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Abstract

Agents with standard, time-separable preferences do not care about the temporal distribution of risk. This is a strong assumption. For example, it seems plausible that a consumer may find persistent shocks to consumption less desirable than uncorrelated fluctuations. Such a consumer is said to exhibit temporal risk aversion. This paper examines the implications of temporal risk aversion for asset prices. The innovation is to work with expected utility preferences that (i) are not time-separable, (ii) exhibit temporal risk aversion, (iii) separate risk aversion from the intertemporal elasticity of substitution, (iv) separate short-run from long-run risk aversion and (v) yield stationary asset pricing implications in the context of an endowment economy. Closed form solutions are derived for the equity premium and the risk free rate. The equity premium depends only on a parameter indexing long-run risk aversion. The risk-free rate instead depends primarily on a separate parameter indexing the desire to smooth consumption over time and the rate of time preference.

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1 Introduction

Agents with standard, time-separable preferences do not care about the temporal distribution of risk. This is a strong assumption. For example, it seems plausible that in reality a consumer may find persistent shocks to consumption less desirable than uncorrelated fluctuations. Such a consumer is said to exhibit temporal risk aversion. This type of risk aversion is lacking from standard preferences, because, with additive separability, expected utility is independent of the correlation of consumption in two different time periods. However, other than mathematical convenience, there is no compelling reason for assuming that temporal risk aversion is exactly zero and that consumers do not care about the temporal distribution of risk.

This paper works with an alternative specification for preferences to investigate the implications of temporal risk aversion for asset prices. Allowing for temporal risk aversion is achieved by abandoning time-separability, while staying within the expected utility framework. This has two additional and closely related consequences. First, a separation of risk aversion from the intertemporal elasticity of substitution is attained. As is well known, with standard preferences the two are tightly linked, yet separating them can be crucial in explaining asset prices. Second, relative risk aversion can depend on the duration of the consumption gamble and this leads to notions of short-run and long-run risk aversion. This is also central to understanding the asset pricing implications. In particular, for the economy studied here, the equity premium is found to depend only on long-run risk aversion.

The preferences are specialized so as to yield stationary asset returns. To study the asset pricing implications, the present paper focuses on an endowment economy with $i.i.d.$ consumption growth. For this case, closed form solutions are derived for the risk-free rate and the equity premium. The risk-free rate depends primarily on a parameter indexing the desire to smooth consumption and the rate of time preference. By contrast, as mentioned, the equity premium depends only the coefficient of long-run relative risk aversion, which is equal to a separate preference parameter.

The rest of this paper is organized as follows. The next section briefly covers the basics of temporal risk aversion. After that, the paper will present and discuss the preferences studied here, derive the pricing kernel for the endowment economy and, finally, present some asset pricing implications. Some of the related literature is discussed throughout the paper.

2 Temporal risk aversion

Following Richard (1975), temporal risk aversion can be defined in the following way.[1] Consider a consumer who lives for two periods and is faced with a choice between two consumption gambles. In the first gamble, consumption in the two periods is either (c_{low}, c_{low}) or, with equal probability, (c_{high}, c_{high}), where $c_{high} > c_{low}$. The second gamble results in either (c_{low}, c_{high}) or (c_{high}, c_{low}), again with equal probability 0.5. If the consumer prefers the second lottery to the first for all values of c_{low} and $c_{high} > c_{low}$, then the consumer is considered to be temporally risk averse. If the first lottery is preferred, then the consumer is said to be temporally risk loving, while temporal risk neutrality applies in the case of indifference. An equivalent definition of temporal risk aversion replaces the second, preferred gamble by independent draws in each period. That is, consumption is uncorrelated over time and, in each period, is either c_{low} or c_{high}, with equal probability. The equivalence of the definitions follows directly from the additive properties of expected utility. A straightforward extension of the definition to T periods is presented in Richard (1975).[2]

It seems reasonable to regard the second gamble as less risky - there is no risk of a 'lifetime of misery' due to permanently low consumption (or less risk of that outcome, in the case of the second definition). In contrast, any consumer with time-separable preferences is indifferent between these two gambles and so is temporally risk neutral, because the serial correlation of consumption does not matter for expected utility under additive separability. (With time-separability expected utility is $E[u(c_1) + v(c_2)]$ which is trivially equal to $Eu(c_1) + Ev(c_2)$.)

Richard (1975) shows that a consumer with a twice differentiable utility function $U(c_1, c_2)$ is temporally risk averse if and only if the cross-partial derivative is negative, i.e. if and only if

$$U_{12} \equiv \frac{\partial^2 U(c_1, c_2)}{\partial c_1 \partial c_2} \leq 0$$

Strict temporal risk aversion holds if the inequality is strict. Temporal risk seeking is equivalent to a positive cross-partial derivative, and temporal risk neutrality to a value of zero. Thus, a utility

[1] See also Bommier (2003), Epstein and Tanny (1980) and Ingersoll (1987, p 43-44). Temporal risk aversion is sometimes also referred to as correlation aversion (Bommier and Epstein and Tanny) or as multivariate risk aversion (Richard), though it is distinct from multivariate risk aversion in the sense of Kihlstrom and Mirman (1974). The next section briefly discusses the relation wth the latter concept, for the type of preferences studied in this paper.

[2] For the T period case, a distinction can be made between pairwise temporal risk aversion (which is specific to two particular periods) and a global concept of temporal risk aversion. See Richard (1975).

function exhibits temporal risk neutrality if and only if it is additively separable.

3 Preferences

Preferences are ordered by

$$\mathcal{U} = W\left(u^{-1}\left(\sum_{t=0}^{T}\beta^t u(c_t)\right)\right) \tag{1}$$

with $\beta > 0$. W and u are strictly monotone real-valued functions and W is also increasing. The transformation W is irrelevant in the absence of uncertainty. In a stochastic setting, it is assumed that the consumer evaluates uncertain consumption streams in accordance with the von Neumann-Morgenstern axioms, so that the consumer maximizes expected utility. At time t, the consumer maximizes $E_t[\mathcal{U}]$. Because the preference ordering $\mathcal{U}(c_0, c_1, ..., c_T)$ is the same in each period, these preferences are time-consistent.

As a consequence of time-consistency, past choices will matter for decisions over current and future consumption, except for some special cases (such as additive separability). This feature is not unusual in the literature on consumption-based asset pricing: it is also present in models with habit or durability in consumption. A recent paper by Kihlstrom (2007) focuses on the alternative case that the consumer ignores past consumption in $\mathcal{U}$. With that approach, current choices are independent of past choices, but that also implies that the preferences are dynamically inconsistent.[3]

Time-separable preferences (and therefore temporal risk neutrality) can be obtained as a special case by setting $W = u$. Otherwise, assuming that W and u are both twice differentiable and checking the cross-partial derivative, it is easy to show that $\mathcal{U}$ is temporally risk averse (loving) if Wu^{-1} is concave (convex). Further algebra shows that this is in turn equivalent to W being more (less) risk averse than u in the sense of Arrow and Pratt. That is, the following statements are equivalent:

(i) $\mathcal{U}$ is temporally risk averse (loving);

(ii) Wu^{-1} is concave (convex); and

[3] Bommier and Rochet (2006) and Eden (2008) examine preferences similar to the ones used here. Bommier and Rochet's focus is on the effect of the planning horizon on risk aversion and portfolio choice that emerges without time-separability. Eden also examines risk aversion, as well as asset pricing implications in a two period setup.

