerprints, the fingerprint data format standard [1] allows for spatial scan resolutions anywhere from 500 to 520 dpi. Image scanning systems are required to capture precisely defined regions on a fingerprint card, however, which means that systems complying with [1] may generate compliant images of arbitrary pixel dimensions (not necessarily divisible by powers of 2). For instance, [27] specifies compliance tests on 520 dpi plain thumb impressions (0.875×1.875 inches) with dimensions of 455×975 pixels. The specification of SET's in [27] for odd-sized images eliminated any need for revisions of [1].

Symmetric extension for one-dimensional, M-channel FIR PR MFB's was discussed by Bamberger [2] as a lead-in to his treatment of symmetric extension methods for nonseparable filter banks with general two-dimensional subsampling lattices. The topic of M-channel SET's has subsequently been treated in more detail in [21, 3, 49, 4, 8, 13, 14, 47, 36]. Bamberger *et al.* [3, 49, 4] also considered M-channel nonuniformly downsampled filter banks. Some of the formulas derived in Section 2 of this paper appear, in slightly different forms, in [36, 4, 14]; readers interested in a more concise overview of the M-channel case are directed to these papers.

One can define smoother extrapolations than symmetric extensions; some such extrapolations appeared in [35]. Nonexpansive transforms based on general linear extrapolation of an input vector of length N_0 have commonly been defined by transforming the extended signal (possibly using asymmetric filters) and simply truncating the resulting subbands to a total combined length of N_0 subband coefficients. The combined effects of such an approach—linear signal extrapolation, multirate filter bank, truncation—is a linear transformation (hopefully nonsingular) of $\mathbf{R}^{N_0}$. When the synthesis filters are FIR, one easily sees that the samples in the "middle" of the signal (i.e., those whose synthesis does not depend on any of the truncated subband coefficients) can be reconstructed by the usual synthesis filter bank. Obtaining perfect reconstruction of the complete signal therefore reduces to reconstructing the samples close to the boundaries whose synthesis is affected by the truncated coefficients. Dropping the use of symmetric extensions and linear phase filters, however, makes the task of reconstructing the boundaries less trivial than it is in the case of SET's.

de Queiroz [21] wrote down an overdetermined system of linear equations for these boundary samples and investigated solution methods such as the pseudoinverse. Bradley and Faber [6, 7] derived square systems of equations for synthesis of the left and right boundaries in the reconstructed signal and studied the condition numbers of these two systems. In the case of symmetric extension combined with Daubechies' orthogonal (nonlinear phase) wavelet filters, their numerical results indicated that simultaneous invertibility of both the left and right boundary problems was sensitive to the supports of the filters and the relative delay between the lowpass and highpass channels. Taking a slightly different approach, McGill and Taswell [42] worked out the computation needed to restore the truncated subband coefficients for a cascaded filter bank in a zero-padding extension scheme and discussed applying that approach with more general linear extension methods. Déprez *et al.* [23] also developed a method of restoring truncated subband coefficients in a zero-padding scheme for use in implementing time-varying filter banks.

Recently, de Queiroz and Rao [22] have continued the development of boundary synthesis techniques utilizing pseudoinverses in the special case of paraunitary MFB's. Their theoretical analysis asserting that paraunitary linear extension transforms are generally invertible appears to contradict a conjecture based on numerical evidence in [7]. This author, however, confesses that he is unable to follow the proof presented in the appendix of [22], nor does he understand why the invertibility of transforms with completely general linear boundary extensions should depend on special properties (i.e., orthogonality) of the associated filter bank. It therefore appears that the question of invertibility for nonexpansive schemes employing general linear extension remains open and must probably be studied on a case-by-case basis.

Linear boundary extension methods can be regarded mathematically as special types of time-varying linear filter banks, and some authors have treated boundary problems via the construction of special "boundary filters." Cohen *et al.* [17], improving on earlier work by Y. Meyer and others, constructed boundary wavelets and associated filter banks in order to produce smooth orthonormal wavelet bases and fast multiresolution transforms for the Hölder spaces, $C^s([0,1])$. Herley [28, 29] and Herley and Vetterli [30] developed both orthogonal and non-orthogonal boundary filters and wavelets for coding finite-duration signals and for implementing time-varying filter banks. A somewhat different method of coding finite-duration signals based on smooth deformation and periodic extension of the input vector was proposed by Wickerhauser [69]. Finally, methods based on computing nonlinear extensions have also appeared; e.g., DeVore *et al.* [24, 25].

All of these general extension approaches require additional computations and/or changes to the filter bank near the boundaries during both analysis and synthesis. The symmetric extension method, in contrast, requires no additional computations to form the signal extension, nor does it require implementation of special boundary filters. Symmetric extension is thus a compromise between the algorithmic simplicity of methods like circular extension or zero-padding with truncation [62] and the greater smoothness obtainable by more complicated methods. Nonetheless, any linear extension method ensures that signal reconstruction will require only linear transformations, where the rank of the boundary synthesis problem is constrained by the order of the synthesis filters and the order of the linear signal extension. This makes general linear extension techniques computationally feasible when used with FIR PR MFB's or with rational (IIR) filter banks having FIR inverses [11].

1.3. Organization and Scope of the Paper.

The author's goal in writing the present paper, which is an outgrowth of [8], is to present a complete treatment of nonexpansive symmetric extension transforms at a level of specificity sufficient to allow applications designers to implement SET systems with a minimum of trouble using given PR MFB's. Accordingly, he has tried to make it as straightforward and implementation-oriented as he can. Moreover, he has also tried to make it useful as a reference on the SET algorithms incorporated in the FBI fingerprint image coding standard [27], a topic covered in Section 3.3. The value of a *complete* classification of SET's will become apparent in Section 4, where we will derive necessary restrictions on input signal lengths and new theoretical results on possible filter symmetries for M-channel PR MFB's when $M > 2$.

Our approach is to fill in the missing details in Figure 2, starting with a given linear phase PR MFB. This is complementary to the approach taken by Bamberger *et al.* [3, 49, 4], in which the authors start with a given symmetric extension and downsampling rates and then derive constraints that a filter bank must be designed to satisfy to produce a nonexpansive SET. The approach taken in the present paper is to assume that a linear phase FIR PR MFB has already been designed, based solely on consideration of its filter characteristics, and then to classify the ways in which it can be *implemented* nonexpansively in a system utilizing symmetric extension. This decouples the filter design problem from the implementation issue. In this paper we will not discuss the actual construction of linear phase PR MFB's; that topic is addressed in the references cited above.

Section 2 contains preliminaries regarding symmetric and antisymmetric extensions and an analysis of expansiveness that shows how the classification given in this paper contains *all* nonexpansive SET's. We consider signal decompositions based on three types of symmetric extension. Since the goal of forming such extensions is to avoid introducing singularities in the input signal, we do not present transforms based on antisymmetric extensions. Recent work by Nuri [48], however, using AR(1) source models indicates that there may be a coding gain advantage to using antisymmetric extensions for transforming antisymmetric subbands in cascaded coding schemes. While this has not yet been studied using real data, the concept is appealing insofar as it adheres to the philosophy of viewing cascaded subband coders (e.g., discrete wavelet transforms) as quantitative approximations of ideal analog multiresolution decompositions. Readers interested in transforms based on antisymmetric extensions are directed to [47, 36]. In the present paper, antisymmetric extensions will be used exclusively for reconstructing antisymmetric subbands during synthesis.

Section 3 classifies all nonexpansive two-channel SET algorithms in terms of signal length, filter symmetry, and filter phase. The necessary parameters are given for implementing the operations described in Section 2. The analysis is based on a new characterization of the group delays in a linear phase PR MFB. The particular SET algorithms incorporated by the author into the FBI fingerprint image coding standard [27] are also described in detail.

Section 4 contains a complete but less exhaustive classification of M-channel SET's. The possible symmetry combinations in M-channel linear phase PR MFB's are not completely understood at present, so we enumerate all possible individual channels in M-channel SET's and compute their expansiveness. Examples of M-channel SET's based on predesigned M-channel filter banks are given for the cases M=3, 4. It is shown that nonexpansive SET's exist for input signals of arbitrary length when $M = 3$; the corresponding question for M=4 is open. Some interesting corollaries of our analysis of expansiveness are several new non-existence results for certain combinations of filter symmetries in FIR PR MFB's, generalizing earlier work of Soman *et al.* [58, 59]. Finally, Section 5 contains concluding remarks.

2. Multirate Signal Processing for Symmetric Signals.

We analyze the effects of standard MFB operations on symmetric signal extensions. Attention is restricted to real-valued signals and filters; this simplifies the analysis while still encompassing most conceivable applications.

2.1. Symmetry Properties and Extension Operators.

2.1.1. Signal and filter symmetries.

There are two ways in which a discrete-time signal can be symmetric: it can be symmetric about one of its samples or about a point midway between two samples (see Figure 3). These cases are referred to as *whole-sample symmetry* (WS) and *half-sample symmetry* (HS) in [38, 41], and we retain the same terminology in this paper because of its mnemonic value. We require the entire signal to be symmetric about a given center (or axis of symmetry); nonetheless, a signal can still have multiple centers. There are obvious antisymmetric analogues in both the whole- and half-sample cases; these are designated *whole-sample antisymmetry* (WA) and *half-sample antisymmetry* (HA). We only declare y to be whole-sample antisymmetric about c if $y(c) = 0$.

The above categories are not mutually exclusive since a signal may have different types of symmetry about different centers; e.g., an alternating sequence like $(-1)^n$ is both WS and HA. Moreover, centers of symmetry for periodic signals come in pairs. For instance, let y be N-periodic with a center at c: $y(c+n) = \pm y(c-n)$. Then $c+N/2$ is also a center for y, and if N is even then y is WS (or WA) about both c and $c+N/2$. If N is odd, y will be WS (or WA) at c and HS (or HA) at $c+N/2$; we refer to such signals as *odd-period symmetric* (OS) or *odd-period antisymmetric* (OA). For illustrations, the reader is referred ahead to Figures 4–7, where we present signal extensions with these various symmetries. Definitions are given in Table 1 in terms of time-domain and equivalent frequency-domain criteria.

To explain the last column in Table 1, we define the *dimension*, ρ, of a class of N-periodic signals sharing some particular symmetry property to be the dimension of the submanifold they span in $\mathbf{R}^N$. This indicates the number of nonredundant samples (or degrees of freedom) in an arbitrary signal from that class, so the dimension tells us how many memory registers are required to store an arbitrary signal from the class. For instance, an HS signal contains at most $N/2$ nonredundant samples, and since this upper bound is attained in Figure 4(d), the dimension of the HS class is $N/2$. Similar arguments yield the figures in the last column of Table 1. Note that centers of whole-sample antisymmetry (which are necessarily zero) do *not* count when computing the dimension of the WA and OA classes. The dimensions of transform subbands will be used to evaluate the storage requirements—the "expansiveness"—of the various transforms we construct.

