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THE EFFECT OF SPHERICAL DISTANCE APPROXIMATIONS UPON OI FORECAST ERROR CORRELATIONS

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Abstract

The accuracy of forecast error correlation functions prescribed in an optimal interpolation (OI) analysis system determines, to a large extent, the analysis accuracy. The height-height forecast error correlation function used in the OI system at NMC is a Gaussian function of approximate spherical distance. One approximation is used equatorward of 70° latitude and another is used poleward of 70°. Since the wind-height and wind-wind correlation functions are derived from the height-height correlation via a geostrophic assumption, they depend on first and second derivatives of the approximate spherical distance. Derivatives of an approximation are generally less accurate than the approximation itself.

We show that at 70° latitude the wind-wind cross-correlation function used operationally equatorward of 70° latitude has roughly twice the amplitude of the same correlation function based on the exact spherical distance. At lower latitudes, the correlations based on exact and approximate spherical distances are more comparable. We have not compared the approximate formulation used operationally poleward of 70° with the exact spherical distance formulation.

The approximate distance formulations have been used operationally for the sake of computational efficiency. We introduce a different approximation to the spherical distance which results in more computational work than the operational formulation, but less work than an exact spherical distance formulation. We prove rigorously that all the correlations based on the new approximation differ from those based on the exact spherical distance by a negligible amount. The new approximate formulation is accurate at all latitudes, and therefore dispenses with the need for separate computations in low and high latitudes.

I. Introduction

The accuracy of forecast error correlation functions prescribed in an optimal interpolation (OI) analysis system determines, to a large extent, the analysis accuracy. In the global OI system at NMC (Bergman, 1979; McPherson, et.al., 1979), the correlation C^{zz} between height errors at two points $P_1=(\lambda_1, \phi_1, p_1)$ and $P_2=(\lambda_2, \phi_2, p_2)$ is specified as a product

$$C^{zz}(P_1; P_2) = H^{zz}(\lambda_1, \phi_1; \lambda_2, \phi_2)\, V^{zz}(p_1; p_2),$$

in which the horizontal function is of the form

$$H^{zz}(\lambda_1, \phi_1; \lambda_2, \phi_2) = e^{-\frac{1}{2} b s^2}.$$

Here b is a positive constant and the distance function s depends on $\lambda_1, \phi_1, \lambda_2$ and ϕ_2. The wind-height and wind-wind correlations are derived from C^{zz} by assuming that forecast errors are geostrophic.

In this note, we derive and compare the correlation functions - height-height, wind-height, and wind-wind - obtained for four different distance functions, denoted by s_0, s_1, s_2 and s_3. These functions are given by

$$\cos s_0 = \cos(\phi_1 - \phi_2) - \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right],$$

$$\frac{1}{2} s_1^2 = \left[1 - \cos(\phi_1 - \phi_2)\right] + \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right],$$

$$s_2^2 = (\phi_1 - \phi_2)^2 + (\lambda_1 - \lambda_2)^2 \cos^2\left(\frac{\phi_1 + \phi_2}{2}\right),$$

$$s_3^2 = (\phi_1 - \phi_2)^2 + (\lambda_1 - \lambda_2)^2 \cos^2 \phi_0 .$$

We refer to s_o as the exact spherical distance between points (λ_1, ϕ_1) and (λ_2, ϕ_2). It is actually the angle subtended at the center of a sphere by the two points; $\underline{a}\ \underline{s}_o$, with $\underline{a}$ being the earth radius, is the spherical, or great-circle, distance between the points. The straight-line distance (through the sphere) is given by $\underline{a}\ \underline{s}_1$. To our knowledge, s_o has never been used in an operational OI system. The straight-line distance s_1 is used by the European Centre for Medium Range Weather Forecasts' OI analysis procedure (Lorenc, personal communication). The function s_2 was introduced by Schlatter (1975), and was used equatorward of 70° in the global OI system at NMC until recently. The approximation s_3, in which ϕ_o is locally constant, is currently used equatorward of 70° in the global system (Kistler, personal communication). A separate approximation, which we do not study, is used at NMC poleward of 70°, since s_2 and s_3 are known to approximate s_o poorly past about 70°. A similar approximation is used by the OI analysis system at the Canadian Meteorological Centre everywhere on its hemispheric polar-stereographic analysis grid (Rutherford, 1976).

The motivation for this work is that if two functions differ by a small amount, it is not necessarily true that their derivatives do also. The wind-height and wind-wind correlations depend on first and second derivatives of $\underline{s}$ as a result of the geostrophic assumption. In fact, we found that the height-height correlations based on each of the functions s_i only differ by a negligible amount, while for the wind-height correlations and wind-wind correlations the values obtained using s_2 or s_3 differ more substantially from those obtained using s_o. The differences are small in low latitudes, and generally increase with latitude. The largest differences were obtained for the wind-wind cross-correlation C^{uv} at 70°. In this case, the correlations based on either s_2 or s_3 are roughly double the correlations based on s_o.

The approximate distance formulas s_2 and s_3 have been used operationally at NMC mostly for the sake of computational efficiency. Indeed, the correlation formulas we derive based on s_0 are somewhat more complicated than those we derive based on s_2 and s_3. We have introduced approximation s_1 as an alternative. We show, first of all, that the correlation formulas to which s_1 leads are of about the same complexity as those based on s_2. We also prove rigorously that the correlations based on s_1 differ from those based on s_0 by a negigible amount: less than 0.00056 for the height-height correlations, 0.0063 for the wind-height correlations, and 0.016 for the wind-wind correlations. These bounds are valid at all latitudes: approximation s_1 is a uniform approximation and dispenses with the need for separate computations in low and high latitudes.

Our derivation of the correlation formulas for s_0, s_1, s_2 and s_3 is based on the general correlation formulas obtained in the companion paper (Cohn and Morone, 1984). Those formulas include the effect of the spatial variability of height-field forecast error variances upon the forecast error correlations themselves. In the present paper we have retained the terms accounting for this effect; otherwise the correlation formulas we derive for s_2 and s_3 are algebraically equivalent to those which have been used operationally. Furthermore, the aforementioned bounds on the differences between correlations based on s_1 and those based on s_0 hold regardless of the size of the contribution due to these terms. Neglect of these terms, as is done in current operational practice, would improve the bounds.

The general formulas from the companion paper upon which the present work is based are summarized in Section II. In Section III we derive the correlation formulas for each of s_0, s_1, s_2 and s_3 in turn, and we prove the bounds on differences between correlations based on s_1 and those based on s_0. Plots of correlation functions based on s_o, s_2 and s_3 are discussed in Section IV. Conclusions follow in Section V. Inequalities needed for the error bounds of Section III are proven in an Appendix.

The reader who is interested primarily in comparing the correlation functions arising from the various distance formulas may examine (3.17, 3.10), which give the formulas for s_o; (3.23, 3.10) for s_1; (3.23, 3.34) for s_2; and (3.23, 3.37) for s_3.

II. Summary of Geostrophic Forecast Error Covariance Relationships

Here we summarize the general relationships among forecast error covariances which were derived in the companion paper. These relationships are based upon one assumption only, that the wind-field forecast errors u_i and v_i, at a point $P_i = (\lambda_i, \phi_i, p_i)$, are related geostrophically to the height-field forecast error Z_i:

$$u_i = \alpha_i \frac{\partial}{\partial \phi_i} z_i \quad , \quad v_i = \beta_i \frac{\partial}{\partial \lambda_i} z_i \tag{2.1a,b}$$

where

$$\alpha_i = -\frac{G_i g}{f_i a} \quad , \quad \beta_i = +\frac{G_i g}{f_i a \cos \phi_i} \tag{2.2a,b}$$

Here λ_i, ϕ_i and p_i are the longitude, latitude and pressure coordinates of point P_i, g is the gravitational acceleration, a is the radius of the earth, f_i is the Coriolis parameter, and G_i is the so-called coefficient of geostrophy.

Under assumption (2.1, 2.2), the wind-field forecast error standard deviations σ_i^u and σ_i^v are given by

$$\sigma_i^u = \sigma_i^z \, |\alpha_i| \left[\lim_{P_j \to P_i} \frac{\partial^2 \log C^{zz}}{\partial \phi_i \, \partial \phi_j} + \left(\frac{\partial \log \sigma_i^z}{\partial \phi_i} \right)^2 \right]^{1/2} \tag{2.3a}$$

$$\sigma_i^v = \sigma_i^z |\beta_i| \left[\lim_{P_j \to P_i} \frac{\partial^2 \log C^{zz}}{\partial \lambda_i \partial \lambda_j} + \left(\frac{\partial \log \sigma_i^z}{\partial \lambda_i} \right)^2 \right]^{1/2}, \tag{2.3b}$$

where σ_i^z is the height-field forecast error standard deviation and $C^{zz} = C^{zz}(P_i; P_j)$ is the three-dimensional correlation between height-field forecast errors at P_i and P_j.

Defining quantities γ_i and δ_i by

$$\gamma_i = (\operatorname{sign} \alpha_i) \left[\lim_{P_j \to P_i} \frac{\partial^2 \log C^{zz}}{\partial \phi_i \partial \phi_j} + \left(\frac{\partial \log \sigma_i^z}{\partial \phi_i} \right)^2 \right]^{-1/2}, \tag{2.4a}$$

$$\delta_i = (\operatorname{sign} \beta_i) \left[\lim_{P_j \to P_i} \frac{\partial^2 \log C^{zz}}{\partial \lambda_i \partial \lambda_j} + \left(\frac{\partial \log \sigma_i^z}{\partial \lambda_i} \right)^2 \right]^{-1/2}, \tag{2.4b}$$

and setting i=1, j=2, the wind-height forecast error correlations C^{uz}, C^{zu}, C^{vz}, C^{zv}, and the wind-wind forecast error correlations C^{uv}, C^{vu}, C^{uu} and C^{vv} are given by

$$C^{uz}/C^{zz} = \gamma_1 \left(\frac{\partial \log C^{zz}}{\partial \phi_1} + \frac{\partial \log \sigma_1^z}{\partial \phi_1} \right), \tag{2.5a}$$

$$C^{zu}/C^{zz} = \gamma_2 \left(\frac{\partial \log C^{zz}}{\partial \phi_2} + \frac{\partial \log \sigma_2^z}{\partial \phi_2} \right), \tag{2.5b}$$

$$C^{vz}/C^{zz} = \delta_1 \left(\frac{\partial \log C^{zz}}{\partial \lambda_1} + \frac{\partial \log \sigma_1^z}{\partial \lambda_1} \right), \tag{2.5c}$$

$$C^{zv}/C^{zz} = \delta_2 \left(\frac{\partial \log C^{zz}}{\partial \lambda_2} + \frac{\partial \log \sigma_2^z}{\partial \lambda_2} \right), \tag{2.5d}$$

$$C^{uv}/C^{zz} = \gamma_1 \delta_2 \frac{\partial^2 \log C^{zz}}{\partial \phi_1 \partial \lambda_2} + \frac{C^{uz}}{C^{zz}} \frac{C^{zv}}{C^{zz}}, \tag{2.5e}$$

$$C^{vu}/C^{zz} = \delta_1 \gamma_2 \frac{\partial^2 \log C^{zz}}{\partial \lambda_1 \partial \phi_2} + \frac{C^{vz}}{C^{zz}} \frac{C^{zu}}{C^{zz}} , \tag{2.5f}$$

$$C^{uu}/C^{zz} = \gamma_1 \gamma_2 \frac{\partial^2 \log C^{zz}}{\partial \phi_1 \partial \phi_2} + \frac{C^{uz}}{C^{zz}} \frac{C^{zu}}{C^{zz}} , \tag{2.5g}$$

$$C^{vv}/C^{zz} = \delta_1 \delta_2 \frac{\partial^2 \log C^{zz}}{\partial \lambda_1 \partial \lambda_2} + \frac{C^{vz}}{C^{zz}} \frac{C^{zv}}{C^{zz}} . \tag{2.5h}$$

