PDE2

Some HPC Issues in Partial Differential Equations — Part 2: Finite Difference Methods

1 Material Balance

Many physical problems can be cast in the form of conservation law PDEs. For simplicity, we begin by discussing conservation law PDEs involving only one space variable.

Suppose that some material quantity M, such as fluid or heat, moves parallel to the x-axis and that the flow field is independent of y and z. Under this assumption of one dimensional flow, it is sufficient to examine the flow in a cylinder of cross-sectional area A whose generators are parallel to the x-axis. Let $r = r(x, t)$ be the linear density of material quantity M (amount of M per unit length per unit area) and let $q = q(x, t)$ denote the flux of material M in the x-direction. The flux is the amount of material M that crosses a unit planar area perpendicular to the direction of flow per unit time.

Consider flow of M through the portion of the cylinder defined by $a < x < b$. If there are no additions (sources) or removals (sinks) of material M from within the interval (a, b), then the following conservation principle holds.

> *The time rate of change of the amount of material M in the interval (a, b) equals the flux of material M into (a, b) at $x = a$ minus the flux of material M out of (a, b) at $x = b$.*

In mathematical terms this statement of conservation of material M can be written as follows.

$$\left(\frac{d}{dt}\int_a^b r(x,t)\,dx\right)A = \left(q(a,t) - q(b,t)\right)A \tag{1}$$

Interchanging the time derivative with the integral on the left side and using the fundamental theorem of calculus to express the right side as an equivalent integral, permits Eq. 1 to be written as

$$\int_a^b \frac{\partial}{\partial t} r(x,t)\,dx = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx \tag{2}$$

If the integrands in the above equation are continuous, then, since the interval (a,b) is arbitrary, we obtain the conservation law PDE

$$\frac{\partial}{\partial t} r(x,t) + \frac{\partial}{\partial x} q(x,t) = 0 \tag{3}$$

2 Constitutive Equations

Suppose that $u = u(x,t)$ is some observable function (field variable) that determines the density of M through a relationship of the form

$$r(x,t) = R(x, u(x,t)) \tag{4}$$

In practice, the constitutive relationship between the field variable u and the flux often takes one of the following forms:

$$q(x,t) = F(x,t,u(x,t)) \tag{5}$$

or

$$q(x,t) = -\sigma \frac{\partial}{\partial x} u(x,t) \tag{6}$$

A mathematical model of traffic flow affords an example of a pair of constituitive relationships of the form of Eq. 4 and Eq. 5. In this case the field variable and the traffic density are the same, with $r(x,t) = u(x,t)$. One example of a flux rule of the form of Eq. 5 for traffic flow would have $q = uv$ where traffic speed v is given by $v = V(1 - u/U)$; V is a constant representing maximum speed (at low density); U represents the traffic density at which traffic stalls. Thus, in this case, Eq. 5 would read $q = V(u - u^2/U)$. Conservation law PDEs with the flux defined by Eq. 5 are of hyperbolic type. A numerical treatment of hyperbolic PDEs requires a thorough understanding of the notion of characteristic curves. A discussion of numerical methods for hyperbolic PDEs is beyond the scope of this brief introduction to the numerical treatment of PDEs by the finite difference method.

In Eq. 6, $\sigma = \sigma(x, u(x,t))$ is a positive quantity representing the flux of material M in response to a unit gradient in the field variable u. Time dependent conservation law PDEs with flux "down gradient" are usually parabolic while steady state conservation laws with such flux are generally elliptic. These parabolic and elliptic conservation law equations will be the vehicles by which a number of important computational issues surrounding PDE problems will be introduced.

If we put the constitutive relationships Eq. 4 and Eq. 6 into the conservation law $r_t + q_x = 0$, the result is the one dimensional parabolic conservation law PDE

$$c\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}(\sigma\frac{\partial u}{\partial x}) = 0 \tag{7}$$

where c denotes $R_u(x, u)$. If density is a linear function of u,

$$r(x,t) = cu(x,t) + b \qquad c, b \text{ consts, } c > 0, \tag{8}$$

and if σ is a constant, then Eq. 7 reduces to the constant coefficient diffusion equation

$$\frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{9}$$

where $a^2 = \sigma/c$.

Example 1 *Illustration of one-dimensional population density.*

Suppose that $u = u(x,t)$ represents a one dimensional population density and assume that the individuals in this population move in the x direction so as to avoid crowding. That is, the flux of population is proportional to the negative of the gradient of population density,

$$q = -\sigma u_x, \qquad \sigma > 0.$$

In this flux expression, the proportionality factor σ determines the rate at which population will relocate in response to a unit gradient in population density. For this example, we have $r(x,t) = u(x,t)$, so the conservation law Equation 7 takes the form

$$u_t - (\sigma u_x)_x = 0$$

For different physical settings, a variety of different expressions for σ might be assumed. For example, if the rate at which individuals respond to crowding is independent of position, time, and population density, then we might assume that σ is a constant. If individuals in the population move at a rate that is dependent on position x and time t, then we could take $\sigma = \sigma(x,t)$. If individuals respond differently to crowding at low population densities than at high population densities, then it would be appropriate to make σ a function of density.

Example 2 *Illustration of one-dimensional heat flow.*

Consider one dimensional heat flow in the direction of the x-axis in a heat conducting material with density ρ and specific heat s. The specific heat represents the number of calories required to raise a unit mass of the material a unit degree of temperature. The units of ρ are mass/volume and the units for s are calories/degree/mass. For simplicity, we assume that s and ρ can be treated as functions of at most only position: $s = s(x)$ and $\rho = \rho(x)$. In reality, s and ρ generally depend on temperature u as well. Let r denote the heat density measured in calories/volume. In accordance with Fourier's law of heat conduction, let $q = -\sigma u_x$ be the heat flux vector. The units of q are calories/time/area. Here σ is called the thermal conductivity, and it represents the rate at which heat moves from a hot region to a cold region in response to a unit gradient of temperature. If we assume that σ is a function of position and temperature, $\sigma = \sigma(x, u)$, then the conservation law

$$r_t + q_x = 0$$

with

$$r = R(x, u) = \rho(x)s(x)u(x, t)$$

and

$$q = -\sigma(x, u)u_x$$

leads to the parabolic conservation law

$$\rho s \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\sigma \frac{\partial u}{\partial x}\right) = 0$$

If the material density ρ, the specific heat s, and the thermal conductivity σ can be treated as constants, then our equation is a constant coefficient linear PDE of second order. If these quantities depend on x or t, but not on u, then this PDE is linear but has variable coefficients. If any of the coefficients are allowed to depend on u, then the PDE is nonlinear.

Analytical methods, using Fourier series integral transforms and superposition, can often be used to develop exact solutions to constant coefficient, linear PDE problems. If the PDE is nonlinear or has variable coefficients, then numerical methods generally must be used to approximate the solution of the PDE. One popular numerical method for dealing with PDEs is introduced in the next section.

3 Material Balance Finite Difference Equations in One Dimension

A widely used numerical method for approximating solutions to PDE problems is the finite difference method. To obtain a finite difference equation (FDE) for a partial differential equation, the continuous independent variables in the PDE (x and t in the previous examples) are restricted to a discrete grid of points, say x_n in the x-direction and t_j in the t-direction. Then the continuous derivatives in the PDE are approximated by difference quotients which involve the approximate solution to the PDE only at the grid points. The analytical solution techniques applied to PDE problems are generally based on calculus techniques and are not usually amenable to implementation on digital computers. In contrast, finite difference problems involve systems of algebraic equations and are well suited for use by a computational scientist.

There are two common methods for deriving finite difference equations. One method begins with a PDE and then uses finite difference formulas obtained from Taylor series expansions to approximate the various derivatives in the PDE. This approach is usually favored when performing a mathematical analysis of the consistency of the FDE with the PDE and convergence of the solution of the FDE to the PDE.