The linear phase FIR filters used in SET's correspond to the four cases of whole- and half-sample symmetry and antisymmetry defined above. These filter classes appear in [50, 65] under the unsuggestive headings "Types 1–4." Note that the group delay, γ, for a linear phase FIR filter is also equal to (the location of) a center of symmetry; γ is an integer for WS/WA filters and an odd multiple of 1/2 for HS/HA filters.

2.1.2. Symmetric extension operators.

To be more precise about the process of making a symmetric extension of a signal, x, of length N_0, we define *extension operators* as linear transformations,

$$E_{sys} : \mathbf{R}^{N_0} \to \mathbf{R}^N \quad , \quad x \mapsto y = E_{sys}x \quad .$$

An extension operator simply replicates values of x according to some symmetry relation to construct an extension with one of the symmetry properties described above. As mentioned in the Introduction, we only consider signal analyses based on symmetric extensions, but we will need antisymmetric extensions in SET synthesis banks. The three extensions shown in Figure 4 are the only ones that will be used for extending signals prior to coding. The reason for this particular choice is that these are the only extensions that produce nonexpansive SET's. For instance, in Section 2.3 we will show that SET's based on the $2N_0$-periodic extension $E_s^{(1,3)}x$ in Figure 5 are always expansive, regardless of the type of filter bank used.

Aside from whether an extension is symmetric or antisymmetric, its main distinguishing feature is the number of times the first and last samples are repeated in the extension. A pair of digits, (i, j), will be used to indicate the number of times the first and last samples occur in a single period of the extension. For instance, in the extension $E_s^{(1,2)}x$ shown in Figure 4(c), $x(0)$ occurs once in each period while $x(N_0 - 1)$ occurs twice. (The open dots represent a second full period of the extension.) Each extension is initialized by setting

$$y(n) = x(n) \;\; ; \;\; n = 0, \ldots, N_0 - 1 \;\; ,$$

then extending symetrically from $n = N_0$ to $N - 1$. All of the extensions we define will have centers at either 0 or $-1/2$. This makes it particularly simple to write down left inverses for the extension operators. Let

$$P_L : \mathbf{R}^N \to \mathbf{R}^L \quad ; \quad (x_0, \ldots, x_{N-1}) \mapsto (x_0, \ldots, x_{L-1})$$

be the projection onto the first L coordinate variables; i.e., a rectangular window of length L. Then $P_{N_0}E_{sys}x = x$.

Finally, we have five extensions that will be used exclusively for reconstructing subbands, $b_i = E_i a_i$, in the synthesis bank. The symmetric extension $b = E_s^{(2,1)}a$ in Figure 6 is equivalent to a shift of the extension $E_s^{(1,2)}a$, so it will not be necessary to consider its use as an analysis extension. Four antisymmetric extensions are shown in Figure 7. Note that ρ is the length of the subband, a, and that operators with an "a" subscript are antisymmetric extensions.

The presence of antisymmetric extensions complicates the initialization procedure slightly. Because we want WA extensions to have centers at 0, signals being extended with whole-sample antisymmetry at 0 need to be delayed by one sample so we can insert the trivial value $b(0) = 0$. We introduce a parameter, η, to keep track of this delay and initialize the subband extensions

$$b(k) = a(k - \eta) \ \ ; \ \ k = \eta, \ldots, \eta + \rho - 1 \ \ .$$

η is zero for all extensions except $E_a^{(1,1)}$ and $E_a^{(1,2)}$, which receive a one-sample delay ($\eta = 1$).

2.2. Subband Decomposition and Reconstruction.

The symmetry properties that result from applying linear phase filters to symmetric signals, y, are readily obtained from frequency-domain symmetry characterizations. A more complicated problem is determining when $M : 1$ decimation of a symmetric filter output, $u = y * h$, results in a symmetric subband, $b = \downarrow_M u$.

2.2.1. Downsampling.

The downsampling (or decimation) operator is defined to be

$$(\downarrow_M u)(n) = u(Mn) \ \ . \tag{2.1}$$

Similarly, we define the upsampling (or interpolation) operator to be

$$(\uparrow_M a)(n) = \begin{cases} a(n/M) \ \text{ if } \ M|n \ \ , \\ 0 \ \ \text{otherwise} \ \ . \end{cases} \tag{2.2}$$

Nishikawa *et al.* [47, 36] consider two different types of decimation, termed "Decimation I," which corresponds to (2.1), and "Decimation II," which amounts to (2.1) preceded by a phase shift of $M/2$ samples (only for M even). Since any such phase shift can be incorporated into the phase of the analysis filter, h, this paper will only use the downsampler (2.1) and will classify SET's in terms of the group delays of the analysis filters.

Given a symmetric extension, y, from Figure 4 and a linear phase filter, h, with group delay, γ, Table 2 lists all conditions under which we obtain a symmetric subband, $b = \downarrow_M (y * h)$. A rigorous derivation of this result is given in [10]. Note that Table 2 does not distinguish even- from odd-periodic cases; the symmetry of b is given generically as, e.g., WS or WA, which should be interpreted as OS or OA when the period of b is odd.

We also need the following intertwining relation, or "noble identity" [65], for interchanging decimators and shifts: if $\delta_M(n)$ is a unit impulse at $n = M$ (an M-sample delay) and $b = \downarrow_M u$, then

$$\downarrow_M (\delta_M \ * \ u) \ = \ \delta_1 \ * \ (\downarrow_M u) \ = \ \delta_1 \ * \ b \ \ . \tag{2.3}$$

When $M|N$, (2.3) also holds for $M : 1$ circular downsampling and circular shifts. We shall always insist that the input length, N_0, the analysis extension, and the decimation rate be constrained so that M divides the period, N, of the extended signal. Without this restriction, $M : 1$ decimation of an N-periodic signal doesn't result in an $M : 1$ reduction in the amount of information in the signal, which is the goal of performing decimation. For instance, this is why two-channel periodic extension transforms don't work with odd-length inputs. It is *not* necessary that $M|N_0$ in the case of SET's, however, only that $M|N$.

2.2.2. Projection and extension operations.

Assume we have generated a subband, b, with one of the symmetries listed in Table 2. We now exploit the symmetry of b to reduce data storage requirements. N is the period of y, so the period of b is $K = N/M$. Since b is symmetric, we only need to save one-half of a period of b; the discarded values can be reconstructed using the appropriate symmetry relation.

Consider an HA subband, b, of period $K = 2\rho$ (compare Figure 7(d)); if b is centered at $-1/2$, then the values $\{b(0), \ldots, b(K/2-1)\}$ comprise exactly one-half of a period of b. We define

$$a = P_{K/2} b \ ,$$

where $P_{K/2}$ is the projection or rectangular window, defined in Section 2.1, on the first $\rho = K/2$ samples in a period of b. Now we can reconstruct b by extending the ρ-point sequence, a:

$$b = E_a^{(2,2)} a \ .$$

The same procedure works when b is HS, using the symmetric extension operator $E_s^{(2,2)}$ for reconstruction. The situation when b is WA with center at 0 (Figure 7(a)) is slightly different because we don't need to save the trivial value $b(0) = 0$. Therefore, b gets advanced by one sample and then projected onto the next $\rho = K/2 - 1$ values:

$$a = P_{K/2-1}(\delta_{-1} * b) \qquad ; \qquad b = E_a^{(1,1)} a \ . \tag{2.4}$$

Note that the antisymmetric extension operator, $E_a^{(1,1)}$, inserts the trivial values $b(0) = 0 = b(K/2)$.

In general, the number of points that need to be saved is given by the dimension of b. When b has dimension ρ and center 0 or $-1/2$, we can save the half-period $a = P_\rho b$ provided we advance b by $\eta = 1$ sample, as in (2.4), if b is WA at 0. The dimension, ρ, is thus the same as the rank of the pertinent projection operator, P_ρ, so we shall also refer to ρ as the "rank" of the subband to avoid confusing ranks (dimensions) of individual subbands with the range-space dimension of the complete SET. One more concern: if b is not centered at 0 or $-1/2$, we first need to advance b by enough samples to move its center up to either 0 or $-1/2$; the necessary shift factor, β, is given in Table 2 as a function of signal and filter symmetry.

2.2.3. The extended subband coder.

We are now able to fill in the missing details in Figure 2. For simplicity, we only display a single channel, and we display the analysis and synthesis banks separately; see Figure 8, which indicates the period of each signal component. In the analysis bank, the input signal is extended by a symmetric extension operator from Figure 4, filtered and decimated, then advanced $\beta_i + \eta_i$ samples before being projected onto coordinates $n = 0, \ldots, \rho_i - 1$ to produce the output channel a_i. In the synthesis bank, a_i is extended by the appropriate operator, E_i, as determined by the symmetry of b_i, delayed β_i samples, upsampled, and filtered. The appropriate synthesis extension operators and shift factors, η_i, are given in Table 3 for each of the six possible subband symmetries.

2.2.4. Causality.

Note that SET's are inherently non-causal systems; i.e., y's "past," the set of values $\{y(n) : n < 0\}$, is defined in terms of y's "future," the values $\{y(n) : n > 0\}$. The crucial consideration affecting

SET implementation is the precise set of filter outputs that need to be computed to provide a complete, nonredundant half-period of each subband for transmission. Any application employing a symmetric extension technique must provide knowledge of a sufficiently long segment of "future" data to allow the computation of those essential filter outputs whose values depend on the extended input's "past."

This does *not* imply that a SET cannot be implemented with causal filters, however. According to Figure 8 and (2.3), changing the filter's group delay by a multiple of M only affects the latency specified by the wait, z^{β_i}, needed before projecting off a complete, nonredundant half-period of the subband b_i. For this reason, we do *not* make any assumptions about filter causality but instead allow the reader to infer system latency from β_i, which is given as a function of group delay, γ_i, in the tables. We will say more about causal implementation of SET's in Section 3.3.2.

2.3. SET Expansiveness.

In Section 1.1 we defined the range-space dimension, N_1, of a SET like the one in Figure 2 to be the sum of the individual subband ranks, $N_1 = \sum \rho_i$, and called a SET *nonexpansive* if $N_1 = N_0$. We now consider the issue of expansiveness in greater detail. Our analysis will concentrate on uniformly decimated PR MFB's of the type shown in Figure 1, although we will mention briefly how to extend the analysis to SET's based on nonuniformly downsampled PR MFB's.