III. Covariance Relationships for Various Spherical Distance Approximations

We derive in this section the relationships among forecast error covariances, based on the assumptions that forecast errors are geostrophic (2.1) and that the height-height forecast error correlation C^{zz} is of the form

$$C^{zz}(\lambda_1, \phi_1, p_1; \lambda_2, \phi_2, p_2) = H^{zz}(\lambda_1, \phi_1; \lambda_2, \phi_2) V^{zz}(p_1; p_2), \tag{3.1}$$

where V^{zz} is an arbitrary correlation function of two pressure levels, and

$$H^{zz} = e^{-\frac{1}{2} b s^2} \tag{3.2}$$

Here the dimensionless constant <u>b</u> is given by

$$b = 2a^2/d_o^2 , \tag{3.3}$$

where $a=6371$ km is the radius of the earth and d_o is the correlation distance. Currently at NMC, $d_o = \frac{1}{\sqrt{2}} \times 10^3$ km, so that

$$b \cong 162 . \tag{3.4}$$

The distance function $s = s(\lambda_1, \phi_1; \lambda_2, \phi_2)$ in (3.2) will be either the angle subtended at the center of the sphere by points (λ_1, ϕ_1) and (λ_2, ϕ_2), or an approximation to that angle. That is, a s will be either the spherical distance between two points or an approximation thereof.

III.1. Exact Spherical Distance

The angle s_o subtended at the center of the earth by two points (λ_1, ϕ_1), (λ_2, ϕ_2) satisfies $0 \leq s_o \leq \pi$ and is given by

$$\cos s_o = \sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2) \tag{3.5a}$$

or

$$\cos s_o = \cos(\phi_1 - \phi_2) - \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right]; \tag{3.5b}$$

this s_o is also the spherical, or great-circle, distance between two points on the unit sphere. We prefer formula (3.5b) from a computational standpoint: it requires less trigonometric function evaluation (or table lookup) to calculate s_o than does (3.5a). The formulas below to which (3.5b) leads are also simplified somewhat.

We calculate in this subsection the standard deviations (2.3) and correlations (2.5) based on the height-height correlation (3.1, 3.2), with s given by $s = s_o$. To do so, we must evaluate the first and second derivatives of $\log C^{zz}$. The form of (3.1) immediately implies that

$$\frac{\partial \log C^{zz}}{\partial \xi} = \frac{\partial \log H^{zz}}{\partial \xi} \tag{3.6a}$$

and

$$\frac{\partial \log C^{zz}}{\partial \eta\, \partial \xi} = \frac{\partial \log H^{zz}}{\partial \eta\, \partial \xi} \tag{3.6b}$$

where ξ and η denote any of the coordinates $\lambda_1, \phi_1, \lambda_2, \phi_2$ since for example:

$$\frac{\partial \log C^{zz}}{\partial \xi} = \frac{1}{C^{zz}} \frac{\partial C^{zz}}{\partial \xi} = \frac{1}{H^{zz}V^{zz}} \frac{\partial H^{zz}V^{zz}}{\partial \xi} = \frac{1}{H^{zz}} \frac{\partial H^{zz}}{\partial \xi} = \frac{\partial \log H^{zz}}{\partial \xi} .$$

With $s = s_o$ given by (3.5b), the calculation is simplified by expressing the derivatives of log C^{zz} in terms of those of cos s_o. From (3.6a) and (3.2), we have

$$\frac{\partial \log C^{zz}}{\partial \xi} = -b s_o \frac{\partial s_o}{\partial \xi} ,$$

and since

$$\frac{\partial \cos s_o}{\partial \xi} = -\sin s_o \frac{\partial s_o}{\partial \xi} ,$$

we find that

$$\frac{\partial \log C^{zz}}{\partial \xi} = b\, q(s_o) \frac{\partial \cos s_o}{\partial \xi} , \tag{3.7a}$$

provided $s_o \neq 0$ or π , where

$$q(s_o) = \frac{s_o}{\sin s_o} . \tag{3.7b}$$

Differentiating (3.7a) gives

$$\frac{\partial^2 \log C^{zz}}{\partial\eta\,\partial\xi} = b\,q(s_o)\,\frac{\partial^2 \cos s_o}{\partial\eta\,\partial\xi} + b\,\frac{\partial q(s_o)}{\partial\eta}\,\frac{\partial \cos s_o}{\partial\xi} .$$

Now from (3.7b),

$$\frac{\partial q}{\partial\eta} = \frac{\sin s_o - s_o\cos s_o}{\sin^2 s_o}\,\frac{\partial s_o}{\partial\eta} = \frac{\sin s_o - s_o\cos s_o}{-\sin^3 s_o}\,\frac{\partial \cos s_o}{\partial\eta} ,$$

so that

$$\frac{\partial^2 \log C^{zz}}{\partial\eta\,\partial\xi} = b\,q(s_o)\,\frac{\partial^2 \cos s_o}{\partial\eta\,\partial\xi} - b\,r(s_o)\,\frac{\partial \cos s_o}{\partial\eta}\,\frac{\partial \cos s_o}{\partial\xi} , \tag{3.8a}$$

where

$$r(s_o) = \frac{\sin s_o - s_o\cos s_o}{\sin^3 s_o} . \tag{3.8b}$$

Formulas (3.7) and (3.8) express the derivatives of log C^{zz} in terms of those of cos s_o. The derivatives of cos s_o are calculated directly from (3.5b), as follows:

$$\frac{\partial \cos s_o}{\partial\phi_1} = F^{uz} , \tag{3.9a}$$

$$\frac{\partial \cos s_o}{\partial\phi_2} = F^{zu} , \tag{3.9b}$$

$$\frac{\partial \cos s_o}{\partial\lambda_1} = F^{vz}\cos\phi_1 , \tag{3.9c}$$

$$\frac{\partial \cos s_o}{\partial\lambda_2} = F^{zv}\cos\phi_2 , \tag{3.9d}$$

$$\frac{\partial^2 \cos s_o}{\partial\phi_1\,\partial\lambda_2} = F^{uv}\cos\phi_2 , \tag{3.9e}$$

$$\frac{\partial^2 \cos s_o}{\partial\lambda_1\,\partial\phi_2} = F^{vu}\cos\phi_1 , \tag{3.9f}$$

$$\frac{\partial^2 \cos s_o}{\partial \phi_1 \partial \phi_2} = F^{uu} , \tag{3.9g}$$

$$\frac{\partial^2 \cos s_o}{\partial \lambda_1 \partial \lambda_2} = F^{vv} \cos \phi_1 \cos \phi_2 , \tag{3.9h}$$

where we have introduced the notation (for reasons to become clear momentarily)

$$F^{uz} = -\sin(\phi_1 - \phi_2) + \sin\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right], \tag{3.10a}$$

$$F^{zu} = +\sin(\phi_1 - \phi_2) + \cos\phi_1 \sin\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right], \tag{3.10b}$$

$$F^{vz} = -\cos\phi_2 \sin(\lambda_1 - \lambda_2) , \tag{3.10c}$$

$$F^{zv} = +\cos\phi_1 \sin(\lambda_1 - \lambda_2) , \tag{3.10d}$$

$$F^{uv} = -\sin\phi_1 \sin(\lambda_1 - \lambda_2) , \tag{3.10e}$$

$$F^{vu} = +\sin\phi_2 \sin(\lambda_1 - \lambda_2) , \tag{3.10f}$$

$$F^{uu} = +\cos(\phi_1 - \phi_2) - \sin\phi_1 \sin\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right], \tag{3.10g}$$

$$F^{vv} = +\cos(\lambda_1 - \lambda_2) . \tag{3.10h}$$

We are now ready to evaluate the variance and correlation relationships (2.3, 2.5). Notice first that, away from the poles, the first six functions $F^{\cdot\cdot}$ above all approach zero as point $P_2 = (\lambda_2, \phi_2)$ approaches point $P_1 = (\lambda_1, \phi_1)$, while F^{uu} and F^{vv} approach one. The first derivatives of $\cos s_o$ (3.9a-d), therefore approach zero as $P_2 \rightarrow P_1$, and since

$$\lim_{s\to 0} q(s) = \lim_{s\to 0} \frac{s}{\sin s} = 1 \tag{3.11a}$$

is finite, (3.7a) shows that the first derivatives of log C^{zz} also approach zero as $P_2 \to P_1$. (Indeed they must: see Eq. (2.10b) of the companion paper.) Now, since

$$\lim_{s\to 0} r(s) = \lim_{s\to 0} \frac{\sin s - s\cos s}{\sin^3 s} = \frac{1}{3} \tag{3.11b}$$

is finite, we have from (3.8a) that

$$\lim_{P_2\to P_1} \frac{\partial^2 \log C^{zz}}{\partial\eta\,\partial\xi} = b \lim_{P_2\to P_1} \frac{\partial^2 \cos s_0}{\partial\eta\,\partial\xi} \quad . \tag{3.12}$$

Equations (3.9e-h, 3.10e-h) then imply that the mixed second derivatives of log C^{zz} approach zero as $P_2 \to P_1$, while

$$\lim_{P_2\to P_1} \frac{\partial^2 \log C^{zz}}{\partial\phi_1\,\partial\phi_2} = b \quad , \tag{3.13a}$$

and

$$\lim_{P_2\to P_1} \frac{\partial^2 \log C^{zz}}{\partial\lambda_1\,\partial\lambda_2} = b\,\cos^2\phi_1 \quad . \tag{3.13b}$$

From (3.13) and (2.2), we finally have for the wind forecast error standard deviations (2.3),

$$\boxed{\begin{aligned} \sigma_i^u &= \sigma_i^z \frac{G_i g \sqrt{b}}{|f_i|\, a}\left[1 + \left(\frac{1}{\sqrt{b}}\,\frac{\partial \log \sigma_i^z}{\partial\phi_i}\right)^2\right]^{1/2} \quad , & (3.14a) \\ \sigma_i^v &= \sigma_i^z \frac{G_i g \sqrt{b}}{|f_i|\, a}\left[1 + \left(\frac{1}{\sqrt{b}}\,\frac{\partial \log \sigma_i^z}{\cos\phi_i\,\partial\lambda_i}\right)^2\right]^{1/2} \quad . & (3.14b) \end{aligned}}$$

