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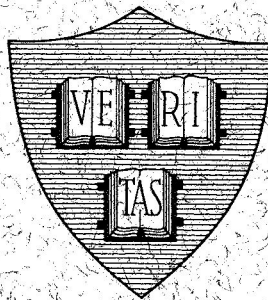
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## SUFFICIENT CONDITIONS FOR NON-NEGATIVITY OF THE SECOND VARIATION IN SINGULAR AND NONSINGULAR CONTROL PROBLEMS



By

D. H. Jacobson

August 1969

Technical Report No. 596

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ABSTRACT

Sufficient conditions for non-negativity of the second variation in singular and nonsingular control problems are presented; these conditions are in the form of equalities and differential inequalities. Control problem examples illustrate the use of the new conditions. The relationships of the new conditions to existing necessary conditions of optimality for singular and nonsingular problems are discussed. When applied to nonsingular control problems, it is shown that the conditions are sufficient to ensure the boundedness of the solution of the well-known matrix Riccati differential equation; these conditions are less stringent than those known heretofore.

## I. Preliminaries

### 1. Introduction

Singular control problems occur often in engineering; for example, in the aerospace industry a number of important problems are singular [1], [2]. Mathematical economics is another field in which singular optimal control problems are common [3]. These and other examples have prompted researchers to inquire into the mathematical properties of singular arcs [4]-[20]. Circa 1964, Kelley [4] discovered a new necessary condition of optimality for singular arcs. This condition was generalized subsequently by Robbins [5], Tait [6], Kelley et al [7] and Goh [8], and is now commonly known as the generalized Legendre-Clebsch condition (or Kelley's condition). In [9] an additional necessary condition of optimality for singular arcs was derived and was shown to be non-equivalent to the generalized Legendre-Clebsch condition. For want of an alternative, we shall refer to this condition as Jacobson's condition.

In this paper we present sufficient conditions for non-negativity of the second variation in singular control problems; in strengthened form these conditions (equalities and inequalities) are sufficient for a weak relative minimum. Both Kelley's and Jacobson's necessary conditions of optimality are derived easily from the new conditions. We show that the conditions are applicable to totally singular, partially singular<sup>†</sup> and nonsingular control functions. Moreover, when applied to nonsingular problems, sufficient conditions for the boundedness of the solution of the well-known matrix Riccati differential equation are obtained; these are less stringent than those known heretofore [21], [26].

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<sup>†</sup> Defined in Section I. 3.

Control problems without terminal constraints are considered first; the results are then generalized to the case where constraints on the terminal states are present. It turns out that the presence of terminal constraints does not complicate unduly the derivation.

## 2. Problem Formulation

We shall consider the class of control problems where the dynamical system is described by the ordinary differential equations:

$$\dot{x} = f(x, u, t) \quad ; \quad x(t_0) = x_0 \quad (1)$$

where, (except for Section V):

$$f(x, u, t) = f_1(x, t) + f_u(x, t)u \quad (2)$$

The performance of the system is measured by the cost functional:

$$V(x_0, t_0) = \int_{t_0}^{t_f} L(x, t)dt + F(x(t_f), t_f) \quad (3)$$

and the terminal states must satisfy

$$\psi(x(t_f), t_f) = 0 \quad . \quad (4)$$

The control function  $u(\cdot)$  is required to satisfy the following constraint:

$$u(\cdot) \in U \quad (5)$$

where the set  $U$  is defined by:

$$U \equiv \{u(\cdot) : |u_i(t)| \leq 1, t \in [t_0, t_f]; i = 1, \dots, m\} \quad . \quad (6)$$

Here,  $x$  is an  $n$ -dimensional state vector and  $u$  is an  $m$ -dimensional control vector.  $f_1$  is an  $n$ -dimensional vector function of  $x$  at time  $t$  and  $f_u$  is an  $n \times m$  matrix function of  $x$  at time  $t$ ; the functions  $L$  and  $F$  are scalar. The terminal constraint function  $\psi$  is an  $s$ -dimensional column vector function of  $x(t_f)$  at  $t_f$ . The functions  $f$ ,  $L$ ,  $F$  and  $\psi$  are assumed to be three times continuously differentiable in each argument. The final time  $t_f$  is assumed to be given explicitly.

The control problem is: determine the control function  $u(\cdot)$  which satisfies (4) and (5) and minimizes  $V(x_0, t_0)$ .

### 3. Totally and Partially Singular Problems

It can be shown that, along an optimal trajectory, the following necessary conditions (Pontryagin's Principle) hold:

$$-\dot{\lambda} = H_x(\bar{x}, \bar{u}, \lambda, t) \quad ; \quad \lambda(t_f) = F_x(\bar{x}(t_f), t_f) + \psi_x^T \nu \quad (7)$$

where

$$\bar{u} = \arg \min_{u \in U} H(\bar{x}, u, \lambda, t) \quad (8)$$

and

$$H(x, u, \lambda, t) = L(x, t) + \lambda^T f(x, u, t) \quad . \quad (9)$$

Here,  $\bar{x}(\cdot)$ ,  $\bar{u}(\cdot)$  denote the candidate state and control functions and  $\lambda(\cdot)$  denotes an n-vector of Lagrange multiplier functions of time.

In general the optimal control function (for the class of problems formulated in Section I. 2) consist of bang-bang sub-arcs and singular sub-arcs.<sup>†</sup> A bang-bang arc is one along which the controls lie on the boundary of U and  $H_{u_i}(\bar{x}, \lambda, t) \neq 0$ ;  $i = 1, \dots, m$  (except at a finite number of switch times where the components of  $\bar{u}$  change sign).

A singular arc [17] is one along which

$$H_{u_i}(\bar{x}, \lambda, t) = 0 \quad ; \quad i = 1, \dots, m \quad (10)$$

for a finite time interval.<sup>‡</sup> Note that this implies that, on a singular arc, H is independent of the control u.

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<sup>†</sup> 'Arc' and 'sub-arc' are used synonymously.

<sup>‡</sup> For simplicity, we shall consider all the controls to be singular simultaneously. If this is not the case, no conceptual difficulties arise.

In the sequel we shall make use of the following definitions:

Defn. 1: A totally singular control function is one along which (10) holds for all  $t \in [t_o, t_f]$ .