2.3.1. Defining expansiveness.

The SET, $x \mapsto \{a_0, \ldots, a_{M-1}\}$, is a linear transformation,

$$\text{SET}: \quad \mathbf{R}^{N_0} \to \mathbf{R}^{\rho_0} \times \cdots \times \mathbf{R}^{\rho_{M-1}} \;\cong\; \mathbf{R}^{N_1} \quad . \tag{2.5}$$

Since the transform is invertible, we have $N_1 \geq N_0$. (Note that the *rank* of a SET is always N_0, hence the mathematically proper but somewhat unwieldy term *range-space dimension* for N_1.) We define the *expansiveness*, ε_{sys}, of a SET to be

$$\varepsilon_{sys} \;=\; N_1 - N_0 \;\geq\; 0 \quad . \tag{2.6}$$

The expansiveness represents the number of additional dimensions (read: "storage registers") needed to hold the transformed signal above and beyond the number required by the original input, x. The nonexpansive case is the one in which $\varepsilon_{sys} = 0$.

The transform (2.5) will be called an *equal-rank* SET if all subbands have the same rank: $\rho_0 = \rho_1 = \cdots = \rho_{M-1} \equiv \rho$. Equal-rank transforms are nice because the details of storing and manipulating output are a little simpler than for systems with different-rank output channels, particularly if the outputs are cascaded back through a SET analysis bank. The equal-rank condition is not necessary, however, for either perfect reconstruction or, as we will show, for nonexpansiveness.

An equal-rank transform has range-space dimension

$$N_1 \;=\; \sum_{i=0}^{M-1} \rho_i \;=\; M\rho \quad .$$

A nonexpansive equal-rank SET therefore requires that $N_0 = N_1 = M\rho$, so we must satisfy the necessary condition $M|N_0$. As a trivial example of the kind of limitations this condition imposes, note that we cannot construct a nonexpansive equal-rank two-channel SET for odd-length signals.

In Section 3 we will construct nonexpansive two-channel SET's for odd-length signals using channels with unequal ranks.

The analysis so far of the operations involved in converting a PR MFB into a SET has focused on individual channels in the coder, without requiring any system-wide considerations. While expansiveness is clearly a property of the overall SET system, we can still define a measure of expansiveness for individual SET channels to give some indication of the efficiency of a single channel. Define the *channel expansiveness*, ε_i, of the i^{th} channel in a SET to be

$$\varepsilon_i = \rho_i - N_0/M \quad . \tag{2.7}$$

This is consistent with definition (2.6) in the sense that

$$\sum_{i=0}^{M-1} \varepsilon_i \;=\; \sum_{i=0}^{M-1} (\rho_i - N_0/M) \;=\; \varepsilon_{sys} \quad , \tag{2.8}$$

although it is possible, under definition (2.7), for a channel to have fractional expansiveness. An advantage of this approach is that it is easy to tabulate the expansiveness of individual channels using (2.7) and then calculate ε_{sys} via (2.8). It also allows us to use the inequality in (2.6) to rule out certain filter combinations for linear phase PR MFB's via the constraint

$$\sum_{i=0}^{M-1} \varepsilon_i \geq 0 \quad . \tag{2.9}$$

If a bank of linear phase filters creates symmetric subbands with $\sum \varepsilon_i < 0$ for some symmetric extension, E_{sys}, then it defines a noninvertible transform of $\mathbf{R}^{N_0}$ and therefore cannot be a PR MFB.

2.3.2. Nonuniformly downsampled filter banks.

Recent work of Bamberger *et al.* on symmetric transform techniques [3, 4] has addressed the possibility of using nonuniformly downsampled PR MFB's; i.e., filter banks that have different decimation ratios, M_i, in different channels. Such a filter bank is *critically* (or *maximally* [18]) *decimated* if

$$\sum_i M_i^{-1} \;=\; 1 \quad .$$

Because of the requirement that $M \mid N$ in any SET scheme, nonuniform downsampling imposes more restrictions on the allowable combinations of input length and extension type than does uniform downsampling since we now need $M_i \mid N$ for all M_i; see Section 4.2 for more on divisibility criteria for M-channel SET's. Nonetheless, we can still employ the above analysis of expansiveness in the case of nonuniform filter banks. For applications in which expansiveness is an issue, it makes sense to restrict attention to critically downsampled filter banks. In the nonuniform case, define the expansiveness of the i^{th} channel to be

$$\varepsilon_i = \rho_i - N_0/M_i \quad ,$$

generalizing (2.7). It follows that the relationship (2.8) also holds for critically downsampled nonuniform MFB's. This means that all of the results in Section 4 concerning individual channels in M-channel SET's are applicable to the design of nonuniformly downsampled SET's, including the analysis of expansiveness.

2.3.3. Completeness of the classification.

The claim was made in Section 2.1 that the three principal symmetric extensions in Figure 4 are the only ones (except for trivial phase shifts) that produce nonexpansive SET's. To make this claim precise, we need to establish the scope of what we consider to be "symmetric extension transforms." Accordingly, we restrict attention to the extensions $E_s^{(i,j)}x$; as mentioned before, these extensions have the advantage that they require no additional computations when extrapolating the signal. Moreover, we only consider algorithms in which the redundancy introduced by symmetric extension is removed by windowing symmetric subbands, b_i. The claim is that the extensions $E_s^{(i,j)}x$ for (i,j)=(1,1), (1,2), or (2,2) are the only ones for which the ranks, ρ_i, of the subbands can add up to exactly N_0. The reason for this is that if either i or j is greater than or equal to 3, then the resulting transform is equivalent to an (unnecessary) extension of the input, x, to a longer input, x', followed by a nonexpansive transform of x'.

For instance, consider the $2N_0$-periodic extension $E_s^{(1,3)}x$ in Figure 5. This is equivalent to $E_s^{(1,1)}x'$, where x' is the signal of length N_0+1 obtained by duplicating the last sample, $c = x(N_0-1)$. Inequality (2.6) shows that any SET based on applying $E_s^{(1,1)}$ to x' will produce subbands with ranks satisfying $\sum \rho_i' \geq N_0 + 1$, unless there is some way to remove the extra sample's worth of redundancy hidden in the transform of x'. To see that this redundancy cannot be removed by simple windowing of the subbands b_i', note that x' can be obtained by one-sample zero-padding of x, followed by the addition of an impulse, $c\delta_{N_0}$:

$$x'(n) \;=\; x^+(n) + c\delta_{N_0}(n) \quad ; \quad n = 0, \ldots, N_0 \quad ,$$

where

$$x^+(n) \;=\; \begin{cases} x(n), & 0 \leq n < N_0 \\ 0, & n = N_0 \end{cases} \quad .$$

Then $y' \equiv E_s^{(1,1)}x' = y^+ + c\delta_{N_0}$, where $y^+ = E_s^{(1,1)}x^+$, so the result of filtering y' by h is

$$y' * h \;=\; y^+ * h \,+\, c\delta_{N_0} * h \quad .$$

The redundancy inherent in the *symmetry* of $b' = \downarrow_M (y' * h)$ can be removed by windowing, as described above, but the redundancy resulting from the one-point extension $x \to x'$ is spread out across the subband by the $(c\delta_{N_0} * h)$ term. In particular, the effects of the one-sample redundancy are *not* manifest in a form (like subband symmetry or endpoint duplication) that can be removed by simple windowing operations.

This same analysis can be applied to other extensions involving multiple duplications of signal endpoints, leading to the following conclusion.

Theorem 1. *All nonexpansive symmetric extension transforms based on signal extrapolations of the form $y = E_s^{(i,j)}x$ are based on the three extensions $E_s^{(1,1)}x$, $E_s^{(1,2)}x$, and $E_s^{(2,2)}x$ (where $E_s^{(2,1)}x$ is dismissed as a trivial phase shift of $E_s^{(1,2)}x$).*

Thus, this paper's scope is limited to SET's based on these three principal extensions.

3. Two-Channel Symmetric Extension Transforms.

This section gives a complete classification of all nonexpansive two-channel SET's, using the extensions $E_s^{(1,1)}x$ and $E_s^{(2,2)}x$. The extension $E_s^{(1,2)}x$ has an odd period and therefore cannot be used with 2:1 downsampling. We first give all possible filter symmetry and phase combinations for two-channel SET filter banks. Next, we tabulate all symmetries and channel expansiveness figures for both admissible analysis extensions. Finally, we put these possibilities together to classify two-channel SET's, including the particular cases incorporated in the FBI standard [27] for gray-scale fingerprint images. Transforms based on the extension $E_s^{(i,j)}$ will be referred to as "(i,j)-SET's."

3.1. Two-Channel Linear Phase PR MFB's.

We begin by describing all possible combinations of filter symmetry for two-channel linear phase PR MFB's like the one in Figure 1. The input-output relationship is

$$\hat{X}(z) = \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) \quad .$$

The coefficient of $X(-z)$, the "aliasing term" in the system function, can be eliminated by defining the synthesis filters according to the following anti-aliasing relations:

$$F_0(z) = \pm z^{-D_s} H_1(-z) \quad , \quad F_1(z) = \mp z^{-D_s} H_0(-z) \quad . \tag{3.1}$$

This produces a linear translation-invariant system with transfer function

$$T(z) \equiv \frac{\hat{X}(z)}{X(z)} = \pm\frac{z^{-D_s}}{2}[H_0(z)H_1(-z) - H_0(-z)H_1(z)] \quad . \tag{3.2}$$

Using equation (3.2), the perfect reconstruction condition (1.1) for the two-channel case reads

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2Az^{-D+D_s} \quad . \tag{3.3}$$

The quantity on the left-hand side of (3.3) is the determinant of the *alias-component (AC) matrix* [55] for the system. Since this determinant is an odd function of z, $D_s - D$ must be odd; define D_a to be the difference, $D_a \equiv D - D_s = 2m+1$. The process of designing a two-channel PR MFB therefore reduces to the following: given an analysis bank whose AC determinant satisfies

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2Az^{-D_a} \quad , \quad D_a \text{ odd}, \tag{3.4}$$

the anti-aliasing relations (3.1) define a PR MFB system with transfer function

$$T(z) = \pm Az^{-D} \quad , \quad D = D_a + D_s \quad . \tag{3.5}$$

This means we can advance or delay the synthesis bank by an arbitrary factor, D_s, with a concomitant advance or delay of the transfer function. In particular, any PR MFB can be made into a zero-delay system by choosing $D_s = -D_a$ in (3.1). From now on, we shall concentrate on DWT analysis banks and assume that the synthesis banks are given by (3.1).

3.1.1. Nontrivial linear phase filter banks.

Consider the possible combinations of filter symmetries that might occur in a SET analysis bank. Table 2 shows that the filters in a two-channel SET must have the same symmetry type; i.e., either both must be H-type or both must be W-type, depending on whether $E_{sys}x$ is HS or WS. Moreover, it is shown in [45] that the only *nontrivial* classes (i.e., classes parameterized by a manifold with positive dimension) of two-channel linear phase FIR PR MFB's are those containing either a WS-WS or an HS-HA filter pair. In the nontrivial cases, an interesting relationship holds between the AC determinantal delay, D_a, in (3.4) and the group delays, γ_0, γ_1, of the individual analysis filters.