Similarly, the factors γ_i and δ_i defined in (2.4) become

$$\gamma_i = (\operatorname{sign} \alpha_i) \frac{1}{\sqrt{b}} \left[1 + \left(\frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\partial \phi_i} \right)^2 \right]^{-1/2} , \tag{3.15a}$$

$$\delta_i = (\operatorname{sign} \beta_i) \frac{1}{\sqrt{b} \cos \phi_i} \left[1 + \left(\frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\cos \phi_i \partial \lambda_i} \right)^2 \right]^{-1/2} . \tag{3.15b}$$

We also define the quantities

$$\gamma_i' = \gamma_i \sqrt{b} = (\operatorname{sign} \alpha_i) \left[1 + \left(\frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\partial \phi_i} \right)^2 \right]^{-1/2} , \tag{3.16a}$$

$$\delta_i' = \delta_i \sqrt{b} \cos \phi_i = (\operatorname{sign} \beta_i) \left[1 + \left(\frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\cos \phi_i \partial \lambda_i} \right)^2 \right]^{-1/2} . \tag{3.16b}$$

Finally, from (3.7a), (3.8a), and (3.9a-h) and (3.16a,b), the forecast error correlations (2.5) become

$$C^{uz}/C^{zz} = \gamma_1' \left[\sqrt{b}\, q(s_0) F^{uz} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \right] , \tag{3.17a}$$

$$C^{zu}/C^{zz} = \gamma_2' \left[\sqrt{b}\, q(s_0) F^{zu} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^z}{\partial \phi_2} \right] , \tag{3.17b}$$

$$C^{vz}/C^{zz} = \delta_1' \left[\sqrt{b}\, q(s_0) F^{vz} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\cos \phi_1 \partial \lambda_1} \right] , \tag{3.17c}$$

$$C^{zv}/C^{zz} = \delta_2' \left[\sqrt{b}\, q(s_0) F^{zv} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^z}{\cos \phi_2 \partial \lambda_2} \right] , \tag{3.17d}$$

$$C^{uv}/C^{zz} = \gamma_1' \delta_2' \left[q(s_0) F^{uv} - r(s_0) F^{uz} F^{zv} \right] + \frac{C^{uz}}{C^{zz}} \frac{C^{zv}}{C^{zz}} \tag{3.17e}$$

$$C^{vu}/C^{zz} = \delta_1'\gamma_2'\left[q(s_o)F^{vu} - r(s_o)F^{vz}F^{zu}\right] + \frac{C^{vz}}{C^{zz}}\frac{C^{zu}}{C^{zz}} \quad , \tag{3.17f}$$

$$C^{uu}/C^{zz} = \gamma_1'\gamma_2'\left[q(s_o)F^{uu} - r(s_o)F^{uz}F^{zu}\right] + \frac{C^{uz}}{C^{zz}}\frac{C^{zu}}{C^{zz}} \quad , \tag{3.17g}$$

$$C^{vv}/C^{zz} = \delta_1'\delta_2'\left[q(s_o)F^{vv} - r(s_o)F^{vz}F^{zv}\right] + \frac{C^{vz}}{C^{zz}}\frac{C^{zv}}{C^{zz}} \quad . \tag{3.17h}$$

In these formulas, γ_i' and δ_i' are given by (3.16a,b), q (s_o) and r (s_o) are given by (3.7b) and (3.8b), and the functions $F^{\cdot\cdot}$ are given by (3.10a-h). Notice that gradients of σ_i^z are needed in the standard deviation formulas (3.14) and corelation formulas (3.17) only in the form

$$\frac{1}{\sqrt{b}}\frac{\partial \log \sigma_i^z}{\partial \phi_i} \quad \text{or} \quad \frac{1}{\sqrt{b}}\frac{\partial \log \sigma_i^z}{\cos\phi_i\, \partial \lambda_i}$$

III.2. Straight-line distance

It is well-known that forecast error correlations are nearly zero past distances on the order of 1000 km. For example, H^{zz} given by (3.2, 3.3) assumes the value $H^{zz} = e^{-4} = 0.018$ at distance $as = 2d_o$. One expects that if $s = s_o$ given by (3.5) is approximated by another function $s = s_1$ which nearly agrees with s_o when s_o is small, then one can obtain standard deviation and correlation formulas that are simpler than (3.14, 3.17) but which yield nearly the same values.

A simple way to do this is to use the fact that

$$\cos s = 1 - \tfrac{1}{2}s^2 + O(s^4) \quad ,$$

so that if we define s_1 by

$$1 - \frac{1}{2} s_1^2 = \cos s_o \ , \tag{3.18a}$$

i.e.,

$$\frac{1}{2} s_1^2 = \left[1 - \cos(\phi_1 - \phi_2)\right] + \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right] \ , \tag{3.18b}$$

then s_1 is a good approximation to s_o when s_o is small. While it is not generally true that if two functions differ by a small amount, then their derivatives also do, we show that when $s = s_1$ in (3.2), the resulting standard deviations and correlations are indeed nearly identical to those given by (3.14, 3.17).

To begin, we point out that s_1 is in fact the straight-line distance (through the sphere) between points $P_1 = (\lambda_1, \phi_1)$ and $P_2 = (\lambda_2, \phi_2)$. The rectangular coordinates of a point on the unit sphere are $(x, y, z) = (r\cos\lambda, r\sin\lambda, \sin\phi)$, where $r = \cos\phi$. The straight-line distance s between P_1 and P_2 is therefore given by

$$\begin{aligned} s^2 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 \\ &= (\cos\phi_1 \cos\lambda_1 - \cos\phi_2 \cos\lambda_2)^2 + (\cos\phi_1 \sin\lambda_1 - \cos\phi_2 \sin\lambda_2)^2 \\ &\qquad + (\sin\phi_1 - \sin\phi_2)^2 \\ &= \cos^2\phi_1 \cos^2\lambda_1 - 2\cos\phi_1 \cos\lambda_1 \cos\phi_2 \cos\lambda_2 + \cos^2\phi_2 \cos^2\lambda_2 \\ &\quad + \cos^2\phi_1 \sin^2\lambda_1 - 2\cos\phi_1 \sin\lambda_1 \cos\phi_2 \sin\lambda_2 + \cos^2\phi_2 \sin^2\lambda_2 \\ &\quad + \sin^2\phi_1 \qquad - 2\sin\phi_1 \sin\phi_2 \qquad + \sin^2\phi_2 \end{aligned}$$

$$= 2\left[1 - \cos\phi_1 \cos\lambda_1 \cos\phi_2 \cos\lambda_2 - \cos\phi_1 \sin\lambda_1 \cos\phi_2 \sin\lambda_2 - \sin\phi_1 \sin\phi_2\right]$$

$$= 2\left[1 - \cos\phi_1 \cos\phi_2 \cos(\lambda_1 - \lambda_2) - \sin\phi_1 \sin\phi_2\right]$$

$$= 2\left\{1 - \cos(\phi_1 - \phi_2) + \cos\phi_1 \cos\phi_2\left[1 - \cos(\lambda_1 - \lambda_2)\right]\right\} ,$$

which is equivalent to (3.18b).

Since s_1 is the straight-line distance, it always underestimates the spherical distance s_o. In fact, from (3.18a) we have

$$s_1 = \sqrt{2(1 - \cos s_o)} = 2 \sin\frac{s_o}{2} \leq 2\left(\frac{s_o}{2}\right) = s_o \qquad (3.19)$$

Since s_1 underestimates s_o, the horizontal height-height correlation $H^{zz}(s_1)$ overestimates $H^{zz}(s_o)$:

$$H^{zz}(s_1) - H^{zz}(s_o) = e^{-1/2\, b s_1^2} - e^{-1/2\, b s_o^2}$$

$$= e^{-b(1-\cos s_o)} - e^{-1/2\, b s_o^2} \geq 0 \,. \qquad (3.20)$$

A Taylor series expansion shows that the maximum difference $H^{zz}(s_1) - H^{zz}(s_o)$ occurs near $s_o = \frac{2}{\sqrt{b}}$ ($\approx 9°$ of arc), where $H^{zz}(s_1) = 0.13589$ and $H^{zz}(s_o) = 0.13534$. A computer program verified that in fact,

$$\max_{0 \leq s_o \leq \pi} \left| H^{zz}(s_1(s_o)) - H^{zz}(s_o) \right| \leq 0.000556 \,, \qquad (3.21)$$

and we conclude that the effect of the approximation $s = s_1$ is negligible for the height-height correlations.

Next we derive the formulas for the wind forecast error variances and for the remaining forecast error correlations, based upon $s = s_1$, and show that the effect of this approximation is still negligible. When $s = s_1$, we have from (3.6a), (3.2), (3.18a) that

$$\begin{aligned} \frac{\partial \log C^{zz}(s_1)}{\partial \xi} &= \frac{\partial}{\partial \xi}\left(-\frac{1}{2} b s_1^2\right) \\ &= b\frac{\partial}{\partial \xi}\left(1-\frac{1}{2}s_1^2\right) = b\,\frac{\partial \cos s_0}{\partial \xi} \end{aligned} \tag{3.22a}$$

and, differentiating again,

$$\frac{\partial^2 \log C^{zz}(s_1)}{\partial \eta\, \partial \xi} = b\,\frac{\partial^2 \cos s_0}{\partial \eta\, \partial \xi} \quad . \tag{3.22b}$$

Comparing (3.7a) and (3.8a) with (3.22a,b) shows that the latter can be obtained from the former simply by setting $q = 1$ and $r = 0$. Since (3.7a) and (3.8a) were the basis for deriving the standard deviations (3.14) and correlations (3.17), the new formulas for standard deviations and correlations are obtained from the old ones merely by setting $q = 1$ and $r = 0$.

The functions q and r do not appear in formulas (3.14), so the wind forecast error standard deviations based upon $s = s_1$ are in fact still given by (3.14). Setting $q = 1$ and $r = 0$ in (3.17) gives the correlation formulas for $s = s_1$:

$$\boxed{C^{uz}/C^{zz} = \gamma_1'\left[\sqrt{b}\, F^{uz} + \frac{1}{\sqrt{b}}\,\frac{\partial \log \sigma_1^z}{\partial \phi_1}\right] \quad ,} \tag{3.23a}$$

$$C^{zu}/C^{zz} = \gamma_2' \left[\sqrt{b}\, F^{zu} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^2}{\partial \phi_2} \right] , \tag{3.23b}$$

$$C^{vz}/C^{zz} = \delta_1' \left[\sqrt{b}\, F^{vz} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^2}{\cos\phi_1 \, \partial \lambda_1} \right] , \tag{3.23c}$$

$$C^{zv}/C^{zz} = \delta_2' \left[\sqrt{b}\, F^{zv} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^2}{\cos\phi_2 \, \partial \lambda_2} \right] , \tag{3.23d}$$

$$C^{uv}/C^{zz} = \gamma_1' \delta_2' F^{uv} + \frac{C^{uz}}{C^{zz}} \frac{C^{zv}}{C^{zz}} , \tag{3.23e}$$

$$C^{vu}/C^{zz} = \delta_1' \gamma_2' F^{vu} + \frac{C^{vz}}{C^{zz}} \frac{C^{zu}}{C^{zz}} , \tag{3.23f}$$

$$C^{uu}/C^{zz} = \gamma_1' \gamma_2' F^{uu} + \frac{C^{uz}}{C^{zz}} \frac{C^{zu}}{C^{zz}} , \tag{3.23g}$$

$$C^{vv}/C^{zz} = \delta_1' \delta_2' F^{vv} + \frac{C^{vz}}{C^{zz}} \frac{C^{zv}}{C^{zz}} , \tag{3.23h}$$

As before, γ_i' and δ_i' are given by (3.16a,b) and the functions $F^{\cdot\cdot}$ are given by (3.10a-h). Formulas (3.23) are certainly simpler than formulas (3.17).

