Introduction to Lie Operators for Accelerator Physics

Yury Eidelman

I greatly appreciate Dr. J. Galayda and Dr. M. Borland for the pleasant opportunity to work at Argonne National Laboratory. I would like to express my gratitude to all staff of Accelerator Systems Division for their attention and interest to these lectures.

ADVANCED PHOTON SOURCE

Table of Contents

BINP

ADVANCED PHOTON SOURCE

Introduction

A typical equation in physics has the following form:

$$\vec{X}'' = \vec{F}(\vec{X}, \vec{X}', t) \ .$$

Here $\vec{X}$ is, for example, the coordinate vector $\vec{r}$ of particle motion or the spin vector $\vec{S}$ of a charge particle, etc.
It is clear that one can write the solution of this equation as

$$\vec{X}^{(f)} = \mathcal{M}\vec{X}^{(i)} \ ,$$

where indices (i) and (f) correspond to the initial and final states of the physical system and **$\mathcal{M}$ is some OPERATOR.**
Using the Lie operator approach one can find this operator with desired order of nonlinearity. But it is a long way. It is necessary to know the following types of operators and their properties:

- **operators that are united into linear algebra;**
- **commutators of these operators (Lie operators);**
- **exponential operators;**
- **commutators of the exponential operators;**
- **adjoint operators;**
- **exponential adjoint operators (Lie transformations).**

Later it will be found that

operator $\mathcal{M}$ is the exponential adjoint operator.

BINP

ADVANCED PHOTON SOURCE

When is the Lie operator approach preferable?

The Lie operator approach is very useful if one would like:

- **to solve analytically your problem with desired order of nonlinearity;**
- **to preserve the symplecticity of your solution of nonlinear equations;**
- **to avoid the nonphysical errors at the numeric calculations;**
- **to spend your time and brain to study this field of mathematics and application of this brilliant technique to different problems in physics.**

ADVANCED PHOTON SOURCE

Lie algebra of the operators

Let us consider the set of operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ..., which are elements of the structure with the following properties:

$\mathcal{L}1$

- **commutativity** and **associativity** of sum operation:

$$\mathcal{A}+\mathcal{B}=\mathcal{B}+\mathcal{A}, \qquad \mathcal{A}+(\mathcal{B}+\mathcal{C})=(\mathcal{A}+\mathcal{B})+\mathcal{C}\ ;$$

- **the zero operator $\mathcal{O}$ exists**:

$$\mathcal{A}+\mathcal{O}=\mathcal{O}+\mathcal{A}=\mathcal{A}\ ;$$

- **the inverse operator $-\mathcal{A}$ exists**:

$$\mathcal{A}+(-\mathcal{A})=(-\mathcal{A})+\mathcal{A}=\mathcal{O}\ .$$

$\mathcal{L}2$

- The **multiplication** operation $\odot$ **with linearity and associativity** is defined:

$$\begin{aligned}
(\mathcal{A}+\mathcal{B})\odot\mathcal{C} &= \mathcal{A}\odot\mathcal{C}+\mathcal{B}\odot\mathcal{C}\ ,\\
\mathcal{A}\odot(\mathcal{B}+\mathcal{C}) &= \mathcal{A}\odot\mathcal{B}+\mathcal{A}\odot\mathcal{C}\ ,\\
\alpha(\mathcal{A}\odot\mathcal{B}) &= (\alpha\mathcal{A})\odot\mathcal{B}=\mathcal{A}\odot(\alpha\mathcal{B})\ ,\\
\mathcal{A}\odot(\mathcal{B}\odot\mathcal{C}) &= (\mathcal{A}\odot\mathcal{B})\odot\mathcal{C} \qquad \text{as well as}
\end{aligned}$$

$\mathcal{L}3$

- the property of **antisymmetry**:

$$\mathcal{A}\odot\mathcal{B}=-\mathcal{B}\odot\mathcal{A} \quad \textbf{and}$$

- **Jacobi condition**:

$$(\mathcal{A}\odot\mathcal{B})\odot\mathcal{C}+(\mathcal{B}\odot\mathcal{C})\odot\mathcal{A}+(\mathcal{C}\odot\mathcal{A})\odot\mathcal{B}=\mathcal{O}\ .$$

The set with properties $\mathcal{L}1, \mathcal{L}2$ forms **linear algebra** and with properties $\mathcal{L}1$–$\mathcal{L}3$ — **Lie algebra**.

ADVANCED PHOTON SOURCE

Lie operators

Let us define the second operation of the **multiplication "$[,]$"** for any operators $\mathcal{A}$, $\mathcal{B}$:

$$[\mathcal{A}, \mathcal{B}] = \mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A} \; .$$

Let us refer to this multiplication as **Lie multiplication** or **commutator** of the operators.
It is very simple to verify the following properties of Lie multiplication:

- **antisymmetry**: $[\mathcal{A}, \mathcal{B}] = -[\mathcal{B}, \mathcal{A}]$;
- **linearity**: $[\mathcal{A}, (\beta\mathcal{B} + \gamma\mathcal{C})] = \beta[\mathcal{A}, \mathcal{B}] + \gamma[\mathcal{A}, \mathcal{C}]$;
- **product rule**: $[\mathcal{A}, (\mathcal{B}\mathcal{C})] = [\mathcal{A}, \mathcal{B}]\,\mathcal{C} + \mathcal{B}[\mathcal{A}, \mathcal{C}]$.

In fact, using the property of associativity, one has that

$$\begin{aligned}
[\mathcal{A}, (\mathcal{B}\mathcal{C})] &= \mathcal{A}(\mathcal{B}\mathcal{C}) - (\mathcal{B}\mathcal{C})\mathcal{A} \\
&= (\mathcal{A}\mathcal{B})\mathcal{C} - (\mathcal{B}\mathcal{A})\mathcal{C} + \mathcal{B}(\mathcal{A}\mathcal{C}) - \mathcal{B}(\mathcal{C}\mathcal{A}) \\
&= (\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A})\mathcal{C} + \mathcal{B}(\mathcal{A}\mathcal{C} - \mathcal{C}\mathcal{A}) = [\mathcal{A}, \mathcal{B}]\,\mathcal{C} + \mathcal{B}[\mathcal{A}, \mathcal{C}] \; .
\end{aligned}$$

Operators with these properties are referred to as **Lie operators.**

Notice: Two operations of multiplication are introduced, but it is not always necessary. For operators, only a commutator as main multiplication operation can be introduced. If this operation satisfies conditions $\mathcal{L}3$, then **the set of** these **Lie operators forms the Lie algebra.**

BINP

Leibnitz rule

A very important property of operators from the Lie algebra is **Leibnitz rule:**

$$\mathcal{A}^n(\mathcal{B}\mathcal{C}) = \sum_{k=0}^{n} C_n^k\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n-k}\mathcal{C}\right) \; .$$

Here C_n^k is a binomial coefficient defined by

$$C_n^k = \frac{n!}{k!(n-k)!}$$

with the following property:

$$\begin{aligned} C_n^{k-1} + C_n^k &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!(k+n-k+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = C_{n+1}^k \; . \end{aligned}$$

Let us prove this rule using the induction method, but the lazy reader can omit it.

For $n = 0$ in both parts of the equality one has the identity operator. Further:

$$\begin{aligned} & \mathcal{A}^{n+1}(\mathcal{B}\mathcal{C}) = \mathcal{A}\sum_{k=0}^{n} C_n^k\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n-k}\mathcal{C}\right) \\ = & \sum_{k=0}^{n} C_n^k\left(\mathcal{A}^{k+1}\mathcal{B}\right)\left(\mathcal{A}^{n-k}\mathcal{C}\right) + \sum_{k=0}^{n} C_n^k\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n-k+1}\mathcal{C}\right) \\ = & \underbrace{\sum_{k=0}^{n-1} C_n^k\left(\mathcal{A}^{k+1}\mathcal{B}\right)\left(\mathcal{A}^{n-k}\mathcal{C}\right)}_{k\to k'=k+1} + \underbrace{C_n^n}_{C_{n+1}^{n+1}}\left(\mathcal{A}^{n+1}\mathcal{B}\right)\left(\mathcal{A}^0\mathcal{C}\right) + \underbrace{C_n^0}_{C_{n+1}^0}\left(\mathcal{A}^0\mathcal{B}\right)\left(\mathcal{A}^{n+1}\mathcal{C}\right) \\ + & \underbrace{\sum_{k=1}^{n} C_n^k\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n-k+1}\mathcal{C}\right)}_{k\to k'} = C_{n+1}^0\left(\mathcal{A}^0\mathcal{B}\right)\left(\mathcal{A}^{n+1}\mathcal{C}\right) + \sum_{k'=1}^{n} C_n^{k'-1}\left(\mathcal{A}^{k'}\mathcal{B}\right)\left(\mathcal{A}^{n+1-k'}\mathcal{C}\right) \\ + & \sum_{k'=1}^{n} C_n^{k'}\left(\mathcal{A}^{k'}\mathcal{B}\right)\left(\mathcal{A}^{n+1-k'}\mathcal{C}\right) + C_{n+1}^{n+1}\left(\mathcal{A}^{n+1}\mathcal{B}\right)\left(\mathcal{A}^0\mathcal{C}\right) \\ = & C_{n+1}^0\left(\mathcal{A}^0\mathcal{B}\right)\left(\mathcal{A}^{n+1}\mathcal{C}\right) + \sum_{k=1}^{n}\underbrace{\left(C_n^{k-1} + C_n^k\right)}_{C_{n+1}^k}\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n+1-k}\mathcal{C}\right) + C_{n+1}^{n+1}\left(\mathcal{A}^{n+1}\mathcal{B}\right)\left(\mathcal{A}^0\mathcal{C}\right) \\ = & \sum_{k=0}^{n+1} C_{n+1}^k\left(\mathcal{A}^k\mathcal{B}\right)\left(\mathcal{A}^{n+1-k}\mathcal{C}\right) \; . \end{aligned}$$

Exponential operators

Let us introduce the **exponential operator** as the following series:

$$e^{\mathcal{A}} = \mathcal{A}^0 + \frac{1}{1!}\mathcal{A}^1 + \frac{1}{2!}\mathcal{A}^2 + \cdots = \sum_{n=0}^{\infty} \frac{\mathcal{A}^n}{n!} .$$

One can simply verify that for this type of operator all properties $\mathcal{L}1$ and $\mathcal{L}2$ are satisfied. It means that a set of these operators is a linear algebra.

Notice: Here the implicit assumption is made that exponential operators form the closed group for multiplication $\odot$, that is, the multiplication of two exponential operators is an exponential as well. It is a very serious assumption that can be proved by the Baker–Campbell-Hausdorff theorem (see later).

The properties of the exponential operators

1°. **Ordinary multiplication** of two operators:

$$\begin{aligned}
e^{\mathcal{A}} e^{\mathcal{B}} &= \left(\sum_{i=0}^{\infty} \frac{\mathcal{A}^i}{i!}\right)\left(\sum_{k=0}^{\infty} \frac{\mathcal{B}^k}{k!}\right) \\
&= \left(\mathcal{A}^0 + \frac{1}{1!}\mathcal{A}^1 + \frac{1}{2!}\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^3 + \cdots\right) \\
&\cdot \left(\mathcal{B}^0 + \frac{1}{1!}\mathcal{B}^1 + \frac{1}{2!}\mathcal{B}^2 + \frac{1}{3!}\mathcal{B}^3 + \cdots\right) \\
&= \mathcal{I} + \underbrace{\frac{1}{1!}\mathcal{A}^1 + \frac{1}{1!}\mathcal{B}^1} + \underbrace{\frac{1}{2!}\mathcal{A}^2 + \frac{1}{1!1!}\mathcal{A}^1\mathcal{B}^1 + \frac{1}{2!}\mathcal{B}^2} \\
&+ \underbrace{\frac{1}{3!}\mathcal{A}^3 + \frac{1}{2!1!}\mathcal{A}^2\mathcal{B}^1 + \frac{1}{1!2!}\mathcal{A}^1\mathcal{B}^2 + \frac{1}{3!}\mathcal{B}^3} + \cdots \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{i} \frac{\mathcal{A}^{i-k}}{(i-k)!}\,\frac{\mathcal{B}^k}{k!} .
\end{aligned}$$

So,

$$\mathbf{e}^{\mathcal{A}}\mathbf{e}^{\mathcal{B}} = \sum_{\mathbf{i}=0}^{\infty}\sum_{\mathbf{k}=0}^{\mathbf{i}} \frac{\mathcal{A}^{\mathbf{i}-\mathbf{k}}}{(\mathbf{i}-\mathbf{k})!}\,\frac{\mathcal{B}^{\mathbf{k}}}{\mathbf{k}!} . \tag{1}$$

2°. Is it true that

$$e^{\mathcal{A}}e^{\mathcal{B}} = e^{\mathcal{B}}e^{\mathcal{A}} \ ?$$

In other words: **is it true that exponential operators commute?** This can be checked using the power series expansion:

$$\begin{aligned}
& e^{\mathcal{A}}e^{\mathcal{B}} - e^{\mathcal{B}}e^{\mathcal{A}} = \\
= & \left[\mathcal{I} + \left(\frac{1}{1!}\mathcal{A}^1 + \frac{1}{1!}\mathcal{B}^1\right) + \left(\frac{1}{2!}\mathcal{A}^2 + \frac{1}{1!1!}\mathcal{A}^1\mathcal{B}^1 + \frac{1}{2!}\mathcal{B}^2\right)\right. \\
+ & \left.\left(\frac{1}{3!}\mathcal{A}^3 + \frac{1}{2!1!}\mathcal{A}^2\mathcal{B}^1 + \frac{1}{1!2!}\mathcal{A}^1\mathcal{B}^2 + \frac{1}{3!}\mathcal{B}^3\right) + \cdots\right] \\
- & \left[\mathcal{I} + \left(\frac{1}{1!}\mathcal{A}^1 + \frac{1}{1!}\mathcal{B}^1\right) + \left(\frac{1}{2!}\mathcal{A}^2 + \frac{1}{1!1!}\mathcal{A}^1\mathcal{B}^1 + \frac{1}{2!}\mathcal{B}^2\right)\right. \\
+ & \left.\left(\frac{1}{3!}\mathcal{A}^3 + \frac{1}{2!1!}\mathcal{A}^2\mathcal{B}^1 + \frac{1}{1!2!}\mathcal{A}^1\mathcal{B}^2 + \frac{1}{3!}\mathcal{B}^3\right) + \cdots\right] \\
= & \frac{1}{1!}(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}) + \frac{1}{2!}(\mathcal{A}^2\mathcal{B} + \mathcal{A}\mathcal{B}^2 - \mathcal{B}^2\mathcal{A} + \mathcal{B}\mathcal{A}^2) + \cdots \\
= & \cdots = [\mathcal{A},\mathcal{B}] + \frac{1}{2!}\left[(\mathcal{A}+\mathcal{B}),[\mathcal{A},\mathcal{B}]\right] + \cdots
\end{aligned}$$

Thus, **if $\mathcal{A}$ and $\mathcal{B}$ do not commute, then the exponentials do not commute as well.**

3°. Is it true that

$$e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}}e^{\mathcal{B}} \ ?$$

Let us check this expression using again the power series expansion:

$$\begin{aligned}
e^{\mathcal{A}+\mathcal{B}} - e^{\mathcal{A}}e^{\mathcal{B}} = & \left[\mathcal{I} + \frac{1}{1!}(\mathcal{A}+\mathcal{B}) + \frac{1}{2!}(\mathcal{A}+\mathcal{B})^2 + \cdots\right] \\
- & \left[\mathcal{I} + \left(\frac{1}{1!}\mathcal{A}^1 + \frac{1}{1!}\mathcal{B}^1\right) + \left(\frac{1}{2!}\mathcal{A}^2 + \frac{1}{1!1!}\mathcal{A}^1\mathcal{B}^1 + \frac{1}{2!}\mathcal{B}^2\right) + \cdots\right] \\
= & \frac{1}{2!}(\mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A}) - \mathcal{A}\mathcal{B} + \cdots = -\frac{1}{2!}[\mathcal{A},\mathcal{B}] + \cdots
\end{aligned}$$

So, **if operators $\mathcal{A}$ and $\mathcal{B}$ commute, then the standard rule of the multiplication of their exponentials is valid.**

ADVANCED PHOTON SOURCE

4°. Let us prove that **linear algebra of the exponential operators has an associativity property:**

$$\left(e^{\mathcal{A}}e^{\mathcal{B}}\right)e^{\mathcal{C}} = e^{\mathcal{A}}\left(e^{\mathcal{B}}e^{\mathcal{C}}\right) .$$

