Impedance of Finite Length Resistor

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DISCLAIMER

IMPEDANCE OF FINITE LENGTH RESISTOR*

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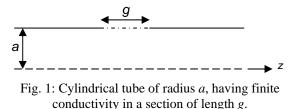R.L. Gluckstern, University of Maryland, College Park, MD 20742, U.S.A.

Abstract

We determine the impedance of a cylindrical metal tube (resistor) of radius a, length g, and conductivity σ, attached at each end to perfect conductors of semi-infinite length. Our main interest is in the asymptotic behavior of the impedance at high frequency, $k >> 1/a$. In the equilibrium regime, $ka^2 << g$, the impedance per unit length is accurately described by the well-known result for an infinite length tube with conductivity σ. In the transient regime, $ka^2 >> g$, we derive analytic expressions for the impedance and wakefield.

INTRODUCTION

We consider the longitudinal impedance of a cylindrical metal tube (resistor) of radius a, length g and conductivity σ attached at each end to perfect conductors of semi-infinite length (Fig. 1). At high frequency there are two regimes: (i) When the Rayleigh range ka^2 corresponding to the tube radius is short compared to the resistor length g, the field pattern settles into an equilibrium in which the field is continually being eaten at the resistor while it is being replenished on axis by the deceleration of the beam [1]. In this case, the impedance per unit length is well approximated by that of an infinite length tube with conductivity σ [2-6]. (ii) When the Rayleigh range ka^2 is short compared to g, equilibrium is not reached and the impedance per unit length differs from that of an infinite tube.

Within the framework of the diffraction model [1], we have the following picture. When the electromagnetic field carried by the relativistic charge passes by the point of discontinuity in conductivity, a diffractive wave is emitted, traveling a distance $\sqrt{z/k}$ towards the cylinder axis as the wave propagates a distance z in the axial direction. The equilibrium regime corresponds to the diffractive wave reaching the axis when $z << g$. The transient regime corresponds to the diffractive wave not reaching the axis until $z >> g$.

Fig. 1: Cylindrical tube of radius a, having finite conductivity in a section of length g.

In this paper, we present an analytic description of the impedance in the transient regime. Our discussion is complementary to the recent work of Ivanyan and Tsakanov [6]. A more comprehensive discussion of much of the work described here can be found in ref. [7].

EQUILIBRIUM REGIME

Let us write the impedance in the form,

$$Z_{\parallel}(a,g;k) = \frac{Z_s(k)g}{2\pi a} G(a,g;k), \qquad (1)$$

where the surface impedance $Z_s(k)$ is defined by (mks units, $Z_0 = 1/\varepsilon_0 c$, and $j = \sqrt{-1}$)

$$Z_s(k) = (1+j)Z_0\sqrt{\frac{k}{2\sigma Z_0}} \quad (k>0). \qquad (2)$$

Branch cuts are chosen such that $Z_s(-k) = Z_s*(k)$ and $Z_{\parallel}(a,g,-k) = Z_{\parallel}*(a,g,k)$. When $ka >> 1$ and $ka^2 << g$, the impedance per unit length is well approximated by that of an infinite cylinder with conductivity σ [2-5],

$$G \cong G_{\infty}(ks_0) = \frac{1}{1 + \frac{1}{4}(j-1)(ks_0)^{3/2}}, \qquad (3)$$

where s_0 is the characteristic length discussed in [3-5],

$$s_0 = \left(\frac{2a^2}{Z_0\sigma}\right)^{1/3}. \qquad (4)$$

The wake field determined from the real part of the impedance via,

$$w_{\parallel}(s) = \frac{2c}{\pi}\int_0^{\infty} dk \cos(ks) \operatorname{Re} Z_{\parallel}(k), \qquad (5)$$

can be expressed as [3]:

$$w_{\parallel}(s) = \frac{g c Z_0}{\pi a^2} W_{\infty}(s/s_0), \qquad (6)$$

where

$$W_{\infty}(s/s_0) = \frac{4}{3}e^{-s/s_0}\cos\left(\frac{\sqrt{3}s}{s_0}\right) - \frac{4\sqrt{2}}{\pi}\int_0^{\infty}\frac{dx\, x^2\, e^{-x^2 s/s_0}}{x^6+8} \qquad (7)$$

is plotted in Fig. 2. $W_{\infty}(0+) = 1$ and

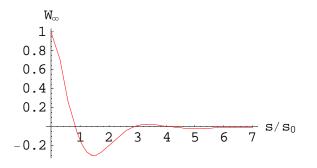$$W_{\infty}(s/s_0) \approx \frac{-1}{4\sqrt{2\pi}}\left(\frac{s_0}{s}\right)^{3/2} \quad (s >> s_0).$$

Fig. 2. The function W_∞ defined in Eq. (7).