(iii) $\frac{-W''(x)}{W'(x)} \geq (\leq) \frac{-u''(x)}{u'(x)}$, for $x = u^{-1}(\Sigma\beta^t u(c_t))$.

As the second statement suggests, this approach follows naturally from Kihlstrom and Mirman's (1974) analysis of risk aversion with multiple goods.

To obtain stationary asset pricing implications in the context of a growth economy, (1) is specialized to iso-elastic functional forms: $u(c) = (1-\gamma)^{-1}c^{1-\gamma}$ and $W(x) = (1-\alpha)^{-1}x^{1-\alpha}$. Then

$$\mathcal{U} = \frac{1}{1-\alpha}\left(\sum_{t=0}^{T}\beta^t c_t^{1-\gamma}\right)^{(1-\alpha)/(1-\gamma)} \tag{2}$$

In a deterministic setting, α is irrelevant for choices and the desire for consumption smoothing is fully determined by γ, with the intertemporal elasticity of substitution equal to $1/\gamma$. If $\alpha = \gamma$, then $\mathcal{U}$ specializes to additively separable utility with constant relative risk aversion $\gamma = \alpha$ (and temporal risk aversion equal to zero). If α exceeds γ, then the consumer is temporally risk averse. The latter is the case we will focus on.

To illustrate the role of the parameter α, suppose consumption is constant over time and equal to $\bar{c}$ (so that $\mathcal{U}$ = constant $\times\ \bar{c}^{1-\alpha}$), then α is the coefficient of relative risk aversion with respect to gambles over $\bar{c}$, i.e. lifetime consumption gambles. For this reason, I will refer to α as long-run (relative) risk aversion.

3.1 Long-run risk aversion

To further understand the role of long-run risk aversion α, it useful to define a lifetime consumption gamble as a lottery that changes consumption in each period by a common factor $1+\tilde{\varepsilon}$, where $\tilde{\varepsilon}$ is a random variable that has zero mean and is orthogonal to the initial consumption process. Formally, let $\{c_t\}_{t=0}^{T}$ be a given stochastic process for consumption without the gamble. With the gamble, consumption is

$$\tilde{c}_t \equiv c_t(1+\tilde{\varepsilon}) \text{ for all } t$$

where $\tilde{\varepsilon} \geq -1$ is independent of the stochastic process $\{c_t\}_{t=0}^{T}$. Without the gamble, (unconditional) expected utility is

$$E\mathcal{U} = E[\frac{1}{1-\alpha}(\Sigma_{t=0}^{T}\beta^t c_t^{1-\gamma})^{(1-\alpha)/(1-\gamma)}]$$

With the gamble, expected utility is

$$E\tilde{\mathcal{U}} \equiv E[\frac{1}{1-\alpha}(\Sigma_{t=0}^{T}\beta^t \tilde{c}_t^{1-\gamma})^{(1-\alpha)/(1-\gamma)}] = E[(1+\tilde{\varepsilon})^{1-\alpha}]E\mathcal{U}$$

(using independence). Hence, regardless of the properties of the initial consumption process, α is the coefficient of relative risk aversion for lifetime consumption gambles. Following the analysis in Pratt (1964), it is straightforward to show that the highest risk premium that the consumer is willing to pay, as a fraction of consumption in each period, to avoid the risk $\tilde{\varepsilon}$ is approximately $(\alpha/2)Var(\tilde{\varepsilon})$. As in Pratt, if higher moments are bounded, the error in the approximation is of smaller order than the variance of $\tilde{\varepsilon}$, so that the approximation is good when this variance is small.

3.2 Short-run risk aversion

It is more common to consider risk aversion with respect to one period consumption gambles. Although with time-separable, iso-elastic utility there is no difference with between this and the long-run concept, with the preferences postulated in (2) there generally is. For this reason, I will use the term short-run risk aversion to refer to risk aversion with respect to one period consumption gambles. To characterize short-run risk aversion, the marginal utility of consumption during period t is:

$$\frac{\partial \mathcal{U}}{\partial c_t} = (x_T)^{\delta} \beta^t c_t^{-\gamma} \tag{3}$$

with

$$x_T \equiv \sum_{t=0}^{T} \beta^t c_t^{1-\gamma}$$

and

$$\delta \equiv (\gamma - \alpha)/(1-\gamma)$$

Differentiating again with respect to c_t and using the result, we obtain for the coefficient of short-run relative risk aversion:[4]

$$\frac{-c_t \partial^2 E_t \mathcal{U}/(\partial c_t)^2}{\partial E_t \mathcal{U}/\partial c_t} = \gamma + (\alpha - \gamma)\frac{E_t[x_T^{\delta-1}]}{E_t[x_T^{\delta}]}\beta^t c_t^{1-\gamma} \tag{4}$$

Short-run risk aversion is the sum of two terms. If the consumer is temporally risk neutral (the time-separable case, with $\alpha = \gamma$), the second term is zero and short-run risk aversion is equal to γ, the standard coefficient of relative risk aversion, which is also the inverse of the intertemporal elasticity of substitution. If the consumer is temporally risk averse ($\alpha > \gamma$), then the second term increases short-run risk aversion beyond γ.

[4] The expectation operator is due to the fact that future consumption (dated $t+1$ onward) may be stochastic.

In that case, it is also clear from equation (4) that short-run relative risk aversion is generally not constant. To gain insight into this, let us first consider the case that future consumption is known at time t. Then

$$\frac{-c_t \partial^2 E_t \mathcal{U}/(\partial c_t)^2}{\partial E_t \mathcal{U}/\partial c_t} = \gamma + (\alpha - \gamma)\frac{\beta^t c_t^{1-\gamma}}{\Sigma_{s=0}^{T} \beta^s c_s^{1-\gamma}}$$

Thus, short-run risk aversion lies strictly between γ and α. If discounted felicity in the current period, $\beta^t c_t^{1-\gamma}$, is a small fraction of lifetime felicity, $\Sigma_{s=0}^{T} \beta^s c_s^{1-\gamma}$, then short-run relative risk aversion in the current period is quite close to γ. This will typically be the case when the horizon T is large. For example, if consumption is constant over time and $\beta = 1$, then short-run risk aversion is $\gamma + (\alpha - \gamma)/(T + 1)$.

The case of stochastic future consumption will be analyzed in more detail after the presentation the endowment economy. It will be shown for that economy that, under certain conditions, short-run risk aversion converges to γ as T goes to infinity.