In fact, one can obtain taking into account the expression (1) that

$$\begin{aligned}
\left(e^{\mathcal{A}}e^{\mathcal{B}}\right)e^{\mathcal{C}} &= \left(\sum_{i=0}^{\infty}\sum_{k=0}^{i}\frac{\mathcal{A}^{i-k}}{(i-k)!}\frac{\mathcal{B}^{k}}{k!}\right)\left(\sum_{n=0}^{\infty}\frac{\mathcal{C}^{n}}{n!}\right) = \cdots \\
&= \sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{n=0}^{k}\frac{\mathcal{A}^{i-k}}{(i-k)!}\frac{\mathcal{B}^{k-n}}{(k-n)!}\frac{\mathcal{C}^{n}}{n!} .
\end{aligned}$$

Let us change indices of summation: $i' \leftarrow i$, $k' \leftarrow i-n$, $n' \leftarrow i-k$ and take into account that then $k-n = k'-n'$ and $n = i'-k'$. For this reason, for fixed $i' = i$ and k in the region $[0 \div i]$ and n in the region $[0 \div k]$, the indices k' and n' will change in the regions $[0 \div i]$ and $[0 \div k']$, respectively. Therefore

$$\begin{aligned}
\left(e^{\mathcal{A}}e^{\mathcal{B}}\right)e^{\mathcal{C}} &= \sum_{i=0}^{\infty}\sum_{k=0}^{i}\sum_{n=0}^{k}\frac{\mathcal{A}^{i-k}}{(i-k)!}\frac{\mathcal{B}^{k-n}}{(k-n)!}\frac{\mathcal{C}^{n}}{n!} \\
&= \sum_{i'=0}^{\infty}\sum_{k'=0}^{i'}\sum_{n'=0}^{k'}\frac{\mathcal{A}^{n'}}{n'!}\frac{\mathcal{B}^{k'-n'}}{(k'-n')!}\frac{\mathcal{C}^{i'-k'}}{(i'-k')!} \\
&= \left(\sum_{i=0}^{\infty}\frac{\mathcal{A}^{i}}{i!}\right)\left(\sum_{k=0}^{\infty}\sum_{n=0}^{k}\frac{\mathcal{B}^{k-n}}{(k-n)!}\frac{\mathcal{C}^{n}}{n!}\right) = e^{\mathcal{A}}\left(e^{\mathcal{B}}e^{\mathcal{C}}\right),
\end{aligned}$$

5°. For associative operators one can introduce their **commutator:**

$$\left[e^{\mathcal{A}}, e^{\mathcal{B}}\right] = e^{\mathcal{A}}e^{\mathcal{B}} - e^{\mathcal{B}}e^{\mathcal{A}}$$

with the obvious **antisymmetry** property. Moreover, the **Jacobi condition** (see above) is valid:

$$\begin{aligned}
&\left[\mathbf{e}^{\mathcal{A}}, \left[\mathbf{e}^{\mathcal{B}}, \mathbf{e}^{\mathcal{C}}\right]\right] + \left[\mathbf{e}^{\mathcal{B}}, \left[\mathbf{e}^{\mathcal{C}}, \mathbf{e}^{\mathcal{A}}\right]\right] + \left[\mathbf{e}^{\mathcal{C}}, \left[\mathbf{e}^{\mathcal{A}}, \mathbf{e}^{\mathcal{B}}\right]\right] = \mathbf{e}^{\mathcal{A}}(\underbrace{e^{\mathcal{B}}e^{\mathcal{C}}}_{\mathbf{1}} - \underbrace{e^{\mathcal{C}}e^{\mathcal{B}}}_{\mathbf{2}}) - (\underbrace{e^{\mathcal{B}}e^{\mathcal{C}}}_{\mathbf{3}} \\
&- \underbrace{e^{\mathcal{C}}e^{\mathcal{B}}}_{4})e^{\mathcal{A}} + e^{\mathcal{B}}(\underbrace{e^{\mathcal{C}}e^{\mathcal{A}}}_{3} - \underbrace{e^{\mathcal{A}}e^{\mathcal{C}}}_{5}) - (\underbrace{e^{\mathcal{C}}e^{\mathcal{A}}}_{6} - \underbrace{e^{\mathcal{A}}e^{\mathcal{C}}}_{2})e^{\mathcal{B}} + e^{\mathcal{C}}(\underbrace{e^{\mathcal{A}}e^{\mathcal{B}}}_{6} - \underbrace{e^{\mathcal{B}}e^{\mathcal{A}}}_{4}) \\
&- (\underbrace{e^{\mathcal{A}}e^{\mathcal{B}}}_{1} - \underbrace{e^{\mathcal{B}}e^{\mathcal{A}}}_{5})e^{\mathcal{C}} = \mathcal{O} .
\end{aligned}$$

So, commutators of the exponential operators satisfy the properties $\mathcal{L}3$. It means that the set of **commutators of the exponential operators** form the Lie algebra, that is, they **are Lie operators** for operations $\oplus$ and $[,]$.

ADVANCED PHOTON SOURCE

Derivatives

Let us consider the operators and their exponentials as a function.
6°. It is convenient to start from the simple relation:

$$\frac{d}{dt}e^{\alpha(t)\mathcal{A}} = \alpha'(t)\mathcal{A}e^{\alpha(t)\mathcal{A}} = e^{\alpha(t)\mathcal{A}}\alpha'(t)\mathcal{A}\ .$$

In fact, one can find that

$$\begin{aligned}
\frac{d}{dt}e^{\alpha(t)\mathcal{A}} &= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}\left[\alpha^n(t)\mathcal{A}\right]^n = \sum_{n=0}^{\infty}\frac{1}{n!}n\alpha'(t)\alpha^{n-1}(t)\mathcal{A}^n \\
&= \alpha'(t)\mathcal{A}\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\alpha^{n-1}(t)\mathcal{A}^{n-1} = \big(n \to k = n-1\big) \\
&= \alpha'(t)\mathcal{A}\sum_{k=0}^{\infty}\frac{1}{k!}\alpha^{k}(t)\mathcal{A}^{k} = \alpha'(t)\mathcal{A}e^{\alpha(t)\mathcal{A}} = e^{\alpha(t)\mathcal{A}}\alpha'(t)\mathcal{A}\ .
\end{aligned}$$

7°. When the operators are functions of parameter t explicitly, i.e., $\mathcal{A} = \mathcal{A}(t)$, then the **Leibnitz rule is valid:**

$$\frac{d}{dt}\mathcal{A}^n(t) = \sum_{k=0}^{n-1}\mathcal{A}^k(t)\mathcal{A}'(t)\mathcal{A}^{n-1-k}(t) \qquad \textbf{for} \quad n \geq 1\ .$$

Let us use the induction method to prove this formula:

$$\begin{aligned}
\frac{d}{dt}\mathcal{A}^{n+1}(t) &= \frac{d}{dt}\left(\mathcal{A}(t)\mathcal{A}^n(t)\right) = \mathcal{A}'(t)\mathcal{A}^n(t) + \mathcal{A}(t)\frac{d}{dt}\mathcal{A}^n(t) \\
&= \underbrace{\mathcal{I}}_{\mathcal{A}^0(t)}\mathcal{A}'(t)\mathcal{A}^n(t) + \mathcal{A}(t)\underbrace{\sum_{k=0}^{n-1}\mathcal{A}^k(t)\mathcal{A}'(t)\mathcal{A}^{n-1-k}(t)}_{k \to k'=k+1} \\
&= \mathcal{A}^0(t)\mathcal{A}'(t)\mathcal{A}^n(t) + \sum_{k'=1}^{n}\mathcal{A}^{k'}(t)\mathcal{A}'(t)\mathcal{A}^{n-k'}(t) = \sum_{k=0}^{n}\mathcal{A}^k(t)\mathcal{A}'(t)\mathcal{A}^{n-k}(t)\ .
\end{aligned}$$

8°. Now one is ready to find the rule to differentiate exponentials. He should not be afraid of the following two expressions:

$$\begin{aligned}
\frac{d}{dt}e^{\mathcal{A}(t)} &= \left(\int_0^1 e^{\tau\mathcal{A}(t)}\mathcal{A}'(t)e^{-\tau\mathcal{A}(t)}d\tau\right)e^{\mathcal{A}(t)} \\
&= e^{\mathcal{A}(t)}\left(\int_0^1 e^{-\tau\mathcal{A}(t)}\mathcal{A}'(t)e^{\tau\mathcal{A}(t)}d\tau\right)\ .
\end{aligned} \tag{2}$$

BINP

The lazy reader can skip this proof again.

In fact, one can find (the argument t is omitted for simplicity) that

$$\frac{d}{dt}e^{\mathcal{A}} = \frac{d}{dt}\left(\mathcal{I} + \sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{A}^n\right) = \sum_{n=1}^{\infty}\frac{1}{n!}\frac{d}{dt}\mathcal{A}^n = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\mathcal{A}^k\mathcal{A}'\mathcal{A}^{n-1-k}$$

$$= \underbrace{\frac{1}{1!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^0}_{n=1;\quad k=0} + \underbrace{\frac{1}{2!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^1 + \frac{1}{2!}\mathcal{A}^1\mathcal{A}'\mathcal{A}^0}_{n=2;\quad k=0,1}$$

$$+ \underbrace{\frac{1}{3!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^2 + \frac{1}{3!}\mathcal{A}^1\mathcal{A}'\mathcal{A}^1 + \frac{1}{3!}\mathcal{A}^2\mathcal{A}'\mathcal{A}^0}_{n=3;\quad k=0,1,2} + \cdots = \begin{pmatrix}\text{regrouping all terms}\\ \text{of this series}\end{pmatrix}$$

$$= \underbrace{\frac{1}{1!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^0}_{n=1;\quad k=0} + \underbrace{\frac{1}{2!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^1}_{n=2;\quad k=0} + \underbrace{\frac{1}{3!}\mathcal{A}^0\mathcal{A}'\mathcal{A}^2}_{n=3;\quad k=0} + \cdots + \underbrace{\frac{1}{2!}\mathcal{A}^1\mathcal{A}'\mathcal{A}^0}_{n=2;\quad k=1} + \underbrace{\frac{1}{3!}\mathcal{A}^1\mathcal{A}'\mathcal{A}^1}_{n=3;\quad k=1} + \cdots$$

$$+ \underbrace{\frac{1}{3!}\mathcal{A}^2\mathcal{A}'\mathcal{A}^0}_{n=3;\quad k=3} + \cdots + \cdots = \mathcal{A}^0\mathcal{A}'\left(\frac{1}{1!}\mathcal{A}^0 + \frac{1}{2!}\mathcal{A}^1 + \frac{1}{3!}\mathcal{A}^2 + \cdots\right)$$

$$+ \mathcal{A}^1\mathcal{A}'\left(\frac{1}{2!}\mathcal{A}^0 + \frac{1}{3!}\mathcal{A}^1 + \cdots\right) + \mathcal{A}^2\mathcal{A}'\left(\frac{1}{3!}\mathcal{A}^0 + \cdots\right) + \cdots$$

$$= \sum_{k=0}^{\infty}\mathcal{A}^k\mathcal{A}'\underbrace{\sum_{n=k+1}^{\infty}\frac{1}{n!}\mathcal{A}^{k-1-n}}_{n\to n'=n-(k+1)} = \sum_{k=0}^{\infty}\mathcal{A}^k\mathcal{A}'\sum_{n'=0}^{\infty}\frac{1}{(n'+k+1)!}\mathcal{A}^{n'+k+1-k-1}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{(n+k+1)!}\mathcal{A}^k\mathcal{A}'\mathcal{A}^n = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{k!n!}{(n+k+1)!}\frac{\mathcal{A}^k}{k!}\mathcal{A}'\frac{\mathcal{A}^n}{n!}\ .$$

Now let us use the relation between the Beta–function $B(k,n)$ and Gamma–function $\Gamma(n)$ and its definition as the first type Euler integral:

$$\frac{k!n!}{(n+k+1)!} = \frac{\Gamma(k+1)\Gamma(n+1)}{\Gamma(n+k+2)} = B(k+1,n+1) = \int_0^1 \tau^k(1-\tau)^n d\tau\ ,$$

and then one can find the required result:

$$\frac{d}{dt}e^{\mathcal{A}} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{\mathcal{A}^k}{k!}\mathcal{A}'\frac{\mathcal{A}^n}{n!}\int_0^1 \tau^k(1-\tau)^n d\tau$$

$$= \int_0^1 \underbrace{\left(\sum_{k=0}^{\infty}\frac{\tau^k\mathcal{A}^k}{k!}\right)}_{e^{\tau\mathcal{A}}}\mathcal{A}'\underbrace{\left(\sum_{0}^{\infty}\frac{(1-\tau)^n\mathcal{A}^n}{n!}\right)}_{e^{(1-\tau)\mathcal{A}}}d\tau = \left(\int_0^1 e^{\tau\mathcal{A}}\mathcal{A}'e^{-\tau\mathcal{A}}d\tau\right)e^{\mathcal{A}}\ .$$

The proof of the second expression for the exponential derivative follows from the symmetry property of the Beta–function: $B(k,n) = B(n,k)$ and is very simple.

ADVANCED PHOTON SOURCE

Adjoint operators

Another important tool that is useful in Lie operator technique is an object $[\mathcal{A},\circ]$ referred to as an **adjoint operator.** Its defined as

$$[\mathcal{A},\circ]\mathcal{B} \stackrel{def.}{=} [\mathcal{A},\mathcal{B}]\ .$$

It means that **the adjoint to $\mathcal{B}$ operator that is generated by $\mathcal{A}$ is simply their commutator $[\mathcal{A},\mathcal{B}]$.**
For this reason adjoint operators satisfy all relations for commutators:

- **antisymmetry:** $[\mathcal{A},\circ]\mathcal{B} = -[\mathcal{B},\circ]\mathcal{A}$;
- **linearity:** $[\mathcal{A},\circ](b\mathcal{B}+c\mathcal{C}) = b[\mathcal{A},\circ]\mathcal{B} + c[\mathcal{A},\circ]\mathcal{C}$;
- **product rule:** $[\mathcal{A},\circ](\mathcal{B}\mathcal{C}) = ([\mathcal{A},\circ]\mathcal{B})\mathcal{C} + \mathcal{B}([\mathcal{A},\circ]\mathcal{C})$;
- **rule of Poisson brackets:**

$$[\mathcal{A},\circ][\mathcal{B},\mathcal{C}] = [[\mathcal{A},\circ]\mathcal{B},\mathcal{C}] + [\mathcal{B},[\mathcal{A},\circ]\mathcal{C}]\ .$$

Now one can define similarly the **exponential adjoint operator** $e^{[\mathcal{A},\circ]}$ as

$$e^{[\mathcal{A},\circ]} \stackrel{def.}{=} \sum_{k=0}^{\infty}\frac{1}{k!}[\mathcal{A},\circ]^k\ .$$

The power of the adjoint operator in this definition means the power of the normal operator. The properties of exponential adjoint operators result from the properties of normal exponentials:

- for each operator $\mathcal{A}$ the **inverse operator $\mathcal{B}$ exists:**

$$\mathcal{B}\cdot e^{[\mathcal{A},\circ]} = e^{[\mathcal{A},\circ]}\cdot\mathcal{B} \equiv \mathcal{I}$$

  and in this case

$$\mathcal{B} = \left(e^{[\mathcal{A},\circ]}\right)^{-1} = e^{-[\mathcal{A},\circ]}\ ,$$

- **linearity:** $e^{[\mathcal{A},\circ]}(\beta\mathcal{B}+\gamma\mathcal{C}) = \beta e^{[\mathcal{A},\circ]}\mathcal{B} + \gamma e^{[\mathcal{A},\circ]}\mathcal{C}$,
- **product rule:** $e^{[\mathcal{A},\circ]}(\mathcal{B}\mathcal{C}) = (e^{[\mathcal{A},\circ]}\mathcal{B})(e^{[\mathcal{A},\circ]}\mathcal{C})$,
- **rule of Poisson brackets:** $e^{[\mathcal{A},\circ]}[\mathcal{B},\mathcal{C}] = [e^{[\mathcal{A},\circ]}\mathcal{B}, e^{[\mathcal{A},\circ]}\mathcal{C}]$.