The second method for obtaining finite difference equations is based on a discrete conservation principle, similar to that introduced in Section 1. In this method, one need not begin with a PDE. All that is needed is a physical problem involving the density of some material and a rule for the flux of that quantity. In applications the quantity under consideration is usually mass, momentum or energy. The conservation law, or material balance, approach to obtaining finite difference equations proceeds by establishing a "control interval" about each of the spatial finite difference grid points. Finite difference equations then result from an application of the conservation principle of Section 1.

In the remainder of this section, we illustrate the material balance approach for developing finite difference equations. Discussion of the Taylor series approach for obtaining FDEs is left to the exercises.

To formulate a FDE for a one dimensional conservation law, we begin by establishing a grid of points on the xt-plane with step size h in the x direction and step size k in the t-direction. Let spatial grid points be denoted by $x_n = x_0 + nh$ and time grid points be $t_j = t_0 + jk$, where n and j are integers and (x_0, t_0) is the origin of the space–time grid. See Figure 1.

In Figure 1, the points ξ_{n-1} and ξ_n are introduced to establish a "control interval". A development of a conservation law FDE proceeds along the same lines as presented in the previous section. We begin with a conservation statement similar to Eq. 1.

$$\int_{\xi_{n-1}}^{\xi_n} (r(x,t_{j+1}) - r(x,t_j))\, dx = \int_{t_j}^{t_{j+1}} (q(\xi_{n-1},t) - q(\xi_n,t))\, dt \tag{10}$$

Equation 10 states that the change in material content of the interval (ξ_{n-1}, ξ_n) from time $t = t_j$ to time $t = t_{j+1}$ is given by the flow of material M into this interval at ξ_{n-1} minus the flow out of the interval at ξ_n from time t_j to time t_{j+1}. Equation 10 is an integral conservation law equation that exactly expresses the conservation of material M in the case that no sources or sinks are present. It is the basis for our development of a variety of conservation law finite difference equations.

To obtain a FDE that is useful in a computational setting requires that the physical variables, density r and flux q, be related to the field variable u. For simplicity, we begin

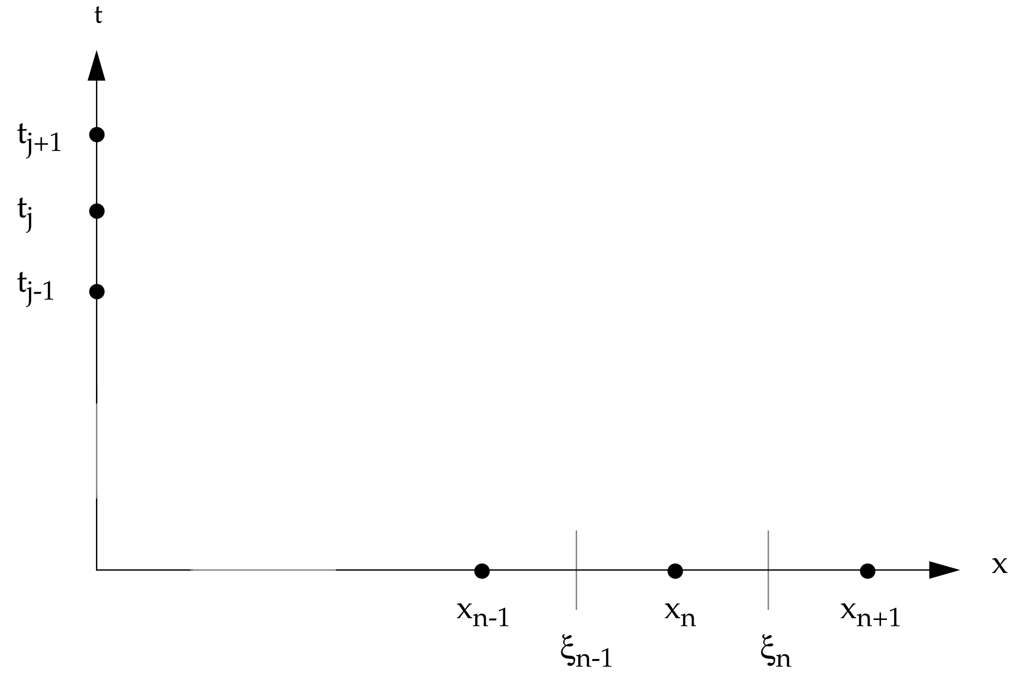

Figure 1: Representative grid points and a control interval.

with the case in which density is assumed to have the form

$$r(x,t) = cu(x,t) + b$$

with c and b constants.

Then the integral conservation law Eq. 10 can be written

$$c\int_{\xi_{n-1}}^{\xi_n} (u(x,t_{j+1}) - u(x,t_j))\; dx = \int_{t_j}^{t_{j+1}} (q(\xi_{n-1},t) - q(\xi_n,t))\; dt \tag{11}$$

We next use the midpoint quadrature rule to approximate the left side of Eq. 11. That is:

$$c\int_{\xi_{n-1}}^{\xi_n} (u(x,t_{j+1}) - u(x,t_j))\; dx \approx c(u(x_n,t_{j+1}) - u(x_n,t_j))h \tag{12}$$

When developing conservation law FDEs, there are three commonly used strategies for using numerical integration to approximate the integral on the right side of Eq. 11:

Left end-point quadrature:

$$\int_{t_j}^{t_{j+1}} (q(\xi_{n-1},t) - q(\xi_n,t))\; dt \approx (q(\xi_{n-1},t_j) - q(\xi_n,t_j))\, k \tag{13}$$

Right end-point quadrature:

$$\int_{t_j}^{t_{j+1}} \left(q(\xi_{n-1},t) - q(\xi_n,t)\right)\, dt \approx \left(q(\xi_{n-1},t_{j+1}) - q(\xi_n,t_{j+1})\right) k \tag{14}$$

Midpoint quadrature:

$$\begin{aligned}\int_{t_j}^{t_{j+1}} \left(q(\xi_{n-1},t) - q(\xi_n,t)\right)\, dt &\approx 0.5\left(q(\xi_{n-1},t_{j+1}) - q(\xi_n,t_{j+1})\right) k \\ &+ 0.5\left(q(\xi_{n-1},t_j) - q(\xi_n,t_j)\right) k \end{aligned}\tag{15}$$

Combining the approximation for the change in storage, Eq. 12, with the respective approximations for net flux into the control interval, Eq. 13, Eq. 14, and Eq. 15, yields:

Explicit Method:

$$c\left(u(x_n,t_{j+1}) - u(x_n,t_j)\right) h \approx \left(q(\xi_{n-1},t_j) - q(\xi_n,t_j)\right) k \tag{16}$$

Implicit Method:

$$c\left(u(x_n,t_{j+1}) - u(x_n,t_j)\right) h \approx \left(q(\xi_{n-1},t_{j+1}) - q(\xi_n,t_{j+1})\right) k \tag{17}$$

Midpoint Method:

$$\begin{aligned}c\left(u(x_n,t_{j+1}) - u(x_n,t_j)\right) h &\approx 0.5\left(q(\xi_{n-1},t_{j+1}) - q(\xi_n,t_{j+1})\right) k \\ &+ 0.5\left(q(\xi_{n-1},t_j) - q(\xi_n,t_j)\right) k \end{aligned}\tag{18}$$

The final step in arriving at a finite difference equation that can be used in a computational setting is to relate the fluxes at ξ_{n-1} and ξ_n to the field variable u.

$$q(x,t) = -\sigma \frac{\partial}{\partial x} u(x,t)$$

First, consider the case that σ is a constant. Using centered finite difference formulas to approximate the fluxes at the control points ξ_{n-1} and ξ_n yields

$$q(\xi_{n-1},t_j) = -\sigma \frac{u(x_n,t_j) - u(x_{n-1},t_j)}{h} \tag{19}$$

and

$$q(\xi_n,t_j) = -\sigma \frac{u(x_{n+1},t_j) - u(x_n,t_j)}{h} \tag{20}$$

with similar formulas for the fluxes at time t_{j+1}.