Defn. 2: A partially singular control function is one along which (10) holds for  $k$  sub-intervals of length  $T_i$ ;  $i = 1, \dots, k$  and where

$$\sum_{i=1}^k T_i < (t_f - t_o).$$

## II. Totally Singular Control Functions, Unconstrained Terminal State

### 1. Existing Necessary Conditions of Optimality

In [7] Kelley et al show that the following (generalized Legendre-Clebsch) condition is necessary for the optimality of a singular arc:

$$(-1)^q \frac{\partial}{\partial u} \left[ \frac{d^{2q}}{dt^{2q}} H_u(\bar{x}, \lambda, t) \right] \geq 0 \quad (11)$$

where the  $q$ -th time derivative of  $H_u$  is the first to contain explicitly the control  $u$ . Kelley et al used special control variations in order to derive this result; see [7]. Recently an additional (Jacobson's) necessary condition was discovered [9]. In order for a singular arc to be optimal it is necessary that

$$\frac{1}{2} f_u^T H_{xu} + \frac{1}{2} H_{ux} f_u + f_u^T Q f_u \geq 0 \quad (12)$$

where

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x \quad ; \quad Q(t_f) = F_{xx}(\bar{x}(t_f), t_f) \quad (13)$$

The partial derivatives  $f_u$ ,  $H_{xu}$ ,  $H_{ux}$  and  $f_x$  are all evaluated along the singular arc  $\bar{x}(\cdot)$ ,  $\bar{u}(\cdot)$ . In [9] the above condition is derived for a scalar control using the technique of Differential Dynamic Programming [22]; in that paper,  $Q(t)$  is shown to be the second partial derivative of  $V(x, t)$  with respect to  $x$  obtained whilst keeping  $u(\cdot) = \bar{u}(\cdot)$ . An alternative

derivation, using the Lagrange multiplier rule, is given in the Appendix of this paper.

Of course, in addition to conditions (11) and (12), Pontryagin's Principle must be satisfied.

## 2. Second Variation ( $\delta^2V$ )

An expression for the second variation is [23]:

$$\delta^2V = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_0}^{t_f} \quad (14)$$

subject to the differential equation:

$$\delta \dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad . \quad (15)$$

In order for the singular (stationary) solution to be minimizing it is necessary that

$$\delta^2V \geq 0 \quad (16)$$

for all  $\delta u(\cdot)$  sufficiently small to justify the second order expansion of  $V$ , and such that

$$\bar{u}(\cdot) + \delta u(\cdot) \in U \quad . \quad (17)$$

Both Kelley's and Jacobson's conditions are necessary for (16) to hold; see [7] and the Appendix of this paper. In Section II.4 we present sufficient conditions for (16) to hold. Note that the auxiliary minimization problem (14), (15) cannot be solved routinely because it is singular.

## 3. Adjoining Linearized System to $\delta^2V$

We now adjoin (15) to (14) using a vector Lagrange multiplier function of time  $\delta\lambda(t)$ :

$$\delta^2\hat{V} = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta\lambda^T [f_x \delta x + f_u \delta u - \delta \dot{x}] \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_0}^{t_f} \quad (18)$$



Integrating  $\delta\lambda^T \delta\dot{x}$  by parts, we obtain:

$$\delta^2\hat{V} = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \delta\lambda^T [f_x \delta x + f_u \delta u] + \delta\dot{\lambda}^T \delta x \right\} dt + \left[ \frac{1}{2} \delta x^T F_{xx} \delta x - \delta\lambda^T \delta x \right] \Big|_{t_0}^{t_f} \quad (19)$$

Let us now choose

$$\delta\lambda(t) = \frac{1}{2} P(t) \delta x \quad (20)$$

where  $P(t)$  is an  $n \times n$  symmetric<sup>†</sup>, time varying, matrix. The second variation becomes:

$$\delta^2\hat{V} = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{P} + H_{xx} + f_x^T P + P f_x) \delta x + \delta u^T (H_{ux} + f_u^T P) \delta x \right\} dt + \left[ \frac{1}{2} \delta x^T F_{xx} \delta x - \frac{1}{2} \delta x^T P \delta x \right] \Big|_{t_0}^{t_f} \quad (21)$$

subject to:

$$\delta\dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (22)$$

Note that  $\delta^2 V = \delta^2 \hat{V}$ , if (22) holds.

#### 4. Sufficient Conditions for Non-negativity of $\delta^2 V$

As remarked in Section II. 2, the auxiliary problem (14), (15) or (21), (22) cannot be solved routinely owing to the fact that it is singular. Our new approach to the problem is to choose a bounded matrix function  $P(\cdot)$  such that:

---

<sup>†</sup> There is no loss of generality in choosing  $P$  to be symmetric; this is so since if  $P$  were chosen to be unsymmetric,  $P + P^T$  would appear in place of  $P$  in (21).

$$H_{ux} + f_u^T P = 0 \quad \forall t \in [t_0, t_f] \quad (23)$$

Here,  $P(t)$  is an  $n \times n$  symmetric matrix function of time and  $H_{ux}$  is  $m \times n$  so that there are cases where (23) can be solved by choosing appropriate values for some of the elements of  $P$ . By choosing  $P$  according to (23) we annihilate the coefficients of the mixed  $\delta u \delta x$  terms in (21). The remaining terms are quadratic forms in  $\delta x(t)$  and  $\delta x(t_f)$ . Clearly, sufficient conditions for  $\delta^2 \hat{V} = \delta^2 V \geq 0$  are that (23) hold and

$$\dot{P} + H_{xx} + f_x^T P + P f_x = M(t) \geq 0 \quad (24)$$

and

$$-P(t_f) + F_{xx}(\bar{x}(t_f), t_f) = G(t_f) \geq 0 \quad (25)$$

Equality (23) together with inequalities (24) and (25) constitute sufficient conditions for  $\delta^2 V \geq 0, \forall \delta x(\cdot)$ .

### 5. Sufficient Conditions for Optimality

Sufficient conditions for a weak relative minimum are obtained by strengthening (24) and (25):

$$\dot{P} + H_{xx} + f_x^T P + P f_x = M(t) > 0 \quad \forall t \in [t_0, t_f] \quad (26)$$

$$-P(t_f) + F_{xx}(\bar{x}(t_f), t_f) = G(t_f) > 0 \quad (27)$$

To see this, note that if (23), (26) and (27) hold, then  $\delta^2 \hat{V} = 0$  if  $\delta x(\cdot) = 0$  almost everywhere including  $t_f$ . However, if  $\delta x(\cdot) = 0$  almost everywhere including  $t_f$ , then by our assumptions on  $L$  and  $F$  -- see Section I.2 -- we have that the total change in cost is:

$$\begin{aligned} \Delta V &= \int_{t_0}^{t_f} L(\bar{x} + \delta x, t) dt - \int_{t_0}^{t_f} L(\bar{x}, t) dt + F(\bar{x}(t_f) + \delta x(t_f), t_f) \\ &\quad - F(\bar{x}(t_f), t_f) \\ &= 0 \end{aligned} \quad (28)$$

i. e. ,

$$\delta^2 \hat{V} = 0 \implies \Delta V = 0 \quad . \quad (29)$$

Thus we can always choose  $\delta x(\cdot) \neq 0$  sufficiently small so that  $\delta^2 \hat{V}$  is the dominant term in the expansion for  $\Delta V$ ; hence we have sufficiency.