ADVANCED PHOTON SOURCE

Lie operators in mechanics (basic idea)

Physicists and astronomers have been using this approach for more than a century. The main idea is as follows. Let's say one has a **problem of the particle nonlinear motion,** i.e., he tries to solve the following equation

$$m\vec{r}'' = f(\vec{r},\vec{r}') \ .$$

If the Hamiltonian formalism is used, it **means that the Hamiltonian includes the terms of the third and higher orders**:

$$\mathcal{H} = h^{(2)}_{ij} X_i X_i + h^{(3)}_{ijk} X_i X_j X_k + \cdots \ .$$

Vector $X_i = (q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n)$, as is known, combines phase space variables. In the Hamiltonian approach the equations of particle motion have a very simple form:

$$X_i' = [\mathcal{H} X_i] \ . \tag{3}$$

Here, the **brackets mean the Poisson brackets** and the following definition is used:

$$[f, \circ]\; g \equiv [f, g] = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \ . \tag{4}$$

The operator in the left part of this expression is known as an adjoint operator. Taking into account this definition one can rewrite equation (3) in the following form:

$$X_i' = [\mathcal{H}, \circ]\, X_i \ ,$$

where **brackets mean the commutator and** at the same time **adjoint operator.** For this reason it is very "simple" to write the solution of this equation in terms of the initial vector $X^{(i)}$:

$$X^{(f)} = e^{-\left[\int_0^t \mathcal{H}(X,t')dt', \circ\right]} X^{(i)} \ .$$

It is possible to simplify this expression if the Hamiltonian does not depend on time explicitly:

$$X^{(f)} = e^{-t\,[\mathcal{H}(X), \circ]} X^{(i)} \ . \tag{5}$$

BINP

ADVANCED PHOTON SOURCE

Nevertheless, the simplicity of this expression is deceitful because it includes the exponential operator:

$$
\begin{aligned}
e^{-t\,[\mathcal{H}(X),\circ]} &= \sum_0^\infty \frac{(-t)^n}{n!}\,[\mathcal{H}(X),\circ]^n \\
&= 1 - \frac{t}{1!}\,[\mathcal{H}(X),\circ] + \frac{t^2}{2!}\,[[\mathcal{H}(X),\circ]\,,\circ] - \frac{t^3}{3!}\,[[[\mathcal{H}(X),\circ]\,,\circ]\,,\circ] + \cdots\,. \qquad (6)
\end{aligned}
$$

This expression **is a definition of the Lie transformation** and is known as an exponential adjoint operator.

Example (standard oscillator)

Let us consider a very simple Hamiltonian for one-dimensional particle motion $(X = (x,p))$:

$$\mathcal{H} = \frac{k}{2}(x^2 + p^2)\;.$$

Then the series (6) can be calculated explicitly:

$$
\begin{aligned}
[\mathcal{H},x]^0 &= 1\;, \\
[\mathcal{H},x]^1 &= \frac{\partial\mathcal{H}}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial\mathcal{H}}{\partial p}\frac{\partial x}{\partial x} = -kp\;, \\
[\mathcal{H},x]^2 &= -k\,[\mathcal{H},p] = -k\left(\frac{\partial\mathcal{H}}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial x}\right) = -k^2x\;, \\
[\mathcal{H},x]^3 &= -k^2\,[\mathcal{H},x] = k^2\left(\frac{\partial\mathcal{H}}{\partial x}\frac{\partial x}{\partial p} - \frac{\partial\mathcal{H}}{\partial p}\frac{\partial x}{\partial x}\right) = k^3p\;, \\
[\mathcal{H},x]^4 &= k^3\,[\mathcal{H},p] = k^3\left(\frac{\partial\mathcal{H}}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial\mathcal{H}}{\partial p}\frac{\partial p}{\partial x}\right) = k^4x\;, \\
&\cdots\qquad .
\end{aligned}
$$

Since $[\mathcal{H},x]^{2n} = (-1)^n k^{2n} x$ and $[\mathcal{H},x]^{2n+1} = (-1)^{n+1}k^{2n+1}p$, one can obtain

$$
\begin{aligned}
\mathbf{e^{-t\,[\mathcal{H},x]}} &= \sum_0^\infty \frac{(-t)^n}{n!}\,[\mathcal{H},x]^n = \sum_0^\infty \frac{(-t)^{2n}}{(2n)!}\,[\mathcal{H},x]^{2n} + \sum_0^\infty \frac{(-t)^{2n+1}}{(2n+1)!}\,[\mathcal{H},x]^{2n+1} \\
&= x\sum_0^\infty \frac{(-t)^{2n}}{(2n)!}k^{2n} + p\sum_0^\infty \frac{(-t)^{2n+1}}{(2n+1)!}(-1)^{n+1}k^{2n+1} \\
&= \mathbf{x}\cos\mathbf{kt} + \mathbf{p}\sin\mathbf{kt}\;.
\end{aligned}
$$

BINP

Similarly, one can find that

$$
\begin{aligned}
\mathbf{e}^{-\mathbf{t}\,[\mathcal{H},\mathbf{p}]} &= \sum_0^\infty \frac{(-t)^n}{n!}\,[\mathcal{H},p]^n \\
&= \sum_0^\infty \frac{(-t)^{2n}}{(2n)!}\,[\mathcal{H},p]^{2n} + \sum_0^\infty \frac{(-t)^{2n+1}}{(2n+1)!}\,[\mathcal{H},p]^{2n+1} \\
&= p\sum_0^\infty \frac{(-t)^{2n}}{(2n)!}k^{2n} + x\sum_0^\infty \frac{(-t)^{2n+1}}{(2n+1)!}(-1)^n k^{2n+1} \\
&= -\mathbf{x}\sin\mathbf{kt} + \mathbf{p}\cos\mathbf{kt}\ .
\end{aligned}
$$

So, in this case the expression (5) takes the well-known matrix form:

$$
X_k^{(f)} = \mathcal{M}_{kl} X_l^{(i)}\ , \quad \textbf{where} \quad \mathcal{M} = \begin{pmatrix} \cos kt & \sin kt \\ -\sin kt & \cos kt \end{pmatrix}\ . \tag{7}
$$

Let us note a very interesting and important result:

> **If the Hamiltonian includes second powers of the phase variables, then action of any power of operator $[\mathcal{H},x]$ on some power of these variables will include the same power of the phase variables.**

It is easy to see that this is true due to the property of the Poisson brackets to keep the powers of phase variables when the Hamiltonian includes their second powers only (see also later).

Lie operator properties (continued)

9°. One can find a very important **similarity** property of the exponential operators:

$$e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}} = e^{[\mathcal{A},\circ]}\mathcal{B}\ .$$

To prove this expression, let us introduce an operator $\mathcal{F}(t) = e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}}$ of parameter t. For this operator $\mathcal{F}(0) = \mathcal{B}$. The derivative of operator $\mathcal{F}(t)$ with respect to the independent parameter t gives:

$$\begin{aligned}\mathcal{F}'(t) &= \mathcal{A}e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}} - e^{t\mathcal{A}}\mathcal{B}\underbrace{\mathcal{A}e^{-t\mathcal{A}}}_{e^{-t\mathcal{A}}\mathcal{A}} = \mathcal{A}e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}} - e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}}\mathcal{A} \\ &= \left[\mathcal{A}, e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}}\right] = [\mathcal{A}, \mathcal{F}(t)] = [\mathcal{A},\circ]\mathcal{F}(t).\end{aligned}$$

So, the operator $\mathcal{F}(t)$ satisfies the differential equation $\mathcal{F}'(t) = [\mathcal{A},\circ]\mathcal{F}(t)$ with the initial condition $\mathcal{F}(0) = \mathcal{B}$. Further, let us consider the operator $\mathcal{G}(t) = e^{t[\mathcal{A},\circ]}\mathcal{B}$ with the same initial condition: $\mathcal{G}(0) = \mathcal{B}$. The derivative of this operator with respect to parameter t equals $\mathcal{G}'(t) = [\mathcal{A},\circ]e^{t[\mathcal{A},\circ]}\mathcal{B} = [\mathcal{A},\circ]\mathcal{G}(t)$. Since this operator satisfies the same equation with the same initial condition, the operators $\mathcal{F}(t)$ and $\mathcal{G}(t)$ equal each other for any t. Then for $t = 1$ one can find the required expression:

$$e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}} = e^{[\mathcal{A},\circ]}\mathcal{B}\ .$$

10°. The property of the similarity allows us to find **two other expressions for derivatives of the exponentials** using the exponential adjoint operators. In fact,

$$\frac{d}{dt}e^{\mathcal{A}(t)} = \frac{e^{[\mathcal{A}(t),\circ]} - \mathcal{I}}{[\mathcal{A}(t),\circ]}\mathcal{A}'(t)e^{\mathcal{A}(t)} = e^{\mathcal{A}(t)}\frac{\mathcal{I} - e^{-[\mathcal{A}(t),\circ]}}{[\mathcal{A}(t),\circ]}\mathcal{A}'(t)\ .$$

The proof is a very easy one. For the first expression one has, using formula (2), that

$$\begin{aligned}\frac{d}{dt}e^{\mathcal{A}(t)} &= \left(\int_0^1 \underbrace{e^{\tau\mathcal{A}(t)}\mathcal{A}'(t)e^{-\tau\mathcal{A}(t)}}_{e^{\tau[\mathcal{A}(t),\circ]}\mathcal{A}'(t)}\,d\tau\right) e^{\mathcal{A}(t)} = \left(\int_0^1 e^{\tau[\mathcal{A}(t),\circ]}\mathcal{A}'(t)d\tau\right) e^{\mathcal{A}(t)} \\ &= \left(\int_0^1 e^{\tau[\mathcal{A}(t),\circ]}d\tau\right)\mathcal{A}'(t)e^{\mathcal{A}(t)} = \frac{e^{[\mathcal{A}(t),\circ]} - \mathcal{I}}{[\mathcal{A}(t),\circ]}\mathcal{A}'(t)e^{\mathcal{A}(t)}\ .\end{aligned}$$

For the second expression one can obtain the proof in a similar way.

11°. Let us find the **logarithmic derivatives** of the multiplication of two exponential operators: $\mathcal{F}(t) = e^{t\mathcal{A}}e^{t\mathcal{B}}$. These derivatives have two forms. For the first of them one can obtain that

$$\begin{aligned}
\mathcal{F}'(t) &= \left(e^{t\mathcal{A}}e^{t\mathcal{B}}\right)' = \mathcal{A}\,\underbrace{e^{t\mathcal{A}}e^{t\mathcal{B}}}_{\mathcal{F}(t)} + e^{t\mathcal{A}}\mathcal{B}e^{t\mathcal{B}} \\
&= \mathcal{A}\mathcal{F}(t) + \underbrace{e^{t\mathcal{A}}\mathcal{B}e^{-t\mathcal{A}}}_{e^{[\mathcal{A}(t),\circ]}\mathcal{B}}\underbrace{e^{t\mathcal{A}}e^{t\mathcal{B}}}_{\mathcal{F}(t)} = \left(\mathcal{A} + e^{[\mathcal{A}(t),\circ]}\mathcal{B}\right)\mathcal{F}(t) \\
\longrightarrow \quad & \mathcal{F}'(\mathbf{t})\mathcal{F}^{-\mathbf{1}}(\mathbf{t}) = \mathcal{A} + \mathbf{e}^{[\mathcal{A}(\mathbf{t}),\circ]}\mathcal{B}\ .
\end{aligned}$$

For the second form one has the similar result:

$$\begin{aligned}
\mathcal{F}'(t) &= \mathcal{A}e^{t\mathcal{A}}e^{t\mathcal{B}} + e^{t\mathcal{A}}\mathcal{B}e^{t\mathcal{B}} = e^{t\mathcal{A}}\underbrace{\left(e^{t\mathcal{B}}e^{-t\mathcal{B}}\right)}_{\mathcal{I}}\mathcal{A}e^{t\mathcal{B}} + e^{t\mathcal{A}}\underbrace{\left(e^{t\mathcal{B}}e^{-t\mathcal{B}}\right)}_{\mathcal{I}}\mathcal{B}e^{t\mathcal{B}} \\
&= \underbrace{e^{t\mathcal{A}}e^{t\mathcal{B}}}_{\mathcal{F}(t)}\underbrace{e^{-t\mathcal{B}}\mathcal{A}e^{t\mathcal{B}}}_{e^{-[\mathcal{B},\circ]}\mathcal{A}} + \underbrace{e^{t\mathcal{A}}e^{t\mathcal{B}}}_{\mathcal{F}(t)}e^{-t\mathcal{B}}\mathcal{B}e^{t\mathcal{B}} \\
&= \mathcal{F}(t)\left(e^{-[\mathcal{B},\circ]}\mathcal{A} + \mathcal{B}\underbrace{\left(e^{-\mathcal{B}}e^{\mathcal{B}}\right)}_{\mathcal{I}}\right) = \mathcal{F}(t)\left(e^{-[\mathcal{B},\circ]}\mathcal{A} + \mathcal{B}\right) \\
\longrightarrow \quad & \mathcal{F}^{-\mathbf{1}}(\mathbf{t})\mathcal{F}'(\mathbf{t}) = \mathbf{e}^{-[\mathcal{B},\circ]}\mathcal{A} + \mathcal{B}\ .
\end{aligned}$$

12°. Finally, let us discuss the **composition** property of the functions of the operators. This is a very important property and is as follows:

$$e^{[\mathcal{A},\circ]}f(\mathcal{B}) = f(e^{[\mathcal{A},\circ]}\mathcal{B})\ .$$

Naturally the functions that can be represented as a power series over their argument are used:

$$f(\mathcal{B}) = \sum_{k=0}^{\infty} a_k\mathcal{B}^k\ .$$

Then using the product rule for exponential adjoint operators, one has the desired result:

$$e^{[\mathcal{A},\circ]}f(\mathcal{B}) = e^{[\mathcal{A},\circ]}\sum_{k=0}^{\infty} a_k\mathcal{B}^k = \sum_{k=0}^{\infty} a_k e^{[\mathcal{A},\circ]}\mathcal{B}^k = \sum_{k=0}^{\infty} a_k\left(e^{[\mathcal{A},\circ]}\mathcal{B}\right)^k = f(e^{[\mathcal{A},\circ]}\mathcal{B})\ .$$

Baker–Campbell-Hausdorff theorem

This is the most important property of the exponential operators. So, **for any operators $\mathcal{A}$ and $\mathcal{B}$ the following expressions are valid:**

$$e^{\mathcal{A}+\mathcal{B}} = e^{\mathcal{A}} e^{\mathcal{B}} e^{\mathcal{C}_2} e^{\mathcal{C}_3} e^{\mathcal{C}_4} \cdots = e^{\mathcal{A}} e^{\mathcal{B}} \left(\prod_{k=2}^{\infty} e^{\mathcal{C}_k} \right)$$

and

$$e^{\mathcal{A}+\mathcal{B}} = \cdots e^{-\mathcal{C}_4} e^{+\mathcal{C}_3} e^{-\mathcal{C}_2} e^{\mathcal{A}} e^{\mathcal{B}} = \left(\prod_{\infty}^{k=2} e^{(-1)^k \mathcal{C}_k} \right) e^{\mathcal{A}} e^{\mathcal{B}} \ ,$$

where

$$\left.\begin{aligned}
\mathcal{C}_2 &= -\frac{1}{2}[\mathcal{A},\mathcal{B}] \ , \\
\mathcal{C}_3 &= \frac{1}{6}[\mathcal{A},[\mathcal{A},\mathcal{B}]] + \frac{1}{3}[\mathcal{B},[\mathcal{A},\mathcal{B}]] \ , \\
\mathcal{C}_4 &= -\frac{1}{24}[\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] - \frac{1}{8}[\mathcal{B},[(\mathcal{A}+\mathcal{B}),[\mathcal{A},\mathcal{B}]]] \ , \\
&\cdots \quad .
\end{aligned}\right\}$$

The inverse theorem gives the following expression:

$$e^{\mathcal{A}} e^{\mathcal{B}} = e^{\mathcal{A}+\mathcal{B}+\mathcal{D}_2+\mathcal{D}_3+\cdots} = \exp\left(\mathcal{A} + \mathcal{B} + \sum_{k=2}^{\infty} \mathcal{D}_k \right) \ ,$$

where

$$\left.\begin{aligned}
\mathcal{D}_2 &= \frac{1}{2}[\mathcal{A},\mathcal{B}] \ , \\
\mathcal{D}_3 &= \frac{1}{12}[(\mathcal{A}-\mathcal{B}),[\mathcal{A},\mathcal{B}]] \ , \\
\mathcal{D}_4 &= \frac{1}{24}[\mathcal{B},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] \ , \\
&\cdots \quad .
\end{aligned}\right\}$$

The advantage of this theorem is due to several circumstances. The most important of them are as follows:

- **Only different commutators of operators $\mathcal{A}$ and $\mathcal{B}$ are included into each $\mathcal{C}_k$ and $\mathcal{D}_k$.** Moreover, the powers of these commutators equal k exactly.

- **The theorem shows how it is necessary to truncate the exponential operator of the sum of two operators** when the presentation of this exponential operator is used in the form of the multiplication of the exponential operators.

