One-Dimensional Special Relativistic Hydrodynamics Code

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- **G. R. effects to be connected to fast flows in different astrophysical scenarios,**
  1. **accreting compact objects**
  2. **stellar collapse**
  3. **coalescing compact binaries**
- **binary neutron stars:**
  1. **sources of gravitational waves**
  2. **G. R. effects in the binary orbit: high precision.**
  3. **gravitational waveform emitted during the inspiral and the final coalescence.**

general relativistic hydrodynamical equations in covariant form

$$\nabla_\mu T^{\mu\nu} = 0,$$

$$\nabla_\mu \mathrm{J}^\mu = 0.$$

$$T^{\mu\nu} = \rho h u^\mu u^\nu + P g^{\mu\nu} \Longrightarrow \text{stress energy tensor}$$

$$\mathrm{J}^\mu = \rho \mathrm{u}^\mu \Longrightarrow \text{current density}$$

$$\mathrm{h} = 1 + \epsilon + \frac{\mathrm{P}}{\rho} \Longrightarrow \text{specific enthalpy}$$

conservative form of equations

$$\frac{\partial \vec{\mathrm{U}}}{\partial \mathrm{t}} + \frac{\partial \vec{\mathrm{F}}^{\mathrm{i}}}{\partial \mathrm{x}^{\mathrm{i}}} = \vec{\mathrm{S}},$$

$\vec{\mathrm{U}} \Longrightarrow$ **vector of conserved quantities**

$\vec{\mathrm{F}}(\vec{\mathrm{U}}) \Longrightarrow$ **flux vectors**

$\vec{\mathrm{S}} \Longrightarrow$ **gravitational source**

$$\vec{\mathrm{U}} = \begin{pmatrix} \mathrm{D} \\ \mathrm{S_j} \\ \tau \end{pmatrix} = \begin{pmatrix} \sqrt{\gamma}\mathrm{W}\rho \\ \sqrt{\gamma}\rho\mathrm{h}\mathrm{W}^2\mathrm{v_j} \\ \sqrt{\gamma}(\rho\mathrm{h}\mathrm{W}^2 - \mathrm{P} - \mathrm{W}\rho) \end{pmatrix}$$

$$\vec{F}^i = \begin{pmatrix} \alpha(v^i - \frac{1}{\alpha}\beta^i)D \\ \alpha\{(v^i - \frac{1}{\alpha}\beta^i)S_j + \sqrt{\gamma}P\delta^i_j\} \\ \alpha\{(v^i - \frac{1}{\alpha}\beta^i)\tau + \sqrt{\gamma}v^iP\} \end{pmatrix}$$

$W = \alpha u^0 = (1 - \gamma_{ij}v^iv^j)^{-1/2} \Longrightarrow$ **Lorentz factor**

the source vector

$$\vec{S} = \begin{pmatrix} 0 \\ \alpha\sqrt{\gamma}T^{\mu\nu}g_{\nu\sigma}\Gamma^{\sigma}_{\mu j} \\ \alpha\sqrt{\gamma}(T^{\mu 0}\partial_{\mu}\alpha - \alpha T^{\mu\nu}\Gamma^{0}_{\mu\nu}) \end{pmatrix}$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_{\mu}g_{\nu\beta} + \partial_{\nu}g_{\mu\beta} - \partial_{\beta}g_{\mu\nu}).$$

the spectral decomposition of Jacobian matrix $\partial\vec{F}^i/\partial\vec{U}$, valid for a generic spatial metric

$$[\frac{\partial\vec{F}^x}{\partial\vec{U}}][\vec{r_i}] = \lambda_i[\vec{r_i}], \quad i = 1, ..., 5.$$

$\vec{r_i}$ five eigenvectors of system with eigenvalues λ_i

EIGENVALUES

$$\lambda_1 = \lambda_2 = \lambda_3 = \alpha v^x - \beta^x.$$

$$\lambda_\pm = \frac{\alpha}{1 - v^2 C_s^2}\{v^x(1 - C_s^2) \pm \sqrt{C_s^2(1 - v^2)[\gamma^{xx}(1 - v^2 C_s^2) - v^x v^x(1 - C_s^2)]}\} - \beta^x$$

