Froissart Bound from Gluon Saturation

[Ferreiro, Iancu, K.I. & McLerran, NP A710 (02) 373]

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Plan:

1. High energy behavior of the hadronic cross section
   Heisenberg, Froissart, Experiments
2. Multiple production of small x gluons and saturation
   Unitarization of the scattering amplitude
3. Formulation of the problem
   Dipole-hadron cross section, Black disk radius,
   Impact parameter dependence of scatt. amplitude
4. Froissart bound from gluon saturation
5. Total $\gamma^* p$ cross section
6. Summary

High energy behavior of hadronic cross section

Old problem: How does the hadronic cross section increase with increasing energy $s \to \infty$? (s: c.m. energy squared)

- W. Heisenberg, Z. Physik 133 (1952) 65
  "Mesonenerzeugung als Stosswellenproblem"

$$\sigma_{\rm tot} \propto \ln^2 s$$

- M. Froissart, Phys. Rev. 123 (1961) 1053
  "Asymptotic Behavior and Subtractions in the Mandelstam Representation"

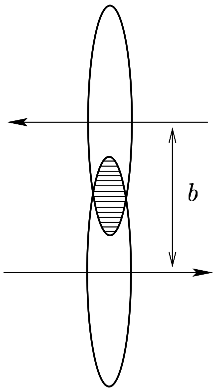
Froissart bound $\sigma_{\rm tot} \le C \ln^2 s$

Constant C was computed later by Martin et al.

- Analyses of experimental data
  - Desgrolard, Giffon, & Martynov, Nuovo Cim. A110 (97) 537
    pp, $p\bar{p}$, $\sqrt{s} > 5$ GeV $\Rightarrow$ $\ln s$ vs $\ln^2 s$: non-distinguishable
  - Cudell et al. Phys. Rev. D65 (2002) 074024
    $pp, p\bar{p}, \pi p, Kp, \gamma p, \gamma\gamma, \Sigma p$, $\sqrt{s} \ge 9$ GeV $\Rightarrow$ $\ln^2 s$ prefered
  - Igi & Ishida, Phys. Rev. D66 (2002) 034023
    πN, $\sqrt{s} \ge 10$ GeV $\Rightarrow$ best fit: $\ln^2 s$

Heisenberg's derivation

see a review by Dosch, Gauron & Nicolescu, hep-ph/0206214



Reaction can occur only if the energy density in the overlap region is high enough to create at least a pion pair:

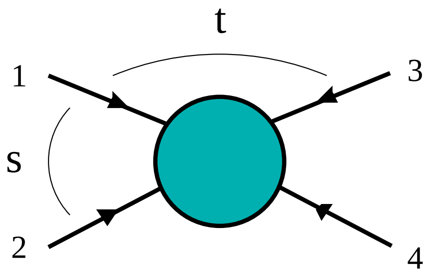
$$\underbrace{\kappa\ \mathrm{e}^{-m_\pi b}}_{\text{overlap of pion cloud}}\ \sqrt{s}\ \geq\ k_0 \quad \text{: average energy of two pions}$$

$$\Rightarrow \quad b_{\max} = \frac{1}{m_\pi} \ln \frac{\sqrt{s}\kappa}{k_0} \quad \text{edge of the reaction zone}$$

Rough estimate of the cross section reads

$$\begin{aligned}\sigma \quad &\simeq \quad \pi b_{\max}^2 = \frac{\pi}{m_\pi^2} \ln^2 \frac{\sqrt{s}\kappa}{k_0} \\ &\propto \quad \ln^2 s \quad (s \to \infty)\end{aligned}$$

Froissart Bound [1/2]



Scattering amplitude $\mathcal{A} = \langle 34|\, T\, |12\rangle$, $\quad S = 1 - T$

1. Lorentz invariance: Mandelstam variables $\mathcal{A} = \mathcal{A}(s,t)$
2. Analiticity (Cauchy's integral formula):
$$\mathcal{A}(s,t) = \frac{1}{2\pi i}\oint \frac{ds'}{s'-s}\mathcal{A}(s',t)$$
(also $\langle 34|\, T\, |12\rangle = \langle 12|\, T\, |34\rangle$ PT inv.)
3. Unitarity: $SS^\dagger = S^\dagger S = \mathbf{1}$
Partial wave exp. ($P_\ell(z_s), Q_\ell(z_s)$: Legendre poly.)