It is possible to represent the result of the Baker–Campbell-Hausdorff theorem in other form using adjoint operators:

$$e^{\mathcal{A}} e^{\mathcal{B}} = e^{\mathcal{C}(\mathcal{A},\mathcal{B})}, \qquad \textbf{where} \qquad \mathcal{C}(\mathcal{A},\mathcal{B}) = \sum_{k=0}^{\infty} \mathcal{C}_k(\mathcal{A},\mathcal{B})$$

and

$$\begin{aligned}
\mathcal{C}_0 &= \mathcal{A}\,; \\
\mathcal{C}_1 &= \frac{[\mathcal{A},\circ]}{\mathcal{I} - e^{-[\mathcal{A},\circ]}}\mathcal{B} = \mathcal{B} + \frac{1}{2}[\mathcal{A},\mathcal{B}] + \frac{1}{12}[\mathcal{A},[\mathcal{A},\mathcal{B}]] + \cdots\,; \\
\mathcal{C}_2 &= -\frac{1}{2}\frac{[\mathcal{A},\circ]}{\mathcal{I} - e^{-[\mathcal{A},\circ]}} \int_0^1 [\mathcal{S}_1(\tau), \mathcal{S}_2(\tau)]\, d\tau = -\frac{1}{12}[\mathcal{B},[\mathcal{A},\mathcal{B}]] + \cdots, \\
& \quad \text{where} \quad \mathcal{S}_1(\tau) = \frac{\mathcal{I} - e^{-\tau[\mathcal{A},\circ]}}{[\mathcal{A},\circ]}\mathcal{C}_1 \text{ and } \mathcal{S}_2(\tau) = e^{-\tau[\mathcal{A},\circ]}\mathcal{C}_1\,; \\
\cdots & \qquad .
\end{aligned}$$

One can see that **in each $\mathcal{C}_k$ only the power k of the operator $\mathcal{B}$ is involved** through different commutators between operators $\mathcal{A}$ and $\mathcal{B}$.

Let us restrict by proof of the direct statement of the Baker–Campbell-Hausdorff theorem only. The lazy reader knows what he can do.
Let us consider two operators $\mathcal{F}(t)$ and $\mathcal{G}(t)$:

$$\mathcal{F}(t) = e^{-t\mathcal{B}}e^{-t\mathcal{A}}e^{t(\mathcal{A}+\mathcal{B})} \quad \text{and} \quad \mathcal{G}(t) = e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}e^{t^4\mathcal{C}_4}\cdots = \prod_{k=2}^{\infty}\exp\left(t^k\mathcal{C}_k\right).$$

Let us fit operators $\mathcal{C}_k$ so that $\mathcal{F} = \mathcal{G}$. Then for $t = 1$, one can find the desired expression. Let us compare the logarithmic derivatives for both operators for arbitrary t. So,

$$\mathcal{G}' = 2t\left(\mathcal{C}_2 e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}e^{t^4\mathcal{C}_4}\cdots\right) + 3t^2\left(e^{t^2\mathcal{C}_2}\mathcal{C}_3e^{t^3\mathcal{C}_3}e^{t^4\mathcal{C}_4}\cdots\right) + 4t^3\left(e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}\mathcal{C}_4e^{t^4\mathcal{C}_4}\cdots\right) + \cdots.$$

But $\mathcal{G}^{-1} = \cdots e^{-t^4\mathcal{C}_4}e^{-t^3\mathcal{C}_3}e^{-t^2\mathcal{C}_2} = \prod_{\infty}^{k=2}\exp\left(-t^k\mathcal{C}_k\right)$, then (for simplicity the terms with orders less than 4 are kept)

$$\begin{aligned}
\mathcal{G}'\mathcal{G}^{-1} &= \Big[2t\left(\mathcal{C}_2 e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}e^{t^4\mathcal{C}_4}\cdots\right) + 3t^2\left(e^{t^2\mathcal{C}_2}\mathcal{C}_3e^{t^3\mathcal{C}_3}e^{t^4\mathcal{C}_4}\cdots\right) \\
&+ 4t^3\left(e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}\mathcal{C}_4e^{t^4\mathcal{C}_4}\cdots\right) + \cdots\Big]\cdot\left(\cdots e^{-t^4\mathcal{C}_4}e^{-t^3\mathcal{C}_3}e^{-t^2\mathcal{C}_2}\right) \\
&\simeq 2t\mathcal{C}_2 + 3t^2e^{t^2\mathcal{C}_2}\mathcal{C}_3e^{-t^2\mathcal{C}_2} + 4t^3e^{t^2\mathcal{C}_2}e^{t^3\mathcal{C}_3}\mathcal{C}_4e^{-t^3\mathcal{C}_3}e^{-t^2\mathcal{C}_2} \\
&= 2t\mathcal{C}_2 + 3t^2e^{t^2[\mathcal{C}_2,\circ]}\mathcal{C}_3 + 4t^3e^{t^2[\mathcal{C}_2,\circ]}e^{t^3[\mathcal{C}_3,\circ]}\mathcal{C}_4 + \cdots \\
&= 2t\mathcal{C}_2 + 3t^2\sum_{k=0}^{\infty}\frac{t^{2k}[\mathcal{C}_2,\circ]^k}{k!}\mathcal{C}_3 + 4t^3\sum_{k=0}^{\infty}\frac{t^{2k}[\mathcal{C}_2,\circ]^k}{k!}\sum_{n=0}^{\infty}\frac{t^{3n}[\mathcal{C}_3,\circ]^n}{n!}\mathcal{C}_4 + \cdots \\
&= 2t\mathcal{C}_2 + 3t^2\left(\mathcal{I} + t^2[\mathcal{C}_2,\circ] + \cdots\right)\mathcal{C}_3 + 4t^3\left(\mathcal{I} + t^2[\mathcal{C}_2,\circ] + \cdots\right)\left(\mathcal{I} + t^3[\mathcal{C}_3,\circ] + \cdots\right)\mathcal{C}_4 + \cdots \\
&= 2t\mathcal{C}_2 + 3t^2\mathcal{C}_3 + 4t^3\mathcal{C}_4 + \cdots.
\end{aligned}$$

For another side

$$\mathcal{F}' = -\mathcal{B}e^{-t\mathcal{B}}e^{-t\mathcal{A}}e^{t(\mathcal{A}+\mathcal{B})} - e^{-t\mathcal{B}}\mathcal{A}e^{-t\mathcal{A}}e^{t(\mathcal{A}+\mathcal{B})} + e^{-t\mathcal{B}}e^{-t\mathcal{A}}(\mathcal{A}+\mathcal{B})e^{t(\mathcal{A}+\mathcal{B})}$$

and $\mathcal{F}^{-1} = e^{-t(\mathcal{A}+\mathcal{B})}e^{t\mathcal{A}}e^{t\mathcal{B}}$. Then

$$\begin{aligned}
\mathcal{F}'\mathcal{F}^{-1} &= \left(-\mathcal{B}e^{-t\mathcal{B}}e^{-t\mathcal{A}}e^{t(\mathcal{A}+\mathcal{B})} - e^{-t\mathcal{B}}\mathcal{A}e^{-t\mathcal{A}}e^{t(\mathcal{A}+\mathcal{B})} + e^{-t\mathcal{B}}e^{-t\mathcal{A}}(\mathcal{A}+\mathcal{B})e^{t(\mathcal{A}+\mathcal{B})}\right)e^{-t(\mathcal{A}+\mathcal{B})}e^{t\mathcal{A}}e^{t\mathcal{B}} \\
&= -\mathcal{B} - e^{-t\mathcal{B}}\mathcal{A}e^{t\mathcal{B}} + e^{-t\mathcal{B}}e^{-t\mathcal{A}}(\mathcal{A}+\mathcal{B})e^{t\mathcal{A}}e^{t\mathcal{B}} \simeq -\mathcal{B} - e^{-t[\mathcal{B},\circ]}\mathcal{A} + e^{-t[\mathcal{B},\circ]}e^{-t[\mathcal{A},\circ]}(\mathcal{A}+\mathcal{B}) \\
&= \mathcal{B} - \left(\mathcal{I} - t[\mathcal{B},\circ] + \frac{t^2}{2}[\mathcal{B},\circ]^2 - \frac{t^3}{6}[\mathcal{B},\circ]^3 + \cdots\right)\mathcal{A} \\
&+ \left(\mathcal{I} - t[\mathcal{B},\circ] + \frac{t^2}{2}[\mathcal{B},\circ]^2 - \frac{t^3}{6}[\mathcal{B},\circ]^3 + \cdots\right)\left(\mathcal{I} - t[\mathcal{A},\circ] + \frac{t^2}{2}[\mathcal{A},\circ]^2 - \frac{t^3}{6}[\mathcal{A},\circ]^3 + \cdots\right)(\mathcal{A}+\mathcal{B})
\end{aligned}$$

$$
\begin{aligned}
&= -\mathcal{B} - \mathcal{A} + t[\mathcal{B},\mathcal{A}] - \frac{t^2}{2}[\mathcal{B},[\mathcal{B},\mathcal{A}]] + \frac{t^3}{6}[\mathcal{B},[\mathcal{B},[\mathcal{B},\mathcal{A}]]] - \cdots \\
&+ \left(\mathcal{I} - t[\mathcal{B},\circ] + \frac{t^2}{2}[\mathcal{B},\circ]^2 - \frac{t^3}{6}[\mathcal{B},\circ]^3 + \cdots\right) \\
&\cdot \left(\mathcal{A}+\mathcal{B} - t[\mathcal{A},(\mathcal{A}+\mathcal{B})] + \frac{t^2}{2}[\mathcal{A},[\mathcal{A},(\mathcal{A}+\mathcal{B})]] - \frac{t^3}{6}[\mathcal{A},[\mathcal{A},[\mathcal{A},(\mathcal{A}+\mathcal{B})]]] + \cdots\right) \\
&= -\mathcal{B} - \mathcal{A} + t[\mathcal{B},\mathcal{A}] - \frac{t^2}{2}[\mathcal{B},[\mathcal{B},\mathcal{A}]] + \frac{t^3}{6}[\mathcal{B},[\mathcal{B},[\mathcal{B},\mathcal{A}]]] - \cdots \\
&+ \left(\mathcal{I} - t[\mathcal{B},\circ] + \frac{t^2}{2}[\mathcal{B},\circ]^2 - \frac{t^3}{6}[\mathcal{B},\circ]^3 + \cdots\right) \\
&\cdot \left(\mathcal{A}+\mathcal{B} - t[\mathcal{A},\mathcal{B}] + \frac{t^2}{2}[\mathcal{A},[\mathcal{A},\mathcal{B}]] - \frac{t^3}{6}[\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] + \cdots\right) \\
&= -\mathcal{B} - \mathcal{A} + t[\mathcal{B},\mathcal{A}] - \frac{t^2}{2}[\mathcal{B},[\mathcal{B},\mathcal{A}]] + \frac{t^3}{6}[\mathcal{B},[\mathcal{B},[\mathcal{B},\mathcal{A}]]] - \cdots \\
&+ \mathcal{A}+\mathcal{B} - t[\mathcal{A},\mathcal{B}] + \frac{t^2}{2}[\mathcal{A},[\mathcal{A},\mathcal{B}]] - \frac{t^3}{6}[\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] + \cdots \\
&- t[\mathcal{B},\mathcal{A}] + t^2[\mathcal{B},[\mathcal{A},\mathcal{B}]] - \frac{t^3}{2}[\mathcal{B},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] + \cdots \\
&+ \frac{t^2}{2}[\mathcal{B},[\mathcal{B},\mathcal{A}]] - \frac{t^3}{2}[\mathcal{B},[\mathcal{B},[\mathcal{A},\mathcal{B}]]] + \cdots - \frac{t^3}{6}[\mathcal{B},[\mathcal{B},[\mathcal{B},\mathcal{A}]]] + \cdots \\
&= -t[\mathcal{A},\mathcal{B}] + \frac{t^2}{2}[\mathcal{A},[\mathcal{A},\mathcal{B}]] - \frac{t^3}{6}[\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] + t^2[\mathcal{B},[\mathcal{A},\mathcal{B}]] \\
&- \frac{t^3}{2}[\mathcal{B},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] - \frac{t^3}{2}[\mathcal{B},[\mathcal{B},[\mathcal{A},\mathcal{B}]]].
\end{aligned}
$$

Comparison of terms with the same powers of t on the right sides of expressions for $\mathcal{G}'\mathcal{G}^{-1}$ and $\mathcal{F}'\mathcal{F}^{-1}$ gives the desired expressions:

$$
\begin{aligned}
\mathcal{C}_2 &= -\frac{1}{2}[\mathcal{A},\mathcal{B}]\ , \\
\mathcal{C}_3 &= \frac{1}{6}[\mathcal{A},[\mathcal{A},\mathcal{B}]] + \frac{1}{3}[\mathcal{B},[\mathcal{A},\mathcal{B}]]\ , \\
\mathcal{C}_4 &= -\frac{1}{24}[\mathcal{A},[\mathcal{A},[\mathcal{A},\mathcal{B}]]] - \frac{1}{8}[\mathcal{B},[(\mathcal{A}+\mathcal{B}),[\mathcal{A},\mathcal{B}]]]\ , \\
\cdots & \quad .
\end{aligned}
$$

Lie operators in accelerator physics (basic idea)

To explain how Lie operators are used in practice let us repeat the previous expression (7):

$$X^{(f)} = \mathcal{M} X^{(i)} \,, \quad \text{if} \quad \mathcal{H} = \frac{k}{2}(x^2 + p^2) \equiv h_2 = h_{ij}^{(2)} X_i X_j \,.$$

Here the second-order polynomial h_2 and its coefficients $h_{ij}^{(2)}$ were introduced to describe the Hamiltonian $\mathcal{H}$ as the mathematical function, and the coordinate vector X includes only two phase variables: $X = (x, p)$. In the language of Lie operators this expression has the following form:

$$X^{(f)} = e^{-t[\mathcal{H},\,\circ]} X^{(i)} = e^{-t[h_2,\,\circ]} X^{(i)} = \mathcal{M} X^{(i)} \quad \longrightarrow \quad \mathbf{e}^{-\mathbf{t}[\mathcal{H},\,\circ]} = \mathcal{M} \,.$$

It is necessary to note that **it is a non-trivial result.** In particular, it is true since one can find from Lie operator theory that the **exponential Lie operator guarantees the symplecticity** of the transformation from the initial vector $X^{(i)}$ to the final vector $X^{(f)}$.

Let us try to take into account the nonlinearity of motion. It means that the Hamiltonian should include the polynomial of the third order at minimum. Therefore **it is necessary to calculate** a more complicated exponential operator:

$$e^{-t([h_2,\,\circ] \,+\, [h_3,\,\circ])}$$

and the definition of the polynomial h_3 is analogous to the polynomial h_2. **What else is necessary now?**

It is necessary:

- to use the Baker–Campbell–Hausdorff theorem to present this expression as a product of exponential operators $e^{-t[h_2,\,\circ]}$, $e^{-t[h_3,\,\circ]}$ and their commutators, at minimum $e^{-t\mathcal{C}_2}$, where $\mathcal{C}_2 = \frac{1}{2}[[h_2, \circ], [h_3, \circ]]$. Recall that the operator $e^{-t[h_2,\,\circ]}$ is a simple matrix $\mathcal{M}$;

- the operator $e^{-t[h_3,\,\circ]}$ must be expanded into the power series, and only a few its first terms must be kept:

  $$e^{-t[h_3,\circ]} = \mathcal{I} - t[h_3, \circ] + \frac{t^2}{2}[h_3, \circ]^2 + \cdots \; .$$

  How many terms it is necessary to keep? Only two if one would like to restrict oneself to terms with powers of the phase variables less than four;

- make the same with the exponential $e^{-t\mathcal{C}_2}$;

- to merge all products of "simple" operators into the corresponding unified operator.

At last one is ready to evaluate the nonlinear motion using the Lie operator technique. But this is a scheme only, and it is necessary to pass a long way to realize this scheme!
So, **the Lie operator technique is a method that allows us to take into account higher orders of the Hamiltonian terms** and calculate the series (6).

