STEERABLE FILTER CASCADES

Jeffery R. Price

University of Tennessee
Knoxville, TN
jrp@utk.edu

Monson H. Hayes III

Georgia Institute of Technology
Atlanta, GA
monson.hayes@ece.gatech.edu

ABSTRACT

In this paper, we present the notion of cascading steerable filters to improve their angular resolution. Additionally, we illustrate that the results of such cascades can be steered themselves. An advantage of this approach is that only a single, relatively small set of steerable filters can be employed to achieve various angular resolutions. Improving angular resolution has previously required an entirely different, larger set of filters.

1. INTRODUCTION AND REVIEW

Oriented filters are useful in many image processing and early vision tasks [1, 2]. Steerable filters are one class of oriented filters that have received significant attention in recent literature [3-7]. After the background information in this section, the remainder of this paper is organized as follows. In Section 2, we describe and demonstrate with an example how cascading appropriately steered filters can provide improved angular resolution. We show that these cascades can be steered themselves in Section 3. In Section 4, we apply steerable cascades to the orientation analysis problem and we conclude in Section 5 with some closing comments.

We provide a concise review of steerability, modeled closely on the excellent presentation of Simoncelli & Farid in [6], but in the frequency domain. Perhaps the simplest example of a steerable filter is the first partial derivative of the Gaussian. In polar frequency coordinates, the horizontal and vertical derivatives are given by

$$G_0^{(1)}(\xi, \phi) = \cos(\phi)(-\xi e^{-\xi^2/2}) \tag{1a}$$

$$G_{\pi/2}^{(1)}(\xi, \phi) = \sin(\phi)(-\xi e^{-\xi^2/2}) \tag{1b}$$

where ξ and ϕ indicate the radial and angular coordinates, respectively. In (1), the subscripts and superscripts of G indicate the orientation and derivative order, respectively. Using trigonometric identities, it is easy to show that the first partial derivative at any orientation, $G_\theta^{(1)}$, can be synthesized by

$$G_\theta^{(1)}(\xi, \phi) = \cos(\theta)G_0^{(1)}(\xi, \phi) + \sin(\theta)G_{\pi/2}^{(1)}(\xi, \phi). \tag{2}$$

Equation (2) embodies the principle of steerability: the directional derivative $G^{(1)}$ can be generated at any arbitrary orientation θ by a linear combination of the *basis filters*, $G_0^{(1)}$ and $G_{\pi/2}^{(1)}$. We refer to the coefficients of this linear summation, $\cos(\theta)$ and $\sin(\theta)$, as the *steering functions*. Since convolution is a linear operation, the result of convolving with an arbitrarily oriented filter can be computed from the results of convolving with only the basis filters.

Steerability is not limited to derivative filters – see [2] for a thorough exposition. We proceed considering only polar separable filters, for which the general *steerability condition* is given by

$$\begin{aligned}
B_\theta(\xi, \phi) &= H(\xi)B(\phi - \theta) \\
&= H(\xi)\sum_{k=1}^{Q} g_k(\theta)B(\phi - \theta_k) \\
&= H(\xi)\sum_{k=1}^{Q} g_k(\theta)B_k(\phi)
\end{aligned} \tag{3}$$

where $H(\cdot)$ is the radial component, $B(\cdot)$ is the angular component, $g_k(\cdot)$ are the steering functions, and θ_k are a fixed set of Q orientations, usually chosen so that $\theta_k = (k-1)/(Q\pi)$ for robustness [2]. The Q filters represented by the $H(\xi)B_k(\phi)$ terms are the basis filters.

We work with filters whose angular component can be expressed as a (finite) Fourier series in polar angle:

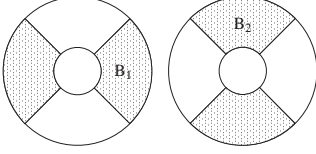$$B(\phi) = \sum_{r=-M}^{M} w(r)e^{jr\phi}. \tag{4}$$

Figure 1: Ideal magnitude response of the two orientation band, steerable filter set from the steerable pyramid. (Shaded regions indicate passbands.)

Additionally, the following two theorems will be important.