3.3 Discussion

With the intertemporal elasticity of substitution (IES) equal to $1/\gamma$, and long-run risk aversion equal to α, these preferences separate, to some degree, consumption smoothing from risk aversion. This contrasts sharply with time-separable preferences for which, as mentioned, the IES is the inverse of the coefficient of relative risk aversion. Epstein-Zin-Weil preferences also provide a separation between the two concepts (see Epstein and Zin (1989) and Weil (1989)). The key difference with the approach taken here is that Epstein-Zin-Weil preferences are non-expected utility preferences. They do not satisfy the axioms of von Neumann and Morgenstern for consumption gambles extending over multiple periods. As discussed by Epstein and Zin, this is manifested by an associated preference for early or late resolution of uncertainty. An open question is to what extent the accomplishments of Epstein-Zin-Weil preferences in the asset pricing literature are due to the fact that they lie outside the traditional expected utility framework.[5] For example, is the preference for early or late resolution of uncertainty crucial? Or is it 'just' the separation of risk aversion from consumption smoothing that matters? As illustrated by the preferences used here, by introducing temporal risk

[5] Some examples of work in asset pricing which fruitfully employs Epstein-Zin-Weil preferences include Bansal and Yaron (2004), Gomes and Michaelides (2008), Hansen, Heaton and Li (2008), Kandel and Stambaugh (1991), Piazzesi and Schneider (2006), Routledge and Zin (2003) and Tallarini (2000).

aversion, it is possible to have that separation without leaving the expected utility framework and, as a consequence, without relying on any preference for early or late resolution of uncertainty. In this sense, the approach taken here is also a smaller departure from the standard time-separable preferences.

4 The Pricing Kernel

Suppose that the consumer reduces consumption in period t by 1 unit, purchases an asset with gross return R_{t+1} and then uses the proceeds to increase consumption in period $t+1$. Equilibrium asset prices have to be such that for the consumer the net marginal effect of this action on expected utility is zero. This perturbation argument implies the following Euler equation:

$$E_t[\partial \mathcal{U}/\partial c_t] = E_t[(\partial \mathcal{U}/\partial c_{t+1})R_{t+1}] \tag{5}$$

Using equation (3) and the law of iterated expectations, the following version of the intertemporal marginal rate of substitution is a valid pricing kernel:[6]

$$M_{t+1}^T = \frac{E_{t+1}[\partial \mathcal{U}/\partial c_{t+1}]}{E_t[\partial \mathcal{U}/\partial c_t]} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma} \frac{E_{t+1}\left[x_T^\delta\right]}{E_t\left[x_T^\delta\right]} \tag{6}$$

The first two factors equal the pricing kernel for the time-separable case ($\alpha = \gamma$); the third factor differs from one only if $\alpha \neq \gamma$ and news about future consumption (dated $t+1$ and onward) is revealed between t and $t+1$. (Recall that $\delta = (\gamma - \alpha)/(1-\gamma)$ and $x_T = \sum_{t=0}^T \beta^t c_t^{1-\gamma}$.) Thus, temporal risk aversion introduces a new, forward-looking factor to the pricing kernel.

The goal is now to obtain a convenient expression for this new factor. It will be possible to do this for an economy with a long horizon, T. Formally, I will consider a sequence of economies indexed by T and then derive a result characterizing the limit of the pricing kernel as T approaches infinity.[7] Focusing on the infinite horizon case in this way will also have the advantage that the pricing kernel will not depend on on the 'time remaining', $T-t$, as a state variable, which would have the undesirable consequence of generating nonstationary asset returns.

[6] By the law of iterated expectations, (5) is equivelent to $E_t[\partial \mathcal{U}/\partial c_t] = E_t[E_{t+1}[(\partial \mathcal{U}/\partial c_{t+1})R_{t+1}]]$, so that $E_t[(E_{t+1}[\partial \mathcal{U}/\partial c_{t+1}]/E_t[\partial \mathcal{U}/\partial c_t])R_{t+1}] = 1$, or $E_t[M_{t+1}^T R_{t+1}] = 1$. The superscript T is included as the horizon T will be varied below.

[7] That is, formally, we will examine $\lim_{T\to\infty}(\arg\max E[\mathcal{U}(c_0, ..., c_T)])$ rather than $\arg\max(E[\lim_{T\to\infty} \mathcal{U}(c_0, ..., c_T)])$.

It is convenient to split up x_T into realized and future terms:

$$x_T = \sum_{s=0}^{t} \beta^s c_s^{1-\gamma} + \sum_{s=t+1}^{T} \beta^s c_s^{1-\gamma} = \beta^t c_t^{1-\gamma}(z_t + {f_T}^{t+1}) \tag{7}$$

where the second step introduces a convenient normalization, with

$$z_t \equiv \frac{\Sigma_{s=0}^{t} \beta^s c_s^{1-\gamma}}{\beta^t c_t^{1-\gamma}} \geq 1 \text{ and } {f_T}^{t+1} \equiv \frac{\Sigma_{s=t+1}^{T} \beta^s c_s^{1-\gamma}}{\beta^t c_t^{1-\gamma}} \tag{8}$$

Denoting the growth rate of consumption by $g_t \equiv c_t/c_{t-1}$, z_t evolves according to

$$z_{t+1} = z_t/(\beta g_{t+1}^{1-\gamma}) + 1$$

or, with $\phi \equiv \beta^{-1} E[g_t^{\gamma-1}]$ and $\varepsilon_{t+1} \equiv z_t(\beta^{-1} g_{t+1}^{\gamma-1} - \phi)$, we have

$$z_{t+1} = \phi z_t + 1 + \varepsilon_{t+1}$$

Now suppose that

$$g_t \equiv c_t/c_{t-1} \text{ is } i.i.d.$$

With that assumption, $E_t[\varepsilon_{t+1}] = 0$. z_t is then a stationary stochastic process (more precisely, non-explosive, due to the initial condition $z_0 = 1$) if and only if

$$\phi \equiv \beta^{-1} E[g_t^{\gamma-1}] < 1 \tag{9}$$

At the same time

$$f_T^{t+1} = \textstyle\sum_{s=0}^{T-(t+1)} \prod_{v=0}^{s} \beta g_{t+1+v}^{1-\gamma} \tag{10}$$

so, exploiting the $i.i.d.$ assumption,

$$E_t[f_T^{t+1}] = \textstyle\sum_{s=0}^{T-(t+1)} \left(E[\beta g_t^{1-\gamma}]\right)^{s+1} = \dfrac{\beta E[g_t^{1-\gamma}]\{1 - (\beta E[g_t^{1-\gamma}])^{T-t}\}}{1 - \beta E[g_t^{1-\gamma}]}$$

To obtain stationary asset return implications, we will be interested in the case where T is large. Since $\beta E[g_t^{1-\gamma}] > \beta(1/E[g_t^{\gamma-1}]) = 1/\phi$, if $\phi < 1$, then $E_t[f_T^{t+1}]$ diverges as T grows. Thus, loosely speaking, if $\phi < 1$ and T is large, then, z_t is small relative to $E_t[f_T^{t+1}]$ for all t, except when t is very close to T.

In contrast, if $\phi > 1$ (more precisely, if $\beta E[g_t^{1-\gamma}] < 1$), then $E_t[f_T^{t+1}]$ remains bounded for arbitrarily large T. At the same time, z_t grows exponentially as time passes. Thus, even if T is

very large, after a few periods, z_t is likely to be large relative to $E_t[f_T^{t+1}]$, as the former diverges in expectation and the latter does not, if $\phi > 1$. This suggests that, at that point in time,

$$\frac{E_{t+1}\left[x_T^\delta\right]}{E_t\left[x_T^\delta\right]} = \frac{E_{t+1}\left[(1+f_T^{t+1}/z_t)^\delta\right]}{E_t\left[(1+f_T^{t+1}/z_t)^\delta\right]} \approx \frac{1}{1} = 1$$

It is apparent from this and expression (6) that the pricing kernel will approach the standard CRRA time-separable one as time passes, if T is large and $\phi > 1$. Intuitively, temporal risk aversion gradually becomes irrelevant, as it is the early periods of consumption that matter the most for lifetime utility in this case.