Theorem 2. *Let $\{h_0, h_1\}$ be a nontrivial linear phase PR MFB with AC determinantal delay, D_a. Then $D_a = \gamma_0 + \gamma_1$; in particular, $\gamma_0 + \gamma_1$ must be an odd integer.*

Proof. Assume that h_0 is always symmetric and that h_1 is either WS or HA:

$$H_0(z^{-1}) = z^{2\gamma_0} H_0(z) \quad , \quad H_1(z^{-1}) = (-1)^{2\gamma_1} z^{2\gamma_1} H_1(z) \quad .$$

Then condition (3.4) implies

$$\begin{aligned} 2Az^{D_a} &= H_0(z^{-1})H_1(-z^{-1}) - H_0(-z^{-1})H_1(z^{-1}) \\ &= z^{2\gamma_0+2\gamma_1}\left[(-1)^{4\gamma_1} H_0(z)H_1(-z) - (-1)^{2\gamma_0+2\gamma_1} H_0(-z)H_1(z)\right] \\ &= z^{2\gamma_0+2\gamma_1}\left[H_0(z)H_1(-z) - H_0(-z)H_1(z)\right] \\ &= 2Az^{-D_a+2\gamma_0+2\gamma_1} \quad , \end{aligned}$$

since $4\gamma_1$ is always even and $2\gamma_0 + 2\gamma_1$ is even in the nontrivial cases. Thus, $D_a = \gamma_0 + \gamma_1$. ∎

As a corollary of Theorem 2, we get the following result characterizing the possible phase shifts that preserve the perfect reconstruction property for a nontrivial linear phase PR MFB.

Corollary 3. *Suppose $\{H_0(z), H_1(z)\}$ is a nontrivial linear phase PR MFB and $H_0'(z) = z^{-m_0}H_0(z)$, $H_1'(z) = z^{-m_1}H_1(z)$. Then $\{H_0'(z), H_1'(z)\}$ is a PR MFB if and only if $m_0 + m_1$ is even, with new AC determinant*

$$2A'z^{-D_a'} = (-1)^{m_1} 2Az^{-D_a-m_0-m_1} \quad .$$

Proof. We are given

$$H_0(z)H_1(-z) - H_0(-z)H_1(z) = 2Az^{-D_a} \quad , \quad D_a \text{ odd,}$$

and

$$H_0'(z)H_1'(-z) - H_0'(-z)H_1'(z) \;=\; (-1)^{-m_1} z^{-m_0-m_1}\left[H_0(z)H_1(-z) - (-1)^{m_1-m_0} H_0(-z)H_1(z)\right] .$$

Necessity. If $m_0 + m_1$ is even then so is $m_1 - m_0$, and the above reduces to

$$H_0'(z)H_1'(-z) - H_0'(-z)H_1'(z) = (-1)^{m_1} 2Az^{-D_a-m_0-m_1} \quad ,$$

which has an odd exponent and thus satisfies (3.4).

Sufficiency. Conversely, if $\{H_0'(z), H_1'(z)\}$ satisfies (3.4) then, applying Theorem 2 to *both* filter banks,

$$D_a' = \gamma_0' + \gamma_1' = D_a + m_0 + m_1 \quad ,$$

so $m_0 + m_1$ must be even since both D_a and D_a' are odd. ■

The reader is cautioned that these results do fail in certain exceptional cases. For instance, Nguyen and Vaidyanathan [45] reported the existence of trivial PR MFB's with both WS and HS filters, e.g.

$$H_0(z) = 1 + z^{-1} \quad , \quad H_1(z) = 1 + z^{-1} + z^{-2} \quad . \tag{3.6}$$

This system satisfies (3.4) with $D_a = 3$ although $\gamma_0 + \gamma_1 = 3/2$, and if H_0' and H_1' are defined as in Corollary 3 then the shifted filter bank retains the PR property for *arbitrary* choices of m_0 and m_1, not just choices whose sum is even, as is necessary in the case of nontrivial filter banks. As we have pointed out, however, mixed-type filter banks like (3.6) cannot be used in two-channel SET's. We also remark that there exist M-channel generalizations of Theorem 2, though we shall not need them in this paper.

3.2. Classification of Two-Channel Symmetric Extension Transforms.

This section enumerates all possible two-channel SET's. The transforms will be classified according to extension, E_{sys}, input signal length, N_0, and analysis filter symmetries and phases. Synthesis filters will not be discussed, except for the FBI implementations, since they are completely determined by (3.1) and do not affect system expansiveness.

3.2.1. The extension $y = E_s^{(1,1)} x$.

We will outline the analysis, listing the results in Table 4. The period of y is $N = 2N_0 - 2$, so the period of a subband, b, is $K = N/2 = N_0 - 1$. Since y is WS, Table 2 shows that H-type filters never produce symmetric subbands when $M = 2$. We therefore do not list any signal/filter combinations in Table 4 involving H-type filters since they can never be part of a $(1,1)$-SET. In general, we will only list symmetry or phase combinations that result in symmetric subbands.

Let N_0 be even (so K is odd), and consider WS filters, h. From Table 2 we see there are two distinct cases, depending on whether the group delay of h is even ($\gamma = 2\nu$) or odd ($\gamma = 2\nu + 1$). Both cases work since M is even, resulting in OS subbands with centers ν and $\nu + 1/2$, respectively, and requiring shift factors, β, of ν and $\nu + 1$. The dimension of an OS subband of period $K = N_0 - 1$ is, according to Table 3,

$$\rho = \frac{K+1}{2} = N_0/2 \ ,$$

so the channel expansiveness is $\varepsilon = \rho - N_0/M = 0$. Table 3 also gives the shift factor $\eta = 0$ (for both phases) and the synthesis extensions $E_s^{(1,2)}$ and $E_s^{(2,1)}$.

This yields the first two lines in Table 4. Note that these are precisely the parameters needed to implement the systems depicted in Figure 8. The next two lines in the table give the analogous results when N_0 is odd. WA filters are not considered since they never occur in 2-channel PR MFB's.

3.2.2. The extension $y = E_s^{(2,2)} x$.

This is an HS extension of period $N = 2N_0$, with subband period $K = N_0$. Table 2 shows that we will not obtain symmetric subbands from $E_s^{(2,2)} x$ if h is WS or WA, so Table 5 lists the possible channels obtained using H-type filters. Derivation is similar to that for Table 4.

3.2.3. (1,1)-Symmetric extension transforms.

According to Table 4, SET's based on the extension $y = E_s^{(1,1)} x$ all use a WS-WS filter pair whose delays, according to Theorem 2, have opposite parities, which we denote $\gamma_0 = 2\nu_0$, $\gamma_1 = 2\nu_1 + 1$. There are only two classes of SET's satisfying these conditions, corresponding to N_0 even or odd.

N_0 **Even:** Using the first two lines in Table 4, the subband ranks and channel expansions are

$$\rho_0 = N_0/2 = \rho_1 \quad , \quad \varepsilon_0 = 0 = \varepsilon_1 \ ,$$

so this is a nonexpansive equal-rank transform.

N_0 **Odd:** The third and fourth lines in Table 4 indicate unequal subband ranks,

$$\rho_0 = (N_0 + 1)/2 \quad , \quad \rho_1 = (N_0 - 1)/2 \quad .$$

Nonetheless, the $(1,1)$-SET for odd-length signals is nonexpansive: $\varepsilon_0 = 1/2$ and $\varepsilon_1 = -1/2$ so $\varepsilon_{sys} = 0$. While one could always duplicate an endpoint on an odd-length signal and apply an even-length transform, such a procedure would necessarily be expansive according to Theorem 1; the transform described here, while it does have unequal subband ranks, provides a nonexpansive alternative for odd-length signals.

3.2.4. (2,2)-Symmetric extension transforms.

According to Theorem 2, with $y = E_s^{(2,2)} x$ we must use HS-HA analysis filters whose delays satisfy either $\gamma_0 = 2\nu_0 + 1/2$ and $\gamma_1 = 2\nu_1 + 1/2$, or else $\gamma_0 = 2\nu_0 + 3/2$ and $\gamma_1 = 2\nu_1 + 3/2$. Combining these two options with the choice of N_0 even or odd gives us four distinct cases.

N_0 **Even,** $\gamma_i = 2\nu_i + 1/2$**:** Let h_0 be HS and h_1 HA; Table 5 indicates unequal ranks,

$$\rho_0 = N_0/2 + 1 \ , \quad \rho_1 = N_0/2 - 1 \ ,$$

but zero expansiveness, $\varepsilon_{sys} = 0$.

N_0 **Even,** $\gamma_i = 2\nu_i + 3/2$**:** This SET has equal subband ranks, $\rho_0 = N_0/2 = \rho_1$, and is nonexpansive.

N_0 **Odd, Both Phases:** Both of these nonexpansive choices have subband ranks

$$\rho_0 = (N_0 + 1)/2 \ , \quad \rho_1 = (N_0 - 1)/2 \ .$$

The difference between the two is in the extension operators needed in the synthesis bank.

3.3. The FBI Fingerprint Image Coding Standard.

The FBI's *WSQ Gray-Scale Fingerprint Image Compression Specification* [27] involves scalar quantization of the subbands resulting from a cascaded two-dimensional (product) filter bank decomposition of digitized fingerprint images; the primitive one-dimensional filtering operations are two-channel SET's. The decomposition involves five levels of cascade, resulting in 64 two-dimensional subbands. Since the FBI specification does not include constraints on the pixel dimensions of an image (e.g., image dimensions need not be divisible by 2^5), the algorithm incorporates SET's for both even- and odd-length signals. This ensures that an image of arbitrary dimensions can be transformed nonexpansively by choosing the appropriate SET algorithm at each stage in the cascade.

The standard dictates a class of SET decompositions that a decoder must be capable of reconstructing. As of the publication of [27] there is just one approved filter bank—a WS-WS lowpass-highpass filter bank, which we write here in noncausal form:

$$H_0(z) = \sum_{n=-4}^{4} h_0(n)z^{-n} \qquad ; \qquad H_1(z) = \sum_{n=-4}^{2} h_1(n)z^{-n} \quad . \tag{3.7}$$

The impulse response taps are listed in [27]; the filters correspond to a pair of smooth, symmetric, biorthogonal wavelet bases constructed in [16, 20].