Symplectic transformations

Let us discuss now in more detail the property of **symplecticity.** The relation between initial and final variables of a dynamical system has the following form of the power series:

$$X_i^{(f)} = \mathcal{K}_i + \mathcal{R}_{ij}X_j^{(i)} + \mathcal{T}_{ijk}X_j^{(i)}Z_k^{(i)} + \mathcal{U}_{ijkl}X_j^{(i)}X_k^{(i)}X_l^{(i)} + \cdots \; .$$

This expression describes the transformation of the initial variables into final form and will be called **transition transformation** or **mapping.** This transformation is characterized by the **Jacobian matrix $\mathbf{M}_{ij}$,** which is defined by the equation

ADVANCED PHOTON SOURCE

$$\mathbf{M}_{ij}(\vec{X}^{(i)}) = \frac{\partial X_i^{(f)}}{\partial X_j^{(i)}} \, .$$

It is clear that in the general case the Jacobian depends on the particle trajectory. That is expressed by the designation $\mathbf{M}_{ij}(\vec{X}^{(i)})$.
Let us introduce a special "unit" matrix $\mathcal{J}$ of the following type:

$$\mathcal{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix} , \quad \text{where} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

For the Hamiltonian equations of motion the Jacobian has the following property:

$$\tilde{\mathbf{M}}(\vec{X}) \, \mathcal{J} \, \mathbf{M}(\vec{X}) = \mathcal{J} \, . \tag{8}$$

Here $\tilde{\mathbf{M}}$ denotes the transpose of $\mathbf{M}$. **Equation (8)** is the condition of the symplecticity for the matrix $\mathbf{M}$. It is necessary to emphasize some items. Firstly, this matrix depends on the vector $\vec{X}$, but for the symplectic transformation the product $\tilde{\mathbf{M}} \, \mathcal{J} \, \mathbf{M}$ does not depend on that. For this reason the relations between coefficients $\mathcal{R}$, $\mathcal{T}$, $\mathcal{U}$, ... must exist for these transformations. Secondly, equation (8) is nonlinear, since the relations between $\mathcal{R}$, $\mathcal{T}$, $\mathcal{U}$, ... are nonlinear as well.
Let us enumerate some properties of the symplectic maps:

- for any symplectic matrix $\mathbf{M}$ the **inverse matrix** $\mathbf{M}^{-1}$ **exists** and $\mathbf{M}^{-1} = -\mathcal{J} \, \tilde{\mathbf{M}} \, \mathcal{J} = \mathcal{J}^{-1} \, \tilde{\mathbf{M}} \, \mathcal{J}$. This **inverse matrix is symplectic** as well, and one can find that $\mathbf{M} \, \mathcal{J} \, \tilde{\mathbf{M}} = \mathcal{J}$;

- **the product of the symplectic matrices is the symplectic matrix;**

- **the determinant of the symplectic matrix M equals** ± 1 ;

- **the symplectic matrix has no zero eigenvalues.** Moreover, if λ is an eigenvalue of the symplectic matrix, then λ^{-1} is its eigenvalue as well. It means that the characteristic polynomial $P(\lambda)$ of the symplectic $n \otimes n$ matrix $\mathbf{M}$, which equals $\det(\mathbf{M} - \lambda \mathbf{I})$, has a property $P(\lambda) = \lambda^{2n} P(1/\lambda)$.

BINP

Poisson brackets and Lie operators

It is very simple to verify that **functions of phase variables in physics are united into the linear algebra.** For these functions the **Poisson brackets** (4) were defined; this means that **one can consider** them **as the adjoint operator** $[f, \circ]$. As is known from analytical mechanics, Poisson brackets have the properties of **antisymmetry**, **linearity**, and satisfy a **product rule** $[f, gh] = [f, g]h + [f, h]g$ and **Jacobi condition** $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$. These properties mean that **Poisson brackets is a Lie operator** for the functions of phase variables. More narrowly, let us use further the Lie operators that are only defined by the Hamiltonian of our physical system, i.e., $f = \mathcal{H}$. It may be useful to be reminded of their main properties:

- **linearity:** $[\mathcal{H}, \circ](ag + bh) = a[\mathcal{H}, \circ]g + b[\mathcal{H}, \circ]$;
- **differentiation rule:** $[\mathcal{H}, \circ](fg) = ([\mathcal{H}, \circ]f)g + ([\mathcal{H}, \circ]g)f$
- and its generalization for the power n (**Leibnitz rule**)**:** $[\mathcal{H}, \circ]^n(fg) = \sum_{k=0}^{n} C_k^n\Big([\mathcal{H}, \circ]^k f\Big)\Big([\mathcal{H}, \circ]^{n-k} g\Big)$;
- **"Jacobi identity":** $[\mathcal{H}, \circ][f, g] = [[\mathcal{H}, \circ]f, g] + [f, [\mathcal{H}, \circ]g]$;
- the **commutator of two Lie operators** $[\mathcal{H}_1, \circ]$ and $[\mathcal{H}_2, \circ]$, **is a Lie operator** as well:

  $$[[\mathcal{H}_1, \circ], [\mathcal{H}_2, \circ]] = [\mathcal{H}, \circ] \qquad \text{where} \qquad \mathcal{H} = [\mathcal{H}_1, \mathcal{H}_2]$$

  and $[,]$ means standard Poisson brackets for functions $\mathcal{H}_1$ and $\mathcal{H}_2$.

To prove this property let us use the Jacobi identity for Poisson brackets for functions $\mathcal{H}_1$ and $\mathcal{H}_2$ and for the arbitrary function f:

$$\begin{aligned}
([[\mathcal{H}_1, \circ], [\mathcal{H}_2, \circ]])\, f &= [\mathcal{H}_1, \circ][\mathcal{H}_2, \circ]f - [\mathcal{H}_2, \circ][\mathcal{H}_1, \circ]f \\
&= [\mathcal{H}_1, [\mathcal{H}_2, f]] - [\mathcal{H}_2, [\mathcal{H}_1, f]] \\
&= [\mathcal{H}_1, [\mathcal{H}_2, f]] + [\mathcal{H}_2, [f, \mathcal{H}_1]] \\
&= -[f, [\mathcal{H}_1, \mathcal{H}_2]] = [[\mathcal{H}_1, \mathcal{H}_2], f] \\
&= ([[\mathcal{H}_1, \mathcal{H}_2], \circ])\, f\ .
\end{aligned}$$

Lie transformations

This paragraph actually is a brief summary of previous material.
A Lie transformation is the exponential adjoint Lie operator (see Eq. (6)):

$$e^{[\mathcal{H},\circ]} \stackrel{def.}{=} \sum_{n=0}^{\infty} \frac{[\mathcal{H},\circ]^n}{n!} \equiv \mathcal{M}_{\mathcal{H}} \ .$$

Naturally $[\mathcal{H},\circ]^0 = \mathcal{I}$ is an identity operator and the Lie operator power $[\mathcal{H},\circ]^n$ is repetition of the n times calculations of the Poisson brackets, for example $[\mathcal{H},\circ]^2 = [\mathcal{H},[\mathcal{H},\circ]]$, etc. Therefore, one can write the Lie transformation $\mathcal{M}_{\mathcal{H}}$ in the following form:

$$\mathcal{M}_{\mathcal{H}} = \mathcal{I} + [\mathcal{H},\circ] + \frac{1}{2!}[\mathcal{H},[\mathcal{H},\circ]] + \frac{1}{3!}[\mathcal{H},[\mathcal{H},[\mathcal{H},\circ]]] \cdots \ .$$

Main properties of Lie transformations are as follows:

- **reversibility,** i.e., existence of the inverse transformation $\mathcal{M}_{\mathcal{H}}^{-1}$ and for this transformation
$$\mathcal{M}_{\mathcal{H}} \cdot \mathcal{M}_{\mathcal{H}}^{-1} = \mathcal{M}_{\mathcal{H}}^{-1} \cdot \mathcal{M}_{\mathcal{H}} = \mathcal{I} \text{ and } \mathcal{M}_{\mathcal{H}}^{-1} = e^{-[\mathcal{H},\circ]} \ ;$$
- **linearity:** $\mathcal{M}_{\mathcal{H}}(af + bg) = a\mathcal{M}_{\mathcal{H}}f + b\mathcal{M}_{\mathcal{H}}g$;
- **product rule:** $\mathcal{M}_{\mathcal{H}}(f \cdot g) = (\mathcal{M}_{\mathcal{H}}f) \cdot (\mathcal{M}_{\mathcal{H}}g)$;
- **rule of "Poisson brackets":** $\mathcal{M}_{\mathcal{H}}[f,g] = [\mathcal{M}_{\mathcal{H}}f, \mathcal{M}_{\mathcal{H}}g]$;
- **rule of the composition of the function of phase variables** $f(\vec{X})$:
$$\mathcal{M}_{\mathcal{H}}f(\vec{X}) = f(\mathcal{M}_{\mathcal{H}}\vec{X}) \ ;$$
- **rule of the composition of the operator function** $F(\mathcal{M}_{\mathcal{G}})$:
$$\mathcal{M}_{\mathcal{H}}f(\mathcal{M}_{\mathcal{G}}) = f(\mathcal{M}_{\mathcal{H}}\mathcal{M}_{\mathcal{G}}) \ ;$$
- **rules of the similarity for Lie transformation** $\mathcal{M}_{\mathcal{H}}$, function of the phase variables $f(\vec{X})$ and its Lie operator $[f,\circ]$:
$$\begin{aligned} \mathcal{M}_{\mathcal{H}} f \mathcal{M}_{\mathcal{H}}^{-1} &= \mathcal{M}_{\mathcal{H}} f \ , \\ \mathcal{M}_{\mathcal{H}}[f,\circ]\mathcal{M}_{\mathcal{H}}^{-1} &= [\mathcal{M}_{\mathcal{H}} f, \circ] \ . \end{aligned}$$

In fact, for arbitrary function g, taking into account the rule of the product one can find that

$$\begin{aligned}\left(\mathcal{M}_{\mathcal{H}}\left(f\mathcal{M}_{\mathcal{H}}^{-1}\right)\right)g &= \mathcal{M}_{\mathcal{H}}\left(f\cdot\mathcal{M}_{\mathcal{H}}^{-1}g\right)\\ &= \left(\mathcal{M}_{\mathcal{H}}f\right)\Big(\underbrace{\mathcal{M}_{\mathcal{H}}\mathcal{M}_{\mathcal{H}}^{-1}}_{\mathcal{I}}g\Big) = (\mathcal{M}_{\mathcal{H}}f)g\ .\end{aligned}$$

Taking into account the rule of Poisson brackets, one can similarly obtain:

$$\begin{aligned}\left(\mathcal{M}_{\mathcal{H}}[f,\circ]\mathcal{M}_{\mathcal{H}}^{-1}\right)g &= \mathcal{M}_{\mathcal{H}}\left([f,\circ]\left(\mathcal{M}_{\mathcal{H}}^{-1}g\right)\right)\\ &= \mathcal{M}_{\mathcal{H}}\left[f,\mathcal{M}_{\mathcal{H}}^{-1}g\right] = [\mathcal{M}_{\mathcal{H}}f,\underbrace{\mathcal{M}_{\mathcal{H}}\mathcal{M}_{\mathcal{H}}^{-1}}_{\mathcal{I}}g]\\ &= [\mathcal{M}_{\mathcal{H}}f,g] = \left([\mathcal{M}_{\mathcal{H}}f,\circ]\right)g\ ;\end{aligned}$$

- **rules of differentiation** when $\mathcal{H}$ is an explicit function of parameter t:

$$\begin{aligned}\frac{d\mathcal{M}_{\mathcal{H}}(t)}{dt} &= \frac{d}{dt}e^{[\mathcal{H}(t),\circ]}\\ &= \left(\int_0^1 e^{\tau[\mathcal{H}(t),\circ]}[\mathcal{H}'(t),\circ]e^{-\tau[\mathcal{H}(t),\circ]}d\tau\right)\mathcal{M}_{\mathcal{H}}(t)\\ &= \mathcal{M}_{\mathcal{H}}(t)\left(\int_0^1 e^{-\tau[\mathcal{H}(t),\circ]}[\mathcal{H}'(t),\circ]e^{\tau[\mathcal{H}(t),\circ]}d\tau\right)\ .\end{aligned}$$

It is possible to present this expression in two other forms:

$$\begin{aligned}\frac{d\mathcal{M}_{\mathcal{H}}(t)}{dt} &= \left[\frac{\mathcal{M}_{\mathcal{H}}(t)-\mathcal{I}}{[\mathcal{H}(t),\circ]}\mathcal{H}'(t),\circ\right]\mathcal{M}_{\mathcal{H}}(t)\\ &= \mathcal{M}_{\mathcal{H}}(t)\left[\frac{\mathcal{I}-\mathcal{M}_{\mathcal{H}}^{-1}(t)}{[\mathcal{H}(t),\circ]}\mathcal{H}'(t),\circ\right]\ .\end{aligned}$$

It is necessary to recall a previous remark regarding Lie transformations and symplectic matrices: it is possible to verify that

- **any symplectic matrix near identity is presented in the form of the corresponding Lie transformation**.

ADVANCED PHOTON SOURCE

Factored products

Let us represent the Hamiltonian of the particle motion as a sum of the polynomials h_2, h_3, ... of the phase variables X_i with the different powers:

$$\begin{aligned} \mathcal{H} &= h^{(2)}_{ij} X_i X_j + h^{(3)}_{ijk} X_i X_j X_k + h^{(4)}_{ijkl} X_i X_j X_k X_l + \cdots \equiv \\ &\equiv h_2 + h_3 + h_4 + \cdots . \end{aligned} \tag{9}$$

The solution of the equations of motion is simply Lie transformation of the initial vector $X^{(i)}$ to final vector $X^{(f)}$:

$$X^{(f)} = e^{-t[\mathcal{H}, \circ]} X^{(i)} = e^{-t[(h_2 + h_3 + h_4 + \cdots), \circ]} X^{(i)} . \tag{10}$$

To calculate this series in practice one can use the following **factored product expansion theorem:**

$$e^{[h_2 + h_3 + h_4 + \cdots, \circ]} = e^{[g_2, \circ]} e^{[g_3, \circ]} e^{[g_4, \circ]} \ \ldots , \tag{11}$$

where all polynomials g_k can be found recursively:

$$\begin{aligned} g_2 &= h_2, \\ g_3 &= \frac{\mathcal{I} - e^{-[h_2, \circ]}}{[h_2, \circ]} h_3, \\ g_4 &= \int_0^1 e^{-\tau [h_2, \circ]} \left(h_4 - \frac{1}{2} \left[\frac{e^{\tau [h_2, \circ]} - \mathcal{I}}{[h_2, \circ]} h_3, h_3 \right] \right) d\tau , \quad \text{etc.} \end{aligned}$$

Let us assume that

$$e^{t[h_2 + h_3 + h_4 + \cdots, \circ]} = e^{[f_2(t), \circ]} e^{[f_3(t), \circ]} e^{[f_4(t), \circ]} \ldots ,$$

then the logarithm derivative with respect to t for the left part of this expression gives directly $[h_2 + h_3 + h_4 + \cdots, \circ]$. But for right part one can find that

$$\left[\frac{e^{[f_2, \circ]} - \mathcal{I}}{[f_2, \circ]} f_2', \circ \right] + e^{[f_2, \circ]} \left[\frac{e^{[f_3, \circ]} - \mathcal{I}}{[f_3, \circ]} f_3', \circ \right] e^{-[f_2, \circ]}$$

$$+ e^{[f_2, \circ]} e^{[f_3, \circ]} \left[\frac{e^{[f_4, \circ]} - \mathcal{I}}{[f_4, \circ]} f_4', \circ \right] e^{-[f_3, \circ]} e^{-[f_2, \circ]} + \prod_{k=2}^{i-1} e^{[f_k, \circ]} \left[\frac{e^{[f_i, \circ]} - \mathcal{I}}{[f_i, \circ]} f_i', \circ \right] \prod_{k=i-1}^{2} e^{-[f_k, \circ]} + \cdots .$$

BINP

The $i-$th term of this series can be transformed using the rule of similarity of the Lie transformations:

$$\prod_{k=2}^{i-1} e^{[f_k,\circ]} \left[\frac{e^{[f_i,\circ]}-\mathcal{I}}{[f_i,\circ]} f_i', \circ\right] \prod_{k=i-1}^{2} e^{-[f_k,\circ]} = \prod_{k=2}^{i-2} e^{[f_k,\circ]} \left[e^{[f_{i-1},\circ]} \frac{e^{[f_i,\circ]}-\mathcal{I}}{[f_i,\circ]} f_i', \circ\right] \prod_{k=i-2}^{2} e^{-[f_k,\circ]}$$

$$= \prod_{k=2}^{i-3} e^{[f_k,\circ]} \left[e^{[f_{i-2},\circ]} e^{[f_{i-1},\circ]} \frac{e^{[f_i,\circ]}-\mathcal{I}}{[f_i,\circ]} f_i', \circ\right] \prod_{k=i-3}^{2} e^{-[f_k,\circ]} = \cdots = \left[\prod_{k=2}^{i-1} e^{[f_k,\circ]} \frac{e^{[f_i,\circ]}-\mathcal{I}}{[f_i,\circ]} f_i', \circ\right].$$

Since one can receive the following relation:

$$[h_2 + h_3 + h_4 + \cdots, \circ] = \left[\frac{e^{[f_2,\circ]}-\mathcal{I}}{[f_2,\circ]} f_2', \circ\right] + \sum_{i=3}^{\infty} \left[\left(\prod_{k=2}^{i-1} e^{[f_k,\circ]}\right) \frac{e^{[f_i,\circ]}-\mathcal{I}}{[f_i,\circ]} f_i', \circ\right].$$