$$\mathcal{A}(s,t) = \sum_{\ell=0}^{\infty}(2\ell+1)a_\ell(s)P_\ell(z_s \equiv \cos\theta_s), \quad |a_\ell(s)| \leq 1$$

$$a_\ell(s) = -g_t(s)Q_\ell(z_s(s, M_t^2)) + \int_{z_s(s,t_0)}^{\infty}\frac{dz'}{\pi}D_t(s,t')Q_\ell(z')$$
$$+(\mathrm{u-pole\ and\ cut})$$

Froissart Bound [2/2]

We evaluate $\mathcal{A}(s,t)$ by the asymptotic form of $a_\ell(s)$

$\ell \to \infty$

$$Q_\ell(z_s) \sim \frac{1}{\sqrt{\ell}} \underbrace{\mathrm{e}^{-(\ell+1/2)\zeta(z_s)}}_{\text{decreasing in } z} < \mathrm{e}^{-\ell\,\zeta(z_s^0)}$$

$$z_s^0 = \min\{z_s(s, M_t^2), z_s(s, t_0)\}$$

$s \to \infty$

$$z_s^0 \sim 1 + \frac{2t_0}{s}, \quad \zeta(z_s^0) \sim 2\sqrt{\frac{t_0}{s}}$$

$$a_\ell(s) \sim f(s)\,\mathrm{e}^{-\ell 2\sqrt{t_0/s}} = \exp\{-2\ell\sqrt{t_0/s} + \ln f(s)\}$$

Partial waves with large ℓ do not contribute.
Upper value of ℓ is ($f(s)$ is at most a power of s.)

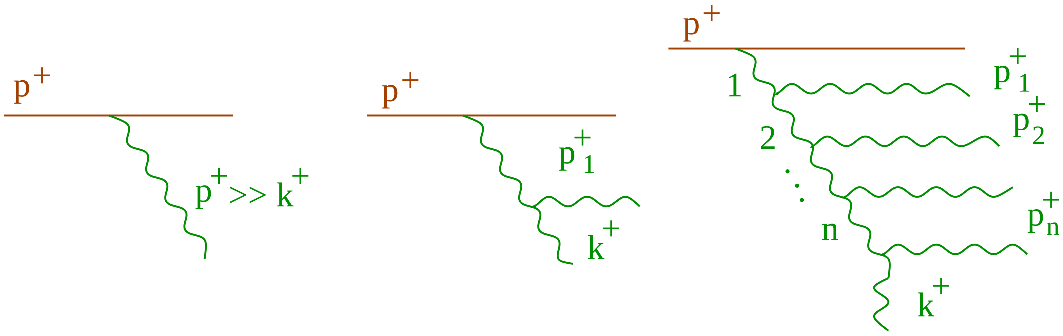
$$\ell_{\max} \sim \frac{1}{2\sqrt{t_0}}\sqrt{s}\ln f(s) \sim C\sqrt{s}\ln s$$

$$\Longrightarrow \quad |\mathcal{A}(s,t)| < \sum_{\ell=0}^{C\sqrt{s}\ln s} (2\ell+1) \sim \# s \ln^2 s$$

Froissart bound $\sigma_{\mathrm{tot}} = \frac{1}{s}\mathrm{Im}\,\mathcal{A}(s,0) < C\ln^2 s$

High energy scattering in QCD

Multiproduction of small x gluons ($x \simeq Q^2/s$ in DIS)



$\sim \bar{\alpha}_s \ln \frac{1}{x}$ $\qquad$ $\sim \frac{1}{n!}\left(\bar{\alpha}_s \ln \frac{1}{x}\right)^n$

Cost of one additional gluon :

$$\frac{\alpha_s N_c}{\pi} \int_{k^+}^{p^+} \frac{dp_1^+}{p_1^+} = \frac{\alpha_s N_c}{\pi} \ln \frac{p^+}{k^+} = \bar{\alpha}_s \ln \frac{1}{x}$$

When $\bar{\alpha}_s \ln(1/x) \sim 1 \Longrightarrow$ Resummation = BFKL

$$xG(x, Q^2) \sim \mathrm{e}^{\omega \bar{\alpha}_s \ln 1/x} = \frac{1}{x^{\omega \bar{\alpha}_s}}, \quad \omega = 4 \ln 2$$