We prove rigorously in the Appendix that the differences between the correlations given by (3.23a-h) and those given by (3.17a-h) are bounded as follows:

$$\left| C^{uz}(s_1) - C^{uz}(s_0) \right| \le T_1(s_0) + \sqrt{b}\, T_2(s_0) , \tag{3.24a}$$

$$\left| C^{zu}(s_1) - C^{zu}(s_0) \right| \le T_1(s_0) + \sqrt{b}\, T_2(s_0) , \tag{3.24b}$$

$$|C^{vz}(s_1) - C^{vz}(s_o)| \leq T_1(s_o) + \sqrt{b}\, T_2(s_o) \quad , \tag{3.24c}$$

$$|C^{zv}(s_1) - C^{zv}(s_o)| \leq T_1(s_o) + \sqrt{b}\, T_2(s_o) \quad , \tag{3.24d}$$

$$|C^{uv}(s_1) - C^{uv}(s_o)| \leq T_1(s_o) + (1+2\sqrt{b})\, T_2(s_o) + \left(\tfrac{1}{2} + \tfrac{1}{\sqrt{3}}\right) T_3(s_o) \, , \tag{3.24e}$$

$$|C^{vu}(s_1) - C^{vu}(s_o)| \leq T_1(s_o) + (1+2\sqrt{b})\, T_2(s_o) + \left(\tfrac{1}{2} + \tfrac{1}{\sqrt{3}}\right) T_3(s_o) \, , \tag{3.24f}$$

$$|C^{uu}(s_1) - C^{uu}(s_o)| \leq T_1(s_o) + (1+2\sqrt{b})\, T_2(s_o) + T_3(s_o) \quad , \tag{3.24g}$$

$$|C^{vv}(s_1) - C^{vv}(s_o)| \leq T_1(s_o) + (1+2\sqrt{b})\, T_2(s_o) + T_3(s_o) \quad . \tag{3.24h}$$

The functions Tj (s_o) are functions of s_o (and b) alone, and are given by

$$T_1(s_o) = |H^{zz}(s_1(s_o)) - H^{zz}(s_o)| \quad , \tag{3.25a}$$

$$T_2(s_o) = |H^{zz}(s_1(s_o)) - q(s_o) H^{zz}(s_o)| \quad , \tag{3.25b}$$

$$T_3(s_o) = [s_1(s_o)]^2 \Big[|r(s_o) H^{zz}(s_o)| + b\, |H^{zz}(s_1(s_o)) - q^2(s_o) H^{zz}(s_o)| \Big] \, . \tag{3.25c}$$

No additional assumptions were made in deriving these bounds. They hold, for example, regardless of the size of gradients of σ^z . The bounds could be improved by making assumptions regarding the size of these gradients, or by sharpening some of the inequalities proven in the Appendix.

No such improvement appears to be necessary. A simple computer program was written to determine the absolute maxima of the bounds (3.24, 3.25). The maximization was carried out over the interval $0 \leq s_o \leq 3$ only, since the factors $\underline{q}$ and $\underline{r}$ become infinite at $s_o = \pi$. (This is simply a minor defect of the correlations (3.17) based on the exact spherical distance: they all become infinite at $s_o = \pi$, and hence are valid correlation functions only on some interval $0 \leq s_o \leq s_{crit}$ with the critical distance s_{crit} being just less than π. The reason for this behavior is that for two points on opposite sides of the globe, i. e., $s_o = \pi$, small displacements of one of the points in opposite directions produce a large change in the great circle arc of minimum length connecting the two points: the arcs lie on opposite sides of the globe. Strictly speaking, for geostrophically derived correlation functions of s_o to be legitimate, $C^{zz}(s_o)$ must approach zero as $s_o \rightarrow \pi$ in such a way as to cancel this singularity. At any rate, correlations (3.23) based on the straight-line distance s_1 do not suffer this technical problem.)

The computation determined that

$$\left| C^{\xi\eta}(s_1) - C^{\xi\eta}(s_o) \right| \leq 0.00624 \tag{3.26a}$$

for the wind-height correlations (3.24a-d),

$$\left| C^{\xi\eta}(s_1) - C^{\xi\eta}(s_o) \right| \leq 0.0159 \tag{3.26b}$$

for the wind-wind cross-correlations (3.24e,f), and

$$\left| C^{\xi\eta}(s_1) - C^{\xi\eta}(s_o) \right| \leq 0.0157 \tag{3.26c}$$

for the wind-wind auto-correlations (3.24g,h). It is unlikely that such small differences as (3.26) would have a perceptible effect on analysis accuracy. Indeed, it is unlikely that one can estimate forecast error correlations to greater accuracy than this by any means in the near future.

III.3. Schlatter's approximation

The next approximation to the spherical distance s_o we consider is $s = s_2$, given by

$$s_2^2 = (\phi_1 - \phi_2)^2 + (\lambda_1 - \lambda_2)^2 \cos^2\left(\frac{\phi_1+\phi_2}{2}\right) \quad . \tag{3.27}$$

This approximation was introduced by Schlatter (1975), and was used equatorward of 70° in the global OI system at NMC until recently.

The order of accuracy of this formula relative to the spherical distance s_o can be obtained by simple Taylor series expansions. It follows from (3.18a) that

$$s_1^2 = s_o^2 + O(s_o^4) \quad , \tag{3.28}$$

while from (3.18b),

$$\begin{aligned} s_1^2 = {} & (\phi_1-\phi_2)^2 + O\left((\phi_1-\phi_2)^4\right) \\ & + \left[(\lambda_1-\lambda_2)^2 + O\left((\lambda_1-\lambda_2)^4\right)\right] \cos\phi_1 \cos\phi_2 \quad . \end{aligned} \tag{3.29a}$$

Using the trigonometric identity

$$\cos\phi_1 \cos\phi_2 = \cos^2\left(\frac{\phi_1+\phi_2}{2}\right) - \sin^2\left(\frac{\phi_1-\phi_2}{2}\right)$$

$$= \cos^2\left(\frac{\phi_1+\phi_2}{2}\right) + O\left((\phi_1-\phi_2)^2\right) \tag{3.29b}$$

(3.29a) becomes

$$s_1^2 = (\phi_1 - \phi_2)^2 + (\lambda_1 - \lambda_2)^2 \cos^2\left(\frac{\phi_1+\phi_2}{2}\right)$$
$$+ O((\phi_1-\phi_2)^4) + O((\lambda_1-\lambda_2)^2(\phi_1-\phi_2)^2) \tag{3.30}$$
$$+ O((\lambda_1-\lambda_2)^4) \cos\phi_1 \cos\phi_2$$

Finally, combining (3.27), (3.28) and (3.30) yields

$$s_2^2 = s_0^2 + O(s_0^4) + O((\phi_1-\phi_2)^4)$$

$$+ O((\lambda_1-\lambda_2)^2(\phi_1-\phi_2)^2) + O((\lambda_1-\lambda_2)^4) \cos\phi_1 \cos\phi_2 \ . \tag{3.31}$$

Equation (3.28) shows that the straight-line distance s_1 is a uniform approximation to s_o, in the sense that the error depends only upon s_o and not directly upon the location of points P_1 and P_2. Equation (3.31) shows, by contrast, that s_2 is a nonuniform approximation. The term $O((\phi_1-\phi_2)^4)$ in (3.31) can be absorbed into the term $O(s_o^4)$ since definition (3.5b) of s_o implies that $|\phi_1-\phi_2| \leq s_o$. The latter two terms in (3.31), however, are not of order $O(s_o^4)$, since small s_o does not imply small $|\lambda_1-\lambda_2|$ as the pole is approached Consequently, s_2 has been used operationally at NMC equatorward of 70° only.

In passing, we point out that the second-to-last term in (3.31) is due to approximating $\cos\phi_1\cos\phi_2$ by $\cos^2\left(\frac{\phi_1+\phi_2}{2}\right)$, while the last term is due to approximating $2[1-\cos(\lambda_1-\lambda_2)]$ by $(\lambda_1-\lambda_2)^2$. Removing either or both of these approximations would yield correlations closer to those resulting from s_o or s_1, but would not simplify substantially the correlation formulas themselves. In fact, the correlation formulas resulting from s_2 are not substantially simpler than the correlation formulas (3.23) based upon s_1.

To derive the correlation formulas for s_2, we use (3.6) and (3.2) to find that

$$\frac{\partial \log C^{zz}}{\partial \xi} = b \frac{\partial}{\partial \xi}\left[-\frac{1}{2} s^2\right] , \tag{3.32a}$$

$$\frac{\partial^2 \log C^{zz}}{\partial \eta \partial \xi} = b \frac{\partial^2}{\partial \eta \partial \xi}\left[-\frac{1}{2} s^2\right] , \tag{3.32b}$$

while, for $s = s_2$,

$$\frac{\partial}{\partial \phi_1}\left[-\frac{1}{2} s^2\right] = F^{uz} , \quad \frac{\partial}{\partial \phi_2}\left[-\frac{1}{2} s^2\right] = F^{zu} , \tag{3.33a,b}$$

$$\frac{\partial}{\partial \lambda_1}\left[-\frac{1}{2} s^2\right] = F^{vz} \cos\phi_1 , \quad \frac{\partial}{\partial \lambda_2}\left[-\frac{1}{2} s^2\right] = F^{zv} \cos\phi_2 , \tag{3.33c,d}$$

$$\frac{\partial^2}{\partial \phi_1 \partial \lambda_2}\left[-\frac{1}{2} s^2\right] = F^{uv} \cos\phi_2 , \quad \frac{\partial^2}{\partial \lambda_1 \partial \phi_2}\left[-\frac{1}{2} s^2\right] = F^{vu} \cos\phi_1 , \tag{3.33e,f}$$

$$\frac{\partial^2}{\partial \phi_1 \partial \phi_2}\left[-\frac{1}{2} s^2\right] = F^{uu} , \quad \frac{\partial^2}{\partial \lambda_1 \partial \lambda_2}\left[-\frac{1}{2} s^2\right] = F^{vv} \cos\phi_1 \cos\phi_2 , \tag{3.33g,h}$$

where now

$$F^{uz} = -(\phi_1 - \phi_2) + \frac{1}{4}(\lambda_1 - \lambda_2)^2 \sin(\phi_1 + \phi_2) \tag{3.34a}$$

$$F^{zu} = +(\phi_1 - \phi_2) + \frac{1}{4}(\lambda_1 - \lambda_2)^2 \sin(\phi_1 + \phi_2) \tag{3.34b}$$

$$F^{vz} = -\frac{1}{2\cos\phi_1}(\lambda_1 - \lambda_2)\left[1 + \cos(\phi_1 + \phi_2)\right] \tag{3.34c}$$

$$F^{zv} = +\frac{1}{2\cos\phi_2}(\lambda_1 - \lambda_2)\left[1 + \cos(\phi_1 + \phi_2)\right] \tag{3.34d}$$

$$F^{uv} = -\frac{1}{2\cos\phi_2}(\lambda_1-\lambda_2)\sin(\phi_1+\phi_2) \tag{3.34e}$$

$$F^{vu} = +\frac{1}{2\cos\phi_1}(\lambda_1-\lambda_2)\sin(\phi_1+\phi_2) \tag{3.34f}$$

$$F^{uu} = +1 + \frac{1}{4}(\lambda_1-\lambda_2)^2\cos(\phi_1+\phi_2) \tag{3.34g}$$

$$F^{vv} = +\frac{1}{2\cos\phi_1\cos\phi_2}\left[1+\cos(\phi_1+\phi_2)\right] \tag{3.34h}$$

In calculating these derivatives, we have used the trigonometric identity

$$\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left[1+\cos\theta\right].$$

to replace (3.27) by

$$s_2^2 = (\phi_1-\phi_2)^2 + \frac{1}{2}(\lambda_1-\lambda_2)^2\left[1+\cos(\phi_1+\phi_2)\right], \tag{3.35}$$

thus resulting in slightly simpler formulas.

