ZETAC Documentation

1. Coordinate system

The terrain-following coordinate is denoted ζ and assumed to range from zero to unity across the model depth. A separable mapping to height z is constructed as follows:

$$z(x, y, \zeta) = h(x, y) + Z(\zeta)[H - h(x, y)], \tag{1}$$

where $h(x, y)$ is the topography, H is the model depth, and Z varies from zero to unity. The advection operator is:

$$\boldsymbol{V} \cdot \nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial \zeta}, \tag{2}$$

where $\omega = d\zeta/dt$ and the horizontal derivatives are at constant ζ. In spherical and terrain-following coordinates, the divergence is

$$\nabla \cdot \boldsymbol{V} = z_\zeta^{-1}\left[\frac{\partial(uz_\zeta)}{\partial x} + (\cos\varphi)^{-1}\frac{\partial(vz_\zeta\cos\varphi)}{\partial y} + \frac{\partial(\omega z_\zeta)}{\partial \zeta}\right], \tag{3}$$

where φ is the latitude and z_ζ is the derivative of height with respect to model coordinate at constant x and y. The ordinary vertical velocity component w can be diagnosed from ω using

$$w = \omega z_\zeta + \boldsymbol{V}_h \cdot \nabla z, \tag{4}$$

where $\boldsymbol{V}_h = (u, v)$ and the horizontal derivatives are at constant ζ.

2. Model equations

Let D represent diffusion. Then the forecast equations are:

$$\frac{\partial u}{\partial t} = -\boldsymbol{V}\cdot\nabla u + \left(f + \frac{u\tan\varphi}{a}\right)v - c_p\theta_v\frac{\partial\pi}{\partial x}\bigg|_z + D_u \tag{5}$$

$$\frac{\partial v}{\partial t} = -\boldsymbol{V}\cdot\nabla v - \left(f + \frac{u\tan\varphi}{a}\right)u - c_p\theta_v\frac{\partial\pi}{\partial y}\bigg|_z + D_v \tag{6}$$

$$z_\zeta\frac{\partial\omega}{\partial t} = -\boldsymbol{V}\cdot\nabla w - \left(g + c_p\theta_v z_\zeta^{-1}\frac{\partial\pi}{\partial\zeta}\right)(1+l)^{-1} - \frac{\partial\boldsymbol{V}_h}{\partial t}\cdot\nabla z + D_\omega \tag{7}$$

$$\frac{\partial\pi}{\partial t} = -\boldsymbol{V}\cdot\nabla\pi - \frac{R}{c_v}\pi\nabla\cdot\boldsymbol{V} + \frac{R}{c_v}\pi\left[\frac{Q}{\theta} + E\left(\frac{1}{1+q_v} + \frac{\varepsilon}{1+\varepsilon q_v}\right)\right] \tag{8a}$$

$$\frac{\partial\theta}{\partial t} = -\boldsymbol{V}\cdot\nabla\theta + Q \tag{9}$$

$$\frac{\partial q_v}{\partial t} = -\boldsymbol{V}\cdot\nabla q_v + E, \tag{10}$$

where l is the total condensate, Q is the diabatic heating (including D_θ), E is the source of water vapor (including D_{q_v}), $g = 9.81\ \mathrm{m/s^2}$, $a = 6371\ \mathrm{km}$ and $\varepsilon = 0.608$. The horizontal pressure gradient is at constant z. The pressure equation (8a) can also be written in flux form, either as

$$\frac{\partial\pi}{\partial t} = \mu^{-1}\pi\left(-\frac{\nabla\cdot\pi^\mu\boldsymbol{V}}{\pi^\mu} + \frac{Q}{\theta} + Er\right), \tag{8b}$$

or as

$$\frac{\partial\pi}{\partial t} = \mu^{-1}\pi\left(-\frac{\nabla\cdot\rho\boldsymbol{V}}{\rho} + \frac{Q - \boldsymbol{V}\cdot\nabla\theta}{\theta} + (E - \boldsymbol{V}\cdot\nabla q_v)r\right), \tag{8c}$$

where $r \equiv (1+q_v)^{-1} + \varepsilon(1+\varepsilon q_v)^{-1}$ and $\mu \equiv c_v/R$. It can be checked that (8b) and (8c) both conserve the quantity π^μ globally when $Q = E = 0$ or $Q - \boldsymbol{V}\cdot\nabla\theta = E - \boldsymbol{V}\cdot\nabla q_v = 0$. This result is a consequence of mass conservation, since π^μ is proportional to $\rho\theta_v$, with ρ the density and $\theta_v = \theta(1+\varepsilon q_v)$ the virtual potential temperature.