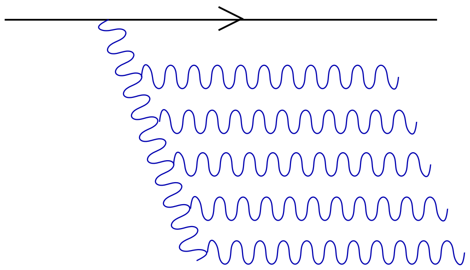
Rapid increase at small x

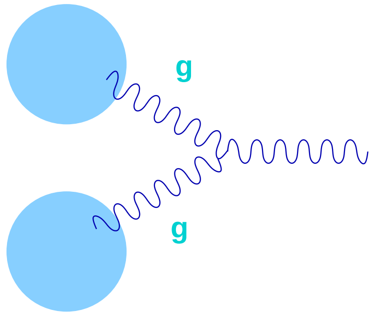
However : violates unitarity $\sigma \sim s^{\omega \bar{\alpha}_s}$

Gluon saturation

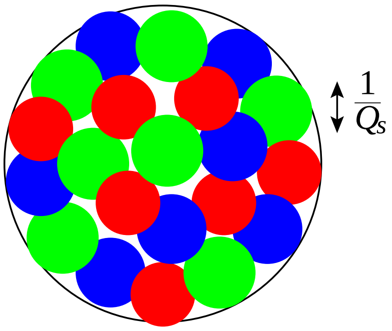
- Density per unit area grows at higher energy
  $\Longrightarrow$ Non-linear effects important!
- High density matter of "saturated gluons"
  - Color Glass Condensate
  - Realized under the balance between

"creation of small x gluons" & "gluon recombination"





- Saturation scale $1/Q_s(x)$:
  - Critical size for strong overlapping in $\perp$ plane
  - Hard scale $Q_s(x) \gg \Lambda_{\rm QCD}$ $\Rightarrow$ perturbative



Unitarization vs. Saturation

- The dipole–hadron scattering amplitude : $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$

$$\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) = 1 - \mathcal{S}_\tau(\boldsymbol{r}, \boldsymbol{b}) \leq 1 \quad \text{Unitarity}$$

$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$: size ; $\boldsymbol{b} = (\boldsymbol{x} + \boldsymbol{y})/2$: impact parameter

$$\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) = \frac{1}{N_c} \left\langle \mathrm{tr}\, U^\dagger(\boldsymbol{x}) U(\boldsymbol{y}) \right\rangle_\tau, \qquad \tau = \ln 1/x$$

$$U^\dagger(\boldsymbol{x}) \equiv \mathrm{Pexp}\left\{ ig \int dx^- A_a^+(x^-, \boldsymbol{x}) t^a \right\}$$

- Single scattering: Weak fields/Low gluon density

$$\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) \simeq -\boldsymbol{r}^2 \frac{\alpha_s C_F}{N_c^2 - 1} xG(x, 1/\boldsymbol{r}^2, \boldsymbol{b})$$

If xG becomes large (BFKL), $\mathcal{N}$ violates unitarity

- Independent multiple scattering: High density

$$\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) \simeq 1 - \exp\left\{ -\boldsymbol{r}^2 \frac{\alpha_s C_F}{N_c^2 - 1} xG(x, 1/\boldsymbol{r}^2, \boldsymbol{b}) \right\} < 1$$

Unitarized! $\boxed{\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) \sim 1 \ \text{ for } \ \boldsymbol{r} \sim 1/Q_s(\tau, \boldsymbol{b})}$

Unitarization of the scattering amplitude

- The Balitsky-Kovchegov (BK) equation ($\tau = \ln 1/x$)

$$\frac{\partial}{\partial\tau}\mathcal{N}_\tau(\boldsymbol{x},\boldsymbol{y}) = \frac{\alpha_s N_c}{\pi}\int \frac{d^2 z_\perp}{2\pi}\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}$$
$$\times\Big\{\underbrace{-\mathcal{N}_\tau(\boldsymbol{x},\boldsymbol{y})+\mathcal{N}_\tau(\boldsymbol{x},\boldsymbol{z})+\mathcal{N}_\tau(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL}}-\underbrace{\mathcal{N}_\tau(\boldsymbol{x},\boldsymbol{z})\mathcal{N}_\tau(\boldsymbol{z},\boldsymbol{y})}_{\text{Non}-\text{linear}}\Big\}$$