Since Lie operators are equal if and only if the functions that created them are equal as well, then

$$h_2 + h_3 + h_4 + \cdots = \frac{e^{[f_2,\circ]}-\mathcal{I}}{[f_2,\circ]} f_2' + e^{[f_2,\circ]} \frac{e^{[f_3,\circ]}-\mathcal{I}}{[f_3,\circ]} f_3' + e^{[f_2,\circ]} e^{[f_3,\circ]} \frac{e^{[f_4,\circ]}-\mathcal{I}}{[f_4,\circ]} f_4' + \cdots. \quad (12)$$

Let us compare the homogeneous terms on each side of this expression. All terms besides the first have a power larger than 2. Gathering the terms with power of 2, one can obtain the following equation:

$$h_2 = \frac{e^{[f_2,\circ]}-\mathcal{I}}{[f_2,\circ]} f_2'.$$

Integration over t gives

$$\int_0^t h_2 dt = t h_2 = \int_0^t \frac{e^{[f_2,\circ]}-\mathcal{I}}{[f_2,\circ]} f_2' dt = \int_0^t \frac{df_2(t)}{dt} dt = f_2(t),$$

and for $t = 1$ one can find the first expression from the theorem: $g_2 \equiv f_2(t=1) = h_2$.
Let us omit the terms of order 2 in (12) and then multiply the result by $e^{-t[h_2,\circ]}$ from left side. Then

$$e^{-t[h_2,\circ]}(h_3 + h_4 + \cdots) = \frac{e^{[f_3,\circ]}-\mathcal{I}}{[f_3,\circ]} f_3' + e^{[f_3,\circ]} \frac{e^{[f_4,\circ]}-\mathcal{I}}{[f_4,\circ]} f_4' + e^{[f_3,\circ]} e^{[f_4,\circ]} \frac{e^{[f_5,\circ]}-\mathcal{I}}{[f_5,\circ]} f_5' + \cdots.$$

The Lie transformation $e^{[h_2,\circ]}$ doesn't change the power of the homogeneity of the each polynomial h_i. For this reason each term $e^{[h_2,\circ]} h_i$ on the left side of the expression has the power i. To collect the terms with the same power of homogeneity on the right side, let us expand the Lie transformation $e^{[f_k,\circ]}$ to series and take into account that

$$\frac{e^u - 1}{u} = \sum_{n=0}^{\infty} \frac{u^n}{(n+1)!}.$$

ADVANCED PHOTON SOURCE

Then one can find the following system of equations:

$$\begin{aligned} e^{-t[h_2,\circ]}h_3 &= f_3', \\ e^{-t[h_2,\circ]}h_4 &= f_4' + \frac{1}{2}[f_3, f_3'], \\ e^{-t[h_2,\circ]}h_5 &= f_5' + [f_3, f_4'] + \frac{1}{6}[f_3, [f_3, f_3']], \\ e^{-t[h_2,\circ]}h_6 &= f_6' + [f_3, f_5'] + \frac{1}{2}[f_4, f_4'] + \frac{1}{2}[f_3, [f_3, f_4']] + \frac{1}{24}[f_3, [f_3, [f_3, f_3']]] \ , \quad \text{etc.} \end{aligned}$$

It is possible to integrate these equations one after the other because each f_k' is expressed only through f_i with $i < k$. In fact, the first equation gives

$$f_3(t) = \int_0^t e^{-\tau[h_2,\circ]}h_3 d\tau = \frac{\mathcal{I} - e^{-t[h_2,\circ]}}{[h_2,\circ]}h_3, \quad \longrightarrow \quad g_3 \equiv f_3(t=1) = \frac{\mathcal{I} - e^{-[h_2,\circ]}}{[h_2,\circ]}h_3 \ .$$

Similarly one can find that

$$\begin{aligned} f_4' &= e^{-t[h_4,\circ]}h_4 - \frac{1}{2}[f_3, f_3'] = e^{-t[h_4,\circ]}h_4 - \frac{1}{2}\left[\frac{\mathcal{I} - e^{-t[h_2,\circ]}}{[h_2,\circ]}h_3, e^{-t[h_2,\circ]}h_3\right] = e^{-t[h_4,\circ]}h_4 \\ &- \frac{1}{2}\left[e^{-t[h_2,\circ]}\frac{e^{t[h_2,\circ]} - \mathcal{I}}{[h_2,\circ]}h_3, e^{-t[h_2,\circ]}h_3\right] = e^{-t[h_4,\circ]}h_4 - \frac{1}{2}e^{-t[h_2,\circ]}\left[\frac{e^{t[h_2,\circ]} - \mathcal{I}}{[h_2,\circ]}h_3, h_3\right] \\ &= e^{-t[h_4,\circ]}\left(h_4 - \frac{1}{2}\left[\frac{e^{t[h_2,\circ]} - \mathcal{I}}{[h_2,\circ]}h_3, h_3\right]\right) \ . \end{aligned}$$

Then

$$g_4 \equiv f_4(t=1) = \int_0^1 e^{-\tau[h_4,\circ]}\left(h_4 - \frac{1}{2}\left[\frac{e^{\tau[h_2,\circ]} - \mathcal{I}}{[h_2,\circ]}h_3, h_3\right]\right) d\tau \ , \qquad \text{etc.}$$

BINP

Truncation problem

How one can truncate the infinite product in (11)? To explain the answer to this question let us introduce a special characteristic – **rank of polynomial function** g_k:

$$r(g_k) \stackrel{def.}{=} k \ .$$

It is clear that the rank of the Poisson brackets for the two polynomials f_m and g_n equals

$$r([f_m, g_n]) = m + n - 2 \ .$$

1°. If Hamiltonian $\mathcal{H}(X)$ includes only second powers of phase variables, that is $\mathcal{H}(x) = h_2$, then for any polynomial g_n one can find that

$$r([\mathcal{H}, g_n]) = r([h_2, g_n]) = n = r(g_n) \ .$$

For this reason the rank of the expression $[h_2, \circ]X$ equals 1 and hence

$$r\left(e^{[h_2,\circ]}X\right) = 1 \quad \longrightarrow \quad e^{[h_2,\circ]}X \propto X \ .$$

This means that **the exponential operator $\mathcal{M}_2 = e^{[h_2,\circ]}$ is simply a matrix $\mathcal{R}$.** This result was obtained earlier when the example with standard oscillator was considered.

2°. It is very simple to verify that $r(g_3) = 3$. Therefore $r\left([h_2, h_3]\right) = 3$ and for any polynomial f_n the following expression is valid:

$$r\left([g_3, f_n]\right) = 3 + n - 2 = n + 1 > n = r(f_n) \ .$$

In other words, **each power of operator $[g_3, \circ]$ increases the rank of the result by one unit!**

3°. Let us choose, for example, the power $n = 3$ of phase variables that we would like take into account in our final results. One can find for operator $\mathcal{M}_3$ that

$$\mathcal{M}_3 X = e^{-[g_3,\circ]}X = X - \frac{[g_3, X]}{1!} + \frac{[g_3, X]^2}{2!} - \frac{[g_3, X]^3}{3!} + \cdots \ .$$

The ranks of each term of this series equal 1, 2, 3, ... correspondingly, and hence one **needs to keep** in his calculations **only the three first terms** of this series!

4°. The next exponential in (11) is $\mathcal{M}_4 = e^{-[g_4,\,\circ]}$. In its expansion to series the rank of each term will equal 1, 3, 4, ... correspondingly. For this reason one can omit all terms of the series besides the first and second! Finally for other exponentials from (11) it is necessary to keep only the first term of their expansions, that is, one can replace each of these exponential operators just by a unit operator: $\mathcal{M}_n = \mathcal{I}$ for $n > 4$.
With the accuracy of the third power of phase variables one can find that

$$\begin{aligned} \mathcal{M} &= e^{[h_2+h_3+h_4+\cdots,\circ]} = \mathcal{M}_2\mathcal{M}_3\mathcal{M}_4\cdots \\ &\approx \mathcal{R}\left(\mathcal{I} + \frac{[g_3,\circ]}{1!} + \frac{[g_3,\circ]^2}{2!}\right)\left(\mathcal{I} + \frac{[g_4,\circ]}{1!}\right) . \end{aligned}$$

So, the **factored product expansion theorem allows one to calculate the operator $\mathcal{M} = e^{-[\mathcal{H},\,\circ]}$ with the desired accuracy of the powers phase vector X** if one will correspondingly truncate all exponential operators in the expression of the theorem.

The discussion of the influence of the truncation of this series on the symplecticity of the operator $\mathcal{M}$ will be continued later.

Equation for operator $\mathcal{M}$

It is possible to obtain the operator $\mathcal{M}$ using another approach. This is a good demonstration of the calculation technique with the operators.
In fact, the solution of the motion equations has the following form:

$$X^{(f)} = \mathcal{M}X^{(i)} .$$

It is clear that the vector $X^{(i)}$ is constant, but $X^{(f)}$ and operator $\mathcal{M}$ are functions of time t. Hence one can rewrite the last equation:

$$X^{(f)} \equiv X(t) = \mathcal{M}(t)X^{(i)} .$$

Suppose g is the arbitrary function of phase variables X. Then, taking into account the rule of the composition of these functions one can find that

$$g(X) = g(\mathcal{M}X^{(i)}) = \mathcal{M}g(X^{(i)}) .$$

Advanced Photon Source

Let us differentiate this expression with respect to t and take into account the equations of motion. Then the derivative $\dot{g}(X)$ will be expressed through the Poisson brackets and the other side will equal $\dot{\mathcal{M}}g(X^{(i)})$:

$$\begin{aligned}\dot{\mathcal{M}}g(X^{(i)}) &= \dot{g}(X) = -\left[\mathcal{H}(X), g(X)\right] \\ &= -\left[\mathcal{H}(\mathcal{M}X^{(i)}), g(\mathcal{M}X^{(i)})\right] = -\left[\mathcal{M}\mathcal{H}(X^{(i)}), \mathcal{M}g(X^{(i)})\right] .\end{aligned}$$

Now let us take into account the rule of the Poisson brackets for Lie transformation and then

$$\begin{aligned}\dot{\mathcal{M}}g(X^{(i)}) &= -\left[\mathcal{M}\mathcal{H}(X^{(i)}), \mathcal{M}g(X^{(i)})\right] \\ &= -\mathcal{M}\left[\mathcal{H}(X^{(i)}), g(X^{(i)})\right] = -\mathcal{M}\left[\mathcal{H}(X^{(i)}), \circ\right] g(X^{(i)}) .\end{aligned}$$

Since the function g is arbitrary, the operator $\mathcal{M}$ obeys the equation

$$\dot{\mathcal{M}} = -\mathcal{M}\left[\mathcal{H}(X^{(i)}), \circ\right] . \tag{13}$$

Earlier the evident solution of this equation was written as

$$X^{(f)} = e^{-t\,[\mathcal{H}(X), \circ]}X^{(i)} . \tag{14}$$

Notice. This result takes place if one assumes that the Hamiltonian $\mathcal{H}$ does not depend on time t implicitly. The situation is more complicated in the opposite case. Here, the nonlazy reader finds the consideration of this.

Let us divide the time interval $0, t$ into N equal subintervals of duration Δt. Introduce intermediate times $t^{(m)}$: $t^{(m)} = m\Delta t$. It is clearly that $t^{(N)} = t$. Also, introduce the shorthand notation

$$\mathcal{H}^{(m)} = \mathcal{H}(X^{(i)}, t^{(m)}) .$$

Then, to lowest order in Δt and taking into account (13), a Taylor expansion gives the result

$$\begin{aligned}\mathcal{M}(t^{(m+1)}) &= \mathcal{M}(t^{(m)} + \Delta t) = \mathcal{M}(t^{(m)}) + \Delta t\dot{\mathcal{M}}(t^{(m)}) \\ &= \mathcal{M}(t^{(m)}) + \mathcal{M} - \Delta t\left[\mathcal{H}^{(m)}, \circ\right] = \mathcal{M}(t^{(m)})\left(\mathcal{I} - \Delta t\left[\mathcal{H}^{(m)}, \circ\right]\right) \\ &= \mathcal{M}(t^{(m)})e^{-\Delta t[\mathcal{H}^{(m)}, \circ]} .\end{aligned}$$

BINP

ADVANCED PHOTON SOURCE

This equation can be solved sequentially to give the result

$$\mathcal{M}(t^{(N)}) = \mathcal{M}(t^{(0)}) e^{-\Delta t[\mathcal{H}^{(0)},\circ]} e^{-\Delta t[\mathcal{H}^{(1)},\circ]} \cdots e^{-\Delta t[\mathcal{H}^{(N-1)},\circ]} .$$

Taking into account that $\mathcal{M}(t^{(0)}) = \mathcal{I}$ and $\mathcal{M}(t^{(N)}) = \mathcal{M}$, the lowest order in Δt gives the formal solution

$$\mathcal{M} = \prod_{k=0}^{N-1} e^{-\Delta t[\mathcal{H}^{(k)},\circ]} . \tag{15}$$

At this point one can make two important remarks.
First, suppose that for any two times t' and t'' the Hamiltonian $\mathcal{H}$ has the commuting property

$$\Big[[\mathcal{H}(X^{(i)}, t')\circ], [\mathcal{H}(X^{(i)}, t'')\circ] \Big] = 0 ,$$

or, equivalently, remembering that the commutator of two Lie operators is again a Lie operator, and this operator can be calculated in terms of Poisson brackets, the property

$$\Big[\mathcal{H}(X^{(i)}, t'), \mathcal{H}(X^{(i)}, t'') \Big] = 0 .$$

By the way, this relation will certainly be satisfied if $\mathcal{H}$ is a time–independent function. Then the various exponentials in the expression for operator $\mathcal{M}$ all commute, and therefore can be combined into one grand exponential to give, to lowest order in Δt, the result

$$\mathcal{M} = e^{-\Delta t \sum_{k=0}^{N-1} [\mathcal{H}^{(k)},\circ]} .$$

Upon taking the limits $N \to \infty$, $\Delta t \to 0$ one obtains the desired exact result

$$\mathcal{M}(t) = e^{-\left[\int_0^t \mathcal{H}(X^{(i)},t')dt',\circ\right]} .$$

For the time-independent Hamiltonian this formulae gives the previous result (14).
The second remark concerns the general noncommuting case. In this case too, by mean of the Baker–Campbell–Hausdorff theorem for manipulating noncommuting exponentials, it is in principle possible to combine the various exponentials in (15). If this were done, the result would in general involve the various Lie operators $[\mathcal{H}^{(m)}, \circ]$ and all their various (multiple) commutators. That is, according to the Baker–Campbell–Hausdorff theorem, products of Lie operators would occur *only* in the form of commutators. It is possible to show that in the general case this will involve (and *only* involve) exponentials of operators that are linear combinations of the operators $[\mathcal{H}(X^{(i)}, t'), \circ]$ at various times t' and their multiple commutators.

BINP

ADVANCED PHOTON SOURCE

Lie operator approach: problems of the practical realization

Let us go to problems of practical realization of the Lie operator approach. The final result of the calculations of series (10) will be present now in the matrix form without details of these calculations:

$$X_k^{(f)} = \mathcal{M}_{kl} X_l^{(i)} + \mathcal{M}_{kl} \mathcal{J}_{lm} \mathcal{F}_{mnp} X_n^{(i)} X_p^{(i)} \, , \tag{16}$$

where matrix $\mathcal{M}$ describes the contribution of the polynomial h_2 (linear optics), the three-dimensional matrix $\mathcal{F}$ corresponds to the polynomial h_3 (second-order effects or aberrations due to sextupoles), and $\mathcal{J}$ is a special symplectic "unit" matrix (as defined earlier). Let us refer to matrix $\mathcal{F}$ as the third-order Lie operator for simplicity. In the **thick lens approach** this operator is calculated with the following expression:

$$\mathcal{F}_{ijk} = \int_0^L h_{lmn}^{(3)} \, \mathcal{M}_{li}(s) \, \mathcal{M}_{mj}(s) \mathcal{M}_{nk}(s) \, ds \, , \tag{17}$$

where all indices correspond to space variables for a single particle motion, i.e., correspond to the set $(x, p_x, y, p_y, \sigma, p_\sigma)$.
The following problems were solved on the way to realizing of the Lie operator approach in practice:

- The Hamiltonian expansion with required accuracy over components of the vector $\vec{X}$ (coefficients $h_{ihk}^{(3)}, \ldots$ in expression (9)) was found.
- The mathematical tool of the special $\mathcal{P}-$, $\mathcal{D}-$ and other functions was developed.
- This tool makes it possible to find the analytical expressions of orbital Lie operators for all types of accelerator and beamline elements in the approach of thick lenses.
- Effective rules of composition of Lie transformations for sequential collider elements were obtained. These rules make it possible to calculate the total one turn map.
- The rule for presenting the exponential of the third-order Lie transformation in the special form was found. This rule helps to keep the symplecticity of the transformation and to avoid the round-off errors during computer calculations.