The quantities defined in (3.34) all have the same limiting behavior as $P_2 \longrightarrow P_1$ as those defined in (3.10), namely that F^{uu} and F^{vv} approach one, while the others approach zero. Therefore the wind forecast error standard deviations based on s_2 are identical to those based on s_o and s_1, namely (3.14a,b). Furthermore, comparison of (3.32a,b) with (3.22a,b) and (3.7a, 3.8a), and comparison of (3.33a-h) with (3.9a-h), shows that the correlation formulas for s_2 are still given by (3.23a-h), but with the functions $F^{\cdot\cdot}$ of (3.10) now replaced by (3.34). Formulas (3.34) are comparable in computational complexity to formulas (3.10).

III.4. Currently operational approximation

Finally we consider the approximation $s = s_3$ currently used equatorward of 70° in the OI system at NMC,

$$s_3^2 = (\phi_1 - \phi_2)^2 + (\lambda_1 - \lambda_2)^2 \cos^2\phi_0 \quad . \tag{3.36}$$

Here the angle ϕ_0 is the latitude of the analysis point, either ϕ_1 or ϕ_2, but is treated as a constant when differentiating s_3. Comparing (3.36) with (3.27), it is clear that the correlation formulas based on s_3 will be much simpler than those based on s_2.

These correlation formulas are easily derived by observing that (3.32), (3.33) still hold for $s = s_3$, but with the quantities $F^{\cdot\cdot}$ now defined by

$$F^{uz} = -(\phi_1 - \phi_2) \quad , \quad F^{zu} = +(\phi_1 - \phi_2) \quad , \tag{3.37a,b}$$

$$F^{vz} = -(\lambda_1 - \lambda_2)\frac{\cos^2\phi_0}{\cos\phi_1} \quad , \quad F^{zv} = +(\lambda_1 - \lambda_2)\frac{\cos^2\phi_0}{\cos\phi_2} \quad , \tag{3.37c,d}$$

$$F^{uv} = 0 \quad , \quad F^{vu} = 0 \quad , \tag{3.37e,f}$$

$$F^{uu} = 1 \quad , \quad F^{vv} = \frac{\cos^2\phi_0}{\cos\phi_1 \cos\phi_2} \quad , \tag{3.37g,h}$$

Again the correct limiting behavior is obtained as $P_2 \rightarrow P_1$, so that the wind forecast error standard deviation formulas (3.14) still hold. As in the previous subsection, the correlation formulas for s_3 are still given by (3.23), but with the functions $F^{\cdot\cdot}$ now given by (3.37).

IV. Comparison of Correlation Functions for Different Distance Approximations

Here we compare graphically the correlation functions arising from the exact spherical distance s_o, the Schlatter approximation s_2, and the operationally-used approximation s_3. We have already proven that the correlation functions based on the straight-line distance s_1 are nearly indistinguishable from those based on s_o.

In all cases shown here we have taken $\frac{\partial \sigma_i^2}{\partial \phi_i} = \frac{\partial \sigma_i^2}{\partial \lambda_i} = 0$, so that there is no contribution to the correlation functions from the forecast error variances. Plots (not shown) in which gradients of σ_i^2 were present showed generally similar errors among s_o-, s_2- and s_3- based correlation functions as in the absence of σ_i^2 gradients. Plots of the correlation functions based on s_o, in the presence of various nonconstant σ^2 fields, were shown and discussed in the companion paper.

We have plotted the correlation functions $C^{zz}(s_i)$, $C^{uz}(s_i)$, $C^{uv}(s_i)$, $C^{uu}(s_i)$ and $C^{vv}(s_i)$ for i=0, 2, 3, at base points with ϕ_1 = 70°N and ϕ_1 = 30°N. In operational practice, while 70° is the dividing latitude, the choice of which distance approximation to use is determined by the location of the analysis point, not of the observations. If the absolute value of the latitude of the analysis gridpoint is greater than or equal to 70°, a polar-sterographic distance approximation (not discussed here) is used, otherwise approximation s_3 is used. Therefore, if the analysis gridpoint is located equatorward of 70° yet two observations to be used to analyze a value at that gridpoint between which a correlation must be computed are located poleward of 70°, formulas derived from approximation s_3 would still be used. Our plots comparing approximations s_2 and s_3 to the exact spherical distance s_1 at ϕ_1 = 70°N can be considered to represent the maximum error that would be possible in operational use. Figures 1a, b, c show

$C^{uv}(s_o)$, $C^{uv}(s_2)$ and $C^{uv}(s_3)$, respectively, for $\phi_1 = 70°N$. The correlations based on s_2 and s_3 are roughly double the correlations based on s_o and are more similar to each other than to the correlations based on s_o. The error fields corresponding to Figs. 1a, b, c, namely $C^{uv}(s_o) - C^{uv}(s_2)$ and $C^{uu}(s_o) - C^{uv}(s_3)$, appear in Figs. 2a and b, respectively. In both cases, the errors are nearly zero along the center line $\lambda_2 = \lambda_1$, and gradually increase with $|\lambda_2 - \lambda_1|$, as expected from (3.31). The maximum error in both cases is about 0.18. We doubt that such a large error is negligible: analyses based on s_o or s_1 would often differ significantly from analyses based on s_2 or s_3.

The differences we have observed in the uv-correlation functions at 70° were the largest of all those examined. For example, we show in Figs. 3a, b, c the functions $C^{uz}(s_o)$, $C^{uz}(s_2)$ and $C^{uz}(s_3)$, respectively still at 70°N, and in Figs. 4a, b the corresponding difference fields $C^{uz}(s_o) - C^{uz}(s_2)$ and $C^{uz}(s_o) - C^{uz}(s_3)$. The maximum error shown in Fig. 4a is less than 0.02, while that in Fig. 4b is about 0.08. Correlation function errors less than about 0.10 are likely to have a significant impact on analysis accuracy only occasionally.

Differences in the remaining correlation functions at 70° were all less than 0.02 for approximation s_2, and were about 0.04, 0.04, and 0.08 for $C^{zz}(s_o) - C^{zz}(s_3)$, $C^{uu}(s_o) - C^{uu}(s_3)$, and $C^{vv}(s_o) - C^{vv}(s_3)$, respectively. At 30°, the differences were all less than 0.02, except for $C^{uv}(s_o) - C^{uv}(s_2)$, $C^{uv}(s_o) - C^{uv}(s_3)$, and $C^{uz}(s_o) - C^{uz}(s_3)$, all of which were between 0.02 and 0.04.

V. Conclusions

We have derived and compared the forecast error correlation functions arising from the use of four different distance formulas: the exact spherical distance s_o, the straight-line distance s_1, the Schlatter approximation s_2, and the operational approximation s_3. We found that some correlations based on s_2 and s_3 differ significantly from those based on s_o. We have proven that all correlations based on s_1 differ from those based on s_o by a negligible amount over the entire sphere. Correlation formulas based on s_1 require less computational work than those based on s_o, about the same work as those based on s_2 and somewhat more work than those based on s_3.

Our operating assumptions, made also in the OI formulation at NMC, were that the height-height correlation is a Gaussian function of (approximate) spherical distance $s = s_i$, and that the wind-height and wind-wind correlations are related geostrophically to the height-height correlation. It is likely that neither of these assumptions reflects in more than a crude way the actual statistics of forecast errors. One should not expect, therefore, the correlations we have derived based on s_1 or s_o to lead to generally more accurate analyses than those based on s_2 or s_3. On the other hand, as refinements in the regional and global OI systems take place at NMC, it is important to bear in mind the potential inaccuracies induced by approximate formulations of the correlation functions.

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Appendix

Here we derive inequalities (3.24a-h) which show that the difference between correlations based on the spherical distance s_0 and those based on the straight-line distance s_1 is negligible. The derivations will make use of the following relationships:

$$|\gamma_i'| \leq 1 \quad , \quad \left| \gamma_i' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\partial \phi_i} \right| \leq 1 \quad , \tag{A.1a,b}$$

$$|\delta_i'| \leq 1 \quad , \quad \left| \delta_i' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_i^z}{\cos\phi_i \, \partial \lambda_i} \right| \leq 1 \quad , \tag{A.1c,d}$$

and for the functions $F^{\cdot\cdot}$ defined in (3.10),

$$|F^{\xi\eta}| \leq 1 \qquad \text{for } \xi, \eta = \text{any pair of u, v, z,} \tag{A.1e}$$

$$|F^{uz} F^{zv}| \leq \left(\frac{1}{2} + \frac{1}{\sqrt{3}} \right) s_1^2 \quad , \tag{A.1f}$$

$$|F^{vz} F^{zu}| \leq \left(\frac{1}{2} + \frac{1}{\sqrt{3}} \right) s_1^2 \quad , \tag{A.1g}$$

$$|F^{uz} F^{zu}| \leq s_1^2 \quad , \tag{A.1h}$$

$$|F^{vz} F^{zv}| \leq s_1^2 \quad . \tag{A.1i}$$

Inequalities (A.1a-e) follow immediately from definitions (3.16a,b) and (3.10a-h). We prove (A.1f-i) at the end of this appendix.