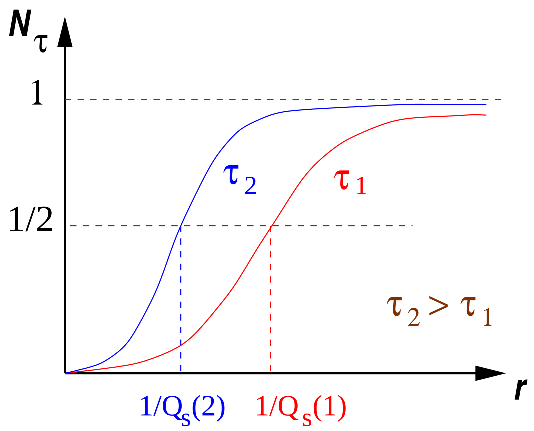
- $\boldsymbol{r}$ dependence of $\mathcal{N}_\tau(\boldsymbol{r},\boldsymbol{b})$ from the BK equation

  Most of them assume homogeneity in $\boldsymbol{b}$
  - Numerical solutions

  [Armesto-Braun(01), Levin-Tuchin(01), Golec=Biernat-Motyka-Staśto(01), Gotsman et al.(01), Levin-Lublinsky(01)]

  - Analytical solutions

  [Mueller(99), Kovchegov(00), Levin-Tuchin(01), Iancu-McLerran(01)]



Situation so far (summary) and ...

We have seen ...

- Hadronic cross section must satisfy the Froissart bound.
- No field-theoretic derivation of Froissart bound. (Nonperturbative?)
- Linear BFKL evolution violates the Froissart bound:
  $\sigma \propto s^{\omega \bar{\alpha}_s} \quad (\omega = 4 \ln 2,\ \bar{\alpha}_s = \alpha_s N_c/\pi)$
- Multiple scattering or non-linear saturation effects lead to unitarization of $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b}) < 1$ at fixed $\boldsymbol{b}$.
- $\boldsymbol{r}$ dependence of $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$ is well-understood.

$\Downarrow$

However, this doesn't necessarily imply the saturation of Froissart bound; We have to sum up $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$ over all possible $b_\perp$ to get cross section.

Need to know the $\boldsymbol{b}$ dependence of $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$.

Formulation of the problem

We compute the dipole-hadron cross section $\sigma_{\rm dipole}(\tau, \boldsymbol{r})$ ($\simeq$ meson-hadron cross section)

$$\sigma_{\rm dipole}(\tau, \boldsymbol{r}) = 2 \int d^2\boldsymbol{b}\, \mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$$

(1) perturbatively, by combining (2) gluon saturation and (3) confinement

(1) Treat only small sized dipole $\boldsymbol{r} \ll 1/\Lambda_{\rm QCD}$
$\Longrightarrow$ perturbatively computable

(2) $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$ obeys the BK equation. (saturation)

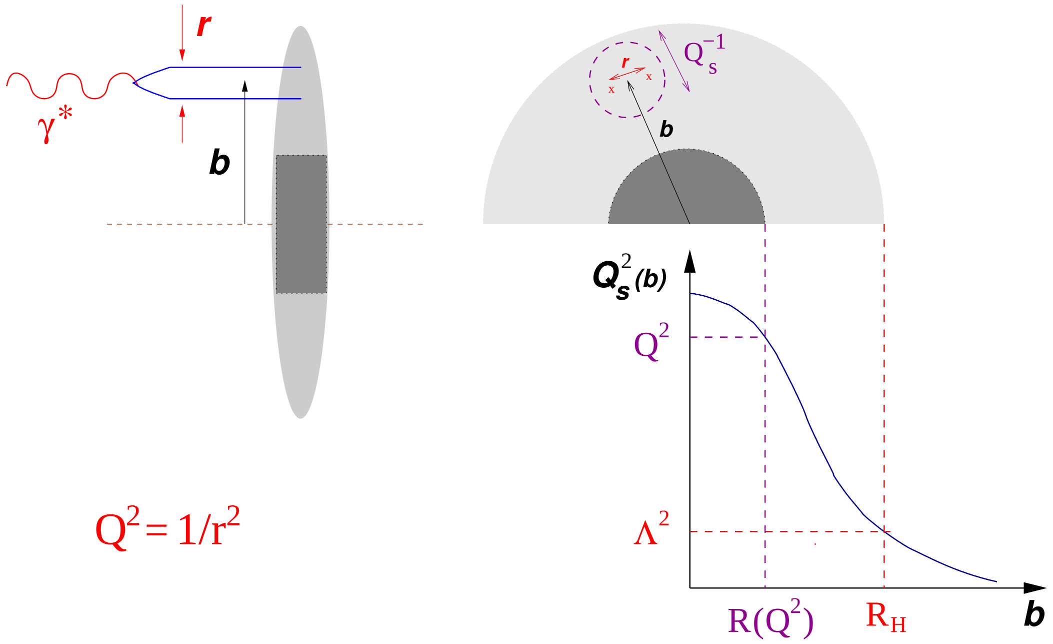
(3) Impact parameter dep. of $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$ at low energy