EIGENVECTORS

$$\vec{r_1} = [\frac{\kappa}{hW(\kappa - \rho C_s^2)}, v_x, v_y, v_z, 1 - \frac{\kappa}{hW(\kappa - \rho C_s^2)}]^T$$

$$\vec{r_2} = [Wv_y, h(\gamma_{xy} + 2W^2 v_x v_y), h(\gamma_{yy} + 2W^2 v_y v_y), h(\gamma_{yz} + 2W^2 v_y v_z), v_y W(2Wh - 1)]^T$$

$$\vec{r_3} = [Wv_z, h(\gamma_{xz} + 2W^2 v_x v_z), h(\gamma_{yz} + 2W^2 v_y v_z), h(\gamma_{zz} + 2W^2 v_z v_z), v_z W(2Wh - 1)]^T$$

$$\vec{r_\pm} = [1, hW(v_x - \frac{v^x - (\lambda_\pm + \beta^x)/\alpha}{\gamma^{xx} - v^x(\lambda_\pm + \beta^x)/\alpha}), hWv_y, hWv_z, \frac{hW(\gamma^{xx} - v^x v^x)}{\gamma^{xx} - v^x(\lambda_\pm + \beta^x)/\alpha} - 1]^T$$

$$C_s = \left.\frac{\partial P}{\partial E}\right|_S. = \frac{\chi}{h} + \frac{P\kappa}{\rho^2 h},$$

where $\chi = \partial P/\partial\rho|_\epsilon$, $\kappa = \partial P/\partial\epsilon|_\rho$

for special relativity,

$$\alpha = 1,\ \beta^x = 0,\ \gamma_{ij} = \delta_{ij}$$

equation of state: $P = P(\rho, \epsilon) = \rho f(\epsilon)$ **for some function** $f(\epsilon)$

for flux splitting method

$$\vec{F}^x(\sigma\vec{U}) = \sigma\vec{F}^x(\vec{U})$$

$\sigma \Longrightarrow$ **any constant**

$$\vec{F}^x = \left(\frac{\partial\vec{F}^x}{\partial\vec{U}}\right)\vec{U}$$

the Jacobian matrix $\partial\vec{F}^x/\partial\vec{U}$

$$\frac{\partial\vec{F}^x}{\partial\vec{U}} = M\Lambda M^{-1}$$

$M \Longrightarrow$ **columns matrix of right eigenvectors**

$\Lambda \Longrightarrow$ **diagonal matrix constructed from the eigenvalues**

flux vector

$$\vec{F}^x = (\vec{F}^x)^+ + (\vec{F}^x)^- = (M\Lambda^+ M^{-1})\vec{U} + (M\Lambda^- M^{-1})\vec{U}$$

here $\Lambda^+ = \frac{1}{2}(\Lambda + |\Lambda|)$, and $\Lambda^- = \frac{1}{2}(\Lambda - |\Lambda|)$

hydrodynamical equations

$$\frac{\partial \vec{U}}{\partial t} + \frac{\partial \vec{F}^x}{\partial x} = 0$$

Discretizing this equation gives

$$\frac{\partial \vec{U}}{\partial t} + \frac{(\vec{f}^*)_{i+1/2} - (\vec{f}^*)_{i-1/2}}{\triangle x} = 0$$

first-order upwind flux

$$(\vec{f}^*_{i+1/2}) = (\vec{F}^x)^+_i + (\vec{F}^x)^-_{i+1}$$

conserved quantities:

$$\vec{U}^{n+1}_i = \vec{U}^n_i - \frac{\triangle t}{\triangle x}[(\vec{F}^x)^+_i + (\vec{F}^x)^-_{i+1} - ((\vec{F}^x)^+_{i-1} + (\vec{F}^x)^-_i)]^n$$

1) **Initial conditions:** ρ, v, P

2) **Initial conditions of conserved quantities:** D, S, τ, h, W, ϵ

3) **Begin the time loop:**

- $dt = \sigma \frac{dx}{|v+c_s|_{max}}$ $\sigma \Longrightarrow$ **Courant number**
- **boundary conditions: outflow, periodic**
- **calculate the Jacobian matrices**
- **update using flux splitting:** D, S^i, **and** τ
- **update:** $P = (\Gamma - 1)\rho\epsilon$
- **calculate the new values:** ρ, v,h, W, ϵ