Example:  $H_{ux} = 0$ ,  $H_{xx} > 0$ ,  $F_{xx} > 0$ . In this case,  $\dot{P}(t) = P(t) = 0$   $\forall t \in [t_0, t_f]$ , and  $G(t_f) = 0$ , satisfies (23), (26), (27).

Note: If the dynamical equations (1) are linear and L and F are quadratic, then (23), (24) and (25) are sufficient conditions for optimality because all variations higher than the second vanish identically.

Example:

$$\left. \begin{array}{ll} \dot{x}_1 = x_2 & x_1(0) = 0 \\ \dot{x}_2 = u & x_2(0) = 0 \end{array} \right\} \quad (30)$$

$$V = \int_0^1 \left( \frac{1}{2} x_1^2 + 2x_1 x_2 + \frac{1}{2} x_2^2 \right) dt \quad (31)$$

$$|u| \leq 1 \quad . \quad (32)$$

Here,  $\bar{u}(\cdot) = 0$  is a totally singular control which satisfies Pontryagin's Principle. We have that:

$$H_{ux} = 0 \quad (33)$$

$$F = 0 \quad (34)$$

and

$$H_{xx} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad (35)$$

Note that  $H_{xx}$  is not positive semi-definite. Equation (23) yields:

$$P_{12}(t) = P_{22}(t) = 0 \quad ; \quad t \in [0, 1] \quad (36)$$

so that the left-hand side of (24) becomes:

$$\begin{bmatrix} \dot{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ P_{11} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad (37)$$

and the left-hand side of (25) becomes:

$$\begin{bmatrix} -P_{11}(t_f) & 0 \\ 0 & 0 \end{bmatrix} \quad (38)$$

Inequalities (24) and (25) are satisfied if we choose

$$\left. \begin{array}{l} \dot{P}_{11} = 0 \\ P_{11} = -2 \end{array} \right\} \quad (39)$$

and since the system dynamics are linear, and the cost quadratic,  $\bar{u}(\cdot) = 0$  is optimal.

#### 6. Relationship to Existing Necessary Conditions

Both Kelley's [7] and Jacobson's [9] conditions can be derived from (23), (24), (25).

##### a) Jacobson's Condition

Let

$$Q + \bar{P} = P \quad , \quad (40)$$

where  $Q$  and  $\bar{P}$  are both  $n \times n$ , symmetric matrix functions of time, then, from (23),

$$H_{ux} + f_u^T Q + f_u^T \bar{P} = 0 \quad (41)$$

so that

$$\frac{1}{2} f_u^T (H_{xu} + Q f_u + \bar{P} f_u) + \frac{1}{2} (H_{ux} + f_u^T Q + f_u^T \bar{P}) f_u = 0 \quad (42)$$

From (24) and (40),

$$-\dot{\bar{P}} - \dot{Q} = H_{xx} + f_x^T (Q + \bar{P}) + (Q + \bar{P}) f_x - M(t) \quad (43)$$

and from (25) and (40)

$$-Q(t_f) - \bar{P}(t_f) + F_{xx}(\bar{x}(t_f), t_f) - G(t_f) = 0 \quad (44)$$

Now set

$$Q(t_f) = F_{xx}(\bar{x}(t_f), t_f) \quad (45)$$

and

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x \quad (46)$$

so that

$$\bar{P}(t_f) = -G(t_f) \quad (47)$$

and

$$-\dot{\bar{P}} = -M(t) + f_x^T \bar{P} + \bar{P} f_x \quad (48)$$

Now, since

$$M(t) \geq 0 \quad \forall t \in [t_0, t_f] \text{ and } G(t_f) \geq 0 \quad (49)$$

we have that

$$\bar{P}(t) \leq 0 \quad \forall t \in [t_0, t_f] \quad (50)$$

Using inequality (50) in (42), we obtain

$$\frac{1}{2} f_u^T H_{xu} + \frac{1}{2} H_{ux} f_u + f_u^T Q f_u \geq 0 \quad (51)$$

Inequality (51) together with (45) and (46) comprise Jacobson's necessary condition.

b) Kelley's Condition (generalized Legendre-Clebsch)

Differentiating (23) with respect to time yields:

$$\dot{H}_{ux} + f_u^T \dot{P} + f_u^T \dot{P} = 0 = \dot{H}_{ux} + f_u^T \dot{P} - f_u^T (H_{xx} + f_x^T P + P f_x - M) = 0 \quad (52)$$

Post multiplying (52) by  $f_u$  and adding its transpose, we obtain

$$\begin{aligned} \dot{H}_{ux} f_u + f_u^T \dot{H}_{ux} + f_u^T P f_u + f_u^T P f_u - 2f_u^T H_{xx} f_u - 2f_u^T f_x^T P f_u \\ - 2f_u^T P f_x f_u + 2f_u^T M f_u = 0 \end{aligned} \quad (53)$$

Using

$$H_{ux} = -f_u^T P \quad (54)$$

in (53):

$$\begin{aligned} \dot{H}_{ux} f_u + f_u^T \dot{H}_{ux} - \dot{f}_u^T H_{xu} - H_{ux} \dot{f}_u - 2f_u^T H_{xx} \dot{f}_u + 2f_u^T f_x^T H_{xu} \\ + 2H_{ux} f_x \dot{f}_u + 2f_u^T M \dot{f}_u = 0 \end{aligned} \quad (55)$$

Rearranging (53),

$$\begin{aligned} -2f_u^T H_{xx} \dot{f}_u + 2H_{ux} (f_x \dot{f}_u - \dot{f}_u) + 2(f_u^T f_x^T - \dot{f}_u^T) H_{xu} \\ + \frac{d}{dt} (H_{ux} f_u) + \frac{d}{dt} (f_u^T H_{xu}) = -2f_u^T M \dot{f}_u \end{aligned} \quad (56)$$

However, we have that

$$M(t) \geq 0 \quad \forall t \in [t_0, t_f] \quad (57)$$

so that

$$\begin{aligned} -f_u^T H_{xx} \dot{f}_u + H_{ux} (f_x \dot{f}_u - \dot{f}_u) + (f_u^T f_x^T - \dot{f}_u^T) H_{xu} \\ + \frac{1}{2} \frac{d}{dt} (H_{ux} f_u) + \frac{1}{2} \frac{d}{dt} (f_u^T H_{xu}) \leq 0 \end{aligned} \quad (58)$$

Now, the left-hand side of (58) is just

$$\frac{\partial}{\partial u} \left[ \frac{d^2}{dt^2} H_u \right] \quad (59)$$

so that

$$(-1) \frac{\partial}{\partial u} \left[ \frac{d^2}{dt^2} H_u \right] \geq 0 \quad (60)$$

This is Kelley's first necessary condition. If this is met with equality,

i. e.,

$$f_u^T M(t) f_u = 0 \quad (61)$$

then (56) is again differentiated with respect to time and (54) and (56) are substituted in. This yields Kelley's second condition, viz.,

$$\frac{\partial}{\partial u} \left[ \frac{d^4}{dt^4} H_u \right] \geq 0 \quad (62)$$

The generalized condition

$$(-1)^q \frac{\partial}{\partial u} \left[ \frac{d^{2q}}{dt^{2q}} H_u \right] \geq 0 \quad (63)$$

is obtained by further differentiations.