3.3.1. FBI filter bank normalization.

The framework of the FBI standard specifies a class of potential encoders based on cascaded two-channel SET decompositions and a single, universal decoder capable of decoding any image produced by a compliant FBI encoder. In particular, a decoder must be capable of synthesizing SET decompositions implemented by any nontrivial two-channel linear phase FIR PR MFB with filters up to 32 taps long. To make this possible, impulse response taps for the filters are transmitted to the decoder as side information in the compressed bit stream. As we saw in Section 3.2, however, there is more than one way to implement a two-channel SET with a given filter bank, so the standard imposes the following conventions for filter bank normalization.

To specify the symmetry properties of the subbands unambiguously, the FBI standard assumes that a WS analysis bank is implemented in (noncausal) minimal phase form with group delays $\gamma_0 = 0$ (lowpass filter) and $\gamma_1 = -1$ (highpass filter). With this convention, the shift factors in Table 4 are $\beta_0 = 0 = \beta_1$ and $\eta_0 = 0 = \eta_1$ for N_0 both even and odd. The symmetries and ranks of the resulting subbands and the extensions used in the synthesis bank are also given in Table 4. Specifically, b_0 is WS at 0 and b_1 is HS at $-1/2$, regardless of whether N_0 is even or odd. When N_0 is even, the first and second lines in Table 4 give equal subband ranks, $\rho_0 = N_0/2 = \rho_1$, and when N_0 is odd the third and fourth lines give subband ranks $\rho_0 = (N_0+1)/2$ and $\rho_1 = (N_0-1)/2$.

With H-type filter banks and (2,2)-SET's, the standard assumes group delays $\gamma_0 = -1/2 = \gamma_1$. In this case the shift factors β_i, η_i are all zero and the subband ranks for the lowpass and highpass channels given by Table 5 are the same as the ranks resulting in the case of the (1,1)-SET with W-type filters.

In both cases, the impulse response taps are normalized so that the amplitude factor in the AC determinant (3.4) is $A = -1$. By Theorem 2, the group delay conventions yield an AC determinantal delay of $D_a = \gamma_0 + \gamma_1 = -1$, so the AC determinant is

$$2Az^{-D_a} = -2z \quad .$$

We therefore set $D_s = 1$ in the anti-aliasing relations (3.1) and use the following sign choice,

$$F_0(z) = -z^{-1}H_1(-z) \quad ; \quad F_1(z) = z^{-1}H_0(-z) \quad , \tag{3.8}$$

to get synthesis filters that yield zero overall distortion, $T(z) = 1$, according to (3.5).

3.3.2. Causal FBI implementations.

The above description of the SET conventions employed in the FBI standard has assumed noncausal filter implementations, which is not physically impossible when the input is a finite-length vector but which is nonetheless an inefficient way of realizing a filter bank in practice. What happens if we delay h_0 and h_1 so that the analysis bank is implemented causally? Consider the filter bank described by (3.7): if we delay h_0 and h_1 by 4 samples each to obtain a causal filter bank, h'_0 and h'_1, then Corollary 3 indicates that the new AC determinant is

$$2A'z^{-D'_a} = -2z^{-7} \quad .$$

If we also set $D'_s = 0$ in the anti-aliasing relations (3.1) to get causal synthesis filters and keep the same sign choice as in (3.8) then, by (3.5), the overall transfer function will be $T'(z) = z^{-7}$, which is easily compensated for in the decoder.

Of particular importance, Table 4 shows that the delayed (causal) filters with group delays $\gamma'_0 = 4$ and $\gamma'_1 = 3$ produce downsampled subbands with the same symmetries (modulo a two-sample delay) and ranks as the noncausal filters: b'_0 is WS (at 2) and b'_1 is HS (at 3/2). These particular symmetries are *assumed* by the decoder when it extends the transmitted subbands prior to running them through the synthesis filter bank, so it is essential to ensure that any modifications made to the analysis filter bank for the sake of efficient system implementation produce transmitted subbands, $a_i(k)$, $0 \le k < \rho_i$, that are identical to those produced by the noncausal implementations specified by the conventions in Section 3.3.1.

For instance, consider a WS-WS analysis filter bank in which h_0 has length 11 and h_1 has length 9. Suppose this filter bank is implemented causally, with $\gamma'_0 = 5$ and $\gamma'_1 = 4$. According to Table 4, when N_0 is even the lowpass subband, b_0, will be *half*-sample symmetric at its left endpoint. This requires the synthesis bank extension $E_0 = E_s^{(2,1)}$, instead of the extension $E_s^{(1,2)}$ that the decoder is expecting to use based on its assumption that a_0 was obtained from a subband filtered with a zero-phase WS filter. When N_0 is odd, the subband ranks also come out wrong and we get $\rho_0 = (N_0 - 1)/2$ and $\rho_1 = (N_0 + 1)/2$, which is opposite from the convention in Section 3.3.1.

The solution is simple: we impose an additional one-sample delay on each filter so that $\gamma''_0 = 6$ and $\gamma''_1 = 5$, which yields subbands with the conventional symmetries and ranks. Implementing such a filter bank is mathematically equivalent to applying a one-sample delay to the input,

$$Y(z) \to z^{-1}Y(z) \quad ,$$

and then using the causal filter bank with $\gamma'_0 = 5$ and $\gamma'_1 = 4$. The new AC determinantal delay is

$$D''_a = \gamma''_0 + \gamma''_1 = 11 \quad .$$

We now generate causal synthesis filters using the anti-aliasing relations

$$\begin{aligned} F''_0(z) &= -zH''_1(-z) \quad (\text{i.e., } \gamma_{F''_0} = 4) \\ F''_1(z) &= zH''_0(-z) \quad (\gamma_{F''_1} = 5). \end{aligned}$$

Since we have used $D_s = -1$, the transfer function for this implementation with causal analysis and synthesis banks is $T''(z) = z^{-10}$.

Similar considerations apply to the causal implementation of H-type filter banks under the FBI specification.

4. M-Channel Symmetric Extension Transforms.

We now extend the analysis given in Section 3 for two-channel SET's to the case of general M-channel systems. The details of implementing the individual channels in such a subband coder are still based on Figure 8 and are thus entirely analogous to the two-channel case. Unlike the two-channel case, however, the difficulties involved in producing numerically tractable design procedures for linear phase M-channel PR MFB's are such that efforts to date have focused on special cases or subclasses of linear phase PR MFB's or on particular values of M. At the time of this writing there is no exhaustive classification of the various filter symmetry/phase combinations for M-channel PR MFB's like we had for the two-channel case in Section 3.1. Therefore, we shall tabulate all possible individual SET channels based on M-channel MFB's; it should be clear to the reader, based on the presentation in Section 3.2, that it's quite easy to design specific SET systems using such tables. Channel expansiveness values are included to facilitate computing the expansiveness of specific systems via (2.8).

4.1. Concentric Filter Banks.

There is one special case of interest about which we can draw some conclusions based on our single-channel analysis. We will call a linear phase filter bank, $\{h_0, \ldots, h_{M-1}\}$, a *concentric* filter bank if all filters have the same center of symmetry. Note that all of the filters in a concentric filter bank must be of the same type, i.e., either all must be H-type or all must be W-type filters, and all have the same group delay. The design of concentric PR MFB's has been studied in [68, 59]. The most useful result to date on symmetries in concentric filter banks is the following theorem from [59] on paraunitary systems; it will allow us to calculate ε_{sys} for concentric paraunitary MFB's.

Theorem 4. *Let $\{h_0, \ldots, h_{M-1}\}$ be a concentric paraunitary filter bank.*

1. *If M is even, there are $M/2$ symmetric filters and $M/2$ antisymmetric filters.*

2. *If M is odd, there are $(M+1)/2$ symmetric filters and $(M-1)/2$ antisymmetric filters.*

As for describing the phase shifts of an M-channel PR MFB that result in another PR MFB, it is clear that the entire analysis bank can be advanced or delayed in unison by an arbitrary amount since this will result in a new analysis/synthesis system whose output differs from that of the original system by a constant delay. It is also easy to see that we can shift any one filter by a multiple of M samples: according to relation (2.3) in Section 2.2, shifting h_i by $M\nu_i$ is equivalent to shifting the downsampled subband, b_i, by ν_i, a process that is easy to compensate for in the synthesis bank. In particular, a linear phase PR MFB in which all filters have group delays that are pairwise congruent modulo M can be reduced to a concentric PR MFB. We shall use these simple facts when discussing the implementation of PR MFB's in SET's.

4.2. Divisibility Constraints on Signal Length.

Unlike the two-channel case, SET's cannot always be applied to input signals, x, of arbitrary length when $M > 2$. Recall from Section 2.2 that we imposed the divisibility constraint

$$M \mid N \quad . \tag{4.1}$$

This prevents us from using the $(2N_0 - 1)$-periodic extension, $E_s^{(1,2)}x$, when M is even. Moreover, while one might expect to use an M-channel coder if the signal length contains powers of M (i.e., if $N_0 = J \cdot M^L$), this is only possible for SET's based on the $2N_0$-periodic extension $E_s^{(2,2)}x$ if $M > 2$. $E_s^{(1,1)}x$ only works for $M = 2$ because if $M \mid (2JM^L - 2)$ then M=2, and the extension $E_s^{(1,2)}x$ is out since $M \nmid (2JM^L - 1)$.

Recall that if $0 \leq r \leq M - 1$ and $N_0 \equiv r \bmod M$, then r is called the *residue of* N_0 mod M:

$$r = \text{res}(N_0, M) \quad .$$

Since N_0 and r satisfy the same equations in arithmetic modulo M, we can now classify the extensions satisfying condition (4.1) in terms of $\text{res}(N_0, M)$ by solving the following simple congruences.

$M \mid 2N_0$, or $2r \equiv 0 \bmod M$:	$r = 0$ (all M), $\quad r = M/2$ (M even)
$M \mid 2N_0 - 1$, or $2r \equiv 1 \bmod M$:	$r = (M+1)/2$ (M odd)
$M \mid 2N_0 - 2$, or $2r \equiv 2 \bmod M$:	$r = 1$ (all M), $\quad r = 1 + M/2$ (M even)

The corresponding constraints are listed in Table 6 and will be used to derive the results in the next subsections. One observation that is immediately clear from Table 6 is that while there are admissible SET extensions for all input lengths when M=2, 3, or 4, there are always excluded congruence classes of input lengths for filter banks with 5 or more channels.

Given a combination satisfying condition (4.1), the next question is whether there exist non-expansive SET's for this choice of N_0, N, and M. As mentioned above, we cannot answer this question in complete generality, but we can say something negative concerning the possibility of nonexpansive SET's with equal subband ranks. In Section 2.3 we showed that such transforms are only possible when $M \mid N_0$. This is equivalent to having $\text{res}(N_0, M) = 0$, so from Table 6 we see that nonexpansive equal-rank SET's are only possible when $M > 2$ using $2N_0$-periodic extensions; the only hope for nonexpansive (1,1)- and (1,2)-SET's is with unequal subband ranks. In light of the distinctions in Tables 2 and 6, the classification of channels will be presented in two parts: first for M even, then for M odd.