The bound on $|C^{uz}(s_1) - C^{uz}(s_0)|$ is derived first. From definitions (3.17a) and (3.23a), we have

$$C^{uz}(s_1) - C^{uz}(s_0) = \gamma_1' \sqrt{b}\, F^{uz}\left[C^{zz}(s_1) - q(s_0)\, C^{zz}(s_0)\right]$$

$$+ \gamma_1' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \left[C^{zz}(s_1) - C^{zz}(s_0)\right] ,$$

so that

$$\left| C^{uz}(s_1) - C^{uz}(s_0) \right| \leq \left| \gamma_1' \sqrt{b}\, F^{uz}\left[C^{zz}(s_1) - q(s_0)\, C^{zz}(s_0)\right] \right|$$

$$+ \left| \gamma_1' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \left[C^{zz}(s_1) - C^{zz}(s_0)\right] \right|$$

$$\leq \sqrt{b} \left| C^{zz}(s_1) - q(s_0)\, C^{zz}(s_0) \right|$$

$$+ \left| C^{zz}(s_1) - C^{zz}(s_0) \right| ,$$

where the triangle inequality and (A.1a, e, b) have been used. From (3.1) and the fact that $|V(p_1; p_2)| \leq 1$, this gives

$$\left| C^{uz}(s_1) - C^{uz}(s_0) \right| \leq \sqrt{b} \left| H^{zz}(s_1) - q(s_0)\, H^{zz}(s_0) \right| + \left| H^{zz}(s_1) - H^{zz}(s_0) \right| . \tag{A.2}$$

A similar proof yields the identical result for the remaining wind-height correlations, thus substantiating (3.24a-d).

We turn next to the wind-wind correlations. From (3.17e) and (3.23e),

$$C^{uv}(s_1) - C^{uv}(s_0) = \gamma_1' \delta_2' F^{uv}\left[C^{zz}(s_1) - q(s_0)\, C^{zz}(s_0)\right]$$

$$+ \gamma_1' \delta_2' r(s_0)\, F^{uz} F^{zv} C^{zz}(s_0)$$

$$+ C^{uz}(s_1)\, C^{zv}(s_1) / C^{zz}(s_1) - C^{uz}(s_0)\, C^{zv}(s_0) / C^{zz}(s_0) .$$

Using the triangle inequality and (A.1a, c, e), we get

$$
\begin{aligned}
\left| C^{uv}(s_1) - C^{uv}(s_0) \right| \leq & \left| C^{zz}(s_1) - q(s_0) C^{zz}(s_0) \right| \\
& + \left| r(s_0) F^{uz} F^{zv} C^{zz}(s_0) \right| \\
& + \left| C^{uz}(s_1) \frac{C^{zv}(s_1)}{C^{zz}(s_1)} - C^{uz}(s_0) \frac{C^{zv}(s_0)}{C^{zz}(s_0)} \right| .
\end{aligned}
\tag{A.3}
$$

According to (3.17a, d) and (3.23a, d),

$$
\begin{aligned}
& C^{uz}(s_1) \frac{C^{zv}(s_1)}{C^{zz}(s_1)} - C^{uz}(s_0) \frac{C^{zv}(s_0)}{C^{zz}(s_0)} \\
&= \gamma_1' \left[\sqrt{b}\, F^{uz} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \right] C^{zz}(s_1)\, \delta_2' \sqrt{b}\, F^{zv} \\
&\quad - \gamma_1' \left[\sqrt{b}\, q(s_0) F^{uz} + \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \right] C^{zz}(s_0)\, \delta_2' \sqrt{b}\, q(s_0) F^{zv} \\
&\quad + \delta_2' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^z}{\cos\phi_2 \partial \lambda_2} \left[C^{uz}(s_1) - C^{uz}(s_0) \right] \\
&= \gamma_1' \delta_2' b F^{uz} F^{zv} \left[C^{zz}(s_1) - q^2(s_0) C^{zz}(s_0) \right] \\
&\quad + \gamma_1' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_1^z}{\partial \phi_1} \delta_2' \sqrt{b}\, F^{zv} \left[C^{zz}(s_1) - q(s_0) C^{zz}(s_0) \right] \\
&\quad + \delta_2' \frac{1}{\sqrt{b}} \frac{\partial \log \sigma_2^z}{\cos\phi_2 \partial \lambda_2} \left[C^{uz}(s_1) - C^{uz}(s_0) \right] ,
\end{aligned}
$$

and with the triangle inequality and (A.1a-e), we have

$$
\begin{aligned}
& \left| C^{uz}(s_1) \frac{C^{zv}(s_1)}{C^{zz}(s_1)} - C^{uz}(s_0) \frac{C^{zv}(s_0)}{C^{zz}(s_0)} \right| \\
&\leq b \left| F^{uz} F^{zv} \right| \left| C^{zz}(s_1) - q^2(s_0) C^{zz}(s_0) \right| \\
&\quad + \sqrt{b} \left| C^{zz}(s_1) - q(s_0) C^{zz}(s_0) \right| + \left| C^{uz}(s_1) - C^{uz}(s_0) \right|
\end{aligned}
$$

Substituting the result and (A.2) into (A.3), and again using the fact that $|V(p_1; p_2)| \leq 1$, we have

$$|C^{uv}(s_1) - C^{uv}(s_0)| \leq |F^{uz}F^{zv}| \Big\{ |r(s_0) H^{zz}(s_0)| + b|H^{zz}(s_1) - q^2(s_0) H^{zz}(s_0)| \Big\}$$

$$+ (1+2\sqrt{b}) |H^{zz}(s_1) - q(s_0) H^{zz}(s_0)|$$

$$+ |H^{zz}(s_1) - H^{zz}(s_0)| . \tag{A.4}$$

Similar derivations show that (A.4) holds also for $|C^{vu}(s_1) - C^{vu}(s_0)|$,

$|C^{uu}(s_1) - C^{uu}(s_0)|$ and $|C^{vv}(s_1) - C^{vv}(s_0)|$

provided that the factor $|F^{uz}F^{zv}|$ is replaced by $|F^{vz}F^{zu}|$,

$|F^{uz}F^{zu}|$ and $|F^{vz}F^{zv}|$, respectively.

With (A.1f-i), this finishes the proof of (3.24e-h).

It remains to verify (A.1f-i). From (3.10c, d) and the trigonometric inequality

$$\sin^2\theta \leq 2(1-\cos\theta) , \tag{A.5}$$

we have

$$
\begin{aligned}
\left| F^{vz} F^{zv} \right| &\leq 2 \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right] \\
&\leq S_1^2 \quad ,
\end{aligned}
$$

where definition (3.18b) has been used, thus verifying (A.1i). To demonstrate (A.1h), we have from (3.10a, b) that

$$
\begin{aligned}
F^{uz} F^{zu} &= -\sin^2(\phi_1 - \phi_2) + \left[\sin\phi_1 \cos\phi_2 - \cos\phi_1 \sin\phi_2 \right] \sin(\phi_1 - \phi_2) \\
&\qquad \cdot \left[1 - \cos(\lambda_1 - \lambda_2) \right] \\
&\quad + \sin\phi_1 \sin\phi_2 \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right]^2 \\
&= -\cos(\lambda_1 - \lambda_2) \sin^2(\phi_1 - \phi_2) \\
&\quad + \sin\phi_1 \sin\phi_2 \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right]^2 \quad ,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left| F^{uz} F^{zu} \right| &\leq \left| \cos(\lambda_1 - \lambda_2) \right| \sin^2(\phi_1 - \phi_2) \\
&\quad + \left| \sin\phi_1 \sin\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right] \right| \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right] \\
&\leq 2 \left[1 - \cos(\phi_1 - \phi_2) \right] + 2 \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2) \right] \\
&= S_1^2 \quad ,
\end{aligned}
$$

where again (A.5) has been used.

Inequalities (A.1f, g) are more tricky. From (3.10a, d) we have

$$F^{uz}F^{zv} = -\sin(\phi_1-\phi_2)\cos\phi_1\sin(\lambda_1-\lambda_2)$$

$$+\cos\phi_1\cos\phi_2\left[1-\cos(\lambda_1-\lambda_2)\right]\sin\phi_1\sin(\lambda_1-\lambda_2) \quad .$$

Using the trigonometric equality

$$\sin(\theta_1-\theta_2)\cos\theta_1 = \sin\theta_1\left[\cos(\theta_1-\theta_2)-1\right]+\left[\sin\theta_1-\sin\theta_2\right] \quad ,$$

this becomes

$$F^{uz}F^{zv} = \sin\phi_1\left[1-\cos(\phi_1-\phi_2)\right]\sin(\lambda_1-\lambda_2)+(\sin\phi_2-\sin\phi_1)\sin(\lambda_1-\lambda_2)$$

$$+\cos\phi_1\cos\phi_2\left[1-\cos(\lambda_1-\lambda_2)\right]\sin\phi_1\sin(\lambda_1-\lambda_2)$$

$$= \frac{1}{2}s_1^2\sin\phi_1\sin(\lambda_1-\lambda_2)+(\sin\phi_2-\sin\phi_1)\sin(\lambda_1-\lambda_2) \quad ,$$

so that

$$\left|F^{uz}F^{zv}\right| \le \frac{1}{2}s_1^2+\left|\sin(\lambda_1-\lambda_2)\left[\sin\phi_1-\sin\phi_2\right]\right| \quad . \qquad \text{(A.6)}$$

Now, from the trigonometric identity

$$(\sin\theta_1-\sin\theta_2)^2 = \left[1-\cos(\theta_1-\theta_2)+2\cos\theta_1\cos\theta_2\right]\left[1-\cos(\theta_1-\theta_2)\right] \quad ,$$

we have

$$\left|\sin(\lambda_1-\lambda_2)\left[\sin\phi_1-\sin\phi_2\right]\right| = |xy| \quad ,$$

where

$$x^2 = \sin^2(\lambda_1-\lambda_2)\left[1-\cos(\phi_1-\phi_2) + 2\cos\phi_1\cos\phi_2\right] \quad ,$$

$$y^2 = 1 - \cos(\phi_1-\phi_2) \quad .$$

Now we use the inequality

$$2|xy| \leq \left(\frac{x}{\varepsilon}\right)^2 + (\varepsilon y)^2 \quad ,$$

valid for all real numbers x, y, ε , which follows from the fact that $\left(\frac{x}{\varepsilon} \pm \varepsilon y\right)^2 \geq 0$. This gives

$$\begin{aligned} 2&\left|\sin(\lambda_1-\lambda_2)[\sin\phi_1-\sin\phi_2]\right| \\ &\leq \frac{1}{\varepsilon^2}\sin^2(\lambda_1-\lambda_2)\left[1-\cos(\phi_1-\phi_2)+2\cos\phi_1\cos\phi_2\right]+\varepsilon^2\left[1-\cos(\phi_1-\phi_2)\right] \\ &= \left[\varepsilon^2+\frac{1}{\varepsilon^2}\sin^2(\lambda_1-\lambda_2)\right]\left[1-\cos(\phi_1-\phi_2)\right]+\frac{2}{\varepsilon^2}\cos\phi_1\cos\phi_2\sin^2(\lambda_1-\lambda_2) \\ &\leq \left(\varepsilon^2+\frac{1}{\varepsilon^2}\right)\left[1-\cos(\phi_1-\phi_2)\right]+\frac{4}{\varepsilon^2}\cos\phi_1\cos\phi_2\left[1-\cos(\lambda_1-\lambda_2)\right] \quad , \end{aligned}$$

where again (A.5) has been used. Choosing

$$\varepsilon^2 + \frac{1}{\varepsilon^2} = \frac{4}{\varepsilon^2} \quad , \quad \text{i.e.,} \quad \varepsilon^2 = \sqrt{3} \quad ,$$

we have finally

$$\left| \sin(\lambda_1 - \lambda_2)\left[\sin\phi_1 - \sin\phi_2\right]\right|$$

$$\le \frac{2}{\sqrt{3}}\left\{\left[1 - \cos(\phi_1 - \phi_2)\right] + \cos\phi_1 \cos\phi_2 \left[1 - \cos(\lambda_1 - \lambda_2)\right]\right\}$$

$$= \frac{1}{\sqrt{3}}\ S_1^2 \quad .$$

Combining this last result with (A.6) yields (A.1f). A similar proof gives (A.1g).