Since I am interested in temporal risk aversion, and since in any case the asset pricing implications of the standard preferences are well understood, I will focus on the alternative case that $\phi < 1$. Note that if $\beta \leq 1$ this is realistic only if $\gamma < 1$, i.e. if the intertemporal elasticity of substitution exceeds unity. Intuitively, $\phi < 1$ means that, unless t is close to T, the future is more important than the past for current decisions.

Epstein and Zin (1989) have critiqued the kind of preferences postulated in (2) by pointing out that, if $\beta < 1$ and without growth, the dependence of the marginal utility of current consumption on past consumption is greater as the past becomes more distant. However, this is not true under condition (9) ($\phi < 1$). This condition can be satisfied even with a constant consumption profile, provided $\beta \geq 1$; with growth β may be less than 1 provided $\gamma < 1$. Interestingly, the parameter restriction under which temporal risk aversion turns out to have novel asset pricing implications also addresses the critique of Epstein and Zin.

Using (7), the new factor in the pricing kernel can be written as

$$\frac{E_{t+1}\left[x_T^\delta\right]}{E_t\left[x_T^\delta\right]} = (\beta g_{t+1}^{1-\gamma})^\delta \frac{E_{t+1}\left[(z_{t+1}+f_T^{t+2})^\delta\right]}{E_t\left[(z_t+f_T^{t+1})^\delta\right]} \tag{11}$$

Intuitively, and informally, for large T, and with $\phi < 1$, we expect $E_t\left[(z_t+f_T^{t+1})^\delta\right]$ and $E_{t+1}\left[(z_{t+1}+f_T^{t+2})^\delta\right]$ to depend very little on z_t and z_{t+1}, respectively, since in this case f diverges in expectation, while z does not. (The *i.i.d.* assumption implies that f has no predictability.) Thus, it is natural to expect that as T goes to infinity, $E_{t+1}\left[(z_{t+1}+f_T^{t+2})^\delta\right]/E_t\left[(z_t+f_T^{t+1})^\delta\right]$ approaches a constant, independent of z_t and z_{t+1}. Assume for a moment that this is true and call the limit Λ. Then

$$\lim_{T\to\infty} \frac{E_{t+1}\left[x_T^\delta\right]}{E_t\left[x_T^\delta\right]} = (\beta g_{t+1}^{1-\gamma})^\delta \Lambda = \beta^\delta g_{t+1}^{\gamma-\alpha} \Lambda$$

Assume further for a moment that there are no issues with interchanging limit and expectation operators. Then taking E_t of both sides yields a simple expression for Λ:

$$\lim_{T\to\infty} E_t\left[\frac{E_{t+1}\left[x_T^\delta\right]}{E_t\left[x_T^\delta\right]}\right] = 1 = E_t\left[\beta^\delta g_{t+1}^{\gamma-\alpha}\right]\Lambda \implies \Lambda = \frac{1}{\beta^\delta E_t[g_{t+1}^{\gamma-\alpha}]}$$

Of course, here $E_t[g_{t+1}^{\gamma-\alpha}]$ may be replaced by $E[g_{t+1}^{\gamma-\alpha}]$ since g is *i.i.d.*

If this informal line of reasoning is correct, then substituting this limiting value into (6) yields a very simple expression for the limiting pricing kernel - see equation (13) below. The following proposition, the main result of the paper, shows that under certain conditions this argument can indeed be formalized.

Theorem 1 *If* $\delta = \frac{\gamma-\alpha}{1-\gamma} < 0$, $\ln g_t \sim i.i.d.\ N(\mu, \sigma^2)$ *for* $g_t \equiv c_t/c_{t-1}$ *and*

$$\ln\beta + (1-\gamma)\mu > (0.5-\delta)(1-\gamma)^2\sigma^2 \tag{12}$$

then the limiting pricing kernel is

$$M_{t+1}^* \equiv \lim_{T\to\infty} M_{t+1}^T = \frac{\beta g_{t+1}^{-\alpha}}{E_t[g_{t+1}^{\gamma-\alpha}]} \tag{13}$$

Proof. See Appendix 1. ■

Remark 1: $\delta < 0$ requires that either $\gamma < 1$ and $\gamma < \alpha$, or $\gamma > 1$ and $\alpha < \gamma$. If (realistically) $\mu > 0$, then, given assumption (12), only the first possibility is consistent with $\beta \leq 1$. As mentioned, $\gamma < \alpha$ implies positive temporal risk aversion.

Remark 2: condition (12) implies $\phi < 1$ (since $\delta < 0$ and $\ln\phi = \ln(\beta^{-1}E[g^{\gamma-1}]) = -(\ln\beta + (1-\gamma)\mu) + 0.5(1-\gamma)^2\sigma^2 < 0$ by (12)).

The proof employs the mean value theorem and the reflection principle for Brownian motion to show that the history of past consumption, summarized in z_t, is asymptotically unimportant in $E_t[x_T^\delta]/(\beta^t c_t^{1-\gamma})^\delta = E_t\left[(z_t + f_T^{t+1})^\delta\right]$. Then, the dominated convergence theorem is used to to interchange the limit and expectations operators. The conditions stated in the theorem are sufficient. I suspect that they are not all necessary, but I have not shown this (except for the conditions outlined in footnote 13 and the following trivial case).

It is worth pointing out that if $\alpha = \gamma$, so that the utility function is time-separable, then equation (13) specializes to the standard pricing kernel for time-separable utility with constant relative risk aversion: $M^*_{t+1} = \beta g^{-\gamma}_{t+1}$.

An increase in long-run risk aversion α makes marginal utility, and therefore the pricing kernel, more responsive to realized growth rates, but without having a big impact on its expected value due to the correction $1/E[g^{\gamma-\alpha}_{t+1}]$. (In the special case $\gamma = 0$, the impact is exactly zero; for positive but small γ the impact is small, as will be shown more explicitly below.) This is important for the asset pricing implications, since, loosely speaking, the equity premium depends on the volatility of the pricing kernel, while the risk free rate depends on its conditional mean.

The following proposition characterizes short-run risk aversion in this economy:

Theorem 2 *Under the conditions stated in theorem 1, short-run risk aversion converges to γ:*

$$\lim_{T\to\infty} \frac{-c_t \partial^2 E_t \mathcal{U}/(\partial c_t)^2}{\partial E_t \mathcal{U}/\partial c_t} = \gamma$$

Proof. *See Appendix 2.* ■

Thus, for this stochastic economy, a similar result applies as for the deterministic case: when the horizon T is large, short-run risk aversion is close to γ. In this sense, the separation between the intertemporal elasticity of substitution and short-run risk aversion vanishes in the limit. This separation remains, however, for long-run risk aversion, which is always equal to α. The next section shows how this affects asset prices.

5 Asset Prices

Using the expression for the limiting pricing kernel in (13), pricing assets is straightforward, using the optimality condition $E_t[M^*_{t+1} R^i_{t+1}] = 1$, where R^i_{t+1} is the gross realized return to any tradeable asset, between period t and $t+1$. In what follows it is assumed that the conditions to theorem 1 are satisfied. This is consistent with temporal risk aversion ($\alpha > \gamma$) only if γ is less than 1, so that the intertemporal elasticity of substitution exceeds unity (and short-run risk aversion is less than 1 in the limit).