$$\mathcal{N}_{\tau_0}(\boldsymbol{r}, \boldsymbol{b}) \propto \mathrm{e}^{-\mu b}$$

μ: scale of confinement ex: $\mu = 2m_\pi$
Non-perturbative input as initial condition

Black disk radius

Dipole-hadron collision
in "dipole frame"



$\boxed{\text{Black disk radius } R(\tau, Q^2)}$

separates "saturated" and "non-saturated" regimes in $\boldsymbol{b}$ for fixed dipole size Q^2.

$R(\tau, Q^2)$ is defined by

$$\mathcal{N}_\tau(Q^2,\ b_\perp = R(Q^2)) \simeq 1$$

$b_\perp < R(Q^2)$: dipole is strongly absorbed, thus "black"
$b_\perp > R(Q^2)$: "gray" area, non-saturated regime.

Froissart bound from gluon saturation

- As the energy is increased, the black disk expands and dominates the dipole cross section.

$$\begin{aligned}\sigma_{\text{dipole}}(\tau, Q^2) &= 2\int d^2\boldsymbol{b}\,\mathcal{N}_\tau(Q^2,\boldsymbol{b}) \\ &\simeq 2\int d^2\boldsymbol{b}\;\Theta(R(\tau,Q^2)-b)\quad [\mathcal{N}_\tau^{\text{black disk}}\simeq 1] \\ &= 2\pi R^2(\tau,Q^2)\end{aligned}$$

- However, for the evolution of a black disk, scattering in the "gray" area is important.

1. Small dipole $\Rightarrow$ perturbative $\Rightarrow$ can use the BK eq.
2. Non-linear effects in BK "kills" the long range contribution $\Rightarrow$ short range scattering $|\boldsymbol{z}-\boldsymbol{b}|\lesssim 1/Q_s(\tau,b)$
   - $\mathcal{N}_\tau(\boldsymbol{x},\boldsymbol{z})\ll 1$ for short range scatt.
     $\Rightarrow$ Linearization $\Rightarrow$ BFKL eq. with boundary
   - Inhomogeneity in $b_\perp$ can be neglected
     $\Rightarrow$ preservation of the initial profile function

$$\mathcal{N}_\tau(Q^2,b_\perp)\approx\mathcal{N}_\tau(Q^2)S(b_\perp),\quad S(b_\perp)=e^{-2m_\pi b}$$

Froissart bound from gluon saturation

- With the homogeneous BFKL solution and the impact parameter dependence $S(b_\perp) = \mathrm{e}^{-2m_\pi b}$, the amplitude is

$$\begin{aligned} \mathcal{N}_\tau(Q^2, b_\perp) &\approx \sqrt{\frac{\Lambda^2}{Q^2}}\, \mathrm{e}^{\omega\bar{\alpha}_s\tau} \times \mathrm{e}^{-2m_\pi b} \\ &= \exp\left\{ -2m_\pi b + \omega\bar{\alpha}_s\tau - \frac{1}{2}\ln\frac{Q^2}{\Lambda^2} \right\} \end{aligned}$$