BINP

ADVANCED PHOTON SOURCE

Hamiltonian of the general magnetic element

In modern elements of beamlines and colliders **it is necessary to take into account simultaneously the curvature K and torsion κ of the particle orbit.** As a rule, the characteristics of the reference orbit inside these elements are constant. In this case the reference orbit consists of portions of screw lines. It is useful to show the expressions for all the components of the vector potential $\vec{A}$ of the magnetic field $\vec{B}$ for the general case of the particle trajectory:

$$\begin{aligned}
A_x &= -\frac{1}{2}\sum_{i,k=0}\frac{\partial^{i+k}B_\tau}{\partial x^i\partial y^k}\Big|_{x,y=0}\frac{x^i}{i!}\frac{y^{k+1}}{(k+1)!}\,,\\
A_y &= \frac{1}{2}\sum_{i,k=0}\frac{\partial^{i+k}B_\tau}{\partial x^i\partial y^k}\Big|_{x,y=0}\frac{x^{i+1}}{(i+1)!}\frac{y^k}{k!}\,,\\
hA_\tau &= -\sum_{i=1}\left[\frac{\partial^{i-1}(hB_y)}{\partial x^{i-1}}\Big|_{x,y=0}\frac{x^i}{i!}-\frac{\partial^{i-1}(B_x)}{\partial y^{i-1}}\Big|_{x,y=0}\frac{y^i}{i!}\right]\\
&- \frac{1}{2}\sum_{i,k=1}\left[\frac{\partial^{i-1+k}(hB_y)}{\partial x^{i-1}\partial y^k}\Big|_{x,y=0}-\frac{\partial^{i+k-1}(hB_x)}{\partial x^i\partial y^{k-1}}\Big|_{x,y=0}\right]\frac{x^i}{i!}\frac{y^k}{k!}\\
&- \frac{\kappa}{2}\left[\sum_{i,k=1}\frac{\partial^{i+k-1}B_\tau}{\partial x^i\partial y^{k-1}}\Big|_{x,y=0}\frac{x^i}{i!}\frac{y^{k+1}}{(k+1)!}+\sum_{i,k=1}\frac{\partial^{i-1+k}B_\tau}{\partial x^{i-1}\partial y^k}\Big|_{x,y=0}\frac{x^{i+1}}{(i+1)!}\frac{y^k}{k!}\right].
\end{aligned}$$

One can see that the derivatives of all the components of the magnetic field used here have different orders. These derivatives determine the standard main parameters of the beamline elements:

$$B_{0i}\equiv\frac{e}{E_0}B_i \qquad \text{– guide field;}$$

$$g=\frac{e}{E_0}\frac{\partial B_y}{\partial x}\Big|_{x,y=0}=\frac{e}{E_0}\frac{\partial B_x}{\partial y}\Big|_{x,y=0} \qquad \text{– quadrupole;}$$

$$q=\frac{e}{2E_0}\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_y}{\partial y}\right)\Big|_{x,y=0} \qquad \text{– skew quadrupole;}$$

$$m_x=\frac{e}{E_0}\frac{\partial^2 B_y}{\partial x^2}\Big|_{x,y=0}\,,\quad m_y=\frac{e}{E_0}\frac{\partial^2 B_y}{\partial x\partial y}\Big|_{x,y=0} \qquad \text{– sextupoles, etc.}$$

BINP

ADVANCED PHOTON SOURCE

Other derivatives of the field components can be expressed in terms of these parameters only, for example:

$$
\begin{aligned}
\frac{e}{E_0}\frac{\partial B_x}{\partial x}\bigg|_{x,y=0} &= q - \frac{1}{2}\left(B'_{0\tau} + KB_{0x}\right) \ , \\
\frac{e}{E_0}\frac{\partial B_y}{\partial y}\bigg|_{x,y=0} &= -q - \frac{1}{2}\left(B'_{0\tau} + KB_{0x}\right) \ , \\
\frac{e}{E_0}\frac{\partial B_\tau}{\partial x}\bigg|_{x,y=0} &= -KB_{0\tau} + B'_{0x} - \kappa B_{0y} \ , \\
\frac{e}{E_0}\frac{\partial B_\tau}{\partial y}\bigg|_{x,y=0} &= B'_{0y} + \kappa B_{0x} \ , \ \text{etc} \ .
\end{aligned}
$$

Here two important representations of Hamiltonian will be shown. The first of them describes the second order of components of the vector $\vec{X}$ general magnetic element taking into account the torsion of the particle orbit:

$$
\begin{aligned}
\mathcal{H} &= h_1 + h_2 = -\Delta B_y x - \Delta B_x y + CK^2\sigma + \frac{p_x^2 + p_y^2}{2} \\
&+ \left(g_x + \frac{r^2}{4}\right)\frac{x^2}{2} + \left(g_y + \frac{r^2}{4}\right)\frac{y^2}{2} - \tilde{q}xy - \frac{r}{2}(xp_y - yp_x) - Kp_\sigma x \ ,
\end{aligned}
\tag{18}
$$

where the Hamiltonian parameters are as follows:

$$
\begin{aligned}
g_x &= g + KB_{0y} - \kappa^2 - \kappa B_{0\tau} \ , & g_y &= -(g + \kappa^2 + \kappa B_{0\tau}) \ , \\
r &= B_{0\tau} + 2\kappa \ , & \tilde{q} &= q + \frac{1}{2}KB_{0x} \ , \\
\Delta B_x &= B_{0x} \ , & \Delta B_y &= K - B_{0y} \ .
\end{aligned}
$$

The term $CK^2\sigma$ takes into account the synchrotron radiation of the particle and $C = 2/3r_e\gamma_0^3$. This Hamiltonian describes the linear coupled motion of the particle in the general case.

For the next Hamiltonian the terms of the third order in $\vec{X}$ are taken into account for the flat orbit inside magnetic element (i.e. $\kappa = 0$):

$$
\begin{aligned}
\mathcal{H} &= h_1 + h_2 + h_3 \ , \quad \textbf{where} \\
h_3 &= \frac{1}{6}(2Kg + m_x)x^3 + \frac{1}{2}(m_y - K + q)x^2 y - \frac{1}{8}B_\tau^2(x^2 + y^2)p_\sigma + \frac{1}{2}Kxp_x^2 \\
&- \frac{1}{2}(m_x + Kg + \frac{1}{2}B''_y)xy^2 - \frac{1}{2}B'_y xyp_y + \frac{1}{2}Kxp_y^2 + \frac{1}{2}B_\tau xp_xp_\sigma \\
&- \frac{1}{2}(p_x^2 + p_y^2)p_\sigma + \frac{1}{4}B'_y p_x y^2 - \frac{1}{2}B_\tau p_x y p_\sigma - \frac{1}{6}m_y y^3 \ .
\end{aligned}
\tag{19}
$$

BINP

Coupled particle motion

It is possible to write the solution for motion equations in the matrix form for Hamiltonian (18).
For **a bending magnet** without combined functions, the parameters are $B_{0y}, K, g \neq 0$, and $\kappa = B_{0x} = B_{0\tau} = q = 0$, so that $r = \tilde{q} = 0$ and $g_x = g + KB_{0y}$, $g_y = -g$. So, the matrix for the orbital motion has the following form:

$$\mathcal{M} = \begin{pmatrix} \mathcal{P}_0(g_x) & \mathcal{P}_1(g_x) & 0 & 0 & 0 & K\mathcal{P}_2(g_x) \\ -g_x\mathcal{P}_1(g_x) & \mathcal{P}_0(g_x) & 0 & 0 & 0 & K\mathcal{P}_1(g_x) \\ 0 & 0 & \mathcal{P}_0(g) & \mathcal{P}_1(g) & 0 & 0 \\ 0 & 0 & -g\mathcal{P}_1(g) & \mathcal{P}_0(g) & 0 & 0 \\ -K\mathcal{P}_1(g_x) & -K\mathcal{P}_2(g_x) & 0 & 0 & 1 & -K^2\mathcal{P}_3(g_x) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can see that new special functions $\mathcal{P}_i$ are used to write this matrix. To explain the reason for introducing of these special functions, let us compare the matrices for **the quadrupole lens** in standard and new forms (L is a length of lens):

$$\mathcal{M} = \begin{pmatrix} \cosh\sqrt{|g|}L & \sqrt{\frac{L}{|g|}}\sinh\sqrt{|g|}L & 0 & 0 & 0 & 0 \\ \sqrt{\frac{|g|}{L}}\sinh\sqrt{|g|}L & \cosh\sqrt{|g|}L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\sqrt{|g|}L & \sqrt{\frac{L}{|g|}}\sin\sqrt{|g|}L & 0 & 0 \\ 0 & 0 & -\sqrt{\frac{|g|}{L}}\sin\sqrt{|g|}L & \cos\sqrt{|g|}L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathcal{P}_0(g,L) & \mathcal{P}_1(g,L) & 0 & 0 & 0 & 0 \\ -g\mathcal{P}_1(g,L) & \mathcal{P}_0(g,L) & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{P}_0(g,L) & \mathcal{P}_1(g,L) & 0 & 0 \\ 0 & 0 & -g\mathcal{P}_1(g,L) & \mathcal{P}_0(g,L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

One can see that the function $\mathcal{P}_0$ is close to cos and cosh; analogously $\mathcal{P}_1$ is close to sin and sinh. It is really so!

At last, let us consider the **spiral magnet,** which is characterized with the following parameters: $K = B_{0y} \neq 0$ and $\kappa \neq 0$, so that $g_x = -\kappa^2 + K^2$, $g_y = -\kappa^2$ and $r = 2\kappa$. In an analogous way one can find its matrix:

$$\mathcal{M} = \begin{pmatrix} \mathcal{D}_0 + \kappa^2\mathcal{D}_2 & \mathcal{D}_1 - \kappa^2\mathcal{D}_3 & \kappa(\mathcal{D}_1 + \kappa^2\mathcal{D}_3) & 2\kappa\mathcal{D}_2 & 0 & K(\mathcal{D}_2 - \kappa^2\mathcal{D}_4) \\ -K^2\mathcal{D}_1 & \mathcal{D}_0 + \kappa^2\mathcal{D}_2 & -\kappa K^2\mathcal{D}_2 & \kappa(\mathcal{D}_1 - g_x\mathcal{D}_3) & 0 & K(\mathcal{D}_1 + \kappa^2\mathcal{D}_3) \\ -\kappa(\mathcal{D}_1 - g_x\mathcal{D}_3) & -2\kappa\mathcal{D}_2 & \mathcal{D}_0 + k^2\mathcal{D}_2 & \mathcal{D}_1 + g_x\mathcal{D}_3 & 0 & -2\kappa K\mathcal{D}_3 \\ \kappa K^2\mathcal{D}_2 & -\kappa(\mathcal{D}_1 + \kappa^2\mathcal{D}_3) & \kappa^2K^2\mathcal{D}_3 & \mathcal{D}_0 + k^2\mathcal{D}_2 & 0 & -\kappa K(\mathcal{D}_2 + \kappa^2\mathcal{D}_4) \\ -K(\mathcal{D}_1 + \kappa^2\mathcal{D}_3) & -K(\mathcal{D}_2 - \kappa^2\mathcal{D}_4) & -\kappa K(\mathcal{D}_2 + \kappa^2\mathcal{D}_4) & -2\kappa K\mathcal{D}_3 & 1 & -K^2(\mathcal{D}_3 - \kappa^2\mathcal{D}_5) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is simple to demonstrate that for the spiral magnet with the field bending by the angle multiple to 360° (in this case $\sqrt{\kappa^2 + K^2}L = 2\pi m$), matrix $\mathcal{M}$ transforms into the expression obtained by Courant. Nevertheless, one can see other special functions $\mathcal{D}_i$. It is very difficult (and maybe impossible) to show the matrix for a spiral magnet in a general case without using the introduced functions. Let us consider these functions.

$\mathcal{P}-$ and $\mathcal{D}-$functions

All functions $\mathcal{P}_i$ **are determined as** the following simple **series:**

$$\mathcal{P}_i(Q,L) = L^i \sum_{j=0}^{\infty} \frac{(Q^2L^2)^j}{(2j+i)!} \,.$$

This series transforms to cos or cosh for $i=0$ when $Q^2 = g$ is larger or smaller than zero, correspondingly; similarly this series transforms to $\dfrac{\sin x}{x}$ or $\dfrac{\sinh x}{x}$ if $i=1$. At once one can see the **advantages of the new form:**

- **the matrix elements can be calculated without the additional analysis of the sign of the gradient value for both directions;**
- **the values close to zero are absent in the denominators;**
- **it is necessary to calculate only two functions.**

The matrix for a bending magnet includes the functions $\mathcal{P}_2$ and $\mathcal{P}_3$. But again it is **necessary to calculate only two series,** because $\mathcal{P}-$functions with different indices are related through the following simple algebraic relation:

$$\mathcal{P}_i(Q,L) = \left[\frac{L^i}{i!} + Q^2\mathcal{P}_{i+2}(Q,L)\right] \,.$$

This expression is very useful for saving computing time, because it allows one to calculate all $\mathcal{P}_i-$functions if series $\mathcal{P}_i$ with the highest odd and even numbers are found. It should be also noted that **these series converge very quickly** due to usually small values of the argument QL in the calculations for real elements. Let us show the relations between those introduced and the usual hyperbolic functions:

$$\begin{aligned}
\mathcal{P}_0(Q,s) &= \cosh Qs \,, \\
\mathcal{P}_1(Q,s) &= \frac{\sinh Qs}{Q} = s\left(\frac{\sinh Qs}{Qs}\right) \,, \\
\mathcal{P}_2(Q,s) &= \frac{1-\cosh Qs}{Q^2} = s^2\left(\frac{1-\cosh Qs}{(Qs)^2}\right) \,, \\
\mathcal{P}_3(Q,s) &= \frac{s - \dfrac{\sinh Qs}{Q}}{Q^2} = s^3\left(\frac{1-\dfrac{\sinh Qs}{Qs}}{(Qs)^2}\right)
\end{aligned}$$

and etc.