Theorem 1. *Let P be the number of nonzero coefficients, $w(r)$, for a filter $H(\xi, \phi)$ whose angular component is expandable in the form of (4). Then P is the minimum number of basis functions necessary to steer $H(\xi, \phi)$ by (3), i.e., Q in (3) must be such that $Q \geq P$ [2].*

Theorem 2. *The steering condition (3) holds for functions with angular components expandable in the form of (4), where only P of the coefficients $w(r)$ are nonzero, if and only if the steering functions $g_k(\theta)$ are solutions of*

$$
\begin{pmatrix}
e^{jr_1\theta_1} & e^{jr_1\theta_2} & \cdots & e^{jr_1\theta_P} \\
e^{jr_2\theta_1} & e^{jr_2\theta_2} & \cdots & e^{jr_2\theta_P} \\
\vdots & & \ddots & \vdots \\
e^{jr_P\theta_1} & e^{jr_P\theta_2} & \cdots & e^{jr_P\theta_P}
\end{pmatrix}
\begin{pmatrix}
g_1(\theta) \\
g_2(\theta) \\
\vdots \\
g_P(\theta)
\end{pmatrix}
=
\begin{pmatrix}
e^{-jr_1\theta} \\
e^{-jr_2\theta} \\
\vdots \\
e^{-jr_P\theta}
\end{pmatrix},
$$

(5)

where $\{r_1, \ldots, r_P\}$, $r_i \in [-M, M]$, indicate the indices of the nonzero $w(r)$ coefficients.

Theorem 1 gives the number of basis filters that are required to provide steerability. Theorem 2, a more general version of that in [2], gives the steering functions, $g_k(\theta)$. Although we have restricted ourselves to polar separable filters for brevity, the theorems above (and our results to follow, except as noted) hold in more general cases [8].

As we are concerned with the angular components of steerable filters, and to limit excessive notation, we will refer to filters by only their angular components. For example, we will refer to the basis filters mentioned above simply as $B_k(\phi)$, dropping the common radial component $H(\xi)$. Note that $B_k(\phi) = B(\phi - \theta_k)$ from (3) can be written using (4) as

$$
B_k(\phi) = B(\phi - \theta_k) = \sum_{r=-M}^{M} [w(r)e^{-jr\theta_k}]e^{jr\phi}
$$

$$
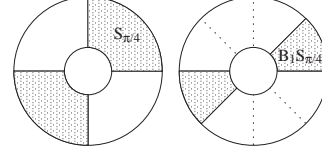= \sum_{r=-M}^{M} w_k(r)e^{jr\phi}.
$$

(6)

Figure 2: Ideal magnitude responses of $S_{\pi/4}$ and the cascade $B_1 S_{\pi/4}$.

2. CASCADES OF STEERED FILTERS

Previously, increasing the angular resolution of steerable filters has been accomplished by increasing the number of basis filters. Here, however, we present a potential alternative. Consider the ideal bandpass nature of the two orientation band, steerable pyramid filters presented in [5] and illustrated in Fig. 1. A filter at any arbitrary rotation can by synthesized by

$$
S_\theta(\phi) = \cos(\theta)B_1(\phi) + \sin(\theta)B_2(\phi).
$$

(7)

Now consider the result of cascading B_1 with the filter $S_{\pi/4}$ as computed by (7). The filter $S_{\pi/4}$ and the (ideal) resulting cascade $B_1 S_{\pi/4}$, which is oriented at $\pi/8$, are shown in Fig. 2. As evident in Fig. 2, the resulting cascade $B_1 S_{\pi/4}$ should provide improved angular resolution in the ideal case. The dashed lines in the $B_1 S_{\pi/4}$ portion of Fig. 2 indicate the four orientation bands one might produce from the four combinations of B_1 and B_2 cascaded with $S_{\pi/4}$ and $S_{-\pi/4}$. We could also envision additional cascade stages to further improve angular resolution. For example, applying $S_{-\pi/8}$ to the $B_1 S_{\pi/4}$ result of Fig. 2 would (ideally) produce a filter oriented at $\pi/16$ with half the angular bandwidth of $B_1 S_{\pi/4}$. It should be evident that this cascading can be applied to steerable sets with an arbitrary number of basis filters, and extended to any number of stages.

We illustrate these ideas with a simple example. We employ a two orientation band, steerable filter set constructed by sampling the first horizontal and vertical derivatives of the Gaussian on a 9×9 lattice. The normalized magnitude responses of B_1, $S_{\pi/4}$, and the cascade $B_1 S_{\pi/4}$ are shown in Fig. 3, where the straight lines indicate the (expected) orientation. The increase in angular resolution of the cascade should be evident in Fig. 3. Note the out-of-band energy orthogonal to the main lobes in $B_1 S_{\pi/4}$. The peak side lobe magnitude is approximately 17% (i.e., -16 dB) of the peak in the directional pass band. Further cascade stages are illustrated in Fig. 4. As more levels are employed, it is evident that the resulting orientation begins to drift significantly from what is expected. The out-of-band energy (about -16 dB) is also still present. In the
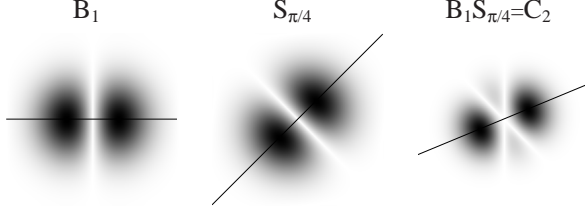
$B_1 \quad\quad S_{\pi/4} \quad\quad B_1 S_{\pi/4} {=} C_2$