It should be stated at the outset that, with *i.i.d.* consumption growth and the asymptotic irrelevance of past consumption to the pricing kernel (see (13)), the model will imply constant

values for the risk-free rate and the equity premium. This is also true for for standard preferences combined with *i.i.d.* consumption growth.

5.1 Risk-free rate

Denote the risk-free one period real interest rate between period t and $t+1$ by R_t^F. In the limiting economy,

$$R_t^F = 1/E_t[M_{t+1}^*] = \beta^{-1} E_t[g_{t+1}^{\gamma-\alpha}]/E_t[g_{t+1}^{-\alpha}]$$

Exploiting lognormality, this yields

$$R_t^F = \beta^{-1} \exp(\gamma\mu - \gamma(\alpha - 0.5\gamma)\sigma^2)$$

Note that for the time-separable case ($\alpha = \gamma$) this simplifies to the standard result: $R_t^{F,ts} = \beta^{-1} \exp(\gamma\mu - 0.5\gamma^2\sigma^2)$ (ts is used to indicate the time-separable case). The general result can also be written as

$$R_t^F = R_t^{F,ts} \exp(-(\alpha - \gamma)\gamma\sigma^2)$$

Recall that temporal risk aversion obtains when $\alpha > \gamma$. Thus, for a given value of $\gamma < 1$, introducing more temporal risk aversion *lowers* the risk free rate. It is tempting to link this with the precautionary savings motive, but more careful analysis is needed to make a precise claim.

The constant short-rate implies a flat real term structure. That is, real yields on all long bonds are constant and equal to the short rate. Again, this is also true for standard preferences with *i.i.d.* consumption growth.

5.2 Consumption claim

Lucas (1978) and Mehra and Prescott (1985) define equity as a claim to aggregate consumption. Deriving the expected return to such a consumption claim using M_{t+1}^* is standard. The resulting 'consumption equity premium' is:

$$E[R_{t+1}^C/R_t^F] - 1 = \exp[\alpha\sigma^2] - 1 \approx \alpha\sigma^2 \tag{14}$$

As can be seen, the risk premium depends only long-run risk aversion. For the preferences used here, temporal risk aversion implies that long-run risk aversion exceeds short-run risk aversion. In this sense, therefore, temporal risk aversion increases the risk premium on a consumption claim.

5.3 Equity

Extending the formulation by Abel (1999), equity is modelled as a claim to dividends equal to $d_t = n^t c_t^\lambda \varepsilon_t$ in period t. The parameter λ is a modeling device that closely approximates the effect of leverage on returns (see Abel (1999)), with positive leverage corresponding to $\lambda > 1$. ε_t is a shock which assumed to be uncorrelated with consumption and is distributed $i.i.d.(1, \sigma_\varepsilon^2)$. It is included because in the data dividend volatility exceeds consumption volatility and because the dividend and consumption growth rates are imperfectly correlated. Finally, $n > 0$ is a convenient way of considering the effect of the duration of the equity claim. $n = 1$ or $n = \exp(-(\lambda-1)\mu - 0.5(\lambda-1)^2\sigma^2)$ are standard choices. The resulting equity premium is:

$$E[R_{t+1}^S / R_t^F] - 1 = \exp[\lambda\alpha\sigma^2] - 1 \approx \lambda\alpha\sigma^2 \tag{15}$$

As finance theory predicts, nonsystematic risk (ε) is not priced. Interpreting $\lambda - 1$ as leverage yields the same result as a straightforward application of Modigliani and Miller's (1958) Proposition II. Because the model has a no term premium (due to $i.i.d.$ consumption growth), the risk premium is independent of the duration parameter n. As for the consumption claim, long-run risk aversion is the only preference parameter that matters for the equity premium. Since the equity premium is increasing in long-run risk aversion, temporal risk aversion increases the equity premium.

Why does the equity premium depend on long-run risk aversion, as opposed the short-run concept? The intuition for this result is that the risk in this economy stems from the $i.i.d.$ shocks to the growth rate of consumption (and therefore dividends). These innovations act as *permanent* shocks to the level of consumption. A positive innovation to the growth rate raises consumption in *all* remaining periods by the same ratio. It is therefore similar to the lifetime consumption gamble discussed in section 3.1. Loosely speaking, it is also more similar the lottery that the consumer dislikes in the definition of temporal risk aversion than the alternative (see section 2).

This result is not driven by the fact that equity is a long-lived claim, as it holds even for a very short-duration equity claim (small n) and even, it can be shown, for a one period equity strip. Rather, it is due to the fact that the economic risk inherent in equity is 'long-run risk' to the level of consumption.

5.4 Numerical examples

As Mehra and Prescott (1985) and others have shown, it is difficult for standard models with time-separable preferences to account for the 6% equity premium and the low risk-free rate with conventional levels for risk aversion.[8] In this subsection, I examine whether it is easier to do so with temporal risk aversion. I use the values calculated by Mehra and Prescott for the mean and standard deviation of the growth rate of real per capita consumption of nondurables and services: $\mu = 0.018$ and $\sigma = 0.036$. Following Abel (1999) and Bansal and Yaron (2004), I set the leverage parameter at $\lambda = 3$. The average U.S. equity premium reported by Mehra and Prescott is 6.2% per annum, and the average real risk-free rate is 0.80%.

First, I ask what equity premium the model can generate subject to matching the risk-free rate exactly and subject to the parameter restrictions needed for theorem 1 to hold (most importantly, condition (12)). Under those restrictions, the model can account for about half the equity premium (3.2%). Parameter values that accomplish this are, for example, $\alpha = 8$, $\gamma = 0.75$ and $\beta = 0.998$. That is, with risk aversion ranging from 0.75 for the short run to 8 for the long run, the model can match the risk-free rate and about half the equity premium. Note that with $\gamma = 0.75$ the intertemporal elasticity of substitution is $1/0.75 \approx 1.3$. Incidentally, the predicted equity premium is well within two standard deviations of the historical average. For comparison, with time-separable preferences the result is an equity premium of only 0.2%.[9] This is the well-know equity premium puzzle.

Alternatively, one can ask how close the model can get to matching the risk-free rate while replicating the point estimate of the equity premium. Using equation (15), to match the equity premium exactly, long-run risk aversion must equal $\alpha = 15.5$. However, that value results in a risk-free rate that is too low, because, as mentioned, temporal risk aversion lowers the risk-free rate. Assuming a high rate of time preference (a low β) would help, but this is ruled out by the parameter restriction to the theorem (12). Under that restriction and with $\alpha = 15.5$, the model can generate a risk-free rate that is about 1 percentage point below the historical average. For example, with permissible parameter values $\alpha = 15.5$, $\gamma = 0.5$ and $\beta = 1.001$, the risk-free rate is $R_t^F = -0.19\%$.[10]

[8] Kandel and Stambaugh (1991) provide a challange to the view that high risk aversion is unreasonable.

[9] Here I follow Mehra and Prescott in requiring $\gamma \leq 10$ and $\beta \leq 1$. Leverage is maintained at $\lambda = 3$.