We have used a lower case u to denote the continuous field variable, $u = u(x,t)$. Note that all of the quadrature and difference formulas involving u are stated as approximate equalities. In each of these approximate equality statements, the amount by which the right side differs from the left side is called the truncation error. If u is a well behaved function (has enough smooth derivatives), then it can be shown that these truncation errors approach zero as the grid spacings, h and k, approach zero.

When we solve a finite difference equation, we obtain approximations to the values of the PDE problem solution at the grid points; that is, approximations to $u(x_n, t_j)$. To emphasize this approximation we write

$$u(x_n, t_j) \approx U_n^j$$

where U_n^j denotes the exact solution of a FDE problem on the grid (x_n, t_j). With this notation in place, we are in a position to state three commonly used finite difference methods for dealing with a conservation law system in which u is the field variable and density and flux are given by

$$r(x,t) = cu(x,t) + b$$

and

$$q(x,t) = -\sigma \frac{\partial}{\partial x} u(x,t)$$

Combining Eq. 12 with the difference quotient formulas of Eq. 19 and Eq. 20 we obtain the following.

Explicit (Forward in Time) Finite Difference Method [for $cu_t = \sigma u_{xx}$]:

$$c\left(U_n^{j+1} - U_n^j\right) h = \left[-\sigma\left(\frac{U_n^j - U_{n-1}^j}{h}\right) + \sigma\left(\frac{U_{n+1}^j - U_n^j}{h}\right)\right] k$$

or, after minor algebraic rearrangement,

$$U_n^{j+1} = rU_{n-1}^j + (1-2r)U_n^j + rU_{n+1}^j \tag{21}$$

where

$$r = \frac{\sigma k}{ch^2}$$

This FDE is referred to as explicit since Eq. 21 provides an explicit formula for calculating the solution to the difference equation at time t_{j+1}, knowing the values at time t_j. A finite difference scheme can be described to be stable if rounding errors do not grow faster than the solution as successive time steps are advanced. It can be shown that Eq. 21 is a stable computational method only if $r \leq \frac{1}{2}$.

If we evaluate the numerical approximations to the flux at time t_{j+1}, we obtain the following.

Implicit (Backward in Time) Finite Difference Method [for $cu_t = \sigma u_{xx}$]:

$$c\left(U_n^{j+1} - U_n^j\right) h = \left[-\sigma\left(\frac{U_n^{j+1} - U_{n-1}^{j+1}}{h}\right) + \sigma\left(\frac{U_{n+1}^{j+1} - U_n^{j+1}}{h}\right)\right] k$$

which can be written

$$-rU_{n-1}^{j+1} + (1+2r)U_n^{j+1} - rU_{n+1}^{j+1} = U_n^j \tag{22}$$

Note that Eq. 22 only implicitly defines the solution at time t_{j+1}, since a system of algebraic equations is required to be satisfied. The solution of this system of equations is discussed in the next section. The implicit method Eq. 22 can be shown to be stable for all values of r. The development of the Crank–Nicolson FDE from Eq. 18 is left to the exercises.

Example 3 *Illustration of a BVP.*

Consider the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} & 0 < x < 1,\ t > 0 \\ u(x,0) = 100 & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \end{array}$$

If N interior grid points are used, then $h = 1/(N+1)$ and $x_n = nh$, $n = 0, 1, 2, \ldots, N+1$. If we select the explicit method Eq. 21, a complete set of FDEs for the example IBVP is

$$\begin{array}{ll} U_n^{j+1} = rU_{n-1}^j + (1-2r)U_n^j + rU_{n+1}^j & n = 1, 2, \ldots, N \\ U_n^0 = 100 & n = 1, 2, \ldots, N \\ U_0^j = 0 = U_{N+1}^j & j = 0, 1, 2, \ldots \end{array}$$

Note how the boundary conditions are incorporated into the $n = 0$ and $n = N+1$ difference equations. In a computer program the difference equations can be dealt with in a single loop if arrays holding U are dimensioned large enough to admit a 0-index and an $(N+1)$-index with zero boundary conditions stored in these positions.

3.1 FDEs with Variable Coefficients

So far, we have developed finite difference equations only for the case where the coefficients c and σ are constants. In practice, these coefficients often depend on space, on time, or on the field variable $u(x,t)$. To indicate some of the many techniques used to deal with variable coefficient FDEs, we consider the following two cases.

Case 1—c and σ depend explicitly on x and t: In this case, if we return to the integral conservation law, Eq. 10, and proceed with similar arguments as in the constant coefficient case, we obtain an explicit FDE of the form

$$c_n^{j+1}U_n^{j+1}h - c_n^jU_n^jh = \left[-\sigma_{n-1/2}^j\left(\frac{U_n^j - U_{n-1}^j}{h}\right) + \sigma_{n+1/2}^j\left(\frac{U_{n+1}^j - U_n^j}{h}\right)\right]k$$

where

$$c_n^{j+1} = c(x_n, t_{j+1}) \qquad c_n^j = c(x_n, t_j)$$

and

$$\sigma_{n-1/2}^j = \sigma(\xi_{n-1}, t_j) \qquad \sigma_{n+1/2}^j = \sigma(\xi_n, t_j)$$

The formulations of the implicit and Crank–Nicolson methods for the case $c = c(x,t)$ and $\sigma = \sigma(x,t)$ are left as exercises.

Case 2—c and σ depend on $u = u(x,t)$: In this case, if we return to the integral conservation law, Eq. 10, and proceed with similar arguments as in the constant coefficient case, we obtain an explicit FDE of the form

$$c_n^{j+1} U_n^{j+1} h - c_n^j U_n^j h = \left[-\sigma_{n-1/2}^j \left(\frac{U_n^j - U_{n-1}^j}{h} \right) + \sigma_{n+1/2}^j \left(\frac{U_{n+1}^j - U_n^j}{h} \right) \right] k$$

where

$$c_n^{j+1} = c\left(U_n^{j+1}\right) \qquad c_n^j = c\left(U_n^j\right)$$

Unless the coefficient of the term U_n^{j+1} is "lagged," by replacing c_n^{j+1} by c_n^j, the left side of this FDE will contain nonlinear terms involving U_n^{j+1} and will not be algebraically explicit in a computational sense. Such lagging of nonlinearities is a common technique in computations.

Since the solution to the FDE is not known at the points ξ_{n-1} and ξ_n, other methods for setting the σ-value at these points must be used. Three commonly used techniques are

Arithmetic Average

$$\sigma_{n-1/2}^j = \frac{\sigma\left(U_{n-1}^j\right) + \sigma\left(U_n^j\right)}{2} \qquad \sigma_{n+1/2}^j = \frac{\sigma\left(U_n^j\right) + \sigma\left(U_{n+1}^j\right)}{2}$$

Geometric Average

$$\sigma_{n-1/2}^j = \sqrt{\sigma\left(U_{n-1}^j\right)\sigma\left(U_n^j\right)} \qquad \sigma_{n+1/2}^j = \sqrt{\sigma\left(U_n^j\right)\sigma\left(U_{n+1}^j\right)}$$

Harmonic Average

$$\sigma_{n-1/2}^j = \frac{2\sigma\left(U_{n-1}^j\right)\sigma\left(U_n^j\right)}{\sigma\left(U_{n-1}^j\right) + \sigma\left(U_n^j\right)} \qquad \sigma_{n+1/2}^j = \frac{2\sigma\left(U_n^j\right)\sigma\left(U_{n+1}^j\right)}{\sigma\left(U_n^j\right) + \sigma\left(U_{n+1}^j\right)}$$

In the steady state limit, flow through a composite medium (a medium with a jump discontinuity in σ) is determined by a harmonic average. See Exercise 27. For implicit finite difference methods involving the flux at time t_{j+1}, it is common practice to lag the σ-values by evaluating them using the U-values from time t_j. The alternative is to deal with a system of algebraically nonlinear difference equations using some iterative procedure such as Newton's method for systems of equations.