Figure 3: Normalized magnitude responses for two stage cascade. (Shown in inverse scale – i.e., black indicates 1 and white indicates 0.)

$C_3{=}C_2 S_{-\pi/8} \quad C_4{=}C_3 S_{-3\pi/16} \quad C_5{=}C_4 S_{-7\pi/32}$
(line at $\pi/16$) \quad\quad (line at $\pi/32$) \quad\quad (line at $\pi/64$)

Figure 4: Normalized magnitude responses for further cascades. (Coordinate scaling is the same as in Fig. 3 for consistency.)

next section, we show how the cascades can be steered, preventing the orientation drift and, furthermore, eliminating the out-of-band energy.

3. STEERABILITY OF CASCADES

We mentioned in the previous section that the two orientation bands of Fig. 1 could be split into four bands using the four possible combinations of B_1 or B_2 cascaded with $S_{\pi/4}$ or $S_{-\pi/4}$. Recalling (7), however, S_θ cascaded with B_1 or B_2 can be written as one of the following:

$$B_1 S_\theta = \cos(\theta) B_1 B_1 + \sin(\theta) B_1 B_2 \quad\quad (8a)$$
$$B_2 S_\theta = \cos(\theta) B_1 B_2 + \sin(\theta) B_2 B_2. \quad\quad (8b)$$

Although we have suggested using the four filters given by $B_1 S_{\pi/4}$, $B_1 S_{-\pi/4}$, $B_2 S_{\pi/4}$, and $B_2 S_{-\pi/4}$, (8) illustrates that these four are composed of linear combinations of only three unique terms: $B_1 B_1$, $B_1 B_2$, and $B_2 B_2$. In this scenario, we would like to determine if we can construct a three band, steerable basis set from these three unique terms. We now address this problem more generally.

Suppose we have a basis set of m filters, each of which can be written as in (6), where only m of the Fourier series coefficients are nonzero. We denote the number of unique filters that result from the n stage cascades of these m filters as $P(m,n)$. For our $m = 2$ example, $P(2,n) = n + 1$ and the unique terms are given by all possible $(B_1)^{l_1}(B_2)^{l_2}$ such that $l_1 + l_2 = n$ with $l_1, l_2 \geq 0$. In the general case,

$$P(m,n) = \binom{n+m-1}{m-1} = \frac{(n+m-1)!}{n!\,(m-1)!} \quad\quad (9)$$

and the unique terms are given by all the possible

$$(B_1)^{l_1}(B_2)^{l_2} \cdots (B_m)^{l_m} \qu\quad\quad (10)$$

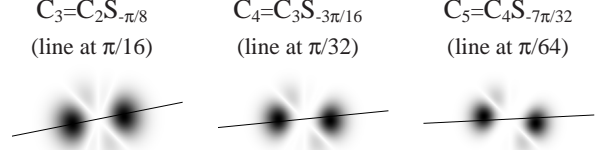such that $\sum_{i=1}^{m} l_i = n$, where $l_i \geq 0$. With this notation, we make the following proposition.

Proposition 1. *Any filter in the form of* (10) *has at most* $P(m,n)$ *nonzero coefficients in its (angular) Four-ier series expansion.*

This can be shown by using (6) to express each of the B_i, for $i = 1, \ldots, m$, as

$$B_i(\phi) = \sum_{p=1}^{m} w_i(r_p) e^{jr_p\phi} \qu\quad\quad (11)$$

where $w_i(r) = w(r)e^{-j\theta_i r}$ and $\{r_1, \ldots, r_m\}$ indicate the indices of the (possibly) nonzero coefficients. Therefore, a filter in the form of (10) can be considered an m term multinomial of degree n, where the m terms are given by $e^{jr_p\phi}$ for $p = 1, \ldots, m$. The maximum number of (monomial) terms in an m term multinomial of degree n is the same as the number of m term monomials of degree n, i.e., (10), and this number is $P(m,n)$ as given by (9), hence Proposition 1.