The matrix for a spiral magnet includes the **functions $\mathcal{D}_i$.** The **definition** of these functions **is as follows:**

$$\mathcal{D}_i(Q_1, Q_2, L) = L^i \sum_{j=0}^{\infty} \frac{1}{(2j+i)!} \sum_{k=0}^{j} (Q_1^2 L^2)^{j-k} (Q_2^2 L^2)^k \ .$$

These functions have many useful properties. One of them is the same as for $\mathcal{P}-$functions: it is **enough to calculate the corresponding series with the highest odd and even numbers only** and after that to use the following recurrent expressions:

$$\begin{aligned} \mathcal{D}_i(Q_1, Q_2, L) &= \mathcal{P}_i(Q_2, L) + Q_1^2 \mathcal{D}_{i+2}(Q_1, Q_2, L) \\ &= \mathcal{P}_i(Q_1, L) + Q_2^2 \mathcal{D}_{i+2}(Q_1, Q_2, L) \ . \end{aligned}$$

The connection of these functions with hyperbolic functions is as follows:

$$\begin{aligned} \mathcal{D}_0(Q_1, Q_2, s) &= \frac{Q_1^2 \cosh Q_1 s - Q_2^2 \cosh Q_2 s}{Q_1^2 - Q_2^2} \ , \\ \mathcal{D}_1(Q_1, Q_2, s) &= \frac{Q_1 \sinh Q_1 s - Q_2 \sinh Q_2 s}{Q_1^2 - Q_2^2} \ , \\ \mathcal{D}_2(Q_1, Q_2, s) &= \frac{-\cosh Q_1 s + \cosh Q_2 s}{Q_1^2 - Q_2^2} \ , \\ \mathcal{D}_3(Q_1, Q_2, s) &= \frac{-\dfrac{\sinh Q_1 s}{Q_1} + \dfrac{\sinh Q_2 s}{Q_2}}{Q_1^2 - Q_2^2} \ \text{and etc.} \end{aligned}$$

At last present the relations between $\mathcal{P}-$ and $\mathcal{D}-$functions only:

$$\begin{aligned} \mathcal{D}_i(Q_1, Q_2, s) &= \frac{Q_1^2 \mathcal{P}_i(Q_1, s) - Q_2^2 \mathcal{P}_i(Q_2, s)}{Q_1^2 - Q_2^2} \ , \\ \mathcal{D}_{i+2}(Q_1, Q_2, s) &= \frac{\mathcal{P}_i(Q_1, s) - \mathcal{P}_i(Q_2, s)}{Q_1^2 - Q_2^2} \ , \end{aligned}$$

and, in particular,

$$\begin{aligned} \mathcal{D}_0(Q, Q, s) &= \mathcal{P}_0(Q, s) + \frac{1}{2} Q^2 s \mathcal{P}_1(Q, s) \qquad \text{and} \\ \mathcal{D}_i(Q, Q, s) &= \frac{1}{2}[s\mathcal{P}_{i-1}(Q, s) + (2-i)\mathcal{P}_i(Q, s)] \ \text{for } i \geq 1 \ . \end{aligned}$$

Let us also show a few important properties of both families of functions that allow us to manipulate them. The first two are simple and describe the **rules of differentiation and integration**:

$$\frac{d\mathcal{P}_i(Q,L)}{dL} = \mathcal{P}_{i-1}(Q,L)\ , \quad \frac{d\mathcal{D}_i(Q_1,Q_2,L)}{dL} = \mathcal{D}_{i-1}(Q_1,Q_2,L)\ ,$$

$$\int_0^L x^j \mathcal{P}_i(Q,x)dx = j!\sum_{k=0}^{j}\frac{(-1)^k}{(j-k)!}L^{j-k}\mathcal{P}_{i+k+1}(Q,L)\ ,$$

$$\int_0^L x^j \mathcal{D}_i(Q_1,Q_2,x)dx = j!\sum_{k=0}^{j}\frac{(-1)^k}{(j-k)!}L^{j-k}\mathcal{D}_{i+k+1}(Q_1,Q_2,L)\ .$$

In practice the integral (17) is used to find the analytical expressions for Lie operators. Therefore, **it is necessary to express the products of $\mathcal{P}-$ and $\mathcal{D}-$functions with different arguments and indices using again these functions.** For the case with indices 0 and 1 it is very simple, and the desired expressions correspond to the rule for sums of hyperbolic functions of different arguments. For example,

$$\begin{aligned}\mathcal{P}_0(Q_1,L)\cdot\mathcal{P}_0(Q_2,L) &= \cosh Q_1L\cdot\cosh Q_2L = \cdots \\ &= \frac{1}{2}\left[\mathcal{P}_0(Q_1+Q_2,L)+\mathcal{P}_0(Q_1-Q_2,L)\right]\ .\end{aligned}$$

But for arbitrary indices the expressions are complicated. For example, for a product of the $\mathcal{P}-$functions with even indices one can find:

$$\mathcal{P}_{2i}(Q_1,L)\cdot\mathcal{P}_{2j}(Q_2,L) = \frac{\mathcal{P}_0(Q_1+Q_2,L)+\mathcal{P}_0(Q_1-Q_2,L)}{2Q_1^{2i}Q_2^{2j}}$$
$$- \frac{1}{Q_1^{2i}Q_2^{2j}}\left[\mathcal{P}_0(Q_1,L)\cdot\sum_{m=0}^{j-1}\frac{(Q_2L)^{2m}}{(2m)!}+\mathcal{P}_0(Q_2,L)\cdot\sum_{m=0}^{i-1}\frac{(Q_1L)^{2m}}{(2m)!}\right.$$
$$\left. + \left(\sum_{m=0}^{i-1}\frac{(Q_1L)^{2m}}{(2m)!}\right)\cdot\left(\sum_{m=0}^{j-1}\frac{(Q_2L)^{2m}}{(2m)!}\right)\right]\ .$$

So, the **tool of $\mathcal{P}-$ and $\mathcal{D}-$functions makes it possible to find the analytical expressions of orbital Lie operators** for all collider elements.

Lie operators for some simple elements

As a rule, the analytical expressions for the Lie operators have a complicated form. For this reason only the expressions for the quadrupole lens and bending magnet will be shown. Here indices 1,2,3,4,5,6 correspond to phase variables $x, p_x, y, p_y, \sigma, p_\sigma$.

Quadrupole lens. Only six Lie operators differ from zero:

$$\mathcal{F}_{116} = \frac{Q_x^2}{2}[\mathcal{P}_1(2Q_x) - L]\,, \quad \mathcal{F}_{126} = \frac{1}{4}[1 - \mathcal{P}_0(2Q_x)]\,,$$

$$\mathcal{F}_{226} = -\frac{1}{2}[\mathcal{P}_1(2Q_x) + L]\,, \quad \mathcal{F}_{336} = \frac{Q_z^2}{2}[\mathcal{P}_1(2Q_z) - L]\,,$$

$$\mathcal{F}_{346} = \frac{1}{4}[1 - \mathcal{P}_0(2Q_z)]\,, \quad \mathcal{F}_{446} = -\frac{1}{2}[\mathcal{P}_1(2Q_z) + L]\,,$$

where $Q_x^2 = -Q_z^2 = g$.

Bending magnet. In this case $K = \frac{e}{E_0}B_y$. Nonzero Lie operators equal

$$\mathcal{F}_{111} = \frac{3}{4}K^3\Big[-\mathcal{P}_1(3K) + \mathcal{P}_1(K)\Big]\,, \quad \mathcal{F}_{112} = \frac{1}{4}K\Big[\mathcal{P}_0(3K) - \mathcal{P}_0(K))\Big]\,,$$

$$\mathcal{F}_{116} = \frac{3}{4}K^2\Big[\mathcal{P}_1(3K) - \mathcal{P}_1(K)\Big]\,, \quad \mathcal{F}_{122} = \frac{1}{4}K\Big[3\mathcal{P}_1(3K) + \mathcal{P}_1(K)\Big]\,,$$

$$\mathcal{F}_{126} = \frac{1}{4}\Big[\mathcal{P}_0(K) - \mathcal{P}_0(3K)\Big]\,, \quad \mathcal{F}_{133} = K\mathcal{P}_1(K)\,,$$

$$\mathcal{F}_{166} = \frac{3}{4}K\Big[\mathcal{P}_1(K) - \mathcal{P}_1(3K)\Big]\,, \quad \mathcal{F}_{222} = \frac{3}{4}K\Big[3\mathcal{P}_2(3K) + \mathcal{P}_2(K)\Big]\,,$$

$$\mathcal{F}_{226} = -\frac{1}{4}\Big[3\mathcal{P}_1(3K) + \mathcal{P}_1(K)\Big]\,, \quad \mathcal{F}_{244} = K\mathcal{P}_2(K)\,,$$

$$\mathcal{F}_{266} = \frac{1}{4}K\Big[\mathcal{P}_2(K) - 9\mathcal{P}_2(3K)\Big]\,, \quad \mathcal{F}_{446} = -\mathcal{P}_1(K)\,,$$

$$\mathcal{F}_{666} = \frac{3}{4}\Big[\mathcal{P}_1(3K) - \mathcal{P}_1(K)\Big]\,.$$

ADVANCED PHOTON SOURCE

Rules of Lie transformation composition

The tool of the Lie operators should be used in accelerator and collider problems because the multiturn effects are very important in this case. Thus it is necessary to merge the Lie operators of selected elements into one common operator. The possibility of this merging increases even more if one takes into account that the one-turn Lie operator saves computing time very effectively. Indeed, if the one-turn operator $\mathcal{L}_1$ is known, then the operator $\mathcal{L}_2 = (\mathcal{L}_1)^2$ allows us to find the transformation in two turns; operator $\mathcal{L}_4 = (\mathcal{L}_2)^2$ gives the transformation in four turns; operator $\mathcal{L}_8 = (\mathcal{L}_4)^2$ gives the transformation in eight turns, etc. Thus, **after n multiplications of the one-turn operator one will have the transformation for 2^n turns.**

Let us obtain the rule for merging the third-order Lie operators of two sequential elements. Transformations through the first and second elements have the following form:

$$\begin{aligned} X_i^{(1)} &= \mathcal{M}_{ij}^{(1)} X_j^{(0)} + \mathcal{M}_{ij}^{(1)} \mathcal{J}_{jk} \mathcal{F}_{klm}^{(1)} X_l^{(0)} X_m^{(0)} , \\ X_i^{(2)} &= \mathcal{M}_{ij}^{(2)} X_j^{(1)} + \mathcal{M}_{ij}^{(2)} \mathcal{J}_{jk} \mathcal{F}_{klm}^{(2)} X_l^{(1)} X_m^{(1)} . \end{aligned}$$

Then one can find that

$$\begin{aligned} X_i^{(2)} &= \mathcal{M}_{ij}^{(2)} \left(\mathcal{M}_{jk}^{(1)} X_k^{(0)} + \mathcal{M}_{ij}^{(1)} \mathcal{J}_{jk} \mathcal{F}_{klm}^{(1)} X_l^{(0)} X_m^{(0)} \right) \\ &+ \mathcal{M}_{ij}^{(2)} \mathcal{J}_{jk} \mathcal{F}_{klm}^{(2)} \left(\mathcal{M}_{ln}^{(1)} X_n^{(0)} + \cdots \right) \left(\mathcal{M}_{mp}^{(1)} X_p^{(0)} + \cdots \right) . \end{aligned}$$

The second terms in each parenthesis can be omitted because their contributions will have the third order over the components of phase variables vector. If one takes into account the symplecticity of the transformation $\mathcal{M}$, i.e., validity of the expression

$$\mathcal{M}\mathcal{J}\tilde{\mathcal{M}} = \mathcal{J} \quad \text{or} \quad \mathcal{M}_{ij}^{(1)} \mathcal{J}_{jk} \tilde{\mathcal{M}}^{(1)}{}_{kl} = \mathcal{J}_{il} ,$$

where $\tilde{}$ means the transposition, then

$$\begin{aligned} X_i^{(2)} &= \left(\mathcal{M}_{ij}^{(2)} \mathcal{M}_{jk}^{(1)} \right) X_k^{(0)} + \left(\mathcal{M}_{ij}^{(2)} \mathcal{M}_{jk}^{(1)} \right) \mathcal{J}_{jk} \mathcal{F}_{klm}^{(1)} X_l^{(0)} X_m^{(0)} \\ &+ \left(\mathcal{M}_{ij}^{(2)} \mathcal{M}_{jk}^{(1)} \right) \mathcal{J}_{kl} \mathcal{F}_{lmn}^{(2)} X_m^{(0)} X_n^{(0)} \\ &= \mathcal{M}_{ij}^{(21)} X_j^{(0)} + \mathcal{M}_{ij}^{(21)} \mathcal{J}_{jk} \mathcal{F}_{klm}^{(21)} X_l^{(0)} X_m^{(0)} , \end{aligned}$$

where $\mathcal{M}^{(21)} = \mathcal{M}^{(2)}\mathcal{M}^{(1)}$ and the desired expression for $\mathcal{F}^{(21)}$ is:

$$\mathcal{F}_{ijk}^{(21)} = \mathcal{F}_{ijk}^{(1)} + \mathcal{F}_{lmn}^{(2)} \mathcal{M}_{li}^{(1)} \mathcal{M}_{mj}^{(1)} \mathcal{M}_{nk}^{(1)} .$$

Symplecticity

Let us return to the **problem of symplecticity of Lie transformations** and briefly discuss this problem again. Recall that the **exponential Lie operators are always symplectic** – this is their fundamental property. But usually the expansions of these operators to series are used. How it is necessary to truncate these series? The question arises since the truncation of the series violates the symplecticity of the total transformation as a rule. Fortunately, **for the third-order Lie operators it is possible to keep their symplecticity with desired accuracy during computer calculations.** One can easily prove that

$$e^{[h_3,\circ]} Z_i = \sum_{n=0}^{\infty} \frac{[h_3,\circ]^n}{n!} Z_i \equiv \sum_{n=0}^{\infty} \frac{\vec{Z}^{(n)}}{n!}\,, \tag{20}$$

where the "vector" $\vec{Z}^{(n)}$ for the step number n is related to its values for the previous steps by the following expression:

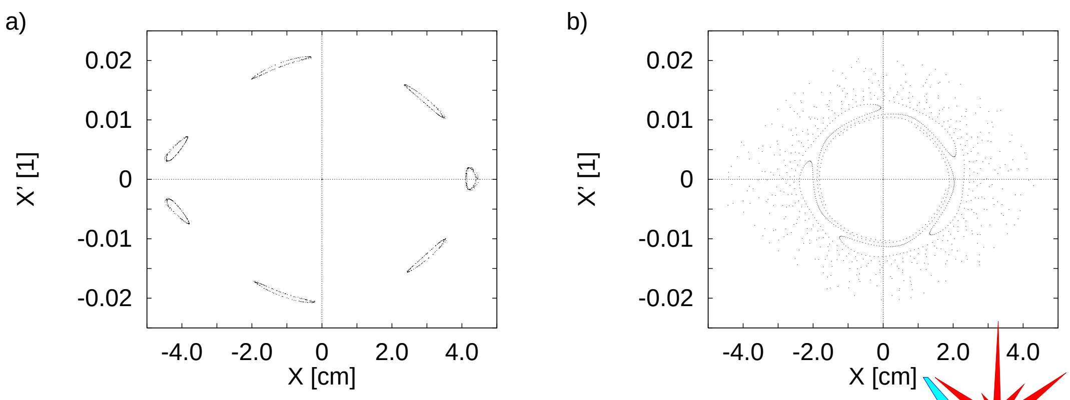
$$Z_i^{(n)} = \sum_{m=0}^{n-1} C_{n-1}^{m}\, h_{ijk}^{(3)}\, Z_j^{(m)}\, Z_k^{(n-m-1)}\,. \tag{21}$$

Here $h_{ijk}^{(3)}$ are the coefficients of polynomial h_3, and C_n^m are the binomial coefficients.

During calculations, these expressions allow us to use such a number of terms in a series that the contributions of all omitted terms will be smaller than the computer accuracy.

The following figure demonstrates the results of the standard tracking using the code TRANSPORT and tracking based on the expressions (20) and (21):

Code SpinLie: **Code TRANSPORT:**



Both trackings are performed for a simple magnetic structure.

Conclusion

The Lie operators approach is a powerful method to solve complicated problems in different fields of physics. The author and his colleagues applied and improved this approach for calculations of spin motion in beamlines, accelerators, and colliders. The code SpinLie allows us to calculate the level of equilibrium polarization for electron and proton colliders and allows one to investigate the spin resonances during acceleration, colliding, and so on.

The author hopes that these lectures will be not useless to readers and thanks them for their attention.

ADVANCED PHOTON SOURCE

References (shortest list)

[1⋆⋆⋆] Alex J. Dragt. Lecture on nonlinear orbit dynamics. American Institute of Physics, Conference Proceedings. AIP 87 (1982) 147–313.

[2⋆⋆⋆] Stanly Steinberg. Lie series, Lie transformations and their applications. Proc. of the CIFMO-CIO Workshop, held at Leon, Mexico, 1985. In: Lie Methods in Optics Springer-Verlag, Berlin, 1986 (Lecture Notes in Physics, 250 (1985) 47–103).

[3⋆⋆⋆] A.J. Dragt, J.M. Finn. Lie series and invariant functions for analytical symplectic maps. J. Math. Phys., 17 (1976) 2215–2227.

[4⋆⋆⋆] A.J. Dragt, E. Forest. Computation of nonlinear behavior of Hamiltonian system using Lie algebra method. J. Math. Phys., 24 (1983) 2734–2744.

[5⋆⋆] Alex J. Dragt. Elementary and advanced Lie algebraic methods with applications to accelerator design, electron microscopes and light optics. Nucl. Instr. Meth., A258 (1987) 339–354.

[6⋆⋆] K. Yokoya. Calculation of the equilibrium polarization of stored electron beams using Lie algebra. Nucl. Instr. Meth., A258 (1987) 149–160.

[7⋆] A.J. Dragt, J.M. Finn. Normal form for mirror machine Hamiltonians. J. Math. Phys., 20 (1979) 2649–2660.

[8⋆] A.J. Dragt. A method of transfer maps for linear and nonlinear beam elements. IEEE Transactions on Nuclear Physics, NS-26 (1979) 3601–3603.

[9⋆] A.J. Dragt. Transfer map approach to the beam-beam interaction. Proc. of Symp. on Nonlinear Dynamics and the Beam–Beam Interaction, Brookhaven National Laboratory (1979).

BINP

[10⋆] **Yu. Eidelman, V. Yakimenko. The application of Lie method to the spin motion in nonlinear collider fields. Part. Accel., 45 (1994) 17–36.**

[11⋆] **Yu. Eidelman, V. Yakimenko. Calculation of orbital and spin Lie operators. Part. Accel., 50 (1995) 261–279.**

[12⋆] **Yu. Eidelman, Yu. Shatunov, V. Yakimenko. Spin tune shifts in storage rings. Nucl. Instr. Meth., A357 (1995) 23–33.**