4 Material Balance Difference Equations in Two and Three Dimensions

The derivation of material balance difference equations in more than one space dimension very closely parallels the derivation for the one space variable case of the previous section. For the case of two space variables, we assume that some material M has a spatial density given by $r(x, y, t)$. Again, for simplicity, we begin by assuming that density is related to a field variable $u = u(x, y, t)$ by a linear constitutive equation of the form

$$r(x, y, t) = cu(x, y, t) + b$$

We assume the flow of material M to be down-gradient with respect to u. That is, the flux $q = q(x, y, t) = (qx(x, y, t), qy(x, y, t))$ is given by

$$q = -\sigma \nabla u \qquad \sigma > 0$$

or

$$q(x, y, t) = -\sigma \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \tag{23}$$

Establish a finite difference grid on the xy-plane with x grid points denoted by x_m and y grid points by y_n. The grid spacing in the x-direction is denoted by hx and in the y-direction by hy. As in the one dimensional case, control points are established midway between the x and y grid points. Using the shaded control region shown in Figure 2, we write a conservation law integral equation similar to Eq. 10. Namely,

$$\begin{aligned} &\textstyle\int_{\eta_n}^{\eta_{n+1}} \int_{\xi_m}^{\xi_{m+1}} \left[r(x, y, t_{j+1}) - r(x, y, t_j) \right] \, dx, dy = \\ &\textstyle\int_{t_j}^{t_{j+1}} \int_{\eta_n}^{\eta_{n+1}} \left[qx(\xi_m, y, t) - qx(\xi_{m+1}, y, t) \right] \, dy \, dt + \int_{t_j}^{t_{j+1}} \int_{\xi_m}^{\xi_{m+1}} \left[qy(x, \eta_n, t) - qy(x, \eta_{n+1}, t) \right] \, dx \, dt \end{aligned} \tag{24}$$

Equation 24 states that the change in M-content in the control region equals the net flux of M across the boundary from t_j to t_{j+1}. If we approximate all the spatial integrals in Eq. 24 using the midpoint quadrature rule, then Eq. 24 yields

$$\begin{aligned} &c \left[u(x_m, y_n, t_{j+1}) - u(x_m, y_n, t_j) \right] \, hxhy \approx \\ &\textstyle\int_{t_j}^{t_{j+1}} \left[qx(\xi_m, y_n, t) - qx(\xi_{m+1}, y_n, t) \right] \, dt \, hy + \int_{t_j}^{t_{j+1}} \left[qy(x_m, \eta_n, t) - qy(x_m, \eta_{n+1}, t) \right] \, dt \, hx \end{aligned} \tag{25}$$

The time integrals can be handled as before. If we use a left hand quadrature rule, an explicit FDE results. The right hand quadrature rule gives the backward in time implicit FDE, and the midpoint rule the Crank–Nicolson FDE.

The fluxes on the faces of the control region are approximated by difference quotients in the same way as in the one dimensional case. That is,

$$qx(\xi_m, y_n, t) \approx -\sigma \frac{u(x_m, y_n, t) - u(x_{m-1}, y_n, t)}{hx}$$

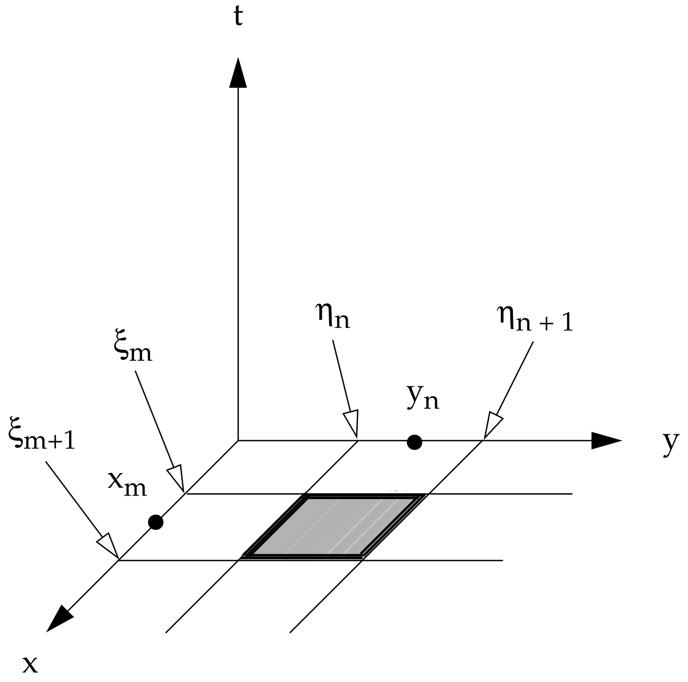

Figure 2: Control region for two dimensional material balance.

$$
\begin{aligned}
qx(\xi_{m+1}, y_n, t) &\approx -\sigma \frac{u(x_{m+1}, y_n, t) - u(x_m, y_n, t)}{hx} \\
qy(x_m, \eta_n, t) &\approx -\sigma \frac{u(x_m, y_n, t) - u(x_m, y_{n-1}, t)}{hy} \\
qy(x_m, \eta_{n+1}, t) &\approx -\sigma \frac{u(x_m, y_{n+1}, t) - u(x_m, y_n, t)}{hy}
\end{aligned}
$$

If we use U^j_{mn} to denote the FDE approximation to $u(x_m, y_n, t_j)$, we then obtain the following finite difference methods.

Explicit (Forward in Time) Finite Difference Method [for $cu_t = \sigma(u_{xx} + u_{yy})$]:

$$
\begin{aligned}
c\left(U^{j+1}_{mn} - U^j_{mn}\right) hxhy &= \left[-\sigma\left(\frac{U^j_{mn} - U^j_{m-1,n}}{hx}\right) + \sigma\left(\frac{U^j_{m+1,n} - U^j_{mn}}{hx}\right)\right] hyk \\
&+ \left[-\sigma\left(\frac{U^j_{mn} - U^j_{m,n-1}}{hy}\right) + \sigma\left(\frac{U^j_{m,n+1} - U^j_{mn}}{hy}\right)\right] hxk
\end{aligned}
$$

or,

$$
\begin{aligned}
U^{j+1}_{mn} &= U^j_{mn} + rxU^j_{m-1,n} - 2rxU^j_{mn} + rxU^j_{m+1,n} \\
&+ ryU^j_{m,n-1} - 2ryU^j_{mn} + ryU^j_{m,n+1} \qquad (26)
\end{aligned}
$$

where

$$rx = \frac{\sigma k}{c(hx)^2}, \qquad ry = \frac{\sigma k}{c(hy)^2}$$

For the explicit method Eq. 21 to be stable, rx and ry must be restricted to satisfy $rx + ry \leq 1/2$. As with the one space variable case, this explicit method provides a simple computational algorithm that is easy to implement efficiently on vector and parallel architectures. However, in many practical problems, its stability restriction often limits its range of applicability.

If we evaluate the numerical approximations to the flux at time t_{j+1}, we obtain the following.