Recalling Theorem 1, Proposition 1 implies that every filter in the form of (10), which results from the n stage cascade of the m basis filters, is steerable with at most $P(m,n)$ basis filters. We would like to know, however, if the $P(m,n)$ different filters that result from our cascades can be used as a basis set to steer themselves.

We begin with a basis set $B_1, \ldots, B_m$, where m is minimal to satisfy the steering condition. With m and n fixed, let $P = P(m,n)$. We will let $C_k(\phi)$ denote the kth unique term of the cascade for $k = 1, \ldots, P$. Each C_k will take the form of (10), where each of those B_k can be written as in (6). The Fourier series expansion of $C_k(\phi)$ can then be written as

$$C_k(\phi) = \sum_{r=-nM}^{nM} a_k(r) e^{jr\phi}, \qu\quad\quad (12)$$

where $a_k(r)$ is nonzero for at most P of the $2nM + 1$ indices by Proposition 1. Let the set of r where possibly $a_k(r) \neq 0$ be given by $\{r_1, \ldots, r_P\}$. Note that this set
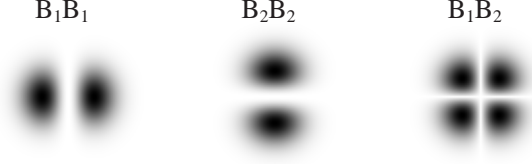
Figure 5: Normalized magnitude response of the three unique cascade terms for the two stage ($n = 2$) cascade of two filter ($m = 2$) basis set. Notice that the cross term B_1B_2 does not have a predominant orientation.

Figure 6: Normalized magnitude response for steered cascade filters. Compare with Fig. 4.

is the same for all k. Now let $C_1(\phi) = [B_1(\phi)]^n$. As noted in Section 1, B_1 is traditionally oriented at zero radians, like $\cos(\phi)$ for example. With this condition, it should be evident that $C_1 = (B_1)^n$ is oriented at zero radians as well, but with a narrower bandwidth than B_1. We would like to use the set of C_k for $k = 1, \ldots, P$ to steer $C_1(\phi)$ to any arbitrary angle θ. In other words, we would like to find, if possible, the set of steering functions $g_k(\theta)$ for $k = 1, \ldots, P$ to solve

$$C_1(\phi - \theta) = \sum_{k=1}^{P} g_k(\theta) C_k(\phi). \qquad (13)$$

If such $g_k(\theta)$ can be found, then our cascades can indeed be used to steer themselves. Employing (12), we can show the following [8].

Proposition 2. *The cascade steering condition given by (13) holds for filters in the form of (12), where only P of the coefficients $a_k(r)$ are nonzero, if and only if the steering functions satisfy*

$$\begin{pmatrix} a_1(r_1) & a_2(r_1) & \cdots & a_P(r_1) \\ a_1(r_2) & a_2(r_2) & \cdots & a_P(r_2) \\ \vdots & & \ddots & \vdots \\ a_1(r_P) & a_2(r_P) & \cdots & a_P(r_P) \end{pmatrix} \begin{pmatrix} g_1(\theta) \\ g_2(\theta) \\ \vdots \\ g_P(\theta) \end{pmatrix}$$
$$= \begin{pmatrix} a_1(r_1)e^{-jr_1\theta} \\ a_1(r_2)e^{-jr_2\theta} \\ \vdots \\ a_1(r_P)e^{-jr_P\theta} \end{pmatrix}, \quad (14)$$

where $\{r_1, \ldots, r_P\}$ indicate the indices of the (possibly) nonzero coefficients $a_k(r)$ for $k = 1, \ldots, P$.

Proposition 2 is just a corollary of Theorem 2. According to Proposition 1, the matrix on the left hand side of (14) always has at least as many columns as rows, lending hope that our cascades can indeed steer themselves. In all the cases we have considered in our research, (14) has been solvable.
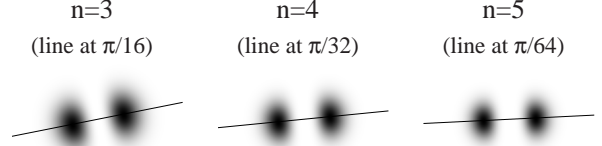
Note that, unlike the original basis filters, where usually $B_i(\phi) = B_1(\phi - \theta_i)$, the cascade filters do not necessarily have a predominant orientation. In other words, $C_k(\phi) \neq C_1(\phi - \theta_k)$ in general. Consider, for example, the two stage cascades of a two band basis set as shown in Fig. 5. The squared terms obviously have a predominant orientation, but notice that the cross term B_1B_2 does not.