4) **calculate the analytic solution: shock tube, periodic function**

5) $\boldsymbol{L_1}$ **error:**

$$\boldsymbol{L_1 = \frac{\Sigma_{i=1}^{npts} |AR(i) - NR(i)|}{npts}}$$

for perfect gas pressure

$$P = (\Gamma - 1)\rho\epsilon$$

$$\Gamma = 5/3 \Longrightarrow \text{adiabatic index}$$

computational domain $0 \leq x \leq 1$

FIRST TEST CASE

$\rho = \sin(2\pi x) + 2.0$, $P = 1$, **and** $v = 0.0$;

$$L_1 \sim O(\triangle x),\ L_1(npts)/L_1(2 * npts) \sim 2$$

FIGURES

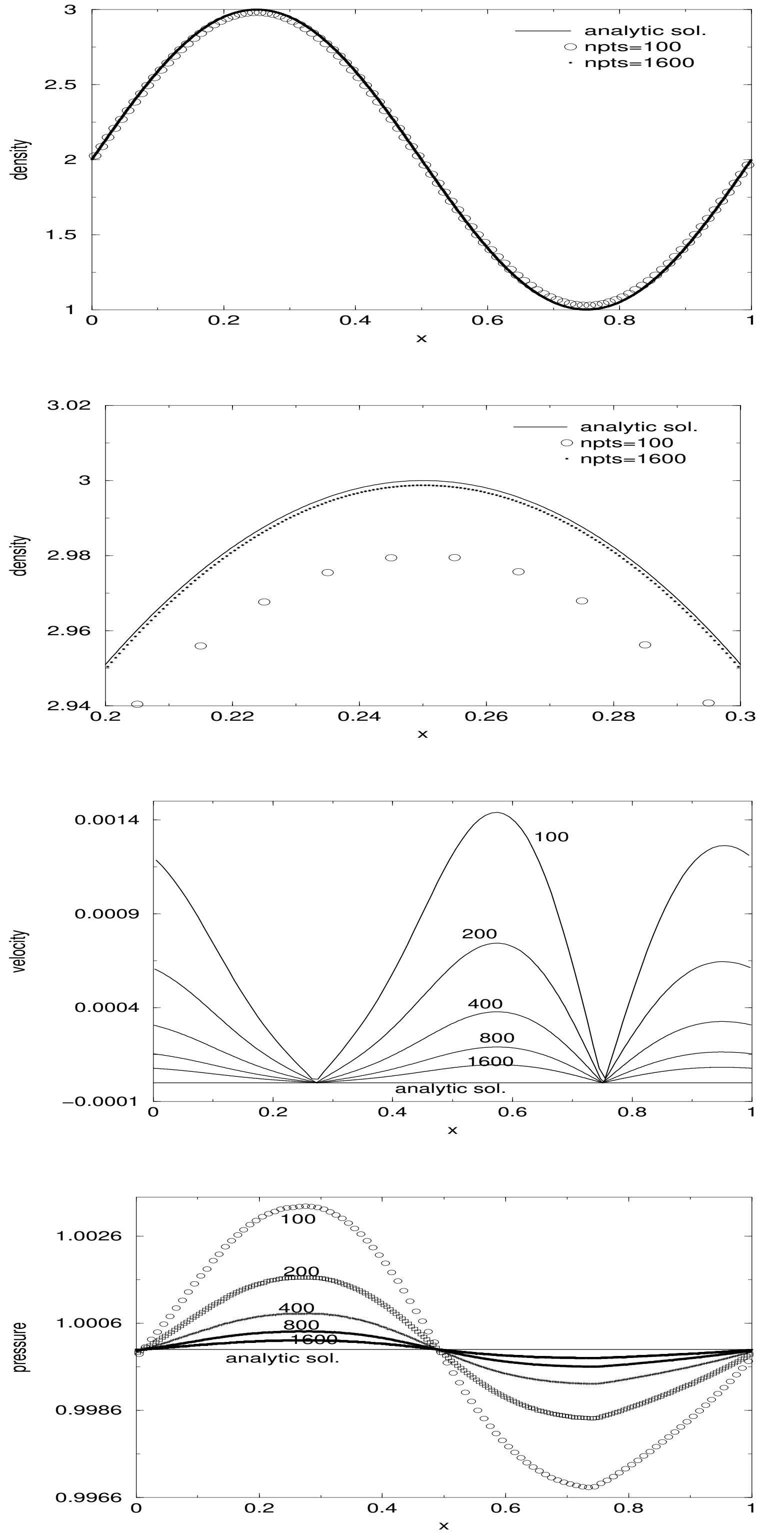

FIG. 1. The density, pressure and velocity profiles are shown for the case $\boldsymbol{\rho = \sin(2\pi x) + 2}$, $\boldsymbol{P = 1}$, $\boldsymbol{v = 0}$ at $\boldsymbol{t = 0.4}$.