Implicit (Backward in Time) Finite Difference Method [for $cu_t = \sigma(u_{xx} + u_{yy})$]:

$$\begin{aligned} c\left(U_{mn}^{j+1} - U_{mn}^{j}\right) hxhy &= \left[-\sigma\left(\frac{U_{mn}^{j+1} - U_{m-1,n}^{j+1}}{hx}\right) + \sigma\left(\frac{U_{m+1,n}^{j+1} - U_{mn}^{j+1}}{hx}\right)\right] hyk \\ &+ \left[-\sigma\left(\frac{U_{mn}^{j+1} - U_{m,n-1}^{j+1}}{hy}\right) + \sigma\left(\frac{U_{m,n+1}^{j+1} - U_{mn}^{j+1}}{hy}\right)\right] hxk \end{aligned}$$

which can be written

$$\begin{aligned} U_{mn}^{j} &= rxU_{m-1,n}^{j+1} + rxU_{m+1,n}^{j+1} + (1 + 2rx + 2ry)U_{mn}^{j+1} \\ &\quad + ryU_{m,n-1}^{j+1} + ryU_{m,n+1}^{j+1} \end{aligned} \tag{27}$$

Since Eq. 17 involves five unknown U-values at the time level t_{j+1}, its solution requires that a system of linear equations be satisfied. The structure of these linear equations and some techniques for dealing with these equations are discussed in the next section. The implicit method Eq. 17 can be shown to provide a stable finite difference scheme for all choices of rx and ry.

We now turn our attention to the case that the coefficients c and σ are not constant. First, we consider the case that c and σ depend explicitly on x, y, and t. To facilitate this discussion, consider a grid labeled as indicated in Figure 3.

Suppose that the finite difference block whose center is at (x_m, y_n) is to be treated as the control block in a material balance argument. Then it is convenient to refer to the sides of the block by reference to the points of the compass. See Figure 4.

With this compass-point notation imposed on the control cell with center (x_m, y_n), various FDEs can be derived from the following statement of conservation:

> *In the absence of sources or sinks, the change of material content in the finite difference cell (x_m, y_n) is equal to the flow of material in across the west face minus the flow out across the east face, plus the flow in across the south face minus the flow out across the north face.*

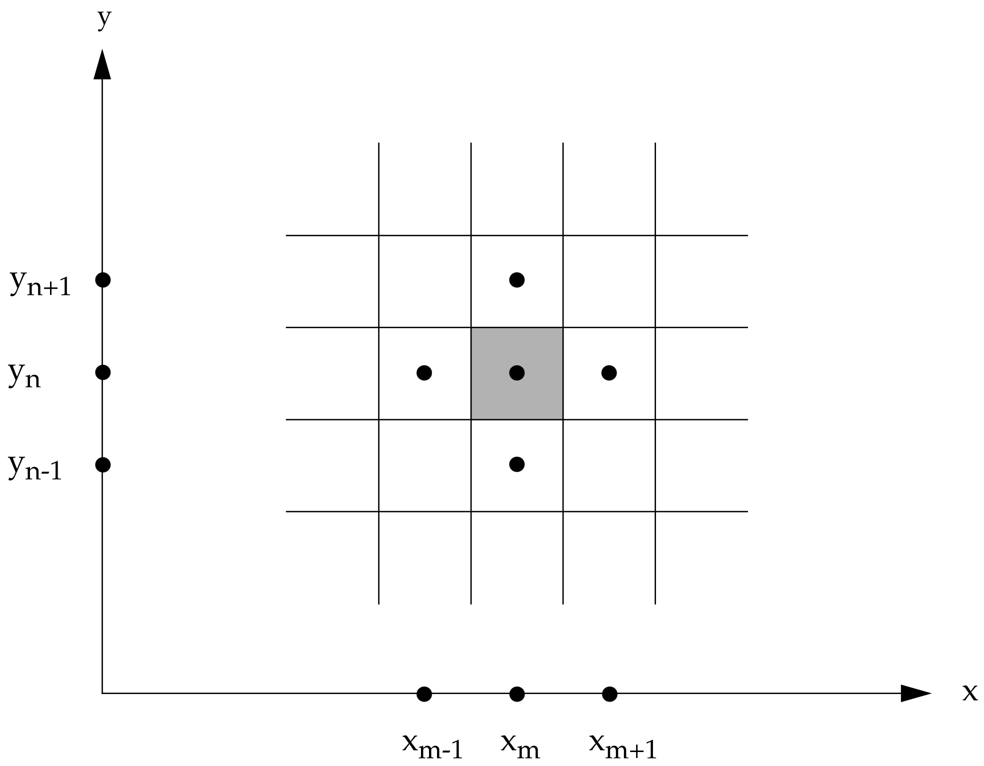

Figure 3: Block centered finite difference grid.

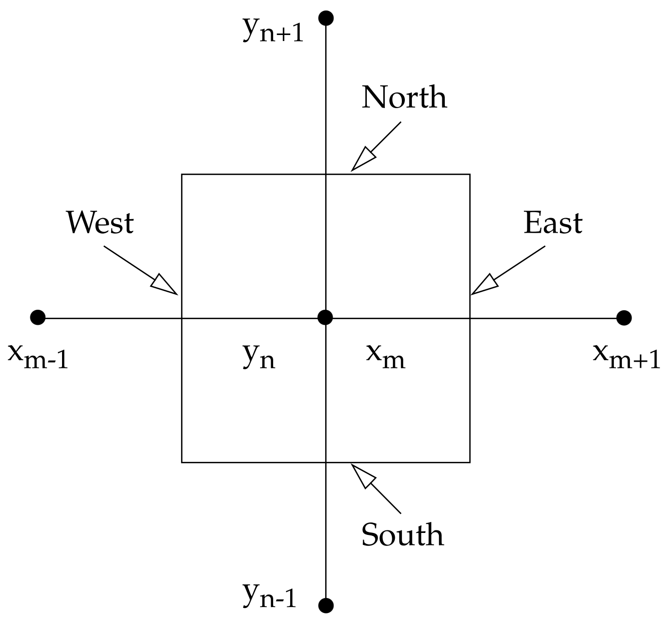

Figure 4: Control region with compass labeled faces.

Applying this conservation principle we obtain the following.

Explicit (Forward in Time) Finite Difference Method [for $cu_t = (\sigma u_x)_x + (\sigma u_y)_y$]:

$$c_{mn}\left(U_{mn}^{j+1} - U_{mn}^{j}\right) hxhy = \tag{28}$$

$$\left[-\sigma w_{mn}^{j}\left(\frac{U_{mn}^{j} - U_{m-1,n}^{j}}{hx}\right) + \sigma e_{mn}^{j}\left(\frac{U_{m+1,n}^{j} - U_{mn}^{j}}{hx}\right)\right] hyk$$

$$+\left[-\sigma s_{mn}^{j}\left(\frac{U_{mn}^{j} - U_{m,n-1}^{j}}{hy}\right) + \sigma n_{mn}^{j}\left(\frac{U_{m,n+1}^{j} - U_{mn}^{j}}{hy}\right)\right] hxk$$

where σw_{mn}, σe_{mn}, σs_{mn}, and σn_{mn} denote values of the coefficient σ to be used in calculating the flux across the west, east, south, and north faces of the control cell whose center is (x_m, y_n). The strategies for choosing the various σ-values on the faces of the control cell are similar to the single space variable case. If $\sigma = \sigma(x, y, t)$, one could set

$$\sigma w_{mn}^{j} = \sigma(\xi_{m-1}, y_n, t_j) \qquad \sigma e_{mn}^{j} = \sigma(\xi_m, y_n, t_j)$$

and

$$\sigma s_{mn}^{j} = \sigma(x_m, \eta_{n-1}, t_j) \qquad \sigma n_{mn}^{j} = \sigma(x_m, \eta_n, t_j)$$