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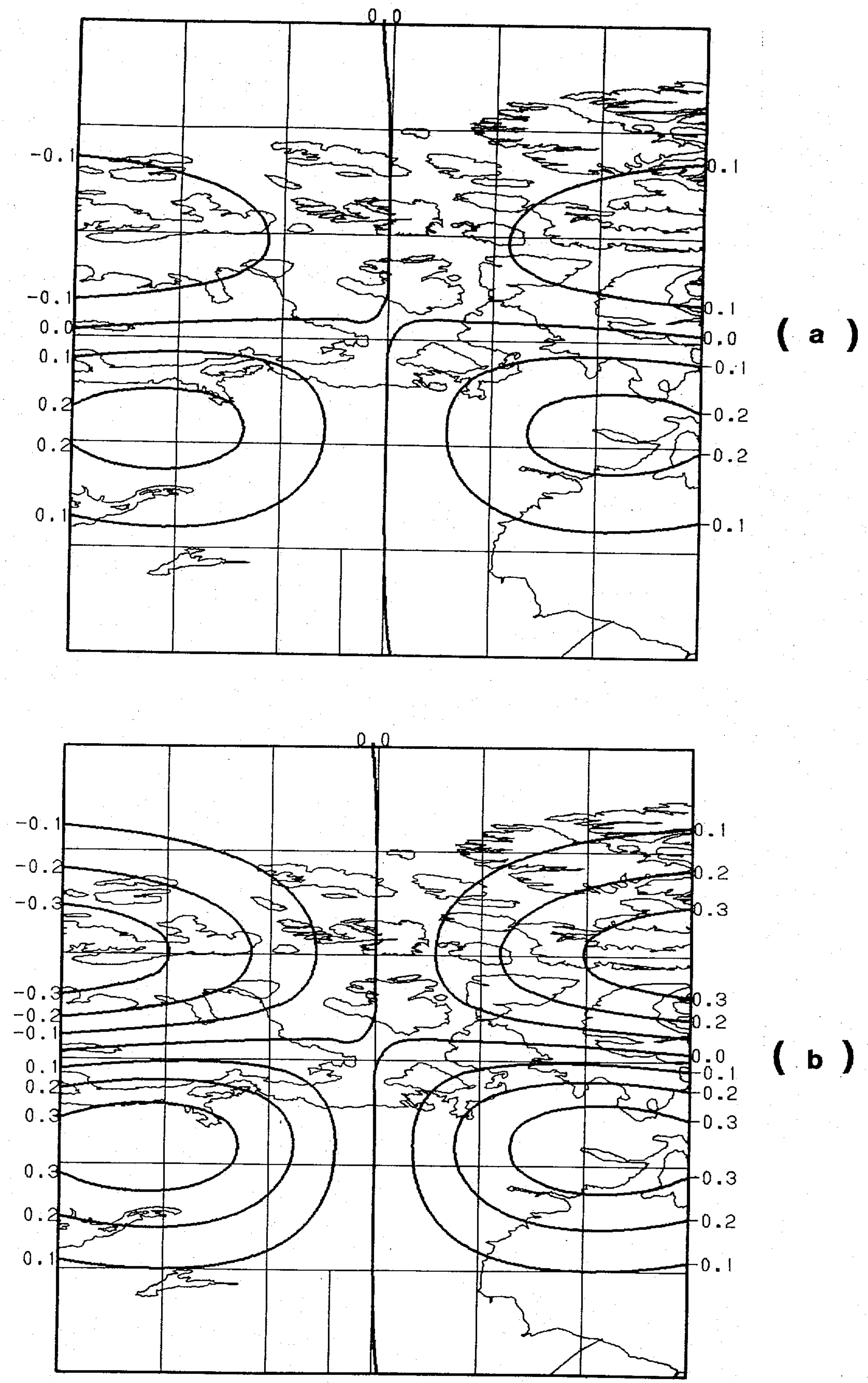
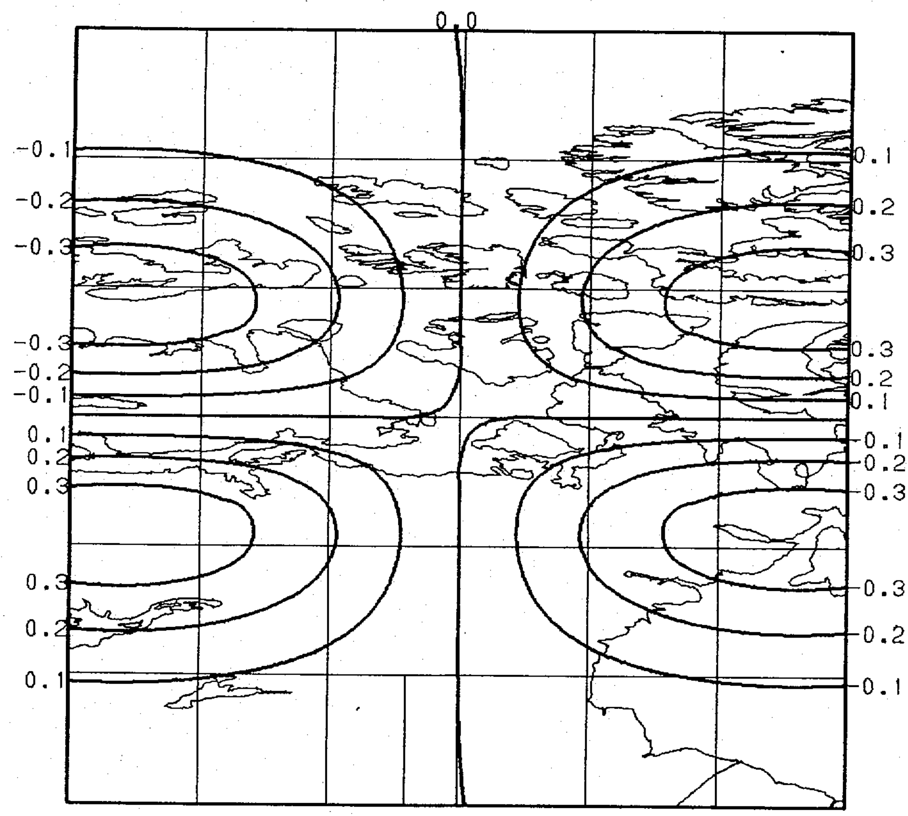


Figure 1. The uv forecast error correlation as computed from (a) the exact spherical distance s_o, (b) Schlatter's distance approximation s_2 and (c) the currently operational distance approximation s_3 at latitude 70°N. The contour interval is 0.1.



0.0
-0.1
-0.2
-0.3
-0.3
-0.2
-0.1
0.1
0.2
0.3
0.3
0.2
0.1
0.1
0.2
0.3
0.3
0.2
0.1
-0.1
-0.2
-0.3
-0.3
-0.2
-0.1

(c)

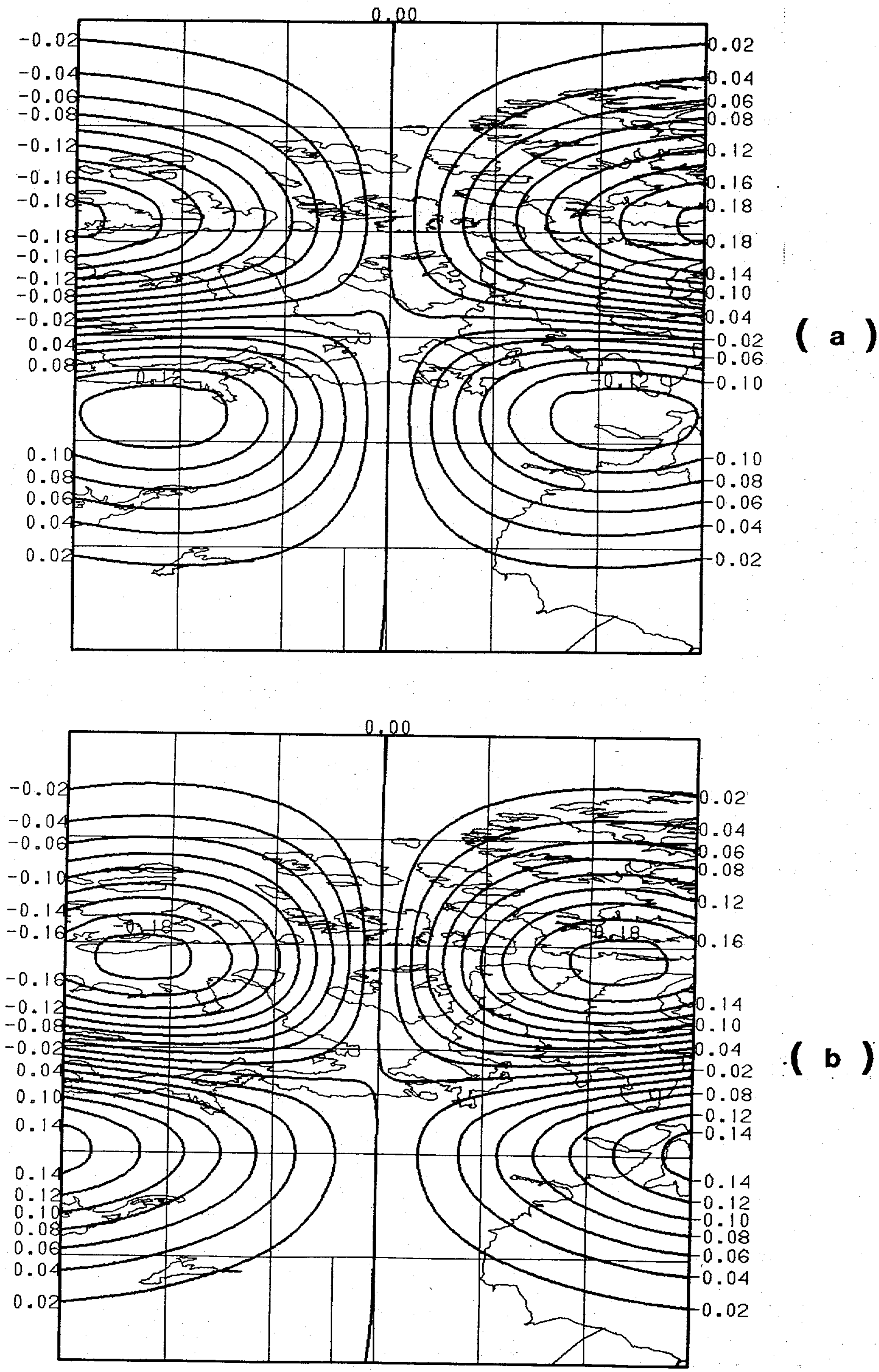


Figure 2. Error fields portraying the difference between uv forecast error correlations computed from the exact spherical distance s_o and (a) Schlatter's distance approximation s_2 and (b) the currently operational distance approximation s_3. The contour interval is 0.02.