$\boldsymbol{npts}$	$\boldsymbol{E(\rho)}$	$\boldsymbol{E(P)}$	$\boldsymbol{E(v)}$
100	1.598E-2	2.016E-3	7.671E-4
200	8.063E-3	1.013E-3	3.925E-4
400	4.050E-3	5.079E-4	1.986E-4
800	2.029E-3	2.545E-4	9.99E-5

$\boldsymbol{npts}$	$\boldsymbol{\frac{L1(npts)}{L1(2*npts)}(\rho)}$	$\boldsymbol{\frac{L1(npts)}{L1(2*npts)}(P)}$	$\boldsymbol{\frac{L1(npts)}{L1(2*npts)}(v)}$
100-200	1.98	1.99	1.95
200-400	1.99	1.99	1.97
400-800	1.99	1.995	1.98

for perfect gas pressure

$$\boldsymbol{P = (\Gamma - 1)\rho\epsilon}$$

$\boldsymbol{\Gamma = 5/3 \Longrightarrow}$ **adiabatic index**

computational domain $\boldsymbol{0 \leq x \leq 1}$

SECOND TEST CASE

$\boldsymbol{\rho = \sin(2\pi x) + 2.0}$, $\boldsymbol{P = 1}$, **and** $\boldsymbol{v = 0.4}$

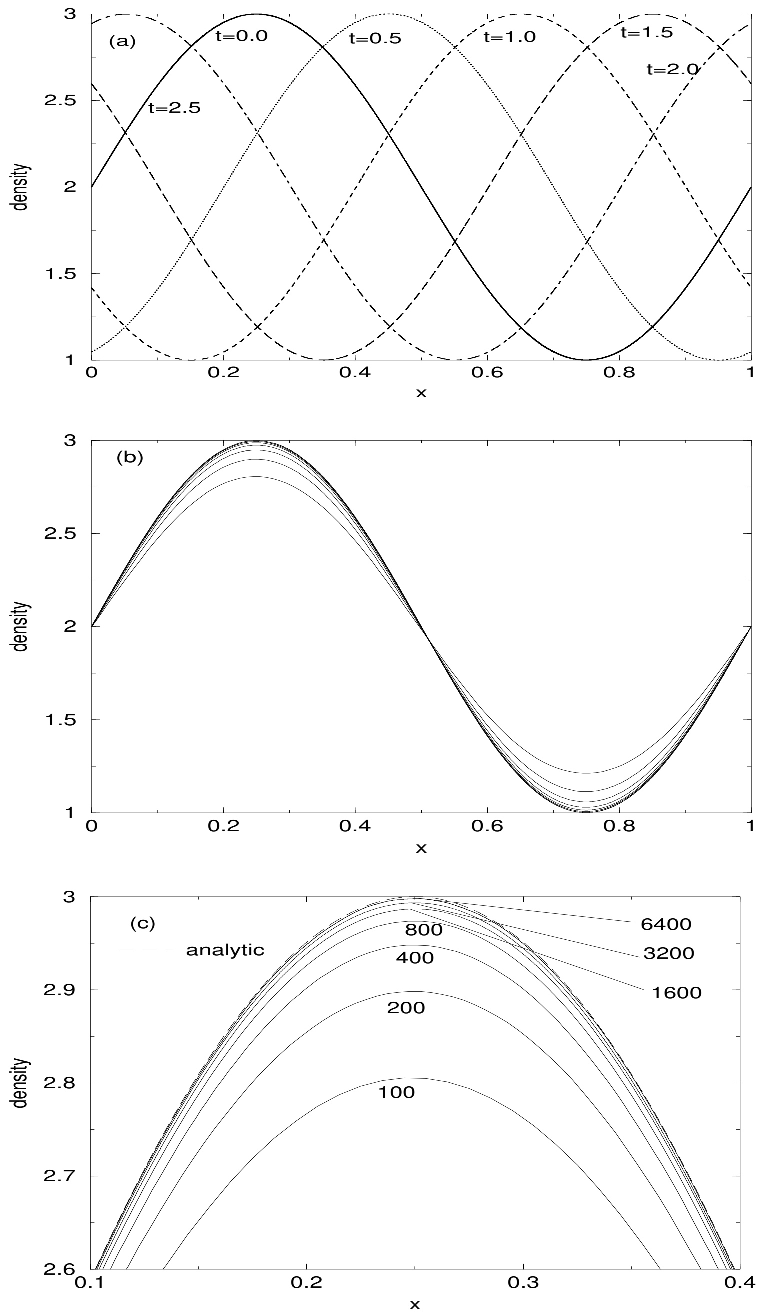

FIG. 2. (a) Density vs. computational domain for $\boldsymbol{npts = 6400}$ at different time steps. (b) Density profiles are shown for different resolutions, $\boldsymbol{npts}$ = 100, 200, 400, 800, 1600, 3200, and 6400. They are compared with the analytic result. (c) A zoomed-in view from (b) near $\boldsymbol{x = 0.25}$.