Note: In Section II. 5, we gave sufficient conditions for optimality; a requirement was that

$$M(t) > 0 \quad \forall t \in [t_0, t_f] \quad (64)$$

However, this condition cannot hold unless  $q = 1$  (see (56); if  $q > 1$ , then  $f_u^T M f_u = 0$ , contradicting (64)).

### III. Totally Singular Control Functions, Constrained Terminal State

#### 1. Second-Variation ( $\delta^2 V^*$ )

We shall allow the terminal constraint

$$\psi(x(t_f), t_f) = 0 \quad (65)$$

where  $\psi$  is an  $s$ -dimensional vector function. As before,  $t_f$  is assumed to be given explicitly.

If  $\psi$  is adjoined to the cost functional by Lagrange multipliers  $\nu$  [23], the second variation is:

$$\delta^2 V^* = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt + \frac{1}{2} \delta x^T (F_{xx} + \nu^T \psi_{xx}) \delta x \Big|_{t_0}^{t_f} \quad (66)$$

subject to<sup>†</sup>

$$\delta \dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (67)$$

and<sup>†</sup>

$$\psi_x \delta x \Big|_{t_f} = 0 \quad (68)$$

## 2. Adjoining Linearized System to $\delta^2 V^*$

As in Section II, we adjoin (67) to (66) by a Lagrange multiplier function  $\delta \lambda(\cdot)$ . We integrate the term  $\delta \lambda^T \delta \dot{x}$  by parts and set

$$\delta \lambda = \frac{1}{2} P(t) \delta x \quad . \quad (69)$$

We obtain finally:

$$\begin{aligned} \delta^2 V^* = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{P} + H_{xx} + f_x^T P + P f_x) \delta x + \delta u^T (H_{ux} + f_u^T P) \delta x \right\} dt \\ & + \frac{1}{2} \delta x^T (F_{xx} + v^T \psi_{xx} - P) \delta x \Big|_{t_f} \end{aligned} \quad (70)$$

subject to  $\psi_x \delta x(t_f) = 0$ .

If  $\psi_x$  has rank  $s$ , then  $s$  components of  $\delta x(t_f)$  -- referred to as  $\delta x^s(t_f)$  -- can be solved for in terms of the remaining  $n - s$  components,  $\delta x^{n-s}(t_f)$ ; for example:<sup>‡</sup>

$$\delta x^s(t_f) = -A_1^{-1} A_2 \delta x^{n-s}(t_f) \quad (71)$$

where

$$s \updownarrow \begin{array}{c} \xrightarrow{n} \\ [ A_1 \mid A_2 ] = \psi_x \\ \xleftarrow{s} \quad \xleftarrow{(n-s)} \end{array} \quad (72)$$

<sup>†</sup> More precisely, we have that  $(\bar{x} + \delta x')$  =  $f(\bar{x} + \delta x', \bar{u} + \delta u, t)$  and  $\psi(\bar{x}(t_f) + \delta x'(t_f), t_f) = 0$ . However, expansions of these which are of higher-order than the first do not influence  $\delta^2 V^*$ .

<sup>‡</sup> If  $A_1$  is singular, then differently partitioned  $\psi_x$  and  $\delta x(t_f)$  must be used.



so that

$$\delta_{\mathbf{x}}(t_f) = \begin{bmatrix} -A_1^{-1} A_2 \delta_{\mathbf{x}}^{n-s}(t_f) \\ \dots \\ \delta_{\mathbf{x}}^{n-s}(t_f) \end{bmatrix} = \begin{bmatrix} -A_1^{-1} A_2 \\ I \end{bmatrix} \delta_{\mathbf{x}}^{n-s}(t_f) = Z \delta_{\mathbf{x}}^{n-s}(t_f) \quad (73)$$

where  $Z$  is  $n \times (n-s)$ .

We now eliminate the constraint

$$\psi_{\mathbf{x}} \delta_{\mathbf{x}} \Big|_{t_f} = 0 \quad (74)$$

from (70) by using (73) in the boundary terms of  $\delta^2 \tilde{V}^*$ :

$$\begin{aligned} \delta^2 \tilde{V} = & \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta_{\mathbf{x}}^T (\dot{P} + H_{\mathbf{xx}} + f_{\mathbf{x}}^T P + P f_{\mathbf{x}}) \delta_{\mathbf{x}} + \delta_{\mathbf{u}}^T (H_{\mathbf{ux}} + f_{\mathbf{u}}^T P) \delta_{\mathbf{x}} \right\} dt \\ & + \frac{1}{2} (\delta_{\mathbf{x}}^{n-s})^T \left\{ Z^T (F_{\mathbf{xx}} + \nu^T \psi_{\mathbf{xx}} - P) Z \right\} \delta_{\mathbf{x}}^{n-s} \Big|_{t_f} . \end{aligned} \quad (75)$$

### 3. Sufficient Conditions for Non-negativity of $\delta^2 \tilde{V}$

Sufficient conditions for  $\delta^2 \tilde{V} \geq 0$  are (by analogy with Section II. 4):

$$\left. \begin{aligned} H_{\mathbf{ux}} + f_{\mathbf{u}}^T P = 0 \\ \dot{P} + H_{\mathbf{xx}} + f_{\mathbf{x}}^T P + P f_{\mathbf{x}} = M(t) \geq 0 \end{aligned} \right\} \forall t \in [t_0, t_f] \quad (76)$$

$$\left. \begin{aligned} \dot{P} + H_{\mathbf{xx}} + f_{\mathbf{x}}^T P + P f_{\mathbf{x}} = M(t) \geq 0 \end{aligned} \right\} \quad (77)$$

and

$$Z^T (F_{\mathbf{xx}} + \nu^T \psi_{\mathbf{xx}} - P) Z \Big|_{t_f} = G(t_f) \geq 0 \quad (78)$$

Note that if  $s = 0$  (no terminal constraints),

$$Z = n \begin{array}{c} \leftarrow n \\ \updownarrow I \\ \rightarrow n \end{array} \quad (79)$$

and (76)-(78) reduce to (23)-(25).