DERIVATION OF INTEGRAL EQUATION

Our analysis [7] of the impedance of a finite length resistor is based on an integral equation for the longitudinal electric field in the resistor. In the transient regime, the kernel of this equation can be simplified, and analytic asymptotic results obtained. In the general case, the integral equation can be solved by expanding the field in a Fourier series in the axial coordinate z, and deriving an infinite set of linear algebraic equations for the Fourier coefficients. Truncating these equations by keeping only a limited number of Fourier components, the equations can be solved numerically [7].

Assuming all time dependence is given by $\exp(j\omega t)$, the fields in the tube can be expressed as:

$$H_\phi = \frac{I_0 e^{-jkz}}{2\pi r} + \frac{jk}{Z_0}\int dq\, e^{-jqz}\,\frac{J_1(Kr)}{KJ_0(Ka)}A(q), \qquad (8)$$

$$E_z = \int dq\, e^{-jqz}\,\frac{J_0(Kr)}{J_0(Ka)}A(q), \qquad (9)$$

where $K = \sqrt{k^2 - q^2}$. The resistive boundary condition is taken to be

$$\begin{aligned}E_z(r=a,z) &= -Z_s(k)H_\phi(r=a,z) \quad (0<z<g)\\ &= 0 \qquad\qquad\qquad (otherwise)\end{aligned}. \qquad (10)$$

Using Eqs. (8) and (9) in (10), we find the integral equation

$$F(z) = 1 - \frac{Z_s(k)k}{Z_0}\int_0^g dz'\, K(z-z')F(z'), \qquad (11)$$

where $E_z(r=a,z) = -\dfrac{Z_s I_0 e^{-jkz}}{2\pi a}F(z)$ and

$$K(z-z') = \sum_{s=1}^\infty \frac{e^{-jb_s|z-z'|/a}}{b_s} e^{jk(z-z')}, \qquad (12)$$

with $b_s^2 = k^2 a^2 - \mu_s^2$ and $J_0(\mu_s) = 0$. The impedance is determined by

$$Z_\|(k) = \frac{Z_s(k)}{2\pi a}\int_0^g dz F(z). \qquad (13)$$

TRANSIENT REGIME

When $ka \gg 1$ and $ka^2 \gg g$, the kernel can be approximated by [8]

$$\begin{aligned}K(z) &\cong \frac{1+j}{2\pi}\sqrt{\frac{\pi}{kz}} \quad (z>0)\\ &\cong 0 \qquad\qquad (z<0)\end{aligned}. \qquad (14)$$

In this case, the integral equation (6) becomes

$$F(x) = 1 - \Lambda\int_0^x dx'\,\frac{F(x')}{\sqrt{x-x'}} \quad (0 \le x \le g/a) \qquad (15)$$

where x=z/a and

$$\Lambda = \frac{(1+j)Z_s(k)\sqrt{ka}}{2\sqrt{\pi}\,Z_0}. \qquad (16)$$

Iterating the kernel, we find the solution

$$F(x) = \exp(-\kappa x) - \frac{j}{\sqrt{\pi}}\sqrt{\kappa x}\,h(\kappa x) \qquad (17)$$

with

$$\kappa = -\pi\Lambda^2 = \frac{k^2 a}{2Z_0\sigma} \qquad (18)$$

and

$$h(y) = \int_0^1 du\,\frac{\exp(-u\,y)}{\sqrt{1-u}} = \sum_{m=0}^\infty \frac{(-)^m 2^{m+1} y^m}{(2m+1)!!}. \qquad (19)$$