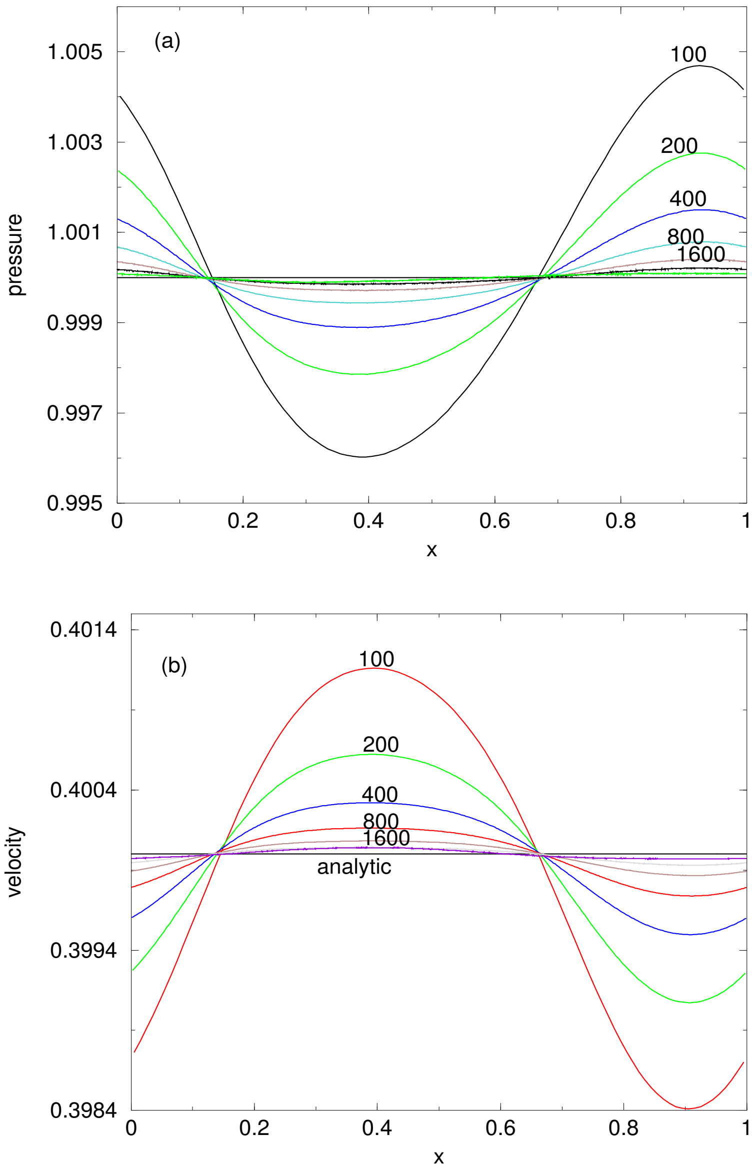

FIG. 3. (a) Pressure vs. computational domain for various resolutions at time t=2.5. (b) Velocity v.s. computational domain for various resolutions at time t=2.5.

$npts$	$E(\rho)$	$E(P)$	$E(v)$
100	0.129	2.767E-3	8.795E-4
200	6.832E-2	1.565E-3	4.972E-4
400	3.514E-2	8.336E-4	2.649E-4
800	1.782E-2	4.305E-4	1.367E-4
1600	8.978E-3	2.188E-4	6.953E-5
3200	4.563E-3	1.117E-4	3.523E-5