#### 4. Sufficient Conditions for Optimality

By strengthening the inequalities in (77) and (78) we obtain

$$\delta^2 \tilde{V} > 0 \quad \forall \delta x(\cdot) \neq 0 \quad (80)$$

with

$$\delta^2 \tilde{V} = 0 \quad \text{if } \delta x(\cdot) = 0 \text{ almost everywhere, including } t_f. \quad (81)$$

The argument of Section II. 5 can be used here to show that (80), (81) imply optimality (weak relative minimum).

Note: As in the case of unconstrained terminal states these strengthened conditions can hold only if the singular arc is first-order (i. e., the generalized Legendre-Clebsch condition holds with strict inequality for  $q = 1$ ).

#### 5. Relationship to Existing Necessary Conditions

As in Section II. 6 it is easy to show that satisfaction of (76)-(78) implies that Kelley's condition is satisfied. Jacobson's condition for problems with constrained terminal state is more complex than for the unconstrained case; see [9]. We shall not derive this condition here, from (76)-(78).

#### 6. Comment on Problems with Constrained Terminal State

When deriving necessary conditions of optimality for problems with terminal constraints by constructing variations of the control function, one is faced with the task of showing that the chosen variation is indeed admissible [7], [9]. This is a formidable task even if the linearized dynamical system is assumed to be completely controllable and  $\psi_x$  is assumed to have rank  $s$ .<sup>†</sup> We remark that the approach taken in this paper does not require arguments of the type referred to above. We need

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<sup>†</sup> These are common assumptions [23].

only assume that it is possible to satisfy  $\psi(x(t_f), t_f) = 0$ , and that  $\psi_x$  has rank  $s$  at  $\bar{x}(t_f), t_f$ . We do not have to construct explicitly admissible control variations.

#### IV. Partially Singular Control Functions<sup>†</sup>

##### 1. First and Second Variation ( $\delta V^* + \delta^2 V^*$ )

As defined in Section I. 3, a partially singular control function may have both singular and nonsingular portions (i. e., sub-intervals of singular and bang-bang control). Along nonsingular arcs  $H_u \neq 0$ , and the condition (Pontryagin's)

$$\min_{u \in U} H(\bar{x}, u, \lambda, t) \quad (82)$$

must hold (this is trivially satisfied along a singular arc). In this case the sum of the first and second variations is:

$$\begin{aligned} \delta V^* + \delta^2 V^* = & \int_{t_0}^{t_f} \{ H_u^T \delta u + \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \} dt \\ & + \frac{1}{2} \delta x^T (F_{xx} + v^T \psi_{xx}) \delta x \Big|_{t_f} \end{aligned} \quad (83)$$

subject to

$$\delta \dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (84)$$

and

$$\psi_x \delta x \Big|_{t_f} = 0 \quad (85)$$

In order to enforce (85) (and  $\psi(x(t_f), t_f) = 0$ ), we have

$$\bar{u}(\cdot) + \delta u(\cdot) \in U_2 \quad (86)$$

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<sup>†</sup> In this section we treat the constrained terminal state problem; the unconstrained problem is a special case.

where

$$U_2 = U \cap U_1 \quad (87)$$

and

$$U_1 \equiv \{u(\cdot) : \psi(x(t_f), t_f) = 0, \dot{x} = f(x, u, t); x(t_0) = x_0\} \quad (88)$$

Note that by (82),

$$H_u^T \delta u \geq 0 \quad , \quad \bar{u}(\cdot) + \delta u(\cdot) \in U_2 \quad (89)$$

with equality holding along singular arcs and at switch times of the bang-bang control arcs. If there are no singular arcs and no switchings of the control (i. e.,  $|H_u| \neq 0, \forall t \in [t_0, t_f]$  so that  $\bar{u}(\cdot) = \text{const.} = +1$  or  $-1$ ) then Pontryagin's Principle is sufficient for optimality because the second-order terms in (83) can be made insignificant (i. e., dominated by  $H_u^T \delta u$ ) for  $\|\delta u(\cdot)\|$  sufficiently small. In the case where bang-bang arcs are present (i. e. where  $\bar{u}(t)$  switches between its upper and lower bounds) one can, by placing a control variation in the immediate vicinity of a switch point, cause  $H_u^T \delta u$  to contribute less to the change in cost  $\delta V^* + \delta^2 V^*$  than the second variation terms.

Clearly, sufficient conditions for  $\delta^2 V^* \geq 0$  are (76)-(78) and sufficient conditions of optimality are these in strengthened form. Less restrictive sufficient conditions for purely bang-bang control functions have been given previously [24], [25]. However, in this section, we allow partially singular (i. e., 'partially bang-bang') control functions and thus embrace a wider class of problems than in [24] and [25].

## V. Problems Nonlinear in Control

### 1. Introduction

In the last section we indicated that our approach to sufficiency is independent of whether the control function is totally singular or partially

singular or, in the purely bang-bang case, nonsingular. In this section we study the more general nonsingular problem where the control  $u$  appears nonlinearly in  $f$  and  $L$ . We show that our approach is indeed applicable and give examples to illustrate our results. As a byproduct of the analysis, we obtain sufficient conditions for the boundedness of the solution of a certain matrix Riccati differential equation; these are less restrictive conditions than those obtained heretofore [21], [26].

We shall consider the following nonlinear optimal control problem:

$$\dot{x} = f(x, u, t) \quad ; \quad x(t_0) = x_0 \quad (90)$$

$$V(x_0, t_0) = \int_{t_0}^{t_f} L(x, u, t) dt + F(x(t_f), t_f) \quad (91)$$

Here it is assumed, for simplicity, that there are no constraints on the control  $u$  or on the terminal state  $x(t_f)$ , though this in no way limits the wider applicability of the analysis (see Section V.5 for constrained terminal state). In this case the second variation is:

$$\begin{aligned} \delta^2 V = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x + \frac{1}{2} \delta u^T H_{uu} \delta u \right\} dt \\ + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} \end{aligned} \quad (92)$$

subject to

$$\delta \dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad (93)$$

Here,

$$H_{uu}(t) \geq 0 \quad \forall t \in [t_0, t_f] \quad (94)$$

is a well-known necessary condition (Legendre-Clebsch) of optimality.