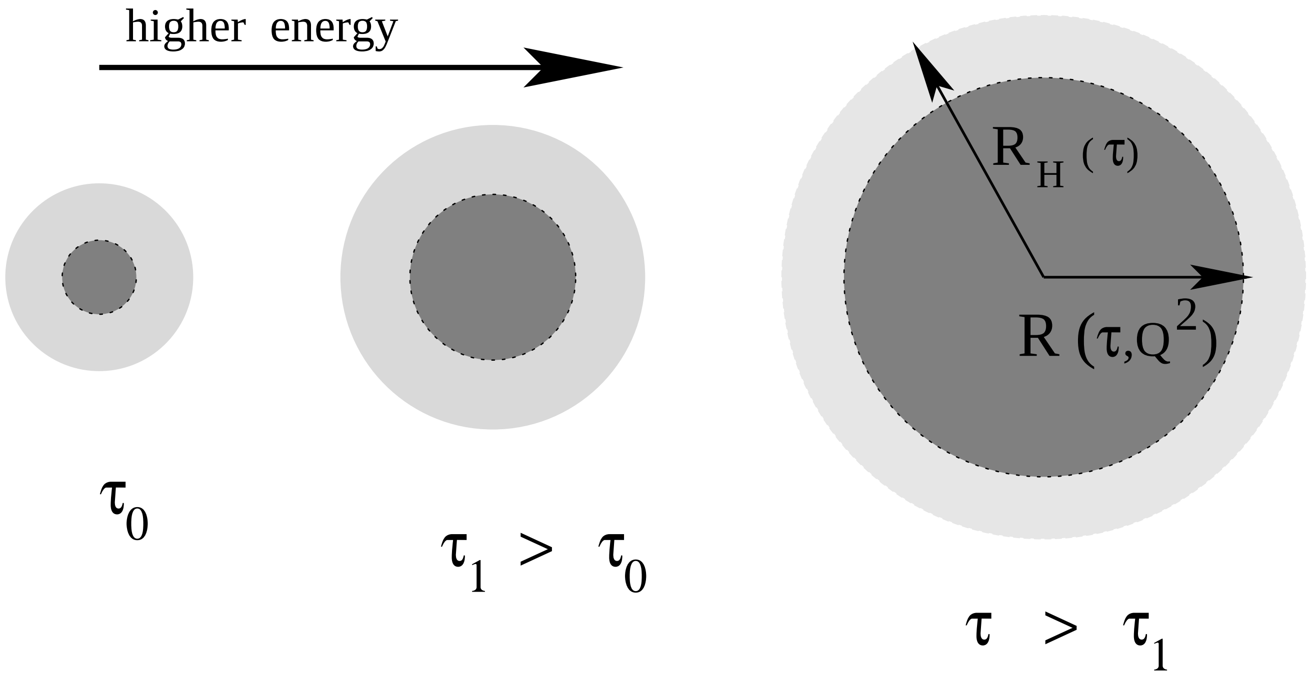
- Therefore, the black disk radius is given by

$$R(\tau, Q^2) \simeq \frac{1}{2m_\pi}\left(\omega\bar{\alpha}_s\tau - \frac{1}{2}\ln\frac{Q^2}{\Lambda^2} \right)$$

giving the cross section consistent with the Froissart bound

$$\begin{aligned} \sigma_{\rm dipole}(\tau, Q^2) &\simeq \frac{\pi}{2m_\pi^2}\left(\omega\bar{\alpha}_s\tau - \frac{1}{2}\ln\frac{Q^2}{\Lambda^2} \right)^2 \\ &\simeq \frac{\pi}{2}\left(\frac{\omega\bar{\alpha}_s}{m_\pi}\right)^2 \ln^2 s \quad \text{as} \quad s \to \infty \end{aligned}$$

Froissart bound from Gluon Saturation

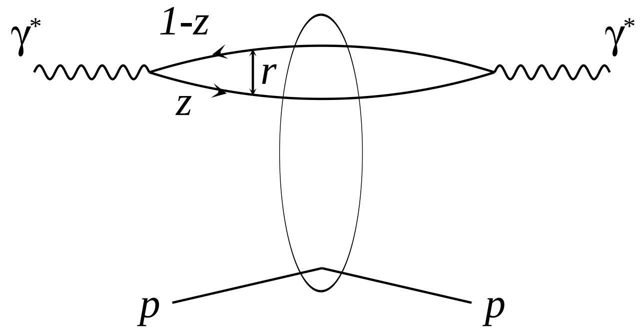


Total $\gamma^* p$ cross section $\sigma_{\rm tot} = \sigma_T + \sigma_L$

At small x $\Psi_{T,L}(z, r_\perp; Q^2)$ LC wavefunction of γ^*

$$\sigma_{T,L}(\tau, Q^2) = \int_0^1 dz \int d^2\boldsymbol{r}\, |\Psi_{T,L}(z, \boldsymbol{r}; Q^2)|^2\, \sigma_{\rm dipole}(\tau, \boldsymbol{r})$$

$\Psi_{T,L}$ $\sigma_{\rm dipole}$ $\Psi^*_{T,L}$



$$\sigma_{\rm dipole}(\tau, \boldsymbol{r}) \simeq \frac{\pi}{2}\left(\frac{\omega\bar{\alpha}_s}{m_\pi}\right)^2 \ln^2 s$$

$$\Longrightarrow \quad \sigma_{\rm tot}^{\gamma^* p} \propto \ln^3 s \quad (s \to \infty)$$

One extra $\ln s$ comes from the LC wavefunction.