$npts$	$\frac{L1(npts)}{L1(2*npts)}(\rho)$	$\frac{L1(npts)}{L1(2*npts)}(P)$	$\frac{L1(npts)}{L1(2*npts)}(v)$
100-200	1.89	1.76	1.76
200-400	1.94	1.87	1.87
400-800	1.97	1.93	1.93
800-1600	1.98	1.96	1.96
1600-3200	1.96	1.95	1.97

for perfect gas pressure

$$P = (\Gamma - 1)\rho\epsilon$$

$\Gamma = 5/3 \Longrightarrow$ **adiabatic index**

computational domain $0 \leq x \leq 1$

THIRD TEST CASE

$P_L = 13.3$, $\rho_L = 10$, $v_L = 0$, **and**

$P_R = 0.66.10^{-6}$, $\rho_R = 1$, $v_R = 0$

$$r = \frac{\log E(\rho)|_{2*npts} - \log E(\rho)|_{npts}}{\log \triangle x|_{2*npts} - \log \triangle x|_{npts}},$$

$$\triangle x = 1/npts$$

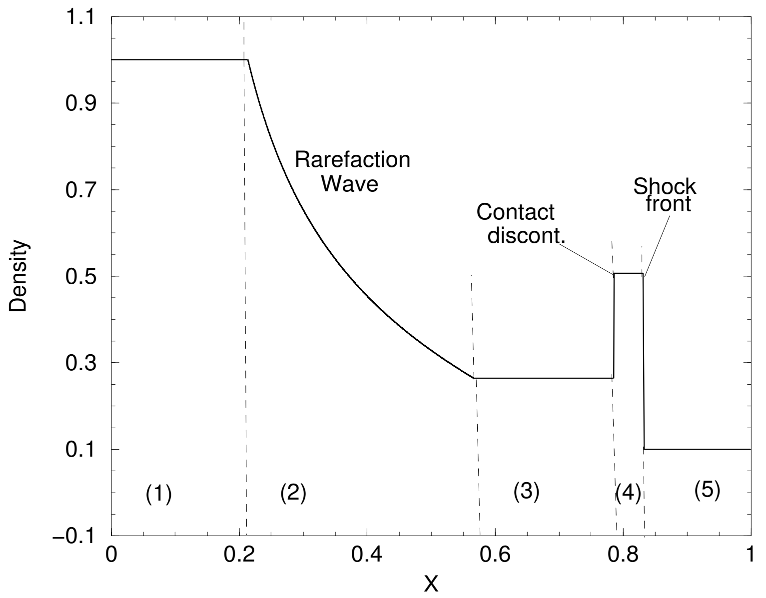

FIG. 4. Schematic representation of the density profile for the Riemann shock tube. We identify five regions: the undisturbed fluid (1) and (5), the rarefaction fan (2), the region bounded by the rarefaction tail and the contact discontinuity(3), and the shock plateau bounded by the contact discontinuity and the shock front (4). This figure is adapted from Ref.[10]

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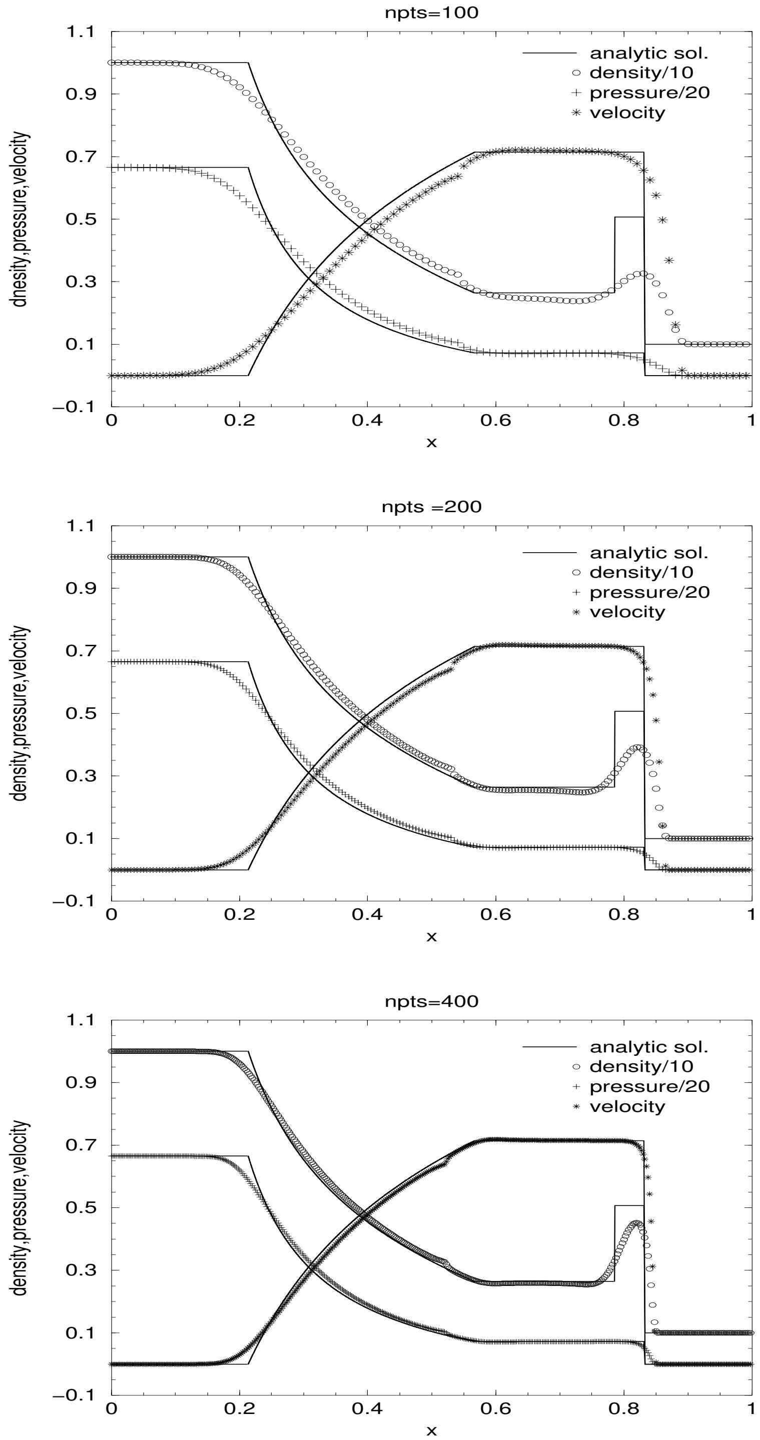