For the problem to be nonsingular, strict inequality must hold, i. e.,

$$H_{uu}(t) > 0 \quad \forall t \in [t_0, t_f] \quad . \quad (95)$$

A known necessary condition of optimality<sup>†</sup> [26] (which together with (95) and Pontryagin's Principle forms sufficient conditions of optimality) is that the solution to the following matrix Riccati differential equation be bounded for  $t \in [t_0, t_f]$ :

$$\begin{aligned} \dot{S} &= H_{xx} + f_x^T S + S f_x - (H_{ux} + f_u^T S)^T H_{uu}^{-1} (H_{ux} + f_u^T S) \quad ; \\ S(t_f) &= F_{xx} \Big|_{t_f} \quad . \end{aligned} \quad (96)$$

Sufficient conditions for the boundedness of  $S(\cdot)$  are known to be [21], [26].

$$\left. \begin{aligned} H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} &\geq 0 \quad \forall t \in [t_0, t_f] \\ F_{xx}(\bar{x}(t_f), t_f) &\geq 0 \\ H_{uu}^{-1}(t) &> 0 \quad \forall t \in [t_0, t_f] \end{aligned} \right\} \quad (97)$$

## 2. Sufficient Conditions for Optimality

Equation (93) can be adjoined to (92) using a vector Lagrange multiplier function of time  $\delta\lambda(\cdot)$ . If, as before, we let

$$\delta\lambda = \frac{1}{2} P(t) \delta x \quad (98)$$

then, the second variation becomes:

$$\begin{aligned} \delta^2 \hat{V} &= \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{P} + H_{xx} + f_x^T P + P f_x) \delta x + \delta u^T (H_{ux} + f_u^T P) \delta x \right. \\ &\quad \left. + \frac{1}{2} \delta u^T H_{uu} \delta u \right\} dt + \frac{1}{2} \delta x^T (F_{xx} - P) \delta x \Big|_{t_f} \quad . \end{aligned} \quad (99)$$

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<sup>†</sup> Classically known as the 'no-conjugate-point condition' [27].

Clearly,  $\delta^2 \hat{V} \geq 0$  if we can choose  $P(t)$  (which has bounded elements) so that

$$H_{ux} + f_u^T P = 0 \quad \forall t \in [t_0, t_f] \quad (100)$$

$$\dot{P} + H_{xx} + f_x^T P + P f_x = M(t) \geq 0 \quad \forall t \in [t_0, t_f] \quad (101)$$

$$-P(t_f) + F_{xx}(\bar{x}(t_f), t_f) = G(t_f) \geq 0 \quad (102)$$

Moreover, because of (95), we have that

$$\delta^2 \hat{V} \geq kN^2[\delta u(\cdot)] \quad \forall \delta u(\cdot) \quad (103)$$

where  $N$  is a suitable norm on  $\delta u(\cdot)$  and  $k > 0$ . Equality (103) indicates that  $\delta^2 \hat{V}$  is strongly positive and, by a theorem of Gelfand and Fomin [27, p. 100], this is sufficient for  $\bar{u}(\cdot)$  to be a minimizing control function (weak relative minimum). Thus conditions (100)-(102) are sufficient for optimality in this nonsingular problem. As an immediate consequence we have the following result: Conditions (100)-(102) imply that the matrix Riccati equation (96) has a bounded solution in the interval  $[t_0, t_f]$  (because the boundedness of  $S(\cdot)$  is a necessary condition of optimality). These conditions are, in general, considerably weaker than (97), as the following example illustrates.

Example:

$$\left. \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \right\} \quad (104)$$

$$V = \int_0^1 \left( \frac{1}{2} x_1^2 + 2x_1 x_2 + \frac{1}{2} x_2^2 + \frac{1}{2} u^2 \right) dt \quad (105)$$

Here,

$$H_{ux} = 0, \quad H_{xx} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad F = 0 \quad (106)$$

These values do not satisfy conditions (97). However, from Section II. 5, we have that:

$$P_{12}(t) = P_{22}(t) = 0 = \dot{P}_{11}(t) \quad \forall t \in [t_0, t_f] \quad (107)$$

and

$$P_{11} \leq -1 \quad (108)$$

satisfy (100)-(102), so that the stationary solution to this problem, obtained from Pontryagin's Principle, is optimal. Note that in this particular case the checking of (100)-(102) is considerably easier than integrating the matrix Riccati differential equation to see whether or not its solution is bounded in the interval  $[0, 1]$ .

### 3. Derivation of Riccati Equation

The Riccati differential equation (96) can be derived directly from (100)-(102) as follows: From (100) and (101),

$$\dot{P} = H_{xx} + f_x^T P + P f_x - M(t) - (H_{ux} + f_u^T P)^T H_{uu}^{-1} (H_{ux} + f_u^T P) \quad (109)$$

Let

$$P = \bar{P} + S \quad , \quad (110)$$

then

$$\begin{aligned} -\dot{\bar{P}} - \dot{S} &= H_{xx} + f_x^T (\bar{P} + S) + (\bar{P} + S) f_x - M(t) \\ &\quad - [H_{ux} + f_u^T (\bar{P} + S)]^T H_{uu}^{-1} [H_{ux} + f_u^T (\bar{P} + S)] \quad (111) \\ &= H_{xx} + f_x^T (\bar{P} + S) + (\bar{P} + S) f_x - M(t) - (H_{ux} + f_u^T S)^T H_{uu}^{-1} (H_{ux} + f_u^T S) \\ &\quad - (H_{ux} + f_u^T S)^T H_{uu}^{-1} f_u^T \bar{P} - \bar{P} f_u H_{uu}^{-1} (H_{ux} + f_u^T S) \\ &\quad - \bar{P} f_u H_{uu}^{-1} f_u^T \bar{P} \quad (112) \end{aligned}$$



Using (100) and (110) in the last three terms of (112) we obtain:

$$\begin{aligned} -\dot{\bar{P}} - \dot{S} = & H_{xx} + f_x^T(S + \bar{P}) + (\bar{P} + S)f_x - (H_{ux} + f_u^T S)^T H_{uu}^{-1} (H_{ux} + f_u^T S) \\ & + \bar{P} f_u H_{uu}^{-1} f_u^T \bar{P} - M(t) \end{aligned} \quad (113)$$

Now choose

$$-\dot{\bar{P}} = -M(t) + f_x^T \bar{P} + \bar{P} f_x + \bar{P} f_u H_{uu}^{-1} f_u^T \bar{P} \quad . \quad (114)$$

From (102) and (110), we have that

$$-\bar{P}(t_f) - S(t_f) + F_{xx} = G(t_f) \geq 0 \quad . \quad (115)$$

Choose,

$$\bar{P}(t_f) = -G(t_f) \quad (116)$$

Now we have that  $\bar{P}(t)$  is bounded in the interval  $[t_0, t_f]$ . This follows from the fact that  $(-\bar{P})$  satisfies a Riccati equation for which conditions (97) hold, viz.,

$$\left. \begin{aligned} M(t) &\geq 0 \\ H_{uu}^{-1}(t) &> 0 \\ G(t_f) &\geq 0 \end{aligned} \right\} \quad \forall t \in [t_0, t_f] \quad (117)$$

Using these results in (113) and (115), we obtain finally:

$$\left. \begin{aligned} -\dot{S} = & H_{xx} + f_x^T S + S f_x - (H_{ux} + f_u^T S)^T H_{uu}^{-1} (H_{ux} + f_u^T S) \\ S(t_f) = & F_{xx}(\bar{x}(t_f), t_f) \end{aligned} \right\} \quad (118)$$

which is the Riccati equation (96). Now since (100)-(102) are satisfied by a matrix function  $P(\cdot)$  which has bounded elements, and since, by (117),  $\bar{P}(\cdot)$  is bounded, we have from (110) the result that  $S(\cdot)$  is bounded.