4.3. M Even.

For both $2N_0$- and $(2N_0 - 2)$-periodic extensions, Table 6 indicates two distinct classes of signal lengths, parameterized by $\text{res}(N_0, M)$, that satisfy (4.1). As in Section 3.2, for each class we list only those combinations of filter symmetry and phase that result in symmetric subbands, b_i.

4.3.1. The extension $y = E_s^{(1,1)}x$.

Since M is even, Table 2 indicates that the $(2N_0 - 2)$-periodic WS extension, $E_s^{(1,1)}x$, produces symmetric subbands only when used with W-type filters, and only for group delays $\gamma = M\nu$ or $\gamma = M\nu + M/2$. Consider the case of signals with $\text{res}(N_0, M) = M/2 + 1$, as given in Table 6; when

h is WS with phase $M\nu$ then, since K is odd in this case, Table 2 shows that b will be OS with center ν. From Table 3, the dimension of b is

$$\rho = \frac{K+1}{2} = \frac{2N_0 - 2 + M}{2M} \quad ,$$

so the expansiveness of this channel is

$$\varepsilon = \rho - \frac{N_0}{M} = \frac{M-2}{2M} \quad .$$

When $\gamma = M\nu + M/2$, b is again OS but with center $\nu + 1/2$. The pertinent results for these two cases are given in the first two lines of Table 7; derivation of the other lines in that table is similar.

Table 7 can be used to rule out the existence of concentric W-type filter banks when M is even. By preceding remarks, such a filter bank could be advanced or delayed to have its center at $M/2$. If it were applied in a $(1,1)$-SET to signals with $\mathrm{res}(N_0, M) = 1$, it would produce a transform with expansiveness

$$\varepsilon_{sys} = M(-1/M) = -1 \quad ,$$

regardless of the mix of WS and WA filters. This contradicts (2.9), implying

Theorem 5. *There are no concentric W-type FIR PR MFB's for M even.*

This generalizes one of the consequences of Theorem 2 in Section 3.1.

4.3.2. The extension $y = E_s^{(2,2)} x$.

Since this is an HS extension, Table 2 indicates that we are constrained to using H-type filters. The possible subband symmetries and expansion factors are listed in Table 8.

Example. ($M = 4$) A four-channel concentric paraunitary filter bank has been constructed in [59] with four H-type filters, each of length eight. The filter bank is causal, with center at $7/2$. Filters h_0 and h_2 are HS while h_1 and h_3 are HA. If the analysis bank is advanced one sample,

$$H_i'(z) = zH_i(z) \quad ,$$

then the filters will have centers $\gamma_i' = 5/2 = (M+1)/2$, so Table 8 shows that we have a nonexpansive equal-rank $(2,2)$-SET when $\mathrm{res}(N_0,4)$=0 and a nonexpansive (2,2)-SET when $\mathrm{res}(N_0,4)$=2. If the analysis bank is advanced three samples,

$$H_i''(z) = z^3 H_i(z) \quad ,$$

then $\gamma_i'' = 1/2$ and the $(2,2)$-SET is nonexpansive for $\mathrm{res}(N_0, 4) = 0,\ 2$. This can be generalized:

Proposition 6. *For M even, any concentric H-type PR MFB yields a nonexpansive equal-rank $(2,2)$-SET for inputs with res$(N_0, M) = 0$ if we shift the analysis bank so that its center is congruent to $(M+1)/2$ mod M. If the analysis bank contains equal numbers of symmetric and antisymmetric filters (e.g., paraunitary filter banks), nonexpansiveness also holds for inputs with res$(N_0, M) = M/2$ and for analysis banks with centers congruent to 1/2 mod M (for both input residue classes 0 and $M/2$).*

When M is even, Tables 7 and 8 show that a given filter bank can be used for SET's on just two distinct residue classes of input lengths, meaning that there will always be excluded input lengths for a given filter bank whenever $M \geq 4$. For M=4, the existence of nonexpansive SET's for signals with $\mathrm{res}(N_0,4)$=1 or 3 depends on the existence of (non-concentric) W-type MFB's for use in (1,1)-SET's.

4.4. M Odd.

The SET possibilities in this case differ markedly from the situation when M is even. Table 6 gives only one residue class of signal lengths satisfying (4.1) for each extension. For either WS or HS signal extensions, however, Table 2 indicates that we get symmetric subbands using both W- and H-type filters. This increases the possible combinations of filter symmetries that can be used in SET's.

4.4.1. The extension $y = E_s^{(1,1)}x$.

When $N = 2N_0 - 2$, the only residue class satisfying condition (4.1) is $\mathrm{res}(N_0, M) = 1$, but we get symmetric subbands with all four classes of filters. The possible symmetry/phase combinations are listed in Table 9; the derivation is via Tables 2 and 3, as in preceding sections. An analogue of Theorem 5 for concentric H-type filter banks when M is odd follows from the last two lines in Table 9: by shifting such a filter bank so it is centered at $M/2$, we could construct a $(1,1)$-SET with expansiveness $\varepsilon_{sys} = -1$. Since this is impossible, we conclude

Theorem 7. *There are no concentric H-type FIR PR MFB's for M odd.*

Remarks. Consider the case of a casusal, concentric PR MFB in which the length, L, of the filters is divisible by M. It follows that the type-1 polyphase matrix, $\mathbf{E}(z)$, for the system has order $N = L/M - 1$. If M is odd then Theorem 7 says we must have a W-type PR MFB, which means that L is odd, implying that N must be even. This result was proven for the special case of paraunitary filter banks by Soman *et al.* ([59], Corollary 1). Thus, Theorems 5 and 7 can be regarded as generalizations of ([59], Corollary 1) for non-paraunitary filter banks and arbitrary values of M, without the restriction $M \mid L$.

Example. ($M = 3$) In [46] the authors construct a causal three-channel PR MFB with the following symmetries:

$$\text{HS},\ \gamma_0 = 27.5 \quad (56\ \text{taps}) \quad ; \quad \text{WS},\ \gamma_1 = 26 \quad (53\ \text{taps}) \quad ; \quad \text{HA},\ \gamma_2 = 27.5 \quad (56\ \text{taps}) \quad . \tag{4.2}$$

We can take full advantage of the choice of filter symmetries in Table 9 to construct a nonexpansive three-channel $(1,1)$-SET using the filter bank (4.2). Delay the bank by one sample,

$$H_i'(z) = z^{-1}H_i(z) \ \ , \ \ \gamma_i' = \gamma_i + 1 \ \ ,$$

so that $\gamma_1' = 27 \equiv 0 \bmod 3$ and $\gamma_0' = \gamma_2' = 28.5 \equiv 3/2 \bmod 3$. When $\mathrm{res}(N_0, 3) = 1$ this new filter bank defines a $(1,1)$-SET with expansiveness $\varepsilon_{sys} = (2/3) + 2(-1/3) = 0$.

When M is odd, there is the possibility of using concentric W-type PR MFB's. Although we do not have an explicit example of such a filter bank construction (a theoretical classification is given in [59]), we can use Theorem 4 and (2.8) to calculate the expansiveness of a $(1,1)$-SET based on a concentric paraunitary W-type filter bank with center congruent to 0 mod M. Using the first two lines in Table 9,

$$\varepsilon_{sys} = \left(\frac{M+1}{2}\right)\left(\frac{M-1}{M}\right) + \left(\frac{M-1}{2}\right)\left(-\frac{M+1}{M}\right) = 0 \ \ .$$

4.4.2. The extension $y = E_s^{(1,2)} x$.

The possible symmetric channels for the $(1,2)$-SET, M odd, are listed in Table 10. The same one-sample delay of the filter bank (4.2) that was used in a (1,1)-SET above now furnishes a (1,2)-SET for inputs with res$(N_0, 3)$=2; the expansiveness is $\varepsilon_{sys} = 2(1/3) - (2/3) = 0$.

Now compute the expansiveness of a (1,2)-SET based on a concentric paraunitary W-type filter bank with center congruent to 0 mod M:

$$\varepsilon_{sys} = \left(\frac{M+1}{2}\right)\left(\frac{M-1}{2M}\right) + \left(\frac{M-1}{2}\right)\left(-\frac{M+1}{2M}\right) = 0 \ .$$

4.4.3. The extension $y = E_s^{(2,2)} x$.

The possible symmetric channels for the $(2,2)$-SET, M odd, are listed in Table 11. The example (4.2) can be incorporated into a nonexpansive $(2,2)$-SET when res$(N_0, 3)$=0 without added phase shifts. This, together with the above examples using (4.2), shows that we can construct nonexpansive SET's for inputs of arbitrary length using the filter bank (4.2) if we use different symmetric extensions for different input length residue classes.

Note that both W-type channels for the $(2,2)$-SET have expansiveness $\varepsilon_i = 0$. Thus, a concentric W-type PR MFB—regardless of the mix of symmetric and antisymmetric filters—furnishes a nonexpansive equal-rank $(2,2)$-SET for signals satisfying res$(N_0, M) = 0$ if the analysis bank's center is congruent to $(M+1)/2$ mod M. We summarize the possibilities for SET's based on concentric W-type filter banks in

Proposition 8. *For M odd, any concentric W-type PR MFB yields a nonexpansive equal-rank $(2,2)$-SET for inputs with res$(N_0, M) = 0$ if its center is congruent to $(M+1)/2$ mod M. If the analysis bank contains $(M+1)/2$ symmetric filters and $(M-1)/2$ antisymmetric filters (e.g., paraunitary filter banks) and the center of the analysis bank is congruent to 0 mod M, we get a nonexpansive (1,1)-SET for inputs satisfying res(N_0, M)=1 and a nonexpansive (1,2)-SET for inputs satisfying res$(N_0, M) = (M+1)/2$. In particular, when M=3 a concentric W-type PR MFB with two symmetric filters and one antisymmetric filter furnishes nonexpansive SET's for all input length residue classes.*

5. Conclusions.

This paper has classified nonexpansive symmetric extension transforms based on the possible combinations of symmetric signal extensions and linear phase filters that produce symmetric subbands in PR MFB's. Detailed algorithms are developed for the implementation of all of these transforms. A general notion of transform expansiveness has been introduced to analyze the extent to which SET's conserve data storage requirements.

In the case of two-channel PR MFB's, we have been able to enumerate all possible SET's explicitly. Nonexpansive SET's based on $(1,1)$- and $(2,2)$-symmetric extensions have been shown to exist for both even- and odd-length signals. The particular SET algorithms employed in the FBI fingerprint image coding specification have been described.