[10] Interestingly, if one uses aggregate real consumption growth, rather than per capita, to calibrate μ and σ, then

The low risk-free rate contrasts sharply with the case of time-separable preferences, for which high levels of risk aversion imply counterfactually high values for the risk-free rate.[11] For example, setting $\gamma = 15.5$ to match the equity premium results in a risk-free rate near 13% with standard preferences, if $\beta \approx 1$. The reason is that with time separability high risk aversion implies a strong desire for consumption smoothing, which in the presence of growth results in a high interest rate in equilibrium.

More important is that the preferences with temporal risk aversion can generate sizable risk premia without suffering from what can be called the 'Lucas-Murphy critique'. As Robert Lucas (1990) has observed,[12] two countries that differ in their growth rate (μ) by 1 percent, differ in their interest rate by γ percent, where γ is the reciprocal of the intertemporal elasticity of substitution (assuming the same preferences and variance of consumption growth). With time separable preferences γ is also risk aversion, so then levels of risk aversion in excess of 4, in Lucas' estimate, would then imply counterfactually large differences in real interest rates across countries (think of South Korea and the U.S.). In his 2003 presidential address, Lucas revisits this argument and concludes that the IES should in fact be close to one. The preferences used in this paper can combine higher long-run risk aversion with a relatively high intertemporal elasticity of substitution (i.e. a low γ), thus avoiding the prediction of enormous cross-country real interest rate differentials.

6 Conclusion

The starting point of this paper has been the idea that consumers may care about the temporal distribution of risk; in particular, they may find persistent shocks to consumption less desirable than uncorrelated fluctuations. I have formulated expected utility preferences that exhibit such temporal risk aversion and studied their asset pricing implications. I found that temporal risk aversion leads naturally to a separation of risk aversion from the intertemporal elasticity of substitution, as well as a distinction between short-run and long-run risk aversion. For an endowment economy with *i.i.d.*

the model can simultaneously match the equity premium and the risk-free rate. The model abstracts from population growth.

[11] Except for very high values of relative risk aversion, when the precautionary effect dominates. Recall that $R_t^F = \beta^{-1} \exp(\mu\gamma - 0.5\gamma^2\sigma^2)$ in the time-separable case. At those very high levels of risk aversion, the quadratic term can approximately cancel with the linear term. However, the 'Lucas-Murphy critique' (explained below) still applies.

[12] Lucas credits Kevin Murphy for making this observation.

consumption growth, I derived a simple expression for the limiting pricing kernel, which yields stationary implications for asset returns. In that economy, closed form solutions show that the equity premium depends only on a parameter indexing long-run risk aversion, while the risk-free rate instead depends primarily on a separate parameter indexing the desire to smooth consumption over time and the rate of time preference. Quantitatively, the model improves upon the ability of standard preferences to simultaneously account for the historical averages of the equity premium and the risk-free rate.

At least two open questions remain. First, is the pricing kernel valid under a wider set of conditions than for which it has been derived here? And, second, what additional asset pricing implications of temporal risk aversion are there for the case of non-*i.i.d.* consumption growth? For example, it seems interesting to investigate the effect of long-run risk (Bansal and Yaron (2004)) in the presence of temporal risk aversion. I leave these questions for future research.

Appendix 1. Proof of Theorem 1

Define

$$\xi^{\delta}(z,\tau) \equiv E[(z+f_{\tau}^{1})^{\delta}|z] \tag{16}$$

Using (7), (10) and the assumption that g_t is *i.i.d.*,

$$E_{t+1}[x_T^{\delta}] = (\beta^{t+1}c_{t+1}^{1-\gamma})^{\delta}\xi^{\delta}(z_{t+1}, T-t-1) \tag{17}$$

so that, introducing a convenient normalization,

$$\frac{E_{t+1}[x_T^{\delta}]}{(\beta^{t+1}c_{t+1}^{1-\gamma})^{\delta}\xi^{\delta}(1,T-t-1)} = \frac{\xi^{\delta}(z_{t+1},T-t-1)}{\xi^{\delta}(1,T-t-1)}$$

Lemma 1, shown and demonstrated below, shows that z_{t+1} is asymptotically irrelevant to this expression. Specifically, equation (22) of the lemma implies that $\lim_{\tau\to\infty}\frac{\xi^{\delta}(z_{t+1},\tau)}{\xi^{\delta}(1,\tau)} = 1$. Hence,

$$\lim_{T\to\infty}\frac{E_{t+1}[x_T^{\delta}]}{(\beta^{t+1}c_{t+1}^{1-\gamma})^{\delta}\xi^{\delta}(1,T-t-1)} = 1 \tag{18}$$

Next, since $E_t[x_T^{\delta}] = E_t[E_{t+1}[x_T^{\delta}]]$, we have, using (17) and a slightly different normalization,

$$\frac{E_t[x_T^{\delta}]}{(\beta^{t}c_{t}^{1-\gamma})^{\delta}\xi^{\delta}(1,T-t-1)} = E_t\left[\tilde{g}_{t+1}^{\delta}\frac{\xi^{\delta}(z_{t+1},T-t-1)}{\xi^{\delta}(1,T-t-1)}\right] \tag{19}$$

where I have introduced the following notation:

$$\tilde{g}_t \equiv \beta g_t^{1-\gamma}$$

Note that $\tilde{g}_{t+1}^{\delta}$ is an integrable random variable (specifically, exploiting the lognormal distribution $E_t[|\tilde{g}_{t+1}^{\delta}|] = E_t[\tilde{g}_{t+1}^{\delta}] = \beta^{\delta}\exp((\gamma-\alpha)\mu + 0.5(\gamma-\alpha)^2\sigma^2) < \infty$). Moreover, since $z_{t+1} \geq 1$ by definition (see (8)) and since $\delta < 0$,

$$0 < \frac{\xi^{\delta}(z_{t+1}, T-t-1)}{\xi^{\delta}(1, T-t-1)} \leq 1$$

Thus, $\tilde{g}_{t+1}^{\delta}\frac{\xi^{\delta}(z_{t+1},T-t-1)}{\xi^{\delta}(1,T-t-1)}$ is a sequence (in T) of integrable random variables with a well-defined limit. The limit follows again from lemma 1, which implies

$$\lim_{T\to\infty}\left(\tilde{g}_{t+1}^{\delta}\frac{\xi^{\delta}(z_{t+1}, T-t-1)}{\xi^{\delta}(1, T-t-1)}\right) = \tilde{g}_{t+1}^{\delta}$$

Hence, we can apply the dominated convergence theorem to this sequence:

$$\lim_{T\to\infty} E_t\left[\tilde{g}_{t+1}^{\delta}\frac{\xi^{\delta}(z_{t+1}, T-t-1)}{\xi^{\delta}(1, T-t-1)}\right] = E_t\left[\lim_{T\to\infty}\left(\tilde{g}_{t+1}^{\delta}\frac{\xi^{\delta}(z_{t+1}, T-t-1)}{\xi^{\delta}(1, T-t-1)}\right)\right] = E_t\left[\tilde{g}_{t+1}^{\delta}\right] \tag{20}$$