4. Another Example

$$\left. \begin{aligned} \dot{x}_1 &= x_2 & ; & & x_1(0) &= x_{10} \\ \dot{x}_2 &= u & ; & & x_2(0) &= x_{20} \end{aligned} \right\} \quad (119)$$

$$V = \int_0^1 \left( -\frac{1}{2}x_1^2 + 2x_2^2 + \frac{1}{2}u^2 \right) dt \quad (120)$$

Here,

$$\left. \begin{aligned} H_{ux} &= 0 \\ H_{xx} &= \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}, \quad F = 0 \end{aligned} \right\} \quad (121)$$

These values do not satisfy conditions (97). Conditions (100)-(102) become

$$[P_{12} \quad P_{22}] = 0 \quad (122)$$

$$\begin{bmatrix} \dot{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ P_{11} & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix} \geq 0 \quad (123)$$

and

$$\begin{bmatrix} -P_{11}(1) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (124)$$

Let us choose  $P_{11}(t_f) = 0$ ; this satisfies (124). From (122),  $P_{12}(t) = P_{22}(t) = 0$ ,  $\forall t \in [t_0, t_f]$ . If we choose:

$$\dot{P}_{11} = 2 \quad \text{and} \quad P_{11}(0) = -2 \quad (125)$$

then (123) becomes

$$\begin{bmatrix} 1 & -2 + 2t \\ -2 + 2t & 4 \end{bmatrix} \geq 0 \quad (126)$$

Inequality (126) holds  $\forall t \in [0, 1]$ . Thus the solution obtained from Pontryagin's Principle is optimal, and the Riccati equation associated with the above control problem has a bounded solution.

### 5. Constrained Terminal State

From Sections III. 3 and V. 2, sufficient conditions for optimality are:

$$\left. \begin{aligned} H_{ux} + f_u^T P = 0 \end{aligned} \right\} \forall t \in [t_0, t_f] \quad (127)$$

$$\left. \begin{aligned} \dot{P} + H_{xx} + f_x^T P + P f_x = M'(t) \geq 0 \end{aligned} \right\} \quad (128)$$

$$Z^T (F_{xx} + \psi_{xx}^T - P) Z \Big|_{t_f} = G'(t_f) \geq 0 \quad (129)$$

and (Legendre-Clebsch)

$$H_{uu}(t) > 0 \quad \forall t \in [t_0, t_f] \quad (130)$$

Example:

$$\left. \begin{aligned} \dot{x}_1 = x_2 & & x_1(0) = x_{10} & ; & x_1(1) = 0 \\ \dot{x}_2 = u & & x_2(0) = x_{20} & ; & x_2(1) = 0 \end{aligned} \right\} \quad (131)$$

$$V = \int_0^1 \left( -\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}u^2 \right) dt \quad (132)$$

Here,

$$H_{ux} = 0 \quad (133)$$

$$H_{xx} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = 0, \quad H_{uu} = 1 \quad (134)$$

In this case, because  $n = s = 2$ , condition (129) disappears. As before, we have that

$$H_{ux} + f_u^T P = [P_{12} \quad P_{22}] = 0 \quad (135)$$

Condition (128) becomes:

$$\begin{bmatrix} \dot{P}_{11} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & P_{11} \\ P_{11} & 0 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \quad (136)$$

Choosing  $P_{11}(0) = -1$  and  $\dot{P}_{11} = 2$ , the left-hand side of (136) becomes:

$$\begin{bmatrix} 1 & -1 + 2t \\ -1 + 2t & 1 \end{bmatrix} \quad (137)$$

which is  $\geq 0 \quad \forall t \in [0, 1]$ , so that the stationary solution obtained from Pontryagin's Principle is optimal. Note that the above sufficiency conditions are, in this case, easier to check than the usual sufficiency conditions for nonsingular, constrained terminal state problems [23]. Moreover, the presence of the terminal constraints actually makes the choice of  $\dot{P}$  and  $P(t_f)$  easier (if in this example there were no terminal constraints, Inequality (129) would be violated by our above choice of  $P_{11}(0)$  and  $\dot{P}_{11}$ ).

## VI. Applicability of the New Conditions

If the conditions

$$\left. \begin{aligned} H_{ux} + f_u^T P &= 0 \\ \dot{P} + H_{xx} + f_x^T P + P f_x &= M'(t) \geq 0 \end{aligned} \right\} \quad \forall t \in [t_0, t_f] \quad (138)$$

$$(139)$$

$$Z^T (F_{xx} + v^T \psi_{xx} - P) Z \Big|_{t_f} = G'(t_f) \geq 0 \quad (120)$$

cannot be satisfied, then no conclusion can be drawn regarding the nature (optimality or nonoptimality) of the stationary control function. This is because the above conditions are sufficient (but probably not necessary).

Example:

$$\dot{x} = u \quad x(t_0) = 0 \quad (141)$$

$$|u| \leq 1 \quad (142)$$

$$V = \int_0^1 \frac{1}{2} x^2 dt - \frac{1}{2} \alpha(t_f) x^2(t_f) \quad (143)$$

Clearly,

$$\bar{u}(\cdot) = 0 \quad (144)$$

is a stationary solution for the above problem. Here,  $H_{ux} = 0$ ,  $H_{xx} = 1$ ,  $F_{xx} = -\alpha(t_f)$  and  $P$  is scalar so that (100) determines

$$P(t) = 0 \quad \forall t \in [0, 1] \quad (145)$$

Condition (101) becomes

$$1 \geq 0 \quad (146)$$

and condition (102) becomes

$$-S \geq 0 \quad (147)$$

Clearly, (101) is satisfied and (102) is satisfied if

$$\alpha(t_f) \leq 0 \quad (148)$$

but is violated if

$$\alpha(t_f) > 0 \quad (149)$$

However, application of Jacobson's necessary condition [9] to this problem shows that if  $\alpha(t_f) > 0$ , the stationary solution (144) is not minimizing.

The above example suggests the following sufficient condition for nonoptimality of a singular control function.

## VII. Sufficient Conditions for Nonoptimality of a Singular Control Function

The second variation for the unconstrained terminal state problem is:

$$\delta^2 \hat{V} = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{P} + H_{xx} + f_x^T P + P f_x) \delta x + \delta u^T (H_{ux} + f_u^T P) \delta x \right\} dt + \frac{1}{2} \delta x^T (F_{xx} - P) \delta x \Big|_{t_f} \quad (150)$$

If it is possible to choose  $P(t)$ ;  $t \in [t_0, t_f]$  such that:

$$\dot{P} + H_{xx} + f_x^T P + P f_x = M''(t) \leq 0 \quad (151)$$

and

$$-P(t_f) + F_{xx}(\bar{x}(t_f), t_f) = G''(t_f) \leq 0 \quad (152)$$

and

$$\frac{1}{2} f_u^T H_{xu} + \frac{1}{2} H_{ux} f_u + f_u^T P f_u < 0 \quad (153)$$

then the singular control is nonoptimal.