In the general M-channel case, the classification of symmetric channels has been applied to the analysis of SET algorithms for linear phase PR MFB's, such as concentric filter banks. Modulo constraints on input length, we have constructed examples of nonexpansive SET's for both even and odd values of M. Algebraic restrictions on allowable combinations of input length and downsample factor prevent any one symmetric extension method from being applicable to signals of arbitrary length when $M > 2$. By using different analysis extensions and phase shifts, however, we have shown how certain 3-channel filter banks can provide nonexpansive SET's for input signals of arbitrary length. We have also shown that a single filter bank never furnishes nonexpansive SET's for input signals of arbitrary length when $M \geq 4$. A reasonably detailed analysis is given of the conditions under which a concentric filter bank furnishes nonexpansive SET's.

As an interesting corollary of our analysis, we have used dimensionality results about SET's to establish the nonexistence of certain classes of linear phase PR MFB's. In particular, we have shown that concentric PR MFB's based on W-type filters are impossible for even M and that concentric PR MFB's based on H-type filters are impossible for odd M.

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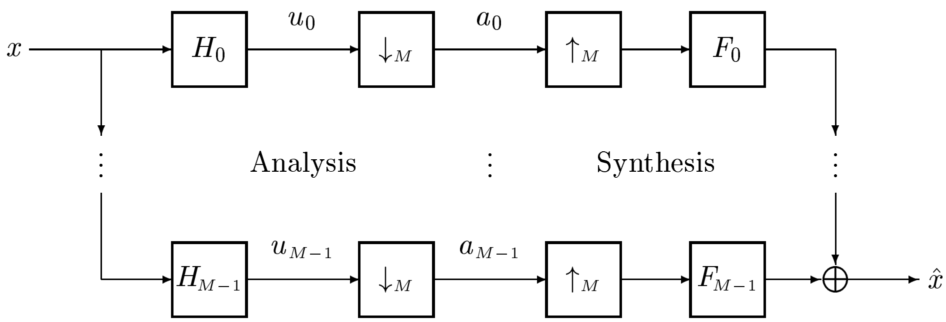

Figure 1: An M-channel multirate filter bank.

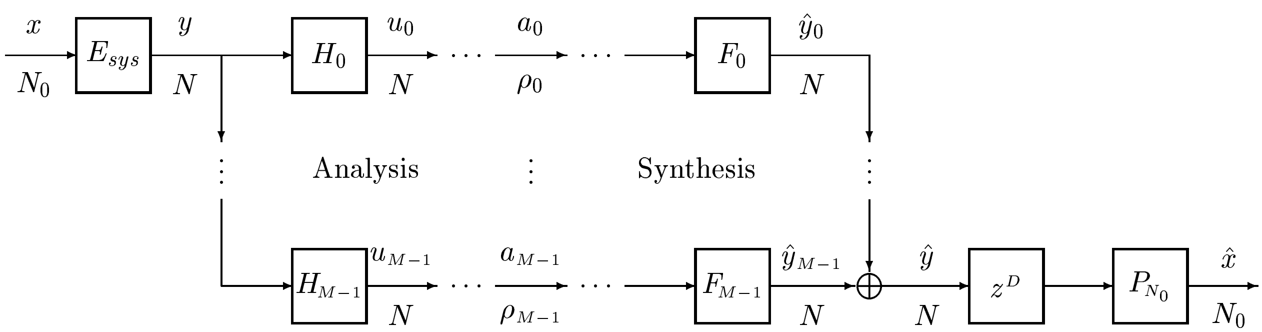

Figure 2: The extended M-channel SET subband coder.

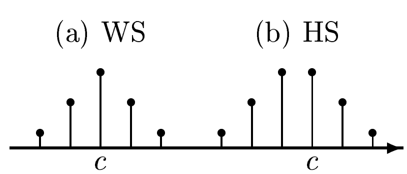

Figure 3: (a) Whole-sample symmetry about c. (b) Half-sample symmetry about $c - 1/2$.

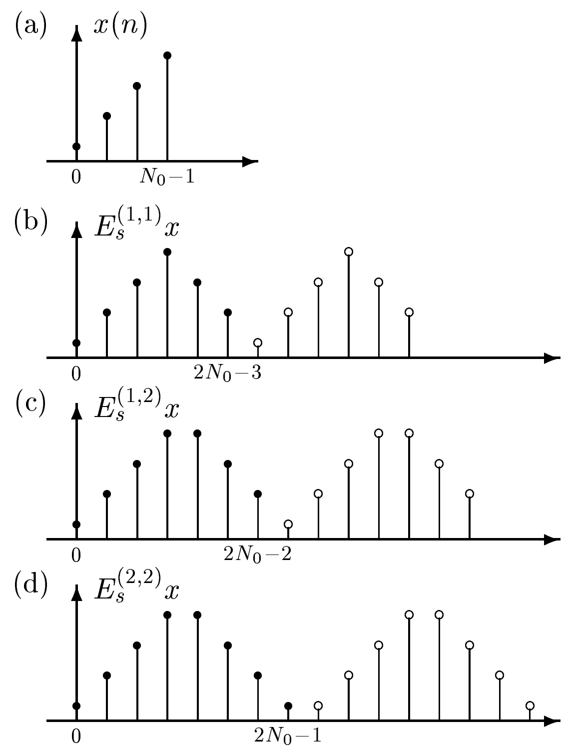

Figure 4: Three symmetric extensions, $y = E_s^{(i,j)}x$, for signal analysis.

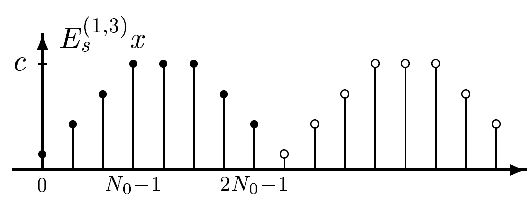

Figure 5: The symmetric extension $y = E_s^{(1,3)}x$.

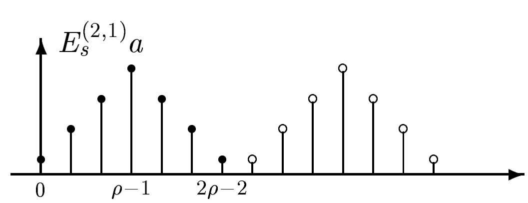

Figure 6: The symmetric extension, $b = E_s^{(2,1)}a$, for subband synthesis.

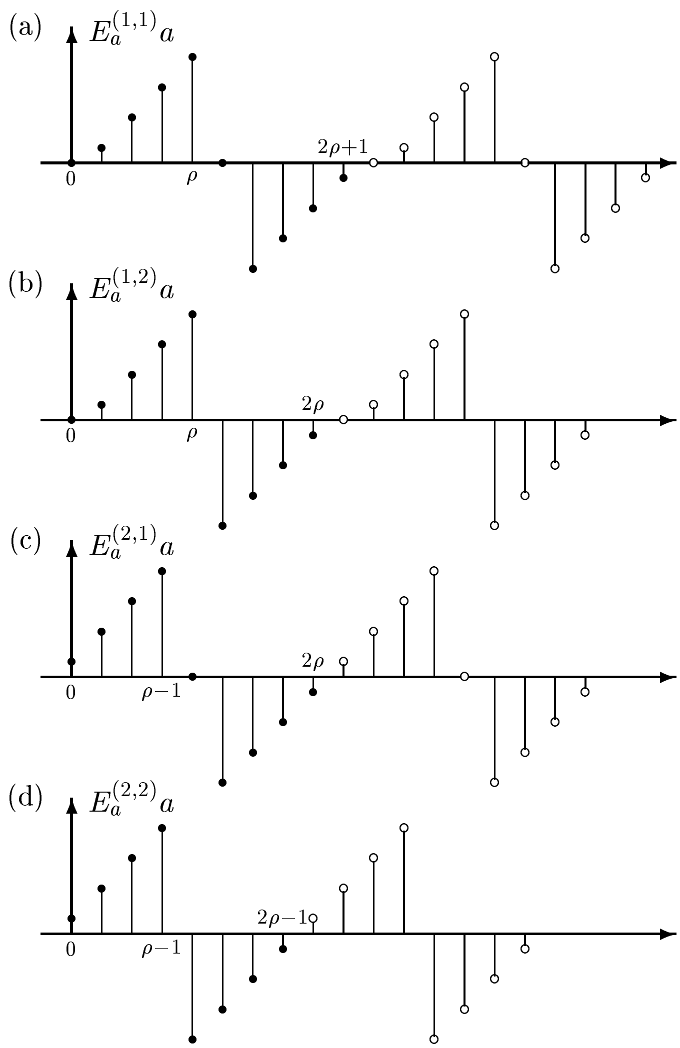

Figure 7: Four antisymmetric extensions, $b = E_a^{(i,j)}a$, for subband synthesis.

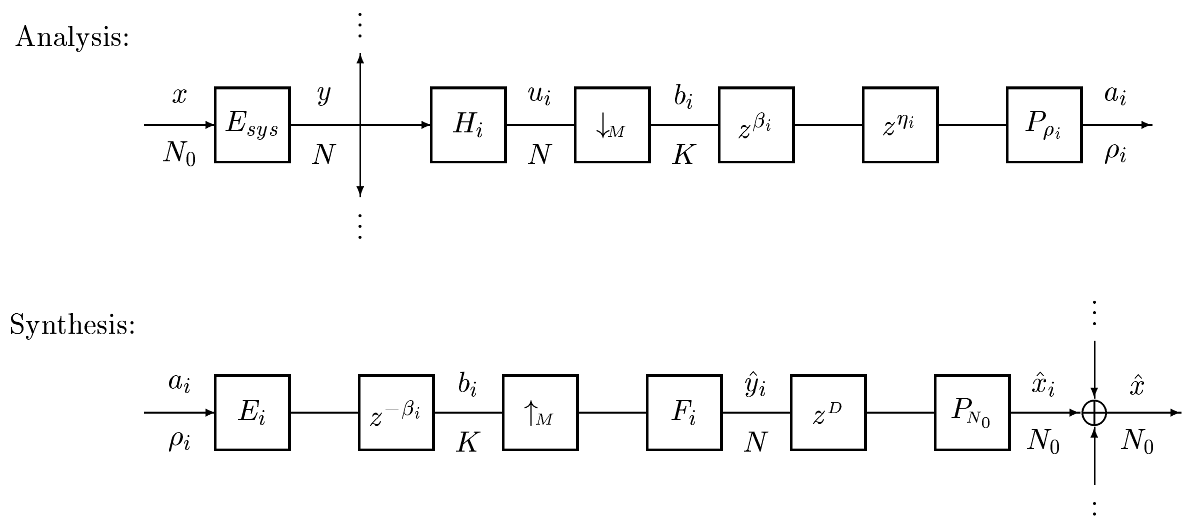

Figure 8: Analysis/synthesis channels for the extended subband coder.