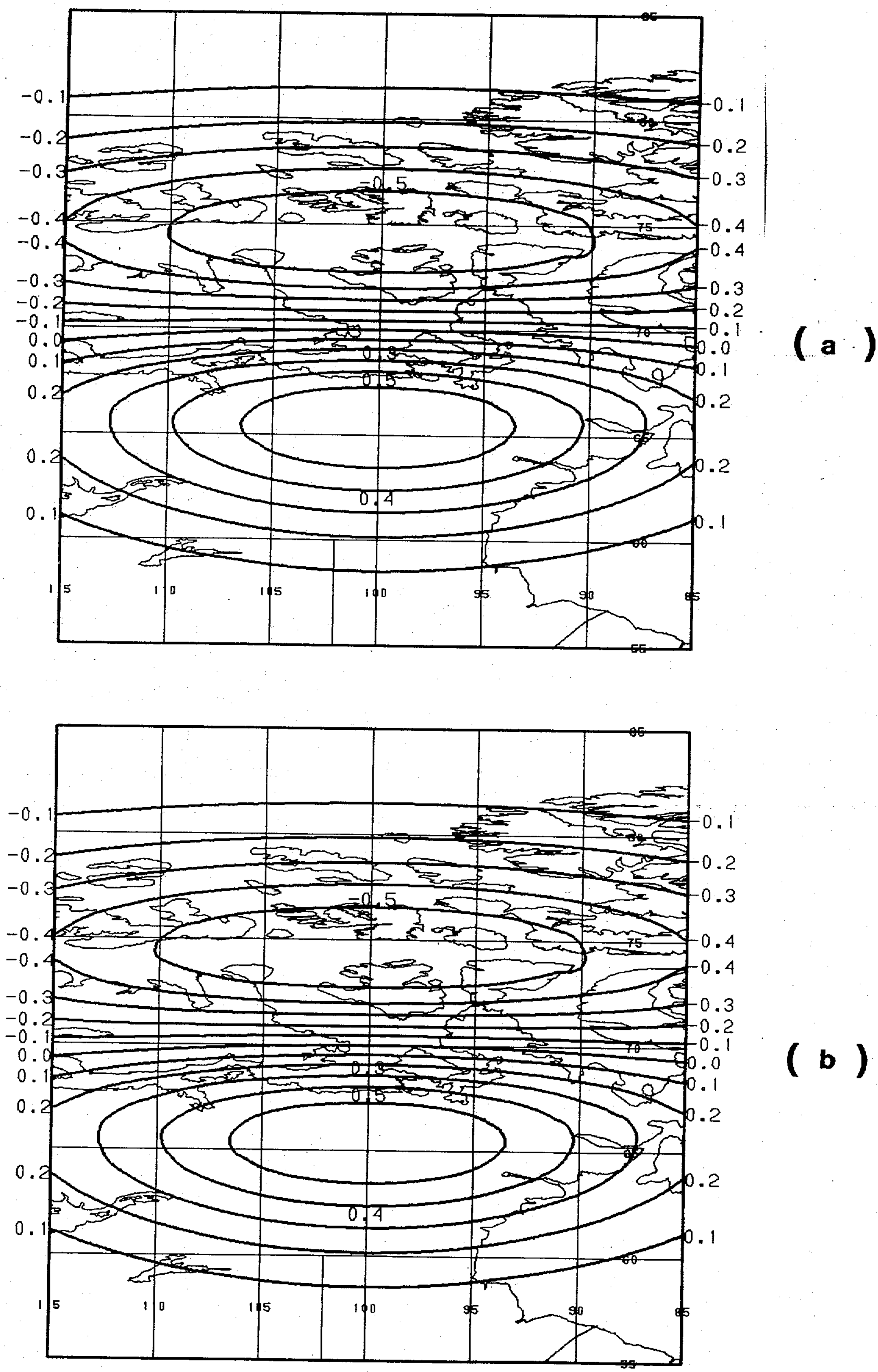
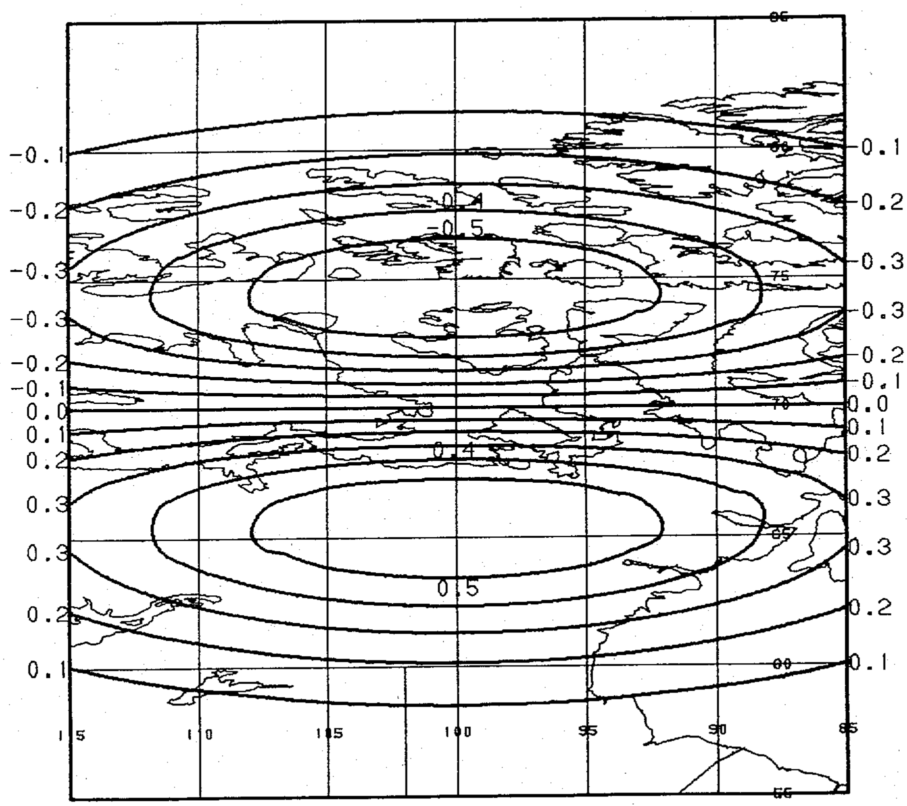
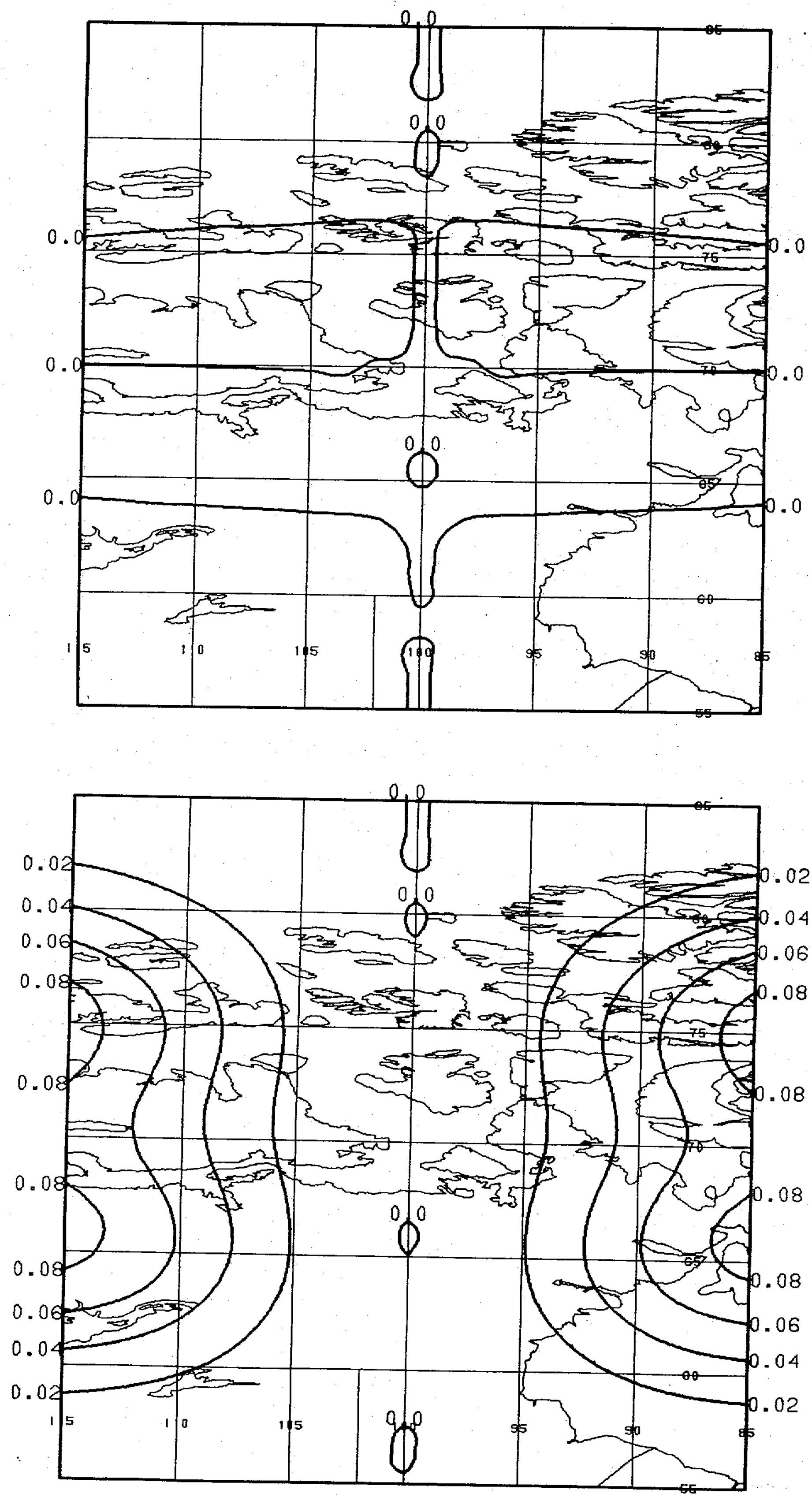


Figure 3. The uz forecast error correlation as computed from (a) the exact spherical distance s_o, (b) Schlatter's distance approximation s_2 and (c) the currently operational distance approximation s_3 at latitude 70°N. The contour interval is 0.1.



-0.1
-0.2
-0.3
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.3
0.2
0.1
-0.4
-0.5
0.4
0.5
0.1
0.2
0.3
0.3
0.2
0.1
0.0
0.1
0.2
0.3
0.3
0.2
0.1

(c)



(a)

(b)

Figure 4. Error fields portraying the difference between uv forecast error correlations computed from the exact spherical distance s_o and (a) Schlatter's distance approximation s_2 and (b) the currently operational distance approximation s_3. The contour interval is 0.02.