Finally, we note that the Fourier series coefficients of the cascade filters, $a_k(r)$ in (12), can be computed easily by discrete convolution of the Fourier series coefficients of the original basis filters, $w_k(r)$ in (4). In other words,

$$a_k(r) = [w_1(r)]^{*l_1} * [w_2(r)]^{*l_2} * \cdots * [w_m(r)]^{*l_m} \quad (15)$$

for $k = 1, \ldots, P(m, n)$, where the mapping between k and the set of l_i is isomorphic, and where $[w_i(r)]^{*l_i}$ indicates $w_i(r)$ convolved with itself l_i times. We note that this approach works only for polar separable filters.

As an example, we steer cascades constructed using the 9×9 first derivative of Gaussian filters used for the examples of Section 2. The magnitude response of the cascade stages for $n = 3, 4, 5$ are shown in Fig. 6. They have been steered to the indicated angles for comparison with Fig. 4. Note that there is no out-of-band energy or orientation drift in the steered cascade filters.

4. APPLICATIONS

We have applied steerable filter cascades to the orientation analysis problem in a manner similar to [2]. As both even and odd parity filters are used in [2], we employ two successive cascade stages for orientation analysis. We begin with two 5×5 filters computed by sampling the first derivatives of the Gaussian. We first use the responses from the two original filters and their two stage cascades. We compute the predominant orientation and oriented energy, as in [2], and refer to them as $\{\theta_1, e_1\}$ and $\{\theta_2, e_2\}$ for the original filters and two stage cascades, respectively. (The subscripts here are used to indicate the number of stages.) The final
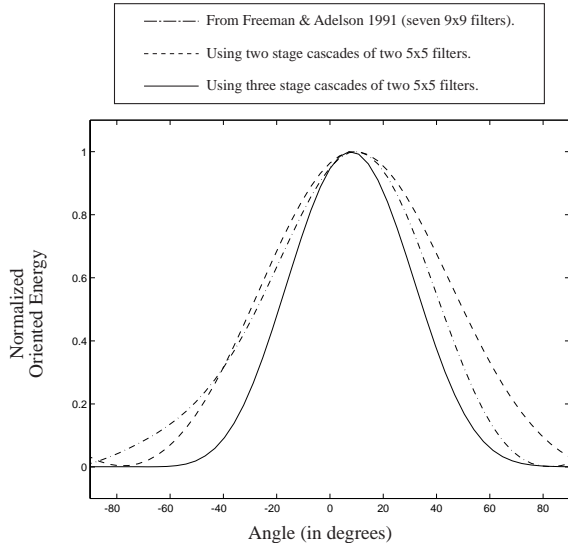
Figure 7: Oriented energy for one pixel in a test image.

predominant orientation is taken to be either θ_1 or θ_2, according to which of e_1 or e_2 is larger (after normalization). For improved angular resolution, we also employ two and three stage cascades in the same manner.

Our one-two and two-three stage analyses were performed on a 170×170 synthetic test image and compared with the method from [2]. The difference between the orientation from our methods and that from [2] was computed at pixels where the oriented energy exceeded 10% of its peak value and the mean absolute difference was calculated. For the one-two stage cascade, this difference was $1.9°$ and for the two-three stage cascade the difference was $2.6°$. The oriented energy from all three methods is plotted against angle in Fig. 7 for a pixel on a structure oriented at approximately $10°$. Note the improved angular resolution in the two-three stage cascade indicated by the narrower bandwidth. A similar experiment was performed using the 512×512 well known "lena" image. In this case, the mean absolute difference between our two-three stage cascade analysis and the method of [2] was $4.4°$.

These results indicate that the orientations computed by our method(s) are nearly identical to those computed by the method from [2]. We, however, begin with only two 5×5 filters, where [2] uses seven 9×9 filters. Practically, of course, the amount of computation is about the same considering the cascading, but we feel that this at least demonstrates the potential of such cascades.

Finally, we mention that cascades within the steerable pyramid [7] have also been investigated and found to have some promise [8]. A brief discussion of this topic should appear in a future endeavor.

5. CONCLUSIONS

In this paper, we introduced steerable filter cascades. It was demonstrated that such cascades can provide improved angular resolution and can be steered themselves. Cascades were applied to the orientation analysis problem and were shown to perform as well as a common approach, but beginning with fewer filters. Additionally it was mentioned , but not discussed, that cascades can be used within the steerable pyramid.

6. REFERENCES

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