3. "Fast" equations

A height-dependent reference state, denoted by an overbar, is chosen so that the following linear terms responsible for acoustic-gravity waves can be subtracted and re-evaluated on a "fast" time step:

$$\frac{\partial u}{\partial t} = -c_p\bar{\theta}_v\frac{\partial}{\partial x}\bigg|_z(\pi - \bar{\pi}) \tag{11}$$

$$\frac{\partial v}{\partial t} = -c_p\bar{\theta}_v\frac{\partial}{\partial y}\bigg|_z(\pi - \bar{\pi}) \tag{12}$$

$$\frac{\partial}{\partial t}(\omega z_\zeta) = \left(g\frac{\theta_v - \bar{\theta}_v}{\bar{\theta}_v} - c_p\bar{\theta}_v z_\zeta^{-1}\frac{\partial}{\partial \zeta}(\pi - \bar{\pi})\right)(1 + l)^{-1} \tag{13}$$

$$\frac{\partial}{\partial t}(\pi - \bar{\pi}) = -\mu^{-1}\bar{\pi}^{1-\mu}\nabla \cdot \bar{\pi}^{\mu}\boldsymbol{V} \tag{14}$$

$$\frac{\partial}{\partial t}(\theta_v - \bar{\theta}_v) = -\boldsymbol{V} \cdot \nabla\bar{\theta}_v . \tag{15}$$

Here we assume that $\bar{\pi}(z)$ is in hydrostatic balance with the virtual potential temperature $\bar{\theta}_v(z)$; thus, $\frac{\partial}{\partial z}\bar{\pi} = -\frac{g}{c_p\bar{\theta}_v}$. The linearization of the pressure equation (14) is based on (8b). What remains on the right-hand side of (5) - (10) after these terms are subtracted is frozen during the fast cycle. The moisture variables l and q_v are also frozen.

Note that the linear terms are subtracted at one model time level (the slow time) and re-evaluated at another (the fast time). So, for example, the full equation for virtual potential temperature, combining (9), (10) and (15), is

$$\frac{\partial \theta_v}{\partial t} = -\boldsymbol{V}_s \cdot \nabla\theta_v - (\boldsymbol{V}_f - \boldsymbol{V}_s) \cdot \nabla\bar{\theta}_v + Q_v , \tag{16}$$

where "s" and "f" refer to "slow" and "fast", and $Q_v = Q(1 + \varepsilon q_v) + E\varepsilon\theta$ (evaluated on the slow time). The size of the fast timestep is constrained by soundwave propagation. In order to relax this constraint without significantly diminishing the accuracy of the solution, the vertical propagation is treated semi-implicitly. Vertical soundwave propagation is due to the vertical derivatives in

(13) and (14). The vertical advection of buoyancy, on the right-hand side of (15), is included in the semi-implicit part of the calculation. This removes the numerical constraint on the fast timestep due to vertical gravity-wave propagation.

4. Mass conservation

The pressure equation (14) is in a convenient form for discretization and for global conservation. However, mass is still not conserved exactly because density must be diagnosed from the exner function and potential temperature, and because the fast-time update is linearized. The first problem can be alleviated by forecasting the density directly with

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \boldsymbol{V} + \rho E \tag{17}$$

and diagnosing π (otherwise numerical errors are transferred dangerously to the equation of state). However, this complicates the coupling with the momentum equation. A partial remedy that is easier to implement is to retain the pressure forecast only for the purpose of updating the velocity, and then overwriting it with a diagnosis from the equation of state and the forecast density. To improve the consistency between the prognosis and diagnosis of pressure, it is then preferable to use (8c) instead of (8b). Note, however, that the assumption about the Eulerian tendencies, $Q - \boldsymbol{V} \cdot \nabla\theta = E - \boldsymbol{V} \cdot \nabla q_v = 0$, may not be as well satisfied on the fast time step as Lagrangian conservation, $Q = E = 0$.

As for the conservation errors introduced by linearization, a partial remedy is to linearize with respect to a recent model state rather than a static reference state. If the subscript "s" refers either to a slowly evolving solution or to the most recent "slow" time, we would subtract the term

$$\frac{\partial \pi}{\partial t} = -\mu^{-1}\frac{\pi_s}{\rho_s}\nabla \cdot \rho_s \boldsymbol{V}. \tag{18}$$

from the pressure equation and intergrate it on the fast time along with the velocity and temperature. The density would be updated similarly on the fast time according to

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho_s \boldsymbol{V} . \tag{19}$$

If the mass divergence is evaluated in the same way in both equations, then (18) and (19) together amount to a linearization of the diagnostic relation $\pi^{\mu} = \rho\theta_v / (\rho_0\theta_0)$, with θ_v held constant. Therefore, to the extent that the fast solution is very close to the slow one and $\theta_v^{-1}|\partial\theta_v / \partial t| \ll \pi^{-1}|\partial\pi / \partial t|$, the diagnostic overwrite of π will be a very small nudge. It is also a localized nudge -- integrations that do not use (19) directly must resort to non-local nudging to keep the total mass from drifting. There is probably nothing to be gained from changing to a time-varying reference atmosphere for the remaining linear terms, and it could even destabilize the integration.