Liboff 3.9

we know $\int_{-\infty}^{\infty} f(x')\,\delta(x - x')\,dx' = f(x)$

a) show $\delta(y) = \delta(-y)$

$\int_{-\infty}^{\infty} f(x')\,\delta(x' - x)\,dx' ==$ $z = -x$ $z' = -x'$ $dz' = -dx'$

$= -\int_{\infty}^{-\infty} f(-z')\,\delta(z - z')\,dz' = \int_{-\infty}^{\infty} f(-z')\,\delta(z - z')\,dz' = f(-z) = f(x)$

b) Show $\delta'(y) = -\delta'(-y)$ Now $\int_{-\infty}^{\infty} f(y)\,\delta'(y)\,dy =$

$f(y)\,\delta(y)\,]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{df(y)}{dy}\,\delta(y)\,dy = -\int_{-\infty}^{\infty} \frac{df(y)}{dy}\,\delta(y)\,dy$

$-\int_{-\infty}^{\infty} f(y)\,\delta'(-y)\,dy = \int_{\infty}^{-\infty} f(y(z))\,\delta'(z)\,dz =$

$-\int_{-\infty}^{\infty} f(y(z))\,\delta'(z)\,dz = -f(y(z))\,\delta(z) + \int_{-\infty}^{\infty} \frac{df(y(z))}{dz}\,\delta(z)\,dz ==$

let $y(z) = -z$ $dy = -dz$

Now $\frac{df(y(z))}{dz} = \frac{df(y)}{dy}\,\frac{dy(z)}{dz} = -\frac{df(y)}{dy}$

$== -\int_{\infty}^{-\infty} \frac{df(y)}{dy}\,\delta(-y)\,(-dy) = -\int_{-\infty}^{\infty} \frac{df(y)}{dy}\,\delta(-y)\,dy = -\int_{-\infty}^{\infty} \frac{df(y)}{dy}\,\delta(y)\,dy$ QED

c) $\int_{-\infty}^{\infty} f(y)\,y\delta(y)\,dy = g(0) = f(0)\,0 = 0$ where $g(y) = f(y)\,y$

d) use i)

e) use i)

f) show $\int_{-\infty}^{\infty} dy\,\delta(a - y)\,\delta(y - b) = \delta(a - b)$

$\int_{-\infty}^{\infty} da\,f(a) \int_{-\infty}^{\infty} dy\,\delta(a - y)\,\delta(y - b) = \int_{-\infty}^{\infty} dy\,\delta(y - b) \int_{-\infty}^{\infty} da\,f(a)\,\delta(a - y) ==$

let $g(y) = \int_{-\infty}^{\infty} da\,f(a)\,\delta(a - y)$

$== \int_{-\infty}^{\infty} dy\,\delta(y - b)\,g(y) = g(b) = \int_{-\infty}^{\infty} da\,f(a)\,\delta(a - b) = f(b)$

Now also

$\int_{-\infty}^{\infty} da\,f(a)\,\delta(a - b) = f(b)$ QED

g) show $f(y)\,\delta(y-a) = f(a)\,\delta(y-a)$

$\int_{-\infty}^{\infty} dy\, f(y)\,\delta(y-a) = f(a)$ and also

$\int_{-\infty}^{\infty} dy\, f(a)\,\delta(y-a) = f(a)\int_{-\infty}^{\infty} dy\,\delta(y-a) = f(a)\,1 = f(a)$ QED

h) Show $y\delta'(y) = -\delta(y)$

$\int_{-\infty}^{\infty} f(y)\, y\delta'(y)\, dy =$

$f(y)\, y\delta(y) - \int_{-\infty}^{\infty}\delta(y)\,\frac{d}{dy}(f(y)\,y)\,dy = -\int_{-\infty}^{\infty}\delta(y)\left[f(y) + y\,\frac{df(y)}{dy}\right] dy ==$

$u = f(y)\,y \quad dv = \delta'(y)$

$== -\int_{-\infty}^{\infty}\delta(y)\,f(y)\,dy - \int_{-\infty}^{\infty}\delta(y)\,y\,\frac{df(y)}{dy}\,dy = -\int_{-\infty}^{\infty}\delta(y)\,f(y)\,dy = -f(0)$ QED

i) Show $f(y)\,\delta(g(y)) = \sum_i \frac{f(y_i)}{|g'(y_i)|}$ where $g(y_i) = 0$

$\frac{dg}{dy} = g' \qquad dy = \frac{dg}{g'}$

$\int_{-\infty}^{\infty} f(y)\,\delta(g(y))\,dy = \int_{-\infty}^{\infty} f(y(g))\,\delta(g)\,\frac{dg}{g'}\ (g' > 0)$ or $\int_{\infty}^{-\infty} f(y(g))\,\delta(g)\,\frac{dg}{g'}\ (g' < 0) ==$

Now if $g'(y_i) < 1$ then we have to reverse the order of integration so we will just use $|g'|$ to take care of this

$== \int_{-\infty}^{\infty} f(y(g))\,\delta(g)\,\frac{dg}{|g'|}$ This only has a contribution if $g = 0$ [i.e. if $g(y_i) = 0$]

So $f(y)\,\delta(g(y)) = \sum \frac{f(y_i)}{|g'(y_i)|}$ QED