Combining equations (18), (19) and (20) yields

$$\lim_{T\to\infty}\frac{E_{t+1}[x_T^{\delta}]}{E_t[x_T^{\delta}]} = \lim_{T\to\infty}\left(\frac{\frac{E_{t+1}[x_T^{\delta}]}{(\beta^{t+1}c_{t+1}^{1-\gamma})^{\delta}\xi^{\delta}(1,T-t-1)}}{\frac{E_t[x_T^{\delta}]}{(\beta^t c_t^{1-\gamma})^{\delta}\xi^{\delta}(1,T-t-1)}}\frac{(\beta^{t+1}c_{t+1}^{1-\gamma})^{\delta}}{(\beta^t c_t^{1-\gamma})^{\delta}}\right) = \frac{\tilde{g}_{t+1}^{\delta}}{E_t\left[\tilde{g}_{t+1}^{\delta}\right]} = \frac{g_{t+1}^{\gamma-\alpha}}{E_t[g_{t+1}^{\gamma-\alpha}]} \tag{21}$$

Finally, combining this with equation (6),

$$\lim_{T\to\infty} M_{t+1}^{T} = \frac{\beta g_{t+1}^{-\alpha}}{E_t[g_{t+1}^{\gamma-\alpha}]}$$

which is the result stated in the theorem, equation (13). As mentioned, here $E_t[g_{t+1}^{\gamma-\alpha}]$ may be replaced by $E[g_{t+1}^{\gamma-\alpha}]$ since g is *i.i.d.* Thus, it only remains to be shown that the following lemma is in fact true:

Lemma 1 *Under the conditions of theorem 1, for all z and $z' \geq 1$,*

$$\lim_{\tau\to\infty}\frac{\xi^{\delta}(z', \tau)}{\xi^{\delta}(z, \tau)} = 1 \tag{22}$$

and

$$\lim_{\tau\to\infty}\frac{\xi^{\delta-1}(z, \tau)}{\xi^{\delta}(z, \tau)} = 0 \tag{23}$$

Proof of lemma 1. The proof proceeds in several steps.

1. [Upper bound on ξ] First, recall the definition

$$\tilde{g}_t \equiv \beta g_t^{1-\gamma}$$

Note that

$$\ln \tilde{g}_t \sim i.i.d.N(\tilde{\mu}, \tilde{\sigma}^2) \quad \text{with} \quad \tilde{\mu} \equiv \ln \beta + (1-\gamma)\mu \ \text{ and } \tilde{\sigma} \equiv |1-\gamma|\sigma$$

We can write, in this notation,

$$\xi^\delta(z,\tau) = E[\{z + \Sigma_{t=1}^{\tau} \Pi_{v=1}^{t} \tilde{g}_v)\}^\delta]$$

Since $\delta < 0$, and exploiting the $i.i.d.$ assumption,

$$\xi^\delta(z,\tau) \leq E[\{\Pi_{v=1}^{\tau} \tilde{g}_v\}^\delta] = (E[\tilde{g}^\delta])^\tau = \exp[(\tilde{\mu}\delta + 0.5\delta^2\tilde{\sigma}^2)\tau] \tag{24}$$

Because assumption (12) implies that $\tilde{\mu}+0.5\delta\tilde{\sigma}^2 > 0$ and since $\xi^\delta(z,\tau) > 0$ and $\delta < 0$, $\lim_{\tau\to\infty} \xi^\delta(z,\tau) = 0$. Thus, both the numerator and the denominator of $\frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)}$ approach zero as $\tau \to \infty$.

2. [Applying the mean value theorem] To proceed, differentiate ξ with respect to z:

$$\partial\xi^\delta(z,\tau)/\partial z = \delta E[(z + f_\tau^1)^{\delta-1}|z] = \delta\xi^{\delta-1}(z,\tau)$$

z and z' are arbitrary numbers weakly larger than 1, but without loss of generality, we can let $z' \geq z$ (we can always relabel them since $\lim_{\tau\to\infty} \frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)} = 1$ is equivalent to $\lim_{\tau\to\infty} \frac{\xi^\delta(z,\tau)}{\xi^\delta(z',\tau)} = 1$.) By the mean value theorem, for any τ, there exists a $\theta_\tau \in [z, z']$, such that

$$\xi^\delta(z',\tau) = \xi^\delta(z,\tau) + (\partial\xi^\delta(\theta_\tau,\tau)/\partial z)(z'-z)$$

Combining these two equations,

$$\frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)} = 1 + \frac{\xi^{\delta-1}(\theta_\tau,\tau)}{\xi^\delta(z,\tau)}\delta(z'-z) \tag{25}$$

Now, since $z \leq \theta_\tau$ and as $\xi^{\delta-1}(z,\tau)$ is decreasing in z,

$$\frac{\xi^{\delta-1}(\theta_\tau,\tau)}{\xi^\delta(z,\tau)} \leq \frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)}$$

The goal now is to show that $\lim_{\tau\to\infty} \frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)} = 0$. Since $\frac{\xi^{\delta-1}(\theta_\tau,\tau)}{\xi^\delta(z,\tau)} > 0$, this would imply that $\lim_{\tau\to\infty} \frac{\xi^{\delta-1}(\theta_\tau,\tau)}{\xi^\delta(z,\tau)} = 0$ and, therefore, by (25), $\lim_{\tau\to\infty} \frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)} = 1$.

3. [Lower bound on ξ][13] To bound the ratio $\xi^{\delta-1}(z,\tau)/\xi^{\delta}(z,\tau)$ from above, a lower bound for $\xi^{\delta}(z,\tau)$ is needed in addition to the upper bound. Using the notation from step 1, we can write $\xi^{\delta}(z,\tau) = E[\{z+\Sigma_{t=1}^{\tau}\exp(\Sigma_{v=1}^{t}\ln\tilde{g}_v)\}^{\delta}]$. Since $\ln\tilde{g}_t \sim i.i.d.N(\tilde{\mu},\tilde{\sigma}^2)$, $\Sigma_{v=1}^{t}\ln\tilde{g}_v \stackrel{d}{=} \tilde{\mu}t+\tilde{\sigma}S_t$, where S_t is the running sum of independent standard normals: $S_t \equiv \Sigma_{v=1}^{t}X_v$ where X_v are $i.i.d.$ $N(0,1)$ random variables. (The notation $\stackrel{d}{=}$ stands for 'is equal in distribution to'.) Thus,

$$\xi^{\delta}(z,\tau) = E[\{z+\Sigma_{t=1}^{\tau}\exp(\tilde{\mu}t+\tilde{\sigma}S_t)\}^{\delta}]$$

Now,

$$\begin{aligned}\Sigma_{t=1}^{\tau}\exp(\tilde{\mu}t+\tilde{\sigma}S_t) &\leq \tau\exp[\max_{1\leq t\leq\tau,t\in\mathbb{N}}(\tilde{\mu}t+\tilde{\sigma}S_t)] \stackrel{d}{=} \tau\exp[\max_{1\leq t\leq\tau,t\in\mathbb{N}}(\tilde{\mu}t+\tilde{\sigma}W_t)]\\ &\leq \tau\exp[\max_{0\leq t\leq\tau,t\in\mathbb{R}}(\tilde{\mu}t+\tilde{\sigma}W_t)]\end{aligned}$$

where W_t is a standard Brownian motion. The second step exploits the equality in distribution of S_t and W_t sampled at integer times, which follows from the properties of standard Brownian motion. The last step follows from the fact that the max is taken over a larger set. Hence, since $\delta<0$,

$$\xi^{\delta}(z,\tau) \geq E[\{z+\tau\exp[\max_{0\leq t\leq\tau,t\in\mathbb{R}}(\tilde{\mu}t+\tilde{\sigma}W_t)]\}^{\delta}]$$