Summary for the second half

To compute the dipole-hadron cross section at high energy one needs

1. information about the increase of scatt. amplitude $\mathcal{N}_\tau(\boldsymbol{r}, \boldsymbol{b})$ with s at fixed $\boldsymbol{b}$
   [must ensure unitarity $\mathcal{N} \leq 1$]
2. information about its decrease with $\boldsymbol{b}$ at fixed s

We solved this problem by using

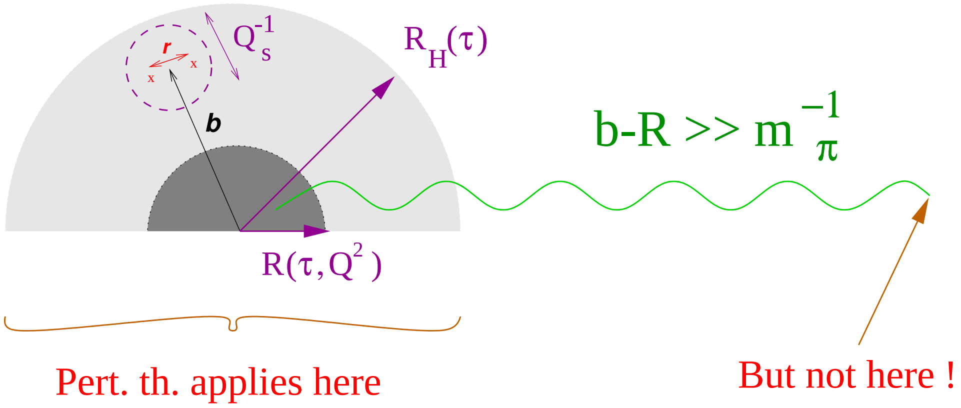
1. perturbative but non-linear evolution equation
   [BK equation which shows saturation]
2. non-perturbative initial condition [confinement, determining the tail behavior at large $\boldsymbol{b}$]

and found that, as a result of conpensation of 1. and 2.,

the dipole-hadron cross section saturates the Froissart bound.

$$\sigma_{\rm dipole}(\tau, Q^2) \simeq \frac{\pi}{2}\left(\frac{\omega\bar{\alpha}_s}{m_\pi}\right)^2 \ln^2 s \quad \text{as} \quad s \to \infty$$

Short and long distance contributions



- Black disk : $\mathcal{N}_\tau(\boldsymbol{r},\boldsymbol{b}) \simeq 1$ for $b \leq R(\tau,Q^2)$
- Grey area $[b > R(\tau,Q^2)]$: $\mathcal{N}_\tau = \mathcal{N}_\tau\big|_{short} + \mathcal{N}_\tau\big|_{long}$

  $\mathcal{N}_\tau\big|_{short} \sim \exp\{\omega\bar{\alpha}_s\tau - 2m_\pi b\}$ (local sources)

  $$\mathcal{N}_\tau\big|_{long} \sim \bar{\alpha}_s \frac{\boldsymbol{r}^2}{2\boldsymbol{b}^4}\int^\tau d\eta\; R^2(\eta,Q^2) \quad \text{(saturated gluons)}$$

- Perturbative grey corona: $R(\tau,Q^2) < b < R_H(\tau)$

  Short-range dominates: $\mathcal{N}_\tau\big|_{short} \gg \mathcal{N}_\tau\big|_{long}$

- Non-perturbative "white region": $b - R_H(\tau) \gg m_\pi^{-1}$

  Long-range formally dominates $\Longrightarrow$ Power-law tails
  $\Longrightarrow$ Unitarity violations [Kovner and Wiedemann, 01]

  Unphysical, since removed by confinement:
  $$1/b^4 \longrightarrow (1/b^4)\,\mathrm{e}^{-2m_\pi(b-R(\tau,Q^2))}$$

- Pert. theory : "Grey" $\to$ "Black" but not "White" $\to$ "Grey"