In the case that $\sigma = \sigma(u(x, y, t))$, if the harmonic averages were chosen for calculating face σ-values, then we would set

$$\sigma w_{mn}^{j} = \frac{2\sigma\left(U_{m-1,n}^{j}\right)\sigma\left(U_{mn}^{j}\right)}{\sigma\left(U_{m-1,n}^{j}\right) + \sigma\left(U_{mn}^{j}\right)} \qquad \sigma e_{mn}^{j} = \frac{2\sigma\left(U_{mn}^{j}\right)\sigma\left(U_{m+1,n}^{j}\right)}{\sigma\left(U_{mn}^{j}\right) + \sigma\left(U_{m+1,n}^{j}\right)}$$

and

$$\sigma s_{mn}^{j} = \frac{2\sigma\left(U_{m,n-1}^{j}\right)\sigma\left(U_{mn}^{j}\right)}{\sigma\left(U_{m,n-1}^{j}\right) + \sigma\left(U_{mn}^{j}\right)} \qquad \sigma n_{mn}^{j} = \frac{2\sigma\left(U_{mn}^{j}\right)\sigma\left(U_{m,n+1}^{j}\right)}{\sigma\left(U_{mn}^{j}\right) + \sigma\left(U_{m,n+1}^{j}\right)}$$

Having assigned four face values of σ for each finite difference cell, we can express the explicit FDE for $cu_t = (\sigma u_x)_x + (\sigma u_y)_y$ as

$$\begin{aligned} U_{mn}^{j+1} &= sw_{mn}^{j} U_{m-1,n}^{j} + se_{mn}^{j} U_{m+1,n}^{j} + ss_{mn}^{j} U_{m,n-1}^{j} + sn_{mn}^{j} U_{m,n+1}^{j} \\ &+ (1 - sw_{mn}^{j} - se_{mn}^{j} - ss_{mn}^{j} - sn_{mn}^{j}) U_{mn}^{j} \end{aligned} \tag{29}$$

where

$$sw_{mn}^{j} = \frac{\sigma w_{mn}^{j} k}{c_{mn}^{j}(hx)^2} \qquad se_{mn}^{j} = \frac{\sigma e_{mn}^{j} k}{c_{mn}^{j}(hx)^2}$$

$$ss_{mn}^{j} = \frac{\sigma s_{mn}^{j} k}{c_{mn}^{j}(hy)^2} \qquad sn_{mn}^{j} = \frac{\sigma n_{mn}^{j} k}{c_{mn}^{j}(hy)^2}$$

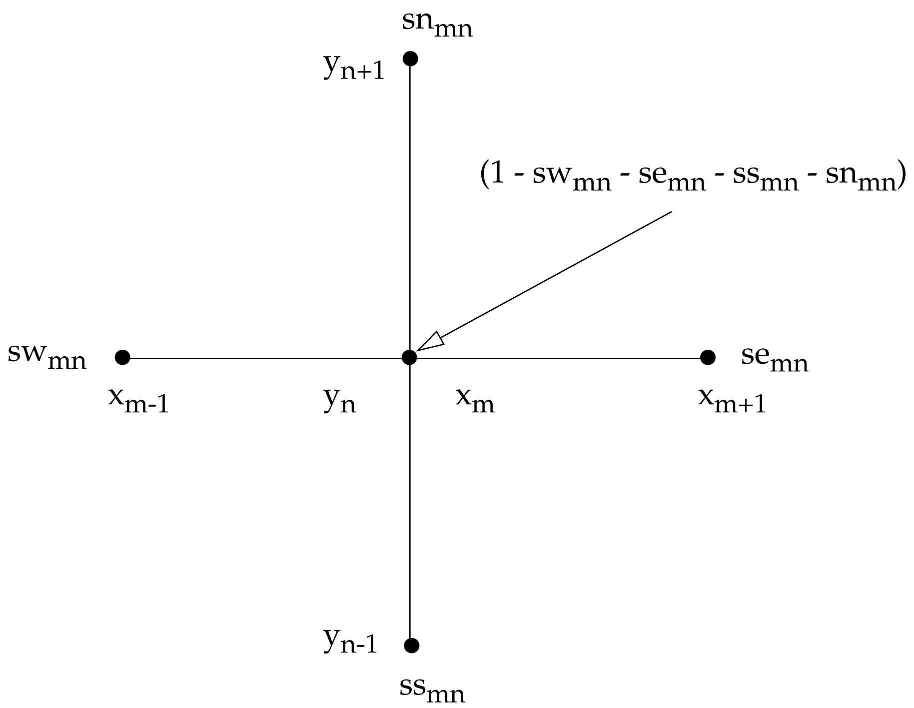

Figure 5: Computational stencil for the Explicit Method.

For stability of Eq. 29, the coefficient of U_{mn} must be nonnegative. In practice, it is often convenient to express equations such as Eq. 29 using a computational stencil as indicated in Figure 5. Note that the explicit FDE can be implemented by overlaying the stencil of Figure 5 on the array U^j_{mn} and forming the linear combination found on the right side of Eq. 29.

The development of the backward in time implicit FDE and the Crank–Nicolson method for the two dimensional, variable coefficient case is straightforward, and is left to the exercises. The extension of these ideas to three dimensions is also left as an exercise. In the following section we discuss some computational issues and techniques for solving the conservation law FDEs developed in this section.

5 Exercises

Exercise 1 *An alternate form of the equation for conservation of material.*

Show that Eq. (10) follows from Eq. (1).

Exercise 2 *Inclusion of a source/sink term in the original form of the material conservation equation.*

Modify Eq. (1) to account for the presence of a source/sink term $S(x,t)$.

Exercise 3 *Inclusion of a source/sink term in the alternate form of the material conservation equation.*

Modify Eq. (10) to account for the presence of a source/sink term $S(x,t)$.

Exercise 4 *Derivation of difference quotient formulas using Taylor's Theorem.*

Taylor's Theorem with remainder states that

$$u(x+h,t) = u(x,t) + u_x(x,t)h + \frac{1}{2!}u_{xx}(\xi,t)h^2 \qquad x < \xi < x+h$$

provided u has two continuous derivatives with respect to x and

$$\begin{aligned} u(x+h,t) &= u(x,t) + u_x(x,t)h + \frac{1}{2!}u_{xx}(x,t)h^2 \\ &+ \frac{1}{3!}u_{xxx}(x,t)h^3 + \frac{1}{4!}u_{xxxx}(\xi,t)h^4 \qquad x < \xi < x+h \end{aligned}$$

provided u has four continuous derivatives with respect to x. Use these Taylor formulas to derive the following difference quotient formulas.

$$\begin{aligned} u_x(x,t) &= \frac{u(x+h,t)-u(x,t)}{h} - \frac{1}{2}u_{xx}(\xi,t)h \qquad x < \xi < x+h \\ u_x(x,t) &= \frac{u(x,t)-u(x-h,t)}{h} + \frac{1}{2}u_{xx}(\xi,t)h \qquad x-h < \xi < x \\ u_{xx}(x,t) &= \frac{u(x-h,t)-2u(x,t)+u(x+h,t)}{h^2} - \frac{1}{12}u_{xxxx}(\xi,t)h^2 \qquad x-h < \xi < x+h \end{aligned}$$

In each formula, the term with argument (x,t) is the truncation error; it represents the error incurred in approximating a derivative with a difference quotient, assuming that all arithmetic is infinite precision. Assuming u has three continuous derivatives with respect to x, use Taylor's Theorem with remainder to show

$$u_x(x,t) = \frac{u(x+h,t)-u(x-h,t)}{2h} - \frac{1}{6}u_{xxx}(\xi,t)h^2 \qquad x-h < \xi < x+h$$

Exercise 5 *An evaluation of the error in the computer implementation of the centered difference formula.*

Since computer arithmetic has finite precision, in addition to the truncation errors of Exercise 4, rounding and precision errors must be considered. Let E_1, E_2, and E_3 denote the rounding/precision errors associated with the evaluations of $u(x-h,t)$, $u(x,t)$, and $u(x+h,t)$, respectively. Show that in a computer implementation of the centered difference formula for $u_x(x,t)$ the sum of the roundoff/precision error and the truncation error has the form

$$Error = \frac{E_3 - E_1}{2h} - \frac{1}{6}u_{xxx}(\xi,t)h^2$$

What does this formula for Error tell you about choosing a large value of h in a calculation? What does it tell you about choosing a very small value of h in a calculation?