FIG. 5. Normalized profiles of density, pressure, and velocity vs. distance for the computed and exact (solid line) solution for time $t = 0.4$. All variables were calculated with a first-order flux splitting method on a uniform grid of npts zones.

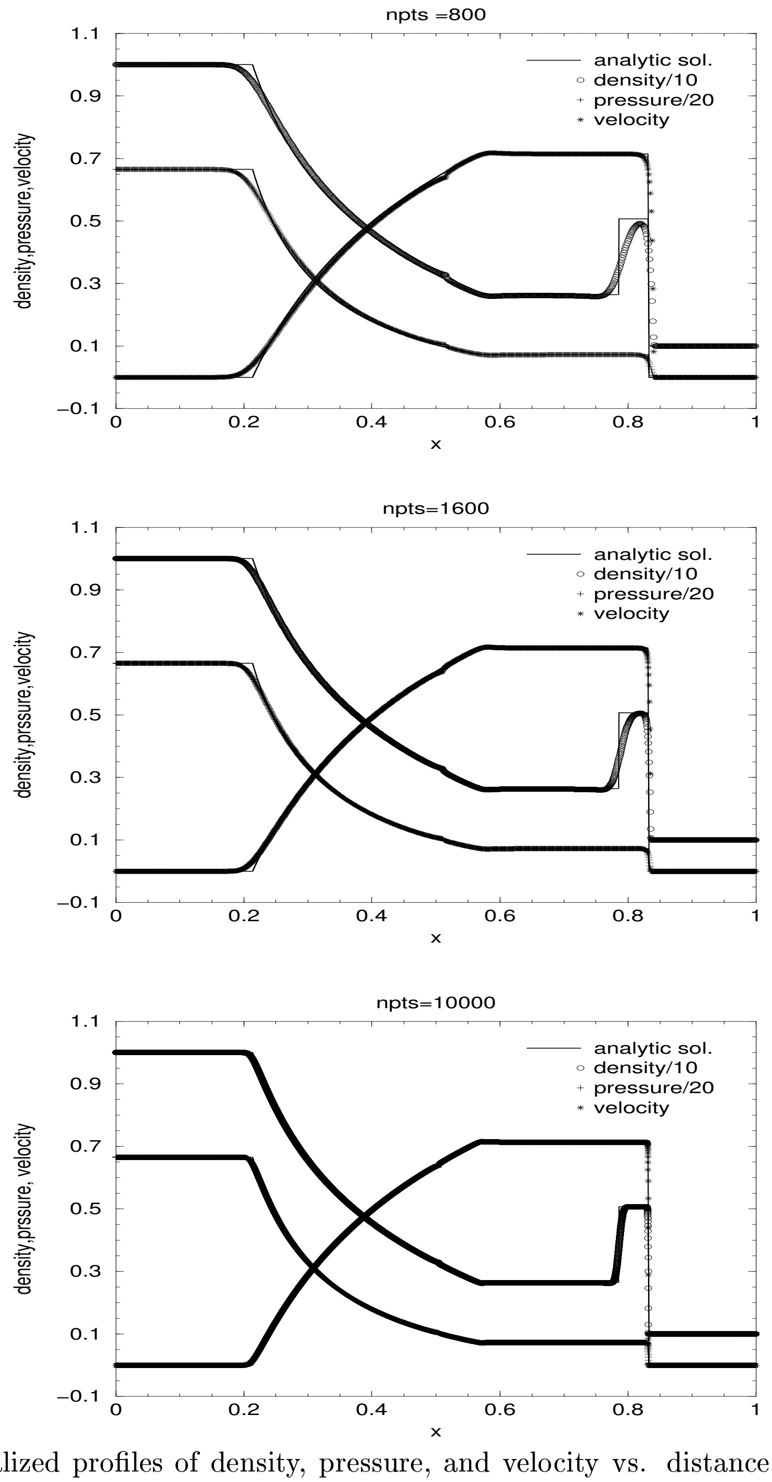

FIG. 6. Normalized profiles of density, pressure, and velocity vs. distance for the computed and exact (solid line) solution for time $\boldsymbol{t = 0.4}$. All variables were calculated with a first-order flux splitting method on a uniform grid of npts zones.

$\boldsymbol{npts}$	$\boldsymbol{E(\rho)}$	$\boldsymbol{E(P)}$	$\boldsymbol{E(v)}$
50	5.125E-01	5.136E-01	7.025E-02
100	3.728E-01	3.400E-01	4.257E-02
200	2.490E-01	2.202E-01	2.683E-02
400	1.632E-01	1.387E-01	1.626E-02
800	1.038E-01	8.550E-02	9.648E-03
1600	6.387E-02	5.145E-02	5.334E-03

$\boldsymbol{npts}$	$\boldsymbol{r(\rho)}$	$\boldsymbol{r(P)}$	$\boldsymbol{r(v)}$
50-100	0.459	0.595	0.722
100-200	0.582	0.626	0.665