Table 1: Symmetry properties of N-periodic signals.

Signal y: Sym.	Center	Symmetry Characterization: Time Domain	DFT Domain	Dimension ρ
WS	c	$y(c+n) = y(c-n)$	$Y^*(k) = e^{j2\pi(2c)k/N}Y(k)$	$N/2+1$
WA	c	$y(c+n) = -y(c-n)$	$Y^*(k) = -e^{j2\pi(2c)k/N}Y(k)$	$N/2-1$
HS	$c-1/2$	$y(c+n) = y(c-1-n)$	$Y^*(k) = e^{j2\pi(2c-1)k/N}Y(k)$	$N/2$
HA	$c-1/2$	$y(c+n) = -y(c-1-n)$	$Y^*(k) = -e^{j2\pi(2c-1)k/N}Y(k)$	$N/2$
OS (N Odd)	WS at c and HS at $c+N/2$, or HS at $c-1/2$ and WS at $c+(N-1)/2$			$(N+1)/2$ (N Odd)
OA (N Odd)	WA at c and HA at $c+N/2$, or HA at $c-1/2$ and WA at $c+(N-1)/2$			$(N-1)/2$ (N Odd)

Table 2: Symmetry properties of down-sampled signals.

Signal y: Sym., Center	Filter h: Sym.	Center γ	M	Subband Symmetry, Center: $u = y * h$	$b = \downarrow_M u$	Shift β
WS, 0	WS/WA	$M\nu$	Any	WS/WA, $M\nu$	WS/WA, ν	ν
		$M\nu + M/2$	Even	WS/WA, $M\nu + M/2$	HS/HA, $\nu + 1/2$	$\nu + 1$
	HS/HA	$M\nu + M/2$	Odd	HS/HA, $M\nu + M/2$	HS/HA, $\nu + 1/2$	$\nu + 1$
HS, $-1/2$	HS/HA	$M\nu + 1/2$	Any	WS/WA, $M\nu$	WS/WA, ν	ν
		$M\nu + (M+1)/2$	Even	WS/WA, $M\nu + M/2$	HS/HA, $\nu + 1/2$	$\nu + 1$
	WS/WA	$M\nu + (M+1)/2$	Odd	HS/HA, $M\nu + M/2$	HS/HA, $\nu + 1/2$	$\nu + 1$

Table 3: Symmetric subband reconstruction.

Subband b: Sym.	Dim. ρ	Center	Shift η	Ext. E
WS	$K/2+1$	0	0	$E_s^{(1,1)}$
WA	$K/2-1$	0	1	$E_a^{(1,1)}$
HS	$K/2$	$-1/2$	0	$E_s^{(2,2)}$
HA	$K/2$	$-1/2$	0	$E_a^{(2,2)}$
OS (K Odd)	$(K+1)/2$	0	0	$E_s^{(1,2)}$
		$-1/2$	0	$E_s^{(2,1)}$
OA (K Odd)	$(K-1)/2$	0	1	$E_a^{(1,2)}$
		$-1/2$	0	$E_a^{(2,1)}$

Table 4: $(1,1)$-Symmetric extension transform channels, $M = 2$.

Signal x: N_0	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
Even (K Odd)	WS	2ν	OS, ν	$N_0/2$	ν	0	$E_s^{(1,2)}$	0
		$2\nu+1$	OS, $\nu+1/2$	$N_0/2$	$\nu+1$	0	$E_s^{(2,1)}$	0
Odd (K Even)	WS	2ν	WS, ν	$(N_0+1)/2$	ν	0	$E_s^{(1,1)}$	$1/2$
		$2\nu+1$	HS, $\nu+1/2$	$(N_0-1)/2$	$\nu+1$	0	$E_s^{(2,2)}$	$-1/2$

Table 5: $(2,2)$-Symmetric extension transform channels, $M = 2$.

Signal x: N_0	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
Even (K Even)	HS	$2\nu+1/2$	WS, ν	$N_0/2+1$	ν	0	$E_s^{(1,1)}$	1
		$2\nu+3/2$	HS, $\nu+1/2$	$N_0/2$	$\nu+1$	0	$E_s^{(2,2)}$	0
	HA	$2\nu+1/2$	WA, ν	$N_0/2-1$	ν	1	$E_a^{(1,1)}$	-1
		$2\nu+3/2$	HA, $\nu+1/2$	$N_0/2$	$\nu+1$	0	$E_a^{(2,2)}$	0
Odd (K Odd)	HS	$2\nu+1/2$	OS, ν	$(N_0+1)/2$	ν	0	$E_s^{(1,2)}$	$1/2$
		$2\nu+3/2$	OS, $\nu+1/2$	$(N_0+1)/2$	$\nu+1$	0	$E_s^{(2,1)}$	$1/2$
	HA	$2\nu+1/2$	OA, ν	$(N_0-1)/2$	ν	1	$E_a^{(1,2)}$	$-1/2$
		$2\nu+3/2$	OA, $\nu+1/2$	$(N_0-1)/2$	$\nu+1$	0	$E_a^{(2,1)}$	$-1/2$

Table 6: Divisibility constraints on signal length, N_0.

M	N	$\mathrm{res}(N_0, M)$	$K = N/M$
Even	$2N_0$	0	Even
		$M/2$	Odd
	$2N_0-2$	1	Even
		$M/2+1$	Odd
Odd	$2N_0$	0	Even
	$2N_0-1$	$(M+1)/2$	Odd
	$2N_0-2$	1	Even

Table 7: $(1,1)$-Symmetric extension transform channels, M even.

Signal x: $\mathrm{res}(N_0, M)$	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
M/2+1 (K Odd)	WS	$M\nu$	OS, ν	$(2N_0-2+M)/2M$	ν	0	$E_s^{(1,2)}$	$(M-2)/2M$
		$M\nu+M/2$	OS, $\nu+1/2$	$(2N_0-2+M)/2M$	$\nu+1$	0	$E_s^{(2,1)}$	$(M-2)/2M$
	WA	$M\nu$	OA, ν	$(2N_0-2-M)/2M$	ν	1	$E_a^{(1,2)}$	$-(M+2)/2M$
		$M\nu+M/2$	OA, $\nu+1/2$	$(2N_0-2-M)/2M$	$\nu+1$	0	$E_a^{(2,1)}$	$-(M+2)/2M$
1 (K Even)	WS	$M\nu$	WS, ν	$(N_0-1)/M+1$	ν	0	$E_s^{(1,1)}$	$(M-1)/M$
		$M\nu+M/2$	HS, $\nu+1/2$	$(N_0-1)/M$	$\nu+1$	0	$E_s^{(2,2)}$	$-1/M$
	WA	$M\nu$	WA, ν	$(N_0-1)/M-1$	ν	1	$E_a^{(1,1)}$	$-(M+1)/M$
		$M\nu+M/2$	HA, $\nu+1/2$	$(N_0-1)/M$	$\nu+1$	0	$E_a^{(2,2)}$	$-1/M$

Table 8: $(2,2)$-Symmetric extension transform channels, M even.

Signal x: $\mathrm{res}(N_0, M)$	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
0 (K Even)	HS	$M\nu+1/2$	WS, ν	$N_0/M+1$	ν	0	$E_s^{(1,1)}$	1
		$M\nu+(M+1)/2$	HS, $\nu+1/2$	N_0/M	$\nu+1$	0	$E_s^{(2,2)}$	0
	HA	$M\nu+1/2$	WA, ν	$N_0/M-1$	ν	1	$E_a^{(1,1)}$	-1
		$M\nu+(M+1)/2$	HA, $\nu+1/2$	N_0/M	$\nu+1$	0	$E_a^{(2,2)}$	0
M/2 (K Odd)	HS	$M\nu+1/2$	OS, ν	$(2N_0+M)/2M$	ν	0	$E_s^{(1,2)}$	$1/2$
		$M\nu+(M+1)/2$	OS, $\nu+1/2$	$(2N_0+M)/2M$	$\nu+1$	0	$E_s^{(2,1)}$	$1/2$
	HA	$M\nu+1/2$	OA, ν	$(2N_0-M)/2M$	ν	1	$E_a^{(1,2)}$	$-1/2$
		$M\nu+(M+1)/2$	OA, $\nu+1/2$	$(2N_0-M)/2M$	$\nu+1$	0	$E_a^{(2,1)}$	$-1/2$

Table 9: $(1,1)$-Symmetric extension transform channels, M odd.

Signal x: $\mathrm{res}(N_0, M)$	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
1 (K Even)	WS	$M\nu$	WS, ν	$(N_0-1)/M+1$	ν	0	$E_s^{(1,1)}$	$(M-1)/M$
	WA	$M\nu$	WA, ν	$(N_0-1)/M-1$	ν	1	$E_a^{(1,1)}$	$-(M+1)/M$
	HS	$M\nu+M/2$	HS, $\nu+1/2$	$(N_0-1)/M$	$\nu+1$	0	$E_s^{(2,2)}$	$-1/M$
	HA	$M\nu+M/2$	HA, $\nu+1/2$	$(N_0-1)/M$	$\nu+1$	0	$E_a^{(2,2)}$	$-1/M$

Table 10: $(1,2)$-Symmetric extension transform channels, M odd.

Signal x: res(N_0, M)	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
$(M+1)/2$ (K Odd)	WS	$M\nu$	OS, ν	$(2N_0-1+M)/2M$	ν	0	$E_s^{(1,2)}$	$(M-1)/2M$
	WA	$M\nu$	OA, ν	$(2N_0-1-M)/2M$	ν	1	$E_a^{(1,2)}$	$-(M+1)/2M$
	HS	$M\nu+M/2$	OS, $\nu+1/2$	$(2N_0-1+M)/2M$	$\nu+1$	0	$E_s^{(2,1)}$	$(M-1)/2M$
	HA	$M\nu+M/2$	OA, $\nu+1/2$	$(2N_0-1-M)/2M$	$\nu+1$	0	$E_a^{(2,1)}$	$-(M+1)/2M$

Table 11: $(2,2)$-Symmetric extension transform channels, M odd.

Signal x: res(N_0, M)	Filter h: Sym.	Center γ	Subband b: Sym., Center	Dim. ρ	Shifts: β	η	Ext. E	Exp. ε
0 (K Even)	WS	$M\nu+(M+1)/2$	HS, $\nu+1/2$	N_0/M	$\nu+1$	0	$E_s^{(2,2)}$	0
	WA	$M\nu+(M+1)/2$	HA, $\nu+1/2$	N_0/M	$\nu+1$	0	$E_a^{(2,2)}$	0
	HS	$M\nu+1/2$	WS, ν	$N_0/M+1$	ν	0	$E_s^{(1,1)}$	1
	HA	$M\nu+1/2$	WA, ν	$N_0/M-1$	ν	1	$E_a^{(1,1)}$	-1