Since $\max_{0\leq t\leq\tau,t\in\mathbb{R}}(\tilde{\mu}t+\tilde{\sigma}W_t) \geq \tilde{\mu}0+\tilde{\sigma}W_0 = 0$, it follows that, for $\tau\geq z$, $z\leq\tau\leq\tau\exp[\max_{0\leq t\leq\tau,t\in\mathbb{R}}(\tilde{\mu}t+$

[13] A straightforward way to derive a lower bound is to apply Jensen's inequality to ξ. Unfortunately, while simpler, this leads to a weaker lower bound than the one derived in the proof, as will be shown below in this footnote. This weaker lower bound does have the virtue of not relying on lognormality, so it is possible to prove the main result without relying on lognormality, albeit under stronger sufficient conditions:

Since $x\to x^{\delta}$ is a convex mapping ($\delta<0$), Jensen's inequality implies that

$$\xi^{\delta}(z,\tau) \geq \left(E[z+f_{\tau}^{1}|z]\right)^{\delta} = \left(z+\textstyle\sum_{s=1}^{\tau}(E[\tilde{g}])^{s}\right)^{\delta} = \left(z+\frac{E[\tilde{g}]}{E[\tilde{g}]-1}((E[\tilde{g}])^{\tau}-1)\right)^{\delta}$$

Thus, combining this with the upper bound,

$$\frac{\xi^{\delta-1}(z,\tau)}{\xi^{\delta}(z,\tau)} \leq \frac{(E[\tilde{g}^{\delta-1}])^{\tau}}{\left(z+\frac{E[\tilde{g}]}{E[\tilde{g}]-1}((E[\tilde{g}])^{\tau}-1)\right)^{\delta}}$$

It is straightforward to show that the right hand side goes to zero as $\tau\to\infty$ if $E[\tilde{g}^{\delta-1}]<E[\tilde{g}]^{\delta}$, in which case $\lim_{\tau\to\infty}\frac{\xi^{\delta-1}(z,\tau)}{\xi^{\delta}(z,\tau)}=0$ follows and the proof goes through without lognormality (provided in addition that $E[\tilde{g}^{\delta}]$ is finite). For the lognormal case, $E[\tilde{g}^{\delta-1}]<E[\tilde{g}]^{\delta}$ requires $\tilde{\mu}>0.5\tilde{\sigma}^2(\delta^2-3\delta+1)$. Unfortunately, this condition is rather easily violated.

$\tilde{\sigma}W_t)]$. Therefore, for $\tau \geq z$,

$$\xi^\delta(z,\tau) \geq E[\{2\tau \exp[\max_{0\leq t\leq\tau, t\in\mathbb{R}}(\tilde{\mu}t + \tilde{\sigma}W_t)]\}^\delta] = (2\tau)^\delta E[\exp[\delta\tilde{\sigma}\max_{0\leq t\leq\tau, t\in\mathbb{R}}((\tilde{\mu}/\tilde{\sigma})t + W_t)]]$$

Standard results on Brownian motion, which are based on the reflection principle, allow for the evaluation of the expectation on the right-hand-side. Applying formula 1.1.3. of Borodin and Salminen (2002, p. 250) yields, after some manipulations, for the expectation:

$$\frac{1}{\tilde{\mu} + 0.5\delta\tilde{\sigma}^2}\left\{(\tilde{\mu} + \delta\tilde{\sigma}^2)\exp[(\delta\tilde{\mu} + 0.5\delta^2\tilde{\sigma}^2)\tau]\Phi\left(\frac{\tilde{\mu} + \delta\tilde{\sigma}^2}{\tilde{\sigma}}\sqrt{\tau}\right) + \tilde{\mu}\Phi\left(-\frac{\tilde{\mu}}{\tilde{\sigma}}\sqrt{\tau}\right)\right\}$$

where Φ denotes the cumulative distribution function of a standard normal ($\Phi(x) \equiv \Pr[N(0,1) \leq x]$). The assumption (12) to the theorem implies that $\tilde{\mu} + \delta\tilde{\sigma}^2 > 0$ and therefore also that $\tilde{\mu} + 0.5\delta\tilde{\sigma}^2 > 0$, $\Phi\left(\frac{\tilde{\mu}+\delta\tilde{\sigma}^2}{\tilde{\sigma}}\sqrt{\tau}\right) > 0.5$ and $\tilde{\mu} > 0$. Using this finally yields the following lower bound for ξ, for $\tau \geq z$:

$$\xi^\delta(z,\tau) \geq (2\tau)^\delta\left(\frac{\tilde{\mu} + \delta\tilde{\sigma}^2}{2\tilde{\mu} + \delta\tilde{\sigma}^2}\right)\exp[(\delta\tilde{\mu} + 0.5\delta^2\tilde{\sigma}^2)\tau] \tag{26}$$

4. [Showing that $\lim_{\tau\to\infty}\frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)} = 0$ and $\lim_{\tau\to\infty}\frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)} = 1$] Applying the lower and upper bounds on ξ ((24) and (26)) yields, for $\tau \geq z$:

$$\begin{aligned}\frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)} &\leq \frac{\exp[(\tilde{\mu}(\delta-1) + 0.5(\delta-1)^2\tilde{\sigma}^2)\tau]}{(2\tau)^\delta\left(\frac{\tilde{\mu}+\delta\tilde{\sigma}^2}{2\tilde{\mu}+\delta\tilde{\sigma}^2}\right)\exp[(\delta\tilde{\mu} + 0.5\delta^2\tilde{\sigma}^2)\tau]} \\ &= (2\tau)^{-\delta}\left(\frac{2\tilde{\mu} + \delta\tilde{\sigma}^2}{\tilde{\mu} + \delta\tilde{\sigma}^2}\right)\exp[-(\tilde{\mu} + (\delta - 0.5)\tilde{\sigma}^2)\tau]\end{aligned}$$

Assumption (12) to the theorem states that $\tilde{\mu} + (\delta - 0.5)\tilde{\sigma}^2 > 0$, so the right hand side of the inequality goes to zero as $\tau \to \infty$ (as the exponential factor dominates). Since $\frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)} \geq 0$, this implies that

$$\lim_{\tau\to\infty}\frac{\xi^{\delta-1}(z,\tau)}{\xi^\delta(z,\tau)} = 0$$

This proves the second claim of the lemma, equation (23). Recalling the conclusion of step 2, it follows that

$$\lim_{\tau\to\infty}\frac{\xi^\delta(z',\tau)}{\xi^\delta(z,\tau)} = 1$$

proving the first claim of the lemma (22). QED.

Appendix 2. Proof of Theorem 2

From equations (4), (7) and (16), and the assumption that g_t is *i.i.d.*,

$$\frac{-c_t \partial^2 E_t \mathcal{U}/(\partial c_t)^2}{\partial E_t \mathcal{U}/\partial c_t} = \gamma + (\alpha - \gamma)\frac{\xi^{\delta-1}(z_t, T-t)}{\xi^{\delta}(z_t, T-t)}$$

Using the result in equation (23) of lemma 1 (see appendix 1), we immediately have

$$\lim_{T\to\infty} \frac{-c_t \partial^2 E_t \mathcal{U}/(\partial c_t)^2}{\partial E_t \mathcal{U}/\partial c_t} = \gamma$$

QED.

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