Exercise 6 *An illustration of the error analysis of the previous exercise.*

To illustrate the error analysis of the previous exercise, approximate the derivative of $\exp(x)$ at $x = 0$ with the centered difference formula and compare with the exact value of the derivative using step sizes ranging from 10^{-10} to 10^{-1}. Use **xmgr** to produce a log–log plot of the error. On your computer, what appears to be a choice of h that minimizes the sum of the rounding/precision error and the truncation error for this calculation? Show how your results change when you use 32 bit, 64 bit, and 128 bit precision in your floating point calculations.

Exercise 7 *The Crank–Nicolson Finite Difference Equation.*

Show that the Crank–Nicolson finite difference equation for $cu_t = \sigma u_{xx}$ can be written as follows:

$$-\frac{r}{2}U_{n-1}^{j+1} + (1+r)U_n^{j+1} - \frac{r}{2}U_{n+1}^{j+1} = \frac{r}{2}U_{n-1}^{j} + (1-r)U_n^{j} + \frac{r}{2}U_{n+1}^{j}$$

where

$$r = \frac{k\sigma}{h^2}$$

Exercise 8 *Derivation of the diffusion-convection conservation law PDE.*

In some physical problems, the flux of material M has both a diffusive (down-gradient) component and a convective component (due to a velocity v) in which case the one dimensional flux can be expressed as

$$q(x,t) = -\sigma\frac{\partial}{\partial x}u(x,t) + vu$$

with v and σ allowed to depend on x, t, and u in general. Use the conservation law, $r_t + q_x = 0$, $r = cu + b$, and the above flux expression to derive the diffusion-convection conservation law PDE $c\frac{du}{dt} = \frac{d}{dx}(\sigma\frac{du}{dx}) - \frac{d(vu)}{dx}$. In the case that c, v, and σ are constants, classify this PDE as hyperbolic, parabolic, or elliptic.

Exercise 9 *Derivation of FDEs for various forms of σ and v.*

Assuming that flux density are as in Exercise 8 and that σ and v are positive constants, use a conservation law argument to derive (a) a centered explicit FDE of the form

$$U_n^{j+1} = (r-s)U_{n-1}^{j} + (1-2r)U_n^{j} + (r+s)U_{n+1}^{j}$$

$$r = \frac{\sigma k}{ch^2} \qquad s = \frac{vk}{2ch}$$

and (b) an "upwind" explicit FDE of the form

$$U_n^{j+1} = (r-s)U_{n-1}^{j} + (1-2r+s)U_n^{j} + rU_{n+1}^{j}$$

$$r = \frac{\sigma k}{ch^2} \qquad s = \frac{vk}{ch}$$

Formulate similar FDEs for the case that σ and v are allowed to depend on x and t; for the case that σ and v are allowed to depend on u.

Exercise 10 *Repeat Exercise 9 deriving FDEs that are implicit, backward in time.*

Exercise 11 *A computer program to solve a set of FDEs.*

Write a computer program that solves the FDEs of Example 3. Use $N = 50$; take $c = 1.0$ and $\sigma = 1$. Experiment with various choices of time step k. Plot your numerical solution at times $t = .1$, $t = .5$, and $t = 1.0$.

Exercise 12 *Explicit FDE used to approximate the solution to an IBVP.*

Use the explicit FDE Eq. (21) to approximate the solution to the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} & 0 < x < 1, \ t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 1 = u(1,t) & t > 0 \end{array}$$

Take $c = 1$ and $\sigma = 1$ and experiment with different values of space step h and time step k. Make some runs with the stability requirement on k/h^2 satisfied and some runs with this requirement violated.

Exercise 13 *Implicit FDE used to approximate the solution to an IBVP.*

Use the implicit FDE Eq. (22) to approximate the solution to the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} & 0 < x < 1, \ t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 1 = u(1,t) & t > 0 \end{array}$$

Take $c = 1$ and $\sigma = 1$ and experiment with different values of space step h and time step k. Hint: The following Gaussian elimination scheme for tridiagonal linear systems may be a useful starting point in constructing a solution to this problem:

```
      program trid
      parameter(n=50)
c   a holds subdiagonal, b the diagonal, and c the superdiagonal
c   d initially holds the right hand side of the linear system
c   d contains the solution at program termination
      dimension a(n), b(n), c(n), d(n)
```

```
      data a /n*-1.0/
      data b /n*2.0/
      data c /n*-1.0/
      data d/n*1.0/
      do 1 i = 2,n
      ratio = a(i)/b(i-1)
      b(i) = b(i) - ratio*c(i-1)
      d(i) = d(i) - ratio*d(i-1)
 1    continue
      d(n) = d(n)/b(n)
      do 2 i = n-1,1,-1
      d(i) = (d(i) - c(i)*d(i+1))/b(i)
 2    continue
      end
```

Exercise 14 *A computer program to factor a tridiagonal matrix.*

Using program trid as a starting point, write a computer program that factors a tridiagonal matrix into the product of a lower triangular and an upper triangular matrix. Then incorporate this matrix factorization into the solution of the implicit FDE method on the IBVP of the previous problem. What is the operation count savings obtained by using the LU factorization?

Exercise 15 *Approximate the solution to a centered difference FDE.*

Use a tridiagonal linear equation solver to approximate the solution to the centered difference FDE for the steady state convection diffusion equation

$$\sigma u_{xx} + v u_x = 0 \qquad 0 < x < 1$$
$$u(0) = 1$$
$$u(1) = 0$$

Use 10, 20, and 100 interior grid points and experiment with values of the "grid peclet number" vh/σ ranging from 0 to 100. In each experiment, use **xmgr** to plot the results of the numerical solution and the exact solution. Repeat using a one sided difference for the first derivative.

Exercise 16 *IBVP solution approximated by an implicit FDE with centered difference.*

Use an implicit FDE with centered difference on the convection term to approximate the solution to the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} + vu & 0 < x < 1, \;\; t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 1 & t > 0 \\ u(1,t) = 0 & t > 0 \end{array}$$

Take $c = 1$ and $\sigma = 1$ and experiment with different values of space step h and time step k.

Exercise 17 *IBVP solution approximated by an explicit FDE.*

Use the explicit FDE Eq. (21) to approximate the solution to the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} & 0 < x < 1, \ \ t > 0 \\ u(x,0) = 0 & 0 < x < 1 \\ u(0,t) = 0 & t > 0 \\ u(1,t) = 100 & t > 0 \end{array}$$

Take $c = 1$ and $\sigma = 1$ for $0 < x < 0.5$ and $\sigma = 10$ for $0.5 < x < 1.0$. Experiment with different values of space step h and time step k. Use **xmgr** to plot your results at various times.

Exercise 18 *Solution to another IBVP approximated by an explicit FDE.*

Use the explicit FDE Eq. (21) to approximate the solution to the IBVP

$$\begin{array}{ll} cu_t = \sigma u_{xx} & 0 < x < 1, \ \ t > 0 \\ u(x,0) = 100 \sin \pi x & 0 < x < 1 \\ u(0,t) = 0 = u(1,t) & t > 0 \end{array}$$

Take $c = 1$ and $\sigma = 1$ and experiment with different values of space step h and time step k. Compare your numerical solution to the exact solution, $u(x,t) = 100 \exp(-\pi^2 t) \sin \pi x$.

Exercise 19 *Graph of a solution of Exercise 18.*

Use **xmgr** to graph the results of one of your calculations from Exercise 18, plotting $u(x, T_j)$ for $0 < x < 1$ and five selected times, $T_1, T_2, ..., T_5$.