We can also express the function h in terms of the imaginary error function

$$h(y) = \sqrt{\frac{\pi}{y}}\,e^{-y}\,erfi(\sqrt{y}). \qquad (20)$$

The impedance is given by

$$Z_\|(k) \cong \frac{Z_s(k)}{2\pi}\int_0^{g/a} dx\left[\exp(-\kappa x) - j\sqrt{\frac{\kappa x}{\pi}}\,h(\kappa x)\right]. \qquad (21)$$

Performing one of the integrations, we find

$$Z_\|(k) \cong \frac{Z_s(k)g}{2\pi a}\,G_T(k\,s_g), \qquad (22)$$

where

$$\begin{aligned}G_T(u) &= \int_0^1 dx\, e^{-u^2 x}\left[1 - \frac{2ju}{\sqrt{\pi}}\sqrt{1-x}\right]\\ &= u^{-2}\left(1 - e^{-u^2} - \frac{2ju}{\sqrt{\pi}} + j\,e^{-u^2}erfi(u)\right)\end{aligned}, \qquad (23)$$

and we have introduced the new length scale

$$s_g = \left(\frac{g}{2Z_0\sigma}\right)^{1/2}. \qquad (24)$$

WAKEFIELD

The wakefield can be expressed in terms of the real part of the impedance using Eq. (5). When $s_g < s_0$, the

impedance of Eq. (22) is a good approximation [7] over the wide range of frequencies given by the inequality $ks_g \gg 4(s_g / s_0)^3$, including values with $k \ll 1/s_g$ as well as $k \gg 1/s_g$. In this case we find,

$$w_\parallel(s) = \frac{cg}{\pi^2 a} \frac{Z_0}{\sqrt{2s_g^3 Z_0 \sigma}} W_T\left(\frac{s}{s_g}\right) , \qquad (25)$$

with (see Fig. 3)

$$W_T(\alpha) = \int_0^\infty du \cos(u\alpha) u^{-3/2}\left(1 - e^{-u^2} + \frac{2u}{\sqrt{\pi}} - e^{-u^2} erfi(u)\right). \quad (26)$$

The behavior for small argument is given by

$$W_T(\alpha) = \sqrt{\frac{2}{\alpha}} - \sqrt{2\pi\alpha} + \frac{2\sqrt{2}}{3}\alpha^{3/2} + O\left(\alpha^{5/2}\right), \quad (27)$$

and the behavior for large argument by

$$W_T(\alpha) \approx \frac{-1}{\sqrt{2}} \sum_{n=0}^\infty \frac{\Gamma(n+3/2)}{\alpha^{n+3/2}\, \Gamma\left(\frac{n}{2}+2\right)} . \qquad (28)$$
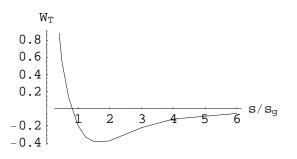

Fig. 3. The function W_T defined in Eq. (27).

DISCUSSION OF RESULTS

When the Rayleigh range of a mode with wave number k and radius a is large compared to the length of the resistor $(ka^2 \gg g)$, the behavior of the impedance can differ significantly from that of a resistor of infinite length. A new length scale s_g [Eq. (24)] enters the problem. For $k \ll 1/s_g$, the longitudinal impedance is given by the low frequency resistive wall impedance,

$$Z_\parallel(k) \cong \frac{Z_s(k)g}{2\pi a} . \qquad (29)$$

For $k \gg 1/s_g$, the high frequency asymptotic behavior can be shown [7] to be twice that given by the diffraction model impedance [1-2, and references there] for a cavity of length g in a beam pipe of radius a,

$$Z_\parallel(k) \cong (1 - j)\frac{2Z_0}{2\pi a}\sqrt{\frac{g}{\pi k}} . \qquad (30)$$

The low frequency resistive wall impedance cannot continue to very high frequencies because the corresponding negative wakefield would result in acceleration of the particles trailing immediately behind the leading particle [2]. Eq. (30) yields a proper retarding wakefield immediately behind the leading particle. Therefore, it is reasonable that at some sufficiently high frequency the diffraction-model-like behavior becomes dominant. The value of k for which the magnitudes of the two asymptotic forms given in (29) and (30) become equal is (up to a constant of the order of one) the inverse of the characteristic length scale s_g.

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