The first two conditions cause the quadratic forms in  $\delta x$  and  $\delta x(t_f)$  in (150) to be nonpositive. If a rectangular pulse variation  $\delta u(\cdot)$  of height  $\eta$  and duration  $\Delta T$  is introduced, then the dominant term (for  $\eta$  and  $\Delta T$  sufficiently small) of

$$\int_{t_0}^{t_f} \delta u^T (H_{ux} + f_u^T P) \delta x dt \quad (154)$$

is

$$\frac{1}{2} \eta^T \left[ \frac{1}{2} f_u^T H_{xu} + \frac{1}{2} H_{ux} f_u + f_u^T P f_u \right] \eta (\Delta T)^2 \quad (155)$$

So that if

$$\frac{1}{2} f_u^T H_{xu} + \frac{1}{2} H_{ux} f_u + f_u^T P f_u < 0 \quad (156)$$

then

$$\delta^2 \hat{V} < 0 \quad (157)$$

and the singular control is not minimizing.

Example:

$$\dot{x} = u \quad ; \quad x(0) = 0 \quad (158)$$

$$V = \int_0^1 \frac{1}{2} x^2 dt - x^2(t_f) \quad (159)$$

In this case, conditions (151) and (152) become

$$\dot{P} + 1 \leq 0 \implies \dot{P} \leq 1 \quad (160)$$

$$-P(t_f) - 2 \leq 0 \implies P(t_f) \geq -2 \quad (161)$$

and

$$\frac{1}{2} f_u^T H_{xu} + H_{ux} f_u + f_u^T P f_u = P \quad (162)$$

Choose,

$$\dot{P} = -1 \quad \text{and} \quad P(t_f) = -2 \quad (163)$$

then conditions (151)-(153) are satisfied and the singular arc is nonoptimal.

### VIII. Conclusion

In this paper sufficient conditions are presented for the second variation to be non-negative in both singular and nonsingular control problems. It is demonstrated that known necessary conditions of optimality for singular problems and the no-conjugate-point condition for nonsingular problems are implied by the new conditions. Simple illustrative examples demonstrate the usefulness of the new conditions. A sufficient condition of optimality for singular problems is obtained by strengthening the inequality conditions; it is shown that these strengthened conditions can only be satisfied by first-order singular problems.

When applied to the nonsingular control problem, the new conditions yield less restrictive sufficient conditions for the boundedness of the solution

of the matrix Riccati differential equation than were known heretofore; this result appears to be useful in its own right.

The derivations presented are carried out for the case of  $u$  an  $n$ -vector, and  $s$ -vector constraints on the terminal state are permitted. Throughout, the final time  $t_f$  is assumed to be given explicitly; the generalization of the conditions to the case where  $t_f$  is given implicitly is straightforward but tedious.

The appendix contains a Lagrange multiplier derivation of a necessary condition of optimality for singular control problems which was derived previously using Differential Dynamic Programming [9].





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Appendix A: Lagrange Multiplier Derivation of Jacobson's Necessary Condition of Optimality for Singular Problems (No Terminal Constraints).

The second variation is:

$$\delta^2 V = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T H_{xx} \delta x + \delta u^T H_{ux} \delta x \right\} dt + \frac{1}{2} \delta x^T F_{xx} \delta x \Big|_{t_f} \quad (\text{A. 1})$$

subject to

$$\delta \dot{x} = f_x \delta x + f_u \delta u \quad ; \quad \delta x(t_0) = 0 \quad . \quad (\text{A. 2})$$

Adjoining (A. 2) to (A. 1) with Lagrange multiplier

$$\delta \lambda = \frac{1}{2} Q(t) \delta x \quad (\text{A. 3})$$

(where  $Q$  is an  $n \times n$  symmetric matrix function of time) and integrating by parts, we obtain

$$\begin{aligned} \delta^2 \hat{V} = \int_{t_0}^{t_f} \left\{ \frac{1}{2} \delta x^T (\dot{Q} + H_{xx} + f_x^T Q + Q f_x) \delta x + \delta u^T (H_{ux} + f_u^T Q) \delta x \right\} dt \\ + \frac{1}{2} \delta x^T (F_{xx} - Q) \delta x \Big|_{t_f} . \end{aligned} \quad (\text{A. 4})$$

Now, choose

$$-\dot{Q} = H_{xx} + f_x^T Q + Q f_x \quad ; \quad Q(t_f) = F_{xx}(\bar{x}(t_f), t_f) \quad (\text{A. 5})$$

then,

$$\delta^2 \hat{V} = \int_{t_0}^{t_f} \delta u^T (H_{ux} + f_u^T Q) \delta x dt \quad (\text{A. 6})$$

Introduce a variation  $\delta u(\cdot)$  which is zero everywhere except, say, in the interval  $[t_1, t_1 + \Delta T]$  where

$$t_1 \text{ and } t_1 + \Delta T \in [t_0, t_f] \quad , \quad (\text{A. 7})$$

and which has constant magnitude  $\eta$  (note that  $\bar{u}(\cdot) + \delta u(\cdot) \in U$ ).

The dominant term of (A. 6) produced by this variation is seen easily to be

$$\frac{1}{2}\eta^T \left[ \frac{1}{2}f_u^T H_{xu} + \frac{1}{2}H_{ux} f_u + f_u^T Q f_u \right] \eta (\Delta T)^2 \quad (\text{A. 8})$$

From (A. 8), for non-negative  $\delta^2 V$ , we must have

$$\frac{1}{2}f_u^T H_{xu} + \frac{1}{2}H_{ux} f_u + f_u^T Q f_u \geq 0 \quad . \quad (\text{A. 9})$$

This inequality, together with (A. 5) comprise the necessary condition of optimality obtained (for the case of scalar control), using Differential Dynamic Programming, in [9].

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13. ABSTRACT

Sufficient conditions for non-negativity of the second variation in singular and nonsingular control problems are presented; these conditions are in the form of equalities and differential inequalities. Control problem examples illustrate the use of the new conditions. The relationships of the new conditions to existing necessary conditions of optimality for singular and nonsingular problems are discussed. When applied to nonsingular control problems, it is shown that the conditions are sufficient to ensure the boundedness of the solution of the well-known matrix Riccati differential equation; these conditions are less stringent than those known heretofore.

KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Singular Arcs Optimal Control Theory Conditions of Optimality Singular Extremals Singular Control Problems						