200-400	0.609	0.666	0.722
400-800	0.653	0.698	0.753
800-1600	0.700	0.732	0.855

- **constructed a one-dimensional general relativistic hydrodynamics, code**

- **written in conservation form,**

- **discretized using a flux splitting method,**

- **test runs for the case of special relativistic flows,**

- **second-order flux splitting method**
- **use nonlinear connection terms, limiters,**
- **code will be extended to three dimensions and add general relativistic source terms**
- **use the other HRSC scheme such as Marquinas method**
- **Paramesh to implement parallelization**
- **run on parallel supercomputers**
- **the hydrodynamic code will be coupled: Drexel relativity group is developing to solve the Einstein equations,**
- **study coalescing compact binaries and the gravitational radiation.**

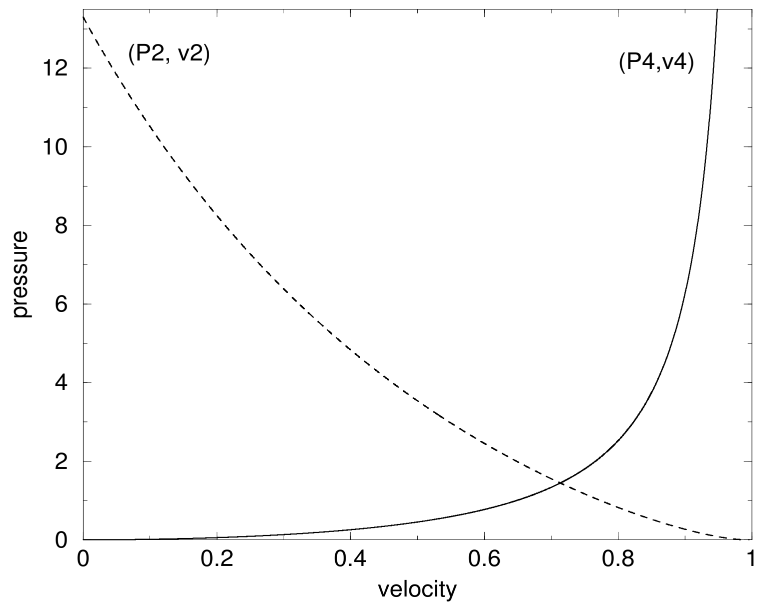

FIG. 7. The curves for $\boldsymbol{P(v)}$ according to Eq.(45) (solid line) and the rarefaction wave (dashed line) are shown.