Exercise 20 *Use PlotMTV to display the results of Exercise 18 as a raster map.*

Exercise 21 *Prescribing the U-values of one or more FDE cells.*

Show that if the left side of the FDE Eq. (29) is zero in a cell, then no change in material content takes place in that cell. Argue that the left side of the FDE Eq. (29) can be made to be zero by either setting c to zero or by setting the difference of the U values at levels j and $j+1$ to zero. Thus, one or many FDE cells can be required to have prescribed U-values by setting appropriate entries of the c-array to zero.

Exercise 22 *The computational technique known as "blasting".*

In Eq. (29), show that, in the limit as c approaches infinity, the flux terms on the right side of the equation become negligible. Argue that if c is very large in a cell, then the U-value in that cell remains essentially constant from time j to time $j+1$. This computational technique of setting c to a very large value in a cell to maintain a fixed U-value in that cell is referred to as "blasting". Blasting is a very important tool in performance programming, since it can be used to eliminate IF statements.

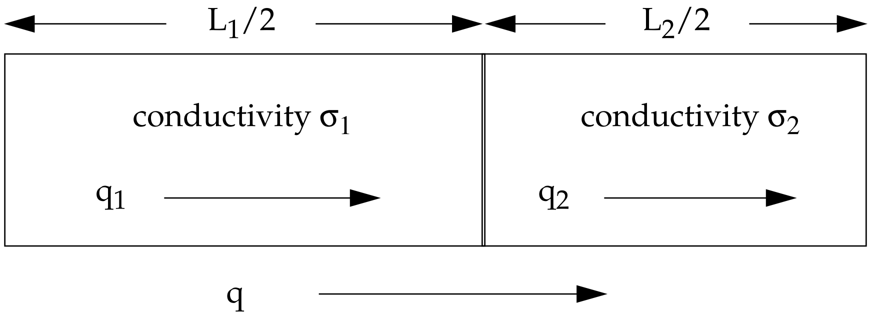

Figure 6: Conductivity.

Exercise 23 *An illustration of blasting.*

To illustrate the technique of blasting, consider the following IBVP

$$
\begin{array}{ll}
cu_t = \sigma u_{xx} & 0 < x < 1, \; t > 0 \\
u(x,0) = 10 & 0 < x < 1 \\
u(0,t) = 0 & t > 0 \\
u(1,t) = 100 & t > 0
\end{array}
$$

Take $\sigma = 1$ at all grid points and take $c = 1$ at all grid points except the middle point where c is "blasted" ($c = 10^{20}$, for example). With 51 interior grid points use the explicit, forward in time, method (with no IF statements) to approximate the solution to this problem. Use **xmgr** to plot your solution at early, intermediate, and late times.

Exercise 24 *Repeat Exercise 23 using the implicit, backward in time method.*

Exercise 25 *An illustration of a "no-flow cell."*

Show that, if harmonic averages are used to compute the fluxes across the faces of a finite difference cell, then that cell becomes a "no-flow cell" in the sense that it is isolated from the flux of material from neighbor cells.

Exercise 26 *An examination of heat flow.*

Consider steady, one dimensional heat flow through a composite medium with conductivity σ_1 on the left half of the medium and conductivity σ_2 on the right half of the medium. See Figure 6. By equating the fluxes q_1, q_2, and q, conclude that the effective conductivity σ of the composite medium is the harmonic average of σ_1 and σ_2.

Exercise 27 *Develop the backward in time FDE for the PDE* $cu_t = \sigma(u_{xx} + u_{yy})$.

Exercise 28 *Develop the Crank–Nicolson FDE for the PDE* $cu_t = \sigma(u_{xx} + u_{yy})$.

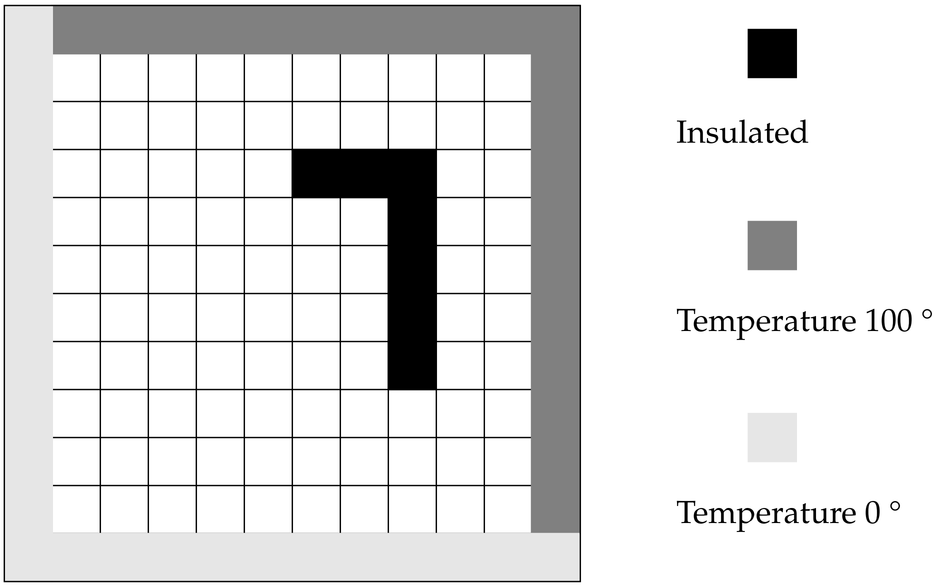

Figure 7: Steady state heat flow in a square sheet of metal.

Exercise 29 *Approximate solution to an IBVP and display of the solution.*

Given the IBVP

$$\begin{array}{ll} cu_t = \sigma(u_{xx} + u_{yy}) & 0 < x, y < 1, \;\; t > 0 \\ u(0, y, t) = 1; \;\; u(1, y, t) = 0 & 0 < y < 1 \quad t > 0 \\ u(x, 0, t) = 1; \;\; u(x, 1, t) = 0 & 0 < x < 1 \quad t > 0 \\ u(x, y, 0) = 0.5 & 0 < x, y < 1, \end{array}$$

use the explicit, forward in time, method with 25 interior grid points in each direction to approximate the solution. Take $c = 1$ and $\sigma = 1$ and compute until steady state (two decimal places) is attained. Use Ximage or plotMTV to display your solution at an early, an intermediate, and a late value of time.

Exercise 30 *A formulation of a system of finite difference equations using the explicit method.*

Given the IBVP of the previous exercise, use a 4x4 block centered finite grid to formulate a system of finite difference equations using the explicit method. Express these 16 FDEs in matrix notation. Repeat using the backward in time implicit method.

Exercise 31 *Developing a system of FDEs for an elliptic BVP.*

As t approaches infinity the solution to the IBVP of Exercise 30 approaches the solution of the following elliptic BVP:

$$\begin{array}{ll} u_{xx} + u_{yy} = 0 & 0 < x, y < 1 \\ u(0, y) = 1; \;\; u(1, y) = 0 & 0 < y < 1 \\ u(x, 0) = 1; \;\; u(x, 1) = 0 & 0 < x < 1 \end{array}$$

(a) Beginning with the solution of Exercise 30, develop a system of 16 FDEs for this BVP. (b) Use a conservation law argument to develop the same system of FDEs for this BVP. (c) Use centered difference formulas for u_{xx} and u_{yy} to develop the system of FDEs found in parts (a) and (b).

Exercise 32 *Write a system of FDEs for a heat flow problem, solve the system, and display the results.*

Consider steady heat flow in a square sheet of metal with an L-shaped insulator as depicted in the Figure 7. Let $\sigma = 1$ in the metal filled finite difference cells and $\sigma = 0$ in the insulated finite difference cell. Using harmonic averages write a system of FDEs for this heat flow problem and then use Gaussian elimination to solve the resulting FDEs. Use PlotMTV or Ximage to display your results.

Exercise 33 *Rework Exercise 32 with the assumption that the L-shaped shaded region has conductivity $\sigma = 1$ and is